A primitive derivation and logarithmic differential forms of Coxeter arrangements

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Abstract

Let W be a finite irreducible real reflection group, which is a Coxeter group. We explicitly construct a basis for the module of differential 1-forms with logarithmic poles along the Coxeter arrangement by using a primitive derivation. As a consequence, we extend the Hodge filtration, indexed by nonnegative integers, into a filtration indexed by all integers. This filtration coincides with the filtration by the order of poles. The results are translated into the derivation case.

1 Introduction and main results

Let V be a Euclidean space of dimension ℓ. Let W be a finite irreducible reflection group (a Coxeter group) acting on V. The **Coxeter arrangement** $\mathcal{A} = \mathcal{A}(W)$ corresponding to W is the set of reflecting hyperplanes. We use [5] as a general reference for arrangements. For each $H \in \mathcal{A}$, choose a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Their product $Q := \prod_{H \in \mathcal{A}} \alpha_H$, which lies in the symmetric algebra $S := \mathrm{Sym}(V^*)$, is a defining polynomial for $\mathcal{A}$. Let $F := S_{(0)}$ be the quotient field of S. Let Ω_S and Ω_F denote the S-module of regular 1-forms on V and the F-vector space of rational 1-forms on V respectively. The action of W on V induces the canonical actions of W on V^*, S, F, Ω_S and Ω_F, which enable us to consider their W-invariant parts. Especially let $R = S^W$ denote the invariant subring of S.

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In [16], Ziegler introduced the S-module of **logarithmic 1-forms** with poles of order m ($m \in \mathbb{Z}_{\geq 0}$) along $\mathcal{A}$ by

$$\Omega(\mathcal{A}, m) := \{\omega \in \Omega_F \mid Q^m \omega \text{ and } (Q/\alpha_H)^m (d\alpha_H \wedge \omega) \text{ are both regular for all } H \in \mathcal{A}\}.$$

Note $\Omega(\mathcal{A}, 0) = \Omega_S$. Define the total module of logarithmic 1-forms by

$$\Omega(\mathcal{A}, \infty) := \bigcup_{m \geq 0} \Omega(\mathcal{A}, m).$$

In this article we study the total module $\Omega(\mathcal{A}, \infty)$ of logarithmic 1-forms and its W-invariant part $\Omega(\mathcal{A}, \infty)^W$ by introducing a geometrically-defined filtration indexed by $\mathbb{Z}$.

Let $P_1, \cdots, P_\ell \in R$ be algebraically independent homogeneous polynomials with $\deg P_1 \leq \cdots \leq \deg P_\ell$, which are called **basic invariants**, such that $R = \mathbb{R}[P_1, \cdots, P_\ell]$ [3, V.5.3, Theorem 3]. Define the **primitive derivation** $D := \partial/\partial P_\ell : F \to F$. Let $T := \{f \in R \mid Df = 0\} = \mathbb{R}[P_1, P_2, \ldots, P_{\ell-1}]$. Consider the T-linear connection (covariant derivative)

$$\nabla_D : \Omega_F \to \Omega_F$$

characterized by $\nabla_D(f\omega) = (Df)\omega + f(\nabla_D \omega)$ $(f \in F, \omega \in \Omega_F)$ and $\nabla_D(d\alpha) = 0$ $(\alpha \in V^*)$.

In Section 2, using the primitive derivation D, we explicitly construct logarithmic 1-forms

$$\omega_1^{(m)}, \omega_2^{(m)}, \ldots, \omega_\ell^{(m)}$$

for each $m \in \mathbb{Z}$ satisfying $\nabla_D \omega_j^{(2k+1)} = \omega_j^{(2k-1)}$ $(k \in \mathbb{Z}, 1 \leq j \leq \ell)$. The 1-forms $\omega_1^{(m)}, \ldots, \omega_\ell^{(m)}$ form a basis for the S-module $\Omega(\mathcal{A}, -m)$ when $m \leq 0$. Thus it is natural to define $\Omega(\mathcal{A}, -m)$ to be the S-module spanned by $\{\omega_1^{(m)}, \omega_2^{(m)}, \ldots, \omega_\ell^{(m)}\}$ for all $m \in \mathbb{Z}$. Let $\mathcal{B}_k := \{\omega_1^{(2k+1)}, \omega_2^{(2k+1)}, \ldots, \omega_\ell^{(2k+1)}\}$ for $k \in \mathbb{Z}$. The following two main theorems will be proved in Section 2:

Theorem 1.1

(1) The R-module $\Omega(\mathcal{A}, 2k-1)^W$ is free with a basis $\mathcal{B}_{-k}$ for $k \in \mathbb{Z}$.

(2) The T-module $\Omega(\mathcal{A}, 2k-1)^W$ is free with a basis $\bigcup_{p \geq -k} \mathcal{B}_p$ for $k \in \mathbb{Z}$.

(3) $\mathcal{B} := \bigcup_{k \in \mathbb{Z}} \mathcal{B}_k$ is a basis for $\Omega(\mathcal{A}, \infty)^W$ as a T-module.

Theorem 1.2

(1) The ∇_D induces a T-linear automorphism $\nabla_D : \Omega(\mathcal{A}, \infty)^W \xrightarrow{\sim} \Omega(\mathcal{A}, \infty)^W$.

(2) Define $\mathcal{F}_0 := \bigoplus_{j=1}^{\ell} T\,(dP_j)$, $\mathcal{F}_{-k} := \nabla_D^k \mathcal{F}_0$ and $\mathcal{F}_k := (\nabla_D^{-1})^k \mathcal{F}_0$ $(k > 0)$. Then $\Omega(\mathcal{A}, \infty)^W = \bigoplus_{k \in \mathbb{Z}} \mathcal{F}_k$.

(3) $\Omega(\mathcal{A}, 2k-1)^W = \mathcal{J}^{(-k)}$, where $\mathcal{J}^{(-k)} := \bigoplus_{p \geq -k} \mathcal{F}_p$ for $k \in \mathbb{Z}$.

Let us briefly discuss our results in connection with earlier researches. Let Der_F denote the F-vector space of $\mathbb{R}$-linear derivations of F to itself. It is dual to Ω_F. The inner product $I : V \times V \to \mathbb{R}$ induces $I^* : V^* \times V^* \to \mathbb{R}$, which is canonically extended to a nondegenerate F-bilinear form $I^* : \Omega_F \times \Omega_F \to F$. Define an F-linear isomorphism

$$I^* : \Omega_F \to \mathrm{Der}_F$$

by $I^*(\omega)(f) := I^*(\omega, df)$ $(f \in F)$. Let $\mathcal{G}_k := I^*(\mathcal{F}_{k-1})$ and $\mathcal{H}^{(k)} := I^*(\mathcal{J}^{(k-1)})$ for $k \in \mathbb{Z}$. Thanks to Theorem 1.2, we have commutative diagrams

$$\begin{array}{ccccccccccccccc}
\cdots & \xrightarrow{\nabla_D} & \mathcal{F}_1 & \xrightarrow{\nabla_D} & \mathcal{F}_0 & \xrightarrow{\nabla_D} & \mathcal{F}_{-1} & \xrightarrow{\nabla_D} & \mathcal{F}_{-2} & \xrightarrow{\nabla_D} & \mathcal{F}_{-3} & \xrightarrow{\nabla_D} & \mathcal{F}_{-4} & \xrightarrow{\nabla_D} & \cdots \\
 & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & \\
\cdots & \xrightarrow{\nabla_D} & \mathcal{G}_2 & \xrightarrow{\nabla_D} & \mathcal{G}_1 & \xrightarrow{\nabla_D} & \mathcal{G}_0 & \xrightarrow{\nabla_D} & \mathcal{G}_{-1} & \xrightarrow{\nabla_D} & \mathcal{G}_{-2} & \xrightarrow{\nabla_D} & \mathcal{G}_{-3} & \xrightarrow{\nabla_D} & \cdots ,
\end{array}$$

$$\begin{array}{ccccccccccccccc}
\cdots & \xrightarrow{\nabla_D} & \mathcal{J}^{(1)} & \xrightarrow{\nabla_D} & \mathcal{J}^{(0)} & \xrightarrow{\nabla_D} & \mathcal{J}^{(-1)} & \xrightarrow{\nabla_D} & \mathcal{J}^{(-2)} & \xrightarrow{\nabla_D} & \mathcal{J}^{(-3)} & \xrightarrow{\nabla_D} & \mathcal{J}^{(-4)} & \xrightarrow{\nabla_D} & \cdots \\
 & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow & & \\
\cdots & \xrightarrow{\nabla_D} & \mathcal{H}^{(2)} & \xrightarrow{\nabla_D} & \mathcal{H}^{(1)} & \xrightarrow{\nabla_D} & \mathcal{H}^{(0)} & \xrightarrow{\nabla_D} & \mathcal{H}^{(-1)} & \xrightarrow{\nabla_D} & \mathcal{H}^{(-2)} & \xrightarrow{\nabla_D} & \mathcal{H}^{(-3)} & \xrightarrow{\nabla_D} & \cdots .
\end{array}$$

in which every ∇_D is a T-linear isomorphism. The objects in the left halves of the diagrams were introduced by K. Saito who called the decomposition $\mathrm{Der}_R = \bigoplus_{k \geq 0} \mathcal{G}_k$ the **Hodge decomposition** and the filtration $\mathrm{Der}_R = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \dots$ the **Hodge filtration** in his groundbreaking work [7, 8]. They are the key to define the flat structure on the orbit space V/W. The flat structure is also called the Frobenius manifold structure from the view point of topological field theory [4].

Our main theorems 1.1 and 1.2 are naturally translated by I^* into the corresponding results concerning the $\mathcal{G}_k$'s and the $\mathcal{H}^{(k)}$'s in Section 3. So we extend the Hodge decomposition and Hodge filtration, **indexed by nonnegative integers**, to the ones **indexed by all integers.** The Hodge filtration $\mathrm{Der}_R = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \dots$ was proved to be equal to the contact-order filtration [13]. On the other hand, Theorem 1.2 (3) asserts that the filtration $\cdots \supset \mathcal{J}^{(-1)} \supset \mathcal{J}^{(0)} = \Omega_R$, indexed by nonpositive integers, coincides with the **pole-order filtration** of the W-invariant part $\Omega(\mathcal{A}, \infty)^W$ of the total module $\Omega(\mathcal{A}, \infty)$ of logarithmic 1-forms. This direction of researches is related with a generalized multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ and the associated logarithmic module $D\Omega(\mathcal{A}, \mathbf{m})$ introduced in [1].

In Section 4, we will give explicit relations of our bases to the bases obtained in [11], [15] and [2].

2 Construction of a basis for $\Omega(\mathcal{A},\infty)$

Let $x_1,\dots,x_\ell$ denote a basis for V^* and $P_1,\dots,P_\ell$ homogeneous basic invariants with $\deg P_1 \le \cdots \le \deg P_\ell$: $S^W = R = \mathbb{R}[P_1,\dots,P_\ell]$. Let $\mathbf{x} := [x_1,\dots,x_\ell]$ and $\mathbf{P} := [P_1,\dots,P_\ell]$ be the corresponding row vectors. Define $A := [I^*(x_i,x_j)]_{1\le i,j\le \ell} \in \mathrm{GL}_\ell(\mathbb{R})$ and $G := [I^*(dP_i,dP_j)]_{1\le i,j\le\ell} \in \mathrm{M}_{\ell,\ell}(R)$. Then $G = J(\mathbf{P})^T A J(\mathbf{P})$, where $J(\mathbf{P}) := \left[\dfrac{\partial P_j}{\partial x_i}\right]_{1\le i,j\le\ell}$ is the Jacobian matrix. It is well-known (e.g., [3, V.5.5, Prop. 6]) that $\det J(\mathbf{P}) \doteq Q$, where $\doteq$ stands for the equality up to a nonzero constant multiple. Let Der_R be the R-module of $\mathbb{R}$-linear derivations of R to itself: $\mathrm{Der}_R = \oplus_{i=1}^{\ell} R\,(\partial/\partial P_i)$. Recall the primitive derivation $D = \partial/\partial P_\ell \in \mathrm{Der}_R$ and $T = \ker(D : R \to R) = \mathbb{R}[P_1,\dots,P_{\ell-1}]$. We will use the notation $D[M] := [D(m_{ij})]_{1\le i,j\le\ell}$ for a matrix $M = [m_{ij}]_{1\le i,j\le\ell} \in \mathrm{M}_{\ell,\ell}(F)$. The next Proposition is due to K. Saito [7, (5.1)] [4, Corollary 4.1]:

Proposition 2.1
$D[G] \in GL_\ell(T)$, that is, $D^2[G] = 0$ and $\det D[G] \in \mathbb{R}^\times$.

Now let us give a key definition of this article, which generalizes the matrices introduced in [11, Lemma 3.3].

Definition 2.2
The matrices $B = B^{(1)}$ and $B^{(k)}$ ($k \in \mathbb{Z}$) are defined by

$$B := J(\mathbf{P})^T A D[J(\mathbf{P})], \quad B^{(k)} := kB + (k-1)B^T.$$

In particular, $D[G] = B + B^T = B^{(k+1)} - B^{(k)}$ for all $k \in \mathbb{Z}$.

Lemma 2.3
$B^{(k)} \in GL_\ell(T)$ for all $k \in \mathbb{Z}$, that is, $D\left[B^{(k)}\right] = 0$ and $\det B^{(k)} \in \mathbb{R}^\times$.

Proof. If $k \ge 1$, then the statement is proved in [11, 3.3 and 3.6] and [13, Lemma 2]. Suppose $k \le 0$. Since

$$B^{(1-k)} = (1-k)B + (-k)B^T = -\{kB + (k-1)B^T\}^T = -(B^{(k)})^T,$$

we obtain $B^{(k)} = -(B^{(1-k)})^T \in \mathrm{GL}_\ell(T)$ because $1 - k \ge 1$. □

The following Lemma is in [11, pp. 670, Lemma 3.4 (iii)]:

Lemma 2.4
(1) $\det J(D^k[\mathbf{x}]) \doteq Q^{-2k}$, *where* $J(D^k[\mathbf{x}]) := \left[\partial D^k(x_j)/\partial x_i\right]_{1\le i,j\le\ell}$ $(k \ge 1)$.
(2) $D[J(\mathbf{P})] = -J(D[\mathbf{x}])J(\mathbf{P})$ *and thus* $\det D[J(\mathbf{P})] \doteq Q^{-1}$.

Definition 2.5
Define $\{R_k\}_{k\in\mathbb{Z}} \subset M_{\ell,\ell}(F)$ *by*

$$\begin{aligned} R_{1-2k} &:= D^k[J(\mathbf{P})] \ (k \geq 0), \\ R_{2k-1} &:= (-1)^k J(D^k[\mathbf{x}])^{-1} D[J(\mathbf{P})] \ (k \geq 1), \\ R_{2k} &:= (-1)^k J(D^k[\mathbf{x}])^{-1} \ (k \geq 0), \\ R_{-2k} &:= D^{k+1}[J(\mathbf{P})] D[J(\mathbf{P})]^{-1} \ (k \geq 0). \end{aligned}$$

In particular, $R_1 = J(\mathbf{P})$, $R_0 = I_\ell$ *and* $R_{-1} = D[J(\mathbf{P})]$.

The following Proposition is fundamental.

Proposition 2.6
For $k \in \mathbb{Z}$, *we have*
(1) $\det R_k \doteq Q^k$,
(2) $R_{2k} = R_{2k-1} D[J(\mathbf{P})]^{-1} = R_{2k-1} B^{-1} J(\mathbf{P})^T A$,
(3) $R_{2k+1} = R_{2k} J(\mathbf{P}) (B^{(k+1)})^{-1} B$,
(4) $R_{2k+1} = R_{2k-1} B^{-1} G (B^{(k+1)})^{-1} B$, *and*
(5) $D[R_{2k+1}] = R_{2k-1}$.

Proof. (2) is immediate from Definition 2.5 because $B^{-1} J(\mathbf{P})^T A = D[J(\mathbf{P})]^{-1}$.

(4) Let $k \geq 1$. Recall the original definition of $B^{(k)}$ in [11, Lemma 3.3] given by

$$B^{(k+1)} = -J(\mathbf{P})^T A J(D^{k+1}[\mathbf{x}]) J(D^k[\mathbf{x}])^{-1} J(\mathbf{P}).$$

Compute

$$\begin{aligned} R_{2k-1}^{-1} R_{2k+1} &= -D[J(\mathbf{P})]^{-1} J(D^k[\mathbf{x}]) J(D^{k+1}[\mathbf{x}])^{-1} D[J(\mathbf{P})] \\ &= -D[J(\mathbf{P})]^{-1} A^{-1} J(\mathbf{P})^{-T} J(\mathbf{P})^T A J(\mathbf{P}) J(\mathbf{P})^{-1} \\ &\qquad J(D^k[\mathbf{x}]) J(D^{k+1}[\mathbf{x}])^{-1} A^{-1} J(\mathbf{P})^{-T} J(\mathbf{P})^T A D[J(\mathbf{P})] \\ &= B^{-1} G (B^{(k+1)})^{-1} B. \end{aligned}$$

Next we will show that

$$D^{k+1}[J(\mathbf{P})] = D^k[J(\mathbf{P})] B^{-1} B^{(1-k)} G^{-1} B$$

for $k \geq 0$ by an induction on k. When $k = 0$ we have

$$J(\mathbf{P}) B^{-1} B^{(1)} G^{-1} B = J(\mathbf{P}) J(\mathbf{P})^{-1} A^{-1} J(\mathbf{P})^{-T} J(\mathbf{P})^T A D[J(\mathbf{P})] = D[J(\mathbf{P})].$$

Next assume $k > 0$. Compute

$$\begin{aligned} D^{k+1}[J(\mathbf{P})] &= D[D^k[J(\mathbf{P})]] = D[D^{k-1}[J(\mathbf{P})] B^{-1} B^{(2-k)} G^{-1} B] \\ &= D^k[J(\mathbf{P})] B^{-1} B^{(2-k)} G^{-1} B + D^{k-1}[J(\mathbf{P})] B^{-1} B^{(2-k)} D[G^{-1}] B \\ &= D^k[J(\mathbf{P})] B^{-1} \{B^{(2-k)} - D[G]\} G^{-1} B \\ &= D^k[J(\mathbf{P})] B^{-1} B^{(1-k)} G^{-1} B, \end{aligned}$$

where, in the above, we used the induction hypothesis

$$D^k[J(\mathbf{P})] = D^{k-1}[J(\mathbf{P})]B^{-1}B^{(2-k)}G^{-1}B,$$

a general formula

$$D[G^{-1}] = -G^{-1}D[G]G^{-1}$$

and

$$D[G] = B + B^T = B^{(2-k)} - B^{(1-k)}.$$

This implies $R_{-2k-1} = R_{-2k+1}B^{-1}B^{(1-k)}G^{-1}B$ which proves (4).

(3) follows from (2) and (4) because $G = J(\mathbf{P})^T AJ(\mathbf{P})$.

(1) Since $\det B^{(k)} \in \mathbb{R}^\times$, $\det J(D^k[\mathbf{x}]) \doteq Q^{-2k}$ and $\det D[J(\mathbf{P})] \doteq Q^{-1}$ by Lemma 2.3 and Lemma 2.4, (1) is proved.

(5) follows from the following computation:

$$\begin{aligned}
D[R_{2k+1}]B^{-1} &= D[R_{2k+1}B^{-1}] = D[R_{2k-1}B^{-1}G(B^{(k+1)})^{-1}] \\
&= \{D[R_{2k-1}]B^{-1}G + R_{2k-1}B^{-1}D[G]\}(B^{(k+1)})^{-1} \\
&= \{R_{2k-3}B^{-1}G + R_{2k-1}B^{-1}(B^{(k+1)} - B^{(k)})\}(B^{(k+1)})^{-1} \\
&= \{R_{2k-1}B^{-1}B^{(k)} + R_{2k-1}B^{-1}(B^{(k+1)} - B^{(k)})\}(B^{(k+1)})^{-1} \\
&= R_{2k-1}B^{-1}.
\end{aligned}$$

□

Definition 2.7
For $m \in \mathbb{Z}$ define $\omega_1^{(m)}, \ldots, \omega_\ell^{(m)} \in \Omega_F$ by

$$[\omega_1^{(m)}, \ldots, \omega_\ell^{(m)}] := [dx_1, \ldots, dx_\ell]R_m.$$

When $m = 2k+1$ $(k \in \mathbb{Z})$, let

$$\mathcal{B}_k := \{\omega_1^{(2k+1)}, \ldots, \omega_\ell^{(2k+1)}\}.$$

For example, $\omega_j^{(1)} = dP_j$ for $1 \le j \le \ell$ and $\mathcal{B}_0 = \{dP_1, \ldots, dP_\ell\}$ because

$$[\omega_1^{(1)}, \ldots, \omega_\ell^{(1)}] = [dx_1, \ldots, dx_\ell]J(\mathbf{P}) = [dP_1, \ldots, dP_\ell].$$

Proposition 2.8
The subset

$$\mathcal{B} := \bigcup_{k\in\mathbb{Z}} \mathcal{B}_k = \{\omega_j^{(2k+1)} \mid 1 \le j \le \ell,\ k \in \mathbb{Z}\}$$

of Ω_F is linearly independent over T.

Proof. Assume

$$\sum_{k\in\mathbb{Z}}[\omega_1^{(2k+1)},\ldots,\omega_\ell^{(2k+1)}]\mathbf{g}^{(2k+1)}=0$$

with $\mathbf{g}^{(2k+1)}=[g_1^{(2k+1)},\ldots g_\ell^{(2k+1)}]^T\in T^\ell,\ k\in\mathbb{Z}$ such that there exist integers d and e such that $d\geq e$, $\mathbf{g}^{(2d+1)}\neq 0, \mathbf{g}^{(2e+1)}\neq 0$ and $\mathbf{g}^{(2k+1)}=0$ for all $k>d$ and $k<e$. Then

$$0=\sum_{k=e}^{d}[dx_1,\ldots,dx_\ell]R_{2k+1}\mathbf{g}^{(2k+1)}$$

implies that

$$0=\sum_{k=e}^{d}R_{2k+1}\mathbf{g}^{(2k+1)}.$$

By Proposition 2.6 (4), there exist $(\ell\times\ell)$-matrices H_{2k+1} $(e\leq k\leq d)$ such that

$$R_{2k+1}=R_{2e+1}H_{2k+1}\ (e\leq k\leq d)$$

and H_{2k+1} can be expressed as a product of $(k-e)$ copies of G and matrices belonging to $\mathrm{GL}_\ell(T)$. Since $\det(R_{2e+1})\neq 0$ by Proposition 2.6 (1),

$$0=\sum_{k=e}^{d}H_{2k+1}\mathbf{g}^{(2k+1)}.$$

Note $D^{d-e}[H_{2k+1}]=0$ $(k<d)$ by Proposition 2.1 and Lemma 2.3. Applying D^{d-e} to the above, we thus obtain

$$D^{d-e}[H_{2d+1}]\mathbf{g}^{(2d+1)}=0.$$

Since the matrix $D^{d-e}[H_{2d+1}]$, which is a product of $(d-e)$ copies of $D[G]$ and matrices in $\mathrm{GL}_\ell(T)$, is nondegenerate, we get $\mathbf{g}^{(2d+1)}=0$, which is a contradiction. □

Proposition 2.9
$\nabla_D\,\omega_j^{(2k+1)}=\omega_j^{(2k-1)}$ $(k\in\mathbb{Z},\ 1\leq j\leq\ell)$.

Proof. By Proposition 2.6 (5) we have

$$\left[\nabla_D\,\omega_1^{(2k+1)},\ldots,\nabla_D\,\omega_\ell^{(2k+1)}\right]=[dx_1,\ldots,dx_\ell]\,D[R_{2k+1}]$$
$$=[dx_1,\ldots,dx_\ell]\,R_{2k-1}=\left[\omega_1^{(2k-1)},\ldots,\omega_\ell^{(2k-1)}\right].\quad\square$$

Recall

$$\begin{aligned}\Omega(\mathcal{A},\infty) : &= \bigcup_{m\geq 0} \Omega(\mathcal{A},m) \\ &= \{\omega \in \Omega_F \mid Q^m\omega \in \Omega_S \text{ for some } m>0 \text{ and} \\ &\qquad\qquad d\alpha_H \wedge \omega \text{ is regular at generic points on } H \\ &\qquad\qquad \text{for each } H \in \mathcal{A}\}.\end{aligned}$$

Lemma 2.10
$\nabla_D(\Omega(\mathcal{A},m)^W) \subseteq \Omega(\mathcal{A},m+2)^W$ *for* $m>0$.

Proof. Choose $H \in \mathcal{A}$ arbitrarily and fix it. Pick an orthonormal basis $\alpha_H = x_1, x_2, \ldots, x_\ell$ for V^*. Let $s = s_H \in W$ be the orthogonal reflection through H. Then $s(x_1) = -x_1$, $s(x_i) = x_i$ $(i \geq 2)$, $s(Q) = -Q$. Let

$$\omega = \sum_{i=1}^{\ell} (f_i/Q^m)dx_i \in \Omega(\mathcal{A},m)^W$$

with each $f_i \in S$. Then

$$\nabla_D\,\omega = \sum_{i=1}^{\ell} D(f_i/Q^m)dx_i$$

is W-invariant with poles of order $m+2$ at most. The 2-form

$$(Q/x_1)^m dx_1 \wedge \omega = \sum_{i=2}^{\ell} (f_i/x_1^m)dx_1 \wedge dx_i$$

is regular because $\omega \in \Omega(\mathcal{A},m)^W$. Let $i \geq 2$. Then $f_i \in x_1^m S$. This implies that $g_i := Q^{m+2}D(f_i/Q^m) \in x_1^{m+1}S$. It is enough to show $g_i \in x_1^{m+2}S$ because

$$(Q/x_1)^{m+2}dx_1 \wedge \nabla_D\,\omega = \sum_{i=2}^{\ell} (g_i/x_1^{m+2})dx_1 \wedge dx_i.$$

When m is odd, we have $s(g_i) = s(Q^{m+2}D(f_i/Q^m)) = -g_i$. Thus $g_i \in x_1^{m+2}S$. When m is even, we have $s(g_i) = s(Q^{m+2}D(f_i/Q^m)) = g_i$. Thus $g_i \in x_1^{m+2}S$. □

Lemma 2.11
$\mathcal{B}_{-k} \subset \Omega(\mathcal{A},2k-1)^W$ *for* $k \geq 1$.

Proof. We will show by an induction on k. Fix $1 \leq j \leq \ell$. Recall $\omega_j^{(-1)} = \nabla_D \, dP_j$ by Proposition 2.9. Since $dP_j \in \Omega(\mathcal{A}, 0)^W$, we have $\nabla_D \, dP_j \in \Omega(\mathcal{A}, 2)^W$ by Lemma 2.10. On the other hand, $\nabla_D \, dP_j$ has poles of order one at most because dP_j is regular. Thus $\omega_j^{(-1)} \in \Omega(\mathcal{A}, 1)^W$. The induction proceeds by Proposition 2.9 and Lemma 2.10. $\square$

We extend the definition of $\Omega(\mathcal{A}, m)$ to the case when m is a negative integer:

$$\Omega(\mathcal{A}, m) := \bigoplus_{j=1}^{\ell} S \, \omega_j^{(-m)} \quad (m < 0).$$

Theorem 2.12
$\Omega(\mathcal{A}, m)$ is a free S-module with a basis $\omega_1^{(-m)}, \omega_2^{(-m)}, \ldots, \omega_\ell^{(-m)}$ for $m \in \mathbb{Z}$.

Proof. *Case 1.* When $m < 0$ this is nothing but the definition.

Case 2. Let $m = 2k - 1$ with $k \geq 1$. Recall $\mathcal{B}_{-k} \subset \Omega(\mathcal{A}, 2k-1)^W$ from Lemma 2.11 and $\det R_{1-2k} \doteq Q^{1-2k}$ by Proposition 2.6 (1). Thus we have

$$\begin{aligned} \omega_1^{(-2k+1)} \wedge \omega_2^{(-2k+1)} \wedge \cdots \wedge \omega_\ell^{(-2k+1)} &= (\det R_{1-2k}) \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell \\ &\doteq Q^{1-2k} (dx_1 \wedge dx_2 \wedge \cdots \wedge dx_\ell). \end{aligned}$$

This shows that $\mathcal{B}_{-k}$ is an S-basis for $\Omega(\mathcal{A}, 2k-1)$ by Saito-Ziegler's criterion [16, Theorem 11].

Case 3. Let $m = 2k$ with $k \geq 0$. When $k = 0$, the assertion is obvious because $\omega_j^{(0)} = dx_j$ and $\Omega(\mathcal{A}, 0) = \Omega_S$. Let $k \geq 1$. By Proposition 2.6 (2) we have

$$\begin{aligned} \left[\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}\right] &= [dx_1, \ldots, dx_\ell] \, R_{-2k} = [dx_1, \ldots, dx_\ell] \, R_{-2k-1} B^{-1} J(\mathbf{P})^T A \\ &= \left[\omega_1^{(-2k-1)}, \ldots, \omega_\ell^{(-2k-1)}\right] B^{-1} J(\mathbf{P})^T A. \end{aligned}$$

This implies that $\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}$ lie in $\Omega(\mathcal{A}, 2k+1)$ by Lemma 2.11. By Proposition 2.6 (3) we have

$$Q^{2k} R_{-2k} = Q^{2k-1} R_{-2k+1} B^{-1} B^{(-k+1)} Q J(\mathbf{P})^{-1}.$$

Since both $Q^{2k-1} R_{-2k+1}$ and $Q J(\mathbf{P})^{-1}$ belong to $M_{\ell,\ell}(S)$, so does $Q^{2k} R_{-2k}$. In other words, the differential forms $\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}$ have poles of order at most $2k$ along $\mathcal{A}$. Since it is easy to see that $\Omega(\mathcal{A}, 2k) = \Omega(\mathcal{A}, 2k+1) \cap (1/Q^{2k}) \Omega_S$, we know that $\omega_j^{(-2k)}$ belongs to $\Omega(\mathcal{A}, 2k)$ for each j. We can apply Saito-Ziegler's criterion [16, Theorem 11] to conclude that $\{\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}\}$

is a basis for $\Omega(\mathcal{A}, 2k)$ over S because $\det R_{-2k} \doteq Q^{-2k}$ by Proposition 2.6 (1).
$\square$

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1.

(1) It is enough to show that $\mathcal{B}_{-k}$ spans $\Omega(\mathcal{A}, 2k-1)^W$ over R. Express an arbitrary element $\omega \in \Omega(\mathcal{A}, 2k-1)^W$ as

$$\omega = \sum_{j=1}^{\ell} f_j \omega_j^{(-2k+1)}$$

with each $f_j \in S$. For any $s \in W$, get

$$0 = \omega - s(\omega) = \sum_{j=1}^{\ell} [f_j - s(f_j)]\, \omega_j^{(-2k+1)}.$$

Since $\mathcal{B}_{-k}$ is linearly independent over F, we obtain $f_j \in S^W = R$.

(2) Let $d_j := \deg P_j$ and $m_j := d_j - 1$ for $1 \leq j \leq \ell$. Let $h := d_\ell$ denote the Coxeter number. Define the degree of a homogeneous rational 1-form by

$$\deg(\sum_{i=1}^{\ell} f_i\, dx_i) = d \Longleftrightarrow f_i = 0 \text{ or } \deg f_i = d \quad (1 \leq i \leq \ell).$$

Then

$$\deg \omega_j^{(2k+1)} = m_j + kh.$$

Recall that $\mathcal{B}$ is linearly independent over T by Proposition 2.8. Let M_{-k} denote the free T-module spanned by $\bigcup_{p \geq -k} \mathcal{B}_p$. Recall that $\Omega(\mathcal{A}, 2k-1)^W$ is a free R-module with a basis $\mathcal{B}_{-k}$ by (1). If $p \geq -k$, then $R_{2p+1} = R_{-2k+1} H$ with a certain matrix $H \in M_{\ell,\ell}(R)$ because of Proposition 2.6 (4). This implies that $M_{-k} \subseteq \Omega(\mathcal{A}, 2k-1)^W$. Use a Poincaré series argument to prove that they are equal:

$$\begin{aligned}
\mathrm{Poin}(M_{-k}, t) &= (1 - t^{d_1})^{-1} \dots (1 - t^{d_{\ell-1}})^{-1} \sum_{p \geq -k} \left(t^{m_1 + ph} + \dots t^{m_\ell + ph}\right) \\
&= (1 - t^{d_1})^{-1} \dots (1 - t^{d_\ell})^{-1} \left(t^{m_1 - kh} + \dots t^{m_\ell - kh}\right) \\
&= \mathrm{Poin}(\Omega(\mathcal{A}, 2k-1)^W, t).
\end{aligned}$$

Therefore $M_{-k} = \Omega(\mathcal{A}, 2k-1)^W$.

(3) Thanks to Proposition 2.8, it is enough to prove that $\mathcal{B}$ spans $\Omega(\mathcal{A}, \infty)^W$ over T. Let $\omega \in \Omega(\mathcal{A}, \infty)$. Then $\omega \in \Omega(\mathcal{A}, 2k-1)^W$ for some $k \geq 1$. By

(2) and (3) we conclude that ω is a linear combination of $\bigcup_{p \geq -k} \mathcal{B}_p$ with coefficients in T. This shows that $\mathcal{B}$ spans $\Omega(\mathcal{A}, \infty)$ over T. $\square$

Proof of Theorem 1.2 (1). By Proposition 2.9,

$$\nabla_D : \Omega(\mathcal{A}, \infty)^W \to \Omega(\mathcal{A}, \infty)^W$$

induces a bijection $\nabla_D : \mathcal{B} \to \mathcal{B}$. Apply Theorem 1.1 (3) to prove that ∇_D is a T-isomorphism. $\square$

Let $\nabla_D^{-1} : \Omega(\mathcal{A}, \infty) \to \Omega(\mathcal{A}, \infty)$ denote the inverse T-isomorphism.

Definition 2.13
For $k \in \mathbb{Z}$, define

$$\mathcal{F}_0 := \bigoplus_{j=1}^{\ell} T \ (dP_j)\,, \quad \mathcal{F}_{-k} := \nabla_D^k(\mathcal{F}_0) \ \ (k > 0), \quad \mathcal{F}_k := \left(\nabla_D^{-1}\right)^k (\mathcal{F}_0) \ \ (k > 0).$$

Thus ∇_D induces a T-isomorphism $\nabla_D : \mathcal{F}_k \tilde{\to} \mathcal{F}_{k-1}$ for each $k \in \mathbb{Z}$. Since ∇_D induces a bijection $\nabla_D : \mathcal{B}_k \to \mathcal{B}_{k-1}$ by Proposition 2.9, each $\mathcal{F}_k$ is a free T-module of rank ℓ with a basis $\mathcal{B}_k = \{\omega_j^{(2k+1)} \mid 1 \leq j \leq \ell\}$.

Proof of Theorem 1.2 (2) and (3).
(2) By Theorem 1.1 (3), $\mathcal{B} = \bigcup_{k \in \mathbb{Z}} \mathcal{B}_k$ is a basis for $\Omega(\mathcal{A}, \infty)^W$ as a T-module. On the other hand, each $\mathcal{F}_k$ has a basis $\mathcal{B}_k$ over T for each $k \in \mathbb{Z}$.
(3) By Theorem 1.1 (2), $\mathcal{J}^{(-k)} = \Omega(\mathcal{A}, 2k-1)^W$. $\square$

Example 2.14
Let $\mathcal{A}$ be the B_2 type arrangement defined by $Q = xy(x+y)(x-y)$ corresponding to the Coxeter group of type B_2. Then $P_1 = (x^2+y^2)/2$, $P_2 = (x^4+y^4)/4$ are basic invariants. Then $T = \mathbb{R}[P_1]$ and $R = \mathbb{R}[P_1, P_2]$. Let

$$\omega = (x^4 + y^4)(\frac{dx}{x} + \frac{dy}{y}) \in \Omega(\mathcal{A}, 1)^W.$$

The unique decomposition of ω corresponding to the decomposition $\Omega(\mathcal{A}, 1)^W = \mathcal{J}^{(-1)} = \mathcal{F}_{-1} \oplus \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \dots$ is explicitly given by:

$$\omega = -8P_1^3\omega_1^{(-1)} + (8/3)P_1^2\omega_2^{(-1)} - 4P_1\omega_1^{(1)} + 2\omega_2^{(1)} \in \mathcal{F}_{-1} \oplus \mathcal{F}_0$$

by an easy calculation.

Corollary 2.15
The $\nabla_D : \Omega(\mathcal{A}, \infty)^W \to \Omega(\mathcal{A}, \infty)^W$ *induces an* T*-isomorphism*

$$\nabla_D : \Omega(\mathcal{A}, 2k-1)^W = \mathcal{J}^{(-k)} \xrightarrow{\sim} \mathcal{J}^{(-k-1)} = \Omega(\mathcal{A}, 2k+1)^W.$$

Concerning the strictly increasing filtration

$$\ldots \Omega(\mathcal{A}, 2k-1) \subset \Omega(\mathcal{A}, 2k) \subset \Omega(\mathcal{A}, 2k+1) \subset \ldots,$$

the following Proposition asserts the W-invariant parts of $\Omega(\mathcal{A}, 2k-1)$ and $\Omega(\mathcal{A}, 2k)$ are equal.

Proposition 2.16
$\Omega(\mathcal{A}, 2k)^W = \Omega(\mathcal{A}, 2k-1)^W = \mathcal{J}^{(-k)}$ *for* $k \in \mathbb{Z}$*. In particular,* $\Omega_R = \Omega_S^W = \Omega(\mathcal{A}, -1)^W$.

Proof. It is obvious that $\Omega(\mathcal{A}, 2k-1) \subseteq \Omega(\mathcal{A}, 2k)$ because $R_{-2k+1} = R_{-2k} J(\mathbf{P})(B^{(1-k)})^{-1} B$ by Proposition 2.6 (3). Thus $\Omega(\mathcal{A}, 2k-1)^W \subseteq \Omega(\mathcal{A}, 2k)^W$.
Let $\omega = \sum_{j=1}^{\ell} f_j\, \omega_j^{(-2k)} \in \Omega(\mathcal{A}, 2k)^W$ with $f_j \in S$. Since

$$(\mathrm{Eq})_k \qquad \left[\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}\right] = \left[\omega_1^{(-2k-1)}, \ldots, \omega_\ell^{(-2k-1)}\right] D[J(\mathbf{P})]^{-1}$$

by Proposition 2.6 (2), we may express

$$\omega = \sum_{j=1}^{\ell} f_j\, \omega_j^{(-2k)} = \sum_{j=1}^{\ell} f_j \left(\sum_{i=1}^{\ell} h_{ij}\, \omega_i^{(-2k-1)} \right) = \sum_{i=1}^{\ell} \left(\sum_{j=1}^{\ell} h_{ij}\, f_j \right) \omega_i^{(-2k-1)},$$

where h_{ij} is the (i,j)-entry of $D[J(\mathbf{P})]^{-1}$. Note that $\omega \in \Omega(\mathcal{A}, 2k+1)^W$ and that $\Omega(\mathcal{A}, 2k+1)^W$ has a basis $\{\omega_1^{(-2k-1)}, \omega_2^{(-2k-1)}, \ldots, \omega_\ell^{(-2k-1)}\}$ over R. Then we know that $\sum_{j=1}^{\ell} h_{ij}\, f_j$ is W-invariant for $1 \le i \le \ell$. Applying $(\mathrm{Eq})_0$ we have

$$\begin{aligned} \omega' := \sum_{j=1}^{\ell} f_j\, dx_j = \sum_{j=1}^{\ell} f_j\, \omega_j^{(0)} &= \sum_{j=1}^{\ell} f_j \sum_{i=1}^{\ell} h_{ij}\, \omega_i^{(-1)} \\ &= \sum_{i=1}^{\ell} \left(\sum_{j=1}^{\ell} h_{ij}\, f_j \right) \omega_i^{(-1)} \in \Omega_S^W. \end{aligned}$$

Recall $\Omega_S^W = \Omega_R = \oplus_{i=1}^{\ell} R\ (dP_i)$ by [9]. Thus there exist $g_i \in R$ $(1 \le i \le \ell)$ such that

$$\omega' = \sum_{i=1}^{\ell} g_i\ (dP_i) = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{\ell} g_i\ (\partial P_i / \partial x_j) \right) dx_j.$$

This implies

$$f_j = \sum_{i=1}^{\ell} g_i \ (\partial P_i/\partial x_j) \quad (1 \le i \le \ell).$$

Since

$$\left[\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}\right] J(\mathbf{P}) = \left[\omega_1^{(-2k+1)}, \ldots, \omega_\ell^{(-2k+1)}\right] B^{-1} B^{(1-k)}$$

by Proposition 2.6 (3), one has

$$\begin{aligned}
\omega &= \sum_{j=1}^{\ell} f_j \, \omega_j^{(-2k)} = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{\ell} g_i \ (\partial P_i/\partial x_j) \right) \omega_j^{(-2k)} \\
&= \sum_{i=1}^{\ell} g_i \left(\sum_{j=1}^{\ell} (\partial P_i/\partial x_j) \, \omega_j^{(-2k)} \right) \in \bigoplus_{i=1}^{\ell} R \ \omega_i^{(-2k+1)} = \Omega(\mathcal{A}, 2k-1)^W.
\end{aligned}$$

This proves $\Omega(\mathcal{A}, 2k)^W \subseteq \Omega(\mathcal{A}, 2k-1)^W$. □

3 The case of derivations

Denote $\partial/\partial x_i$ and $\partial/\partial P_i$ simply by ∂_{x_i} and ∂_{P_i} respectively. Then

$$\mathrm{Der}_S = \bigoplus_{j=1}^{\ell} S \, \partial_{x_j}, \quad \mathrm{Der}_R = \bigoplus_{j=1}^{\ell} R \, \partial_{P_j}, \quad \mathrm{Der}_F = \bigoplus_{j=1}^{\ell} F \, \partial_{x_j}.$$

In this section we translate the results in the previous section by the F-isomorphism

$$I^* : \Omega_F \to \mathrm{Der}_F$$

defined by $I^*(\omega)(f) = I^*(\omega, df)$ for $f \in F$ and $\omega \in \Omega_F$. Explicitly we can express

$$I^*(\sum_{j=1}^{\ell} f_j \ dx_j) = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{\ell} I^*(dx_i, dx_j) \, f_i \right) \partial_{x_j}$$

for $f_j \in F \quad (1 \le j \le \ell)$.

Definition 3.1
Define $\eta_j^{(m)} := I^*(\omega_j^{(m)})$ *for* $m \in \mathbb{Z}, \ 1 \le j \le \ell$.

Then

$$[\eta_1^{(m)}, \ldots, \eta_\ell^{(m)}] = [\partial_{x_1}, \ldots, \partial_{x_\ell}] A R_m.$$

In particular,

$$[\eta_1^{(1)}, \ldots, \eta_\ell^{(1)}] = [\partial_{x_1}, \ldots, \partial_{x_\ell}]AJ(\mathbf{P}) = [I^*(dP_1), \ldots, I^*(dP_\ell)],$$

$$\begin{aligned}[\eta_1^{(-1)}, \ldots, \eta_\ell^{(-1)}] &= [\partial_{x_1}, \ldots, \partial_{x_\ell}]AD[J(\mathbf{P})] = [\partial_{x_1}, \ldots, \partial_{x_\ell}]J(\mathbf{P})^{-T}B \\ &= [\partial_{P_1}, \ldots, \partial_{P_\ell}]B.\end{aligned}$$

Definition 3.2
Define

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S \cdot \alpha_H^m \ \text{ for all } H \in \mathcal{A}\}$$

for $m \geq 0$ which is the S-module of **logarithmic derivations** *along $\mathcal{A}$ of contact order m. When $m < 0$ define*

$$D(\mathcal{A}, m) := \bigoplus_{1 \leq j \leq \ell} S\, \eta_j^{(m)}.$$

Lastly define

$$D(\mathcal{A}, -\infty) := \bigcup_{m \in \mathbb{Z}} D(\mathcal{A}, m).$$

Theorem 3.3
$D(\mathcal{A}, m)$ is a free S-module with a basis $\eta_1^{(m)}, \eta_2^{(m)}, \ldots, \eta_\ell^{(m)}$ for $m \in \mathbb{Z}$.

Proof. *Case 1.* When $m < 0$ this is nothing but the definition.

Case 2. Let $m \geq 0$. For a canonical contraction $\langle\ ,\ \rangle : \mathrm{Der}_F \times \Omega_F \to F$, define the $(\ell \times \ell)$-matrix

$$Y_m := [\langle \omega_i^{(-m)}, \eta_j^{(m)} \rangle]_{1 \leq i,j \leq \ell} = R_{-m} A R_m$$

for $m \geq 0$. Since the two S-modules $\Omega(\mathcal{A}, m)$ and $D(\mathcal{A}, m)$ are dual each other (see [16]) , it is enough to show that $\det Y_m \in \mathrm{GL}_\ell(S)$. It follows from the following Proposition 3.6. □

Corollary 3.4
$I^*(\Omega(\mathcal{A}, m)) = D(\mathcal{A}, -m)$ for $m \in \mathbb{Z}$ and $I^*(\Omega(\mathcal{A}, \infty)) = D(\mathcal{A}, -\infty)$.

Corollary 3.5
$\Omega(\mathcal{A}, -m) = \{\omega \in \Omega_S \mid I^*(\omega, d\alpha_H) \in S \cdot \alpha_H^m$ for any $H \in \mathcal{A}\}$ for $m > 0$.

Proposition 3.6
(1) $Y_{2k-1} = (-1)^{k+1} B^T (B^{(k)})^{-1} B \in GL_\ell(T)$ for $k \in \mathbb{Z}$,
(2) $Y_{2k} = (-1)^k A \in GL_\ell(\mathbb{R})$ for $k \in \mathbb{Z}$.

Proof.

(1) *Case 1.1.* Let $m = 2k - 1$ with $k \geq 1$. We prove by an induction on k. When $k = 1$,

$$Y_1 = R_{-1}^T A R_1 = D[J(\mathbf{P})]^T A J(\mathbf{P}) \quad = \quad B^T \in \mathrm{GL}_\ell(T).$$

Assume that $k > 1$ and prove by induction. By using Proposition 2.6 (5) and (4), we obtain

$$\begin{aligned} Y_{2k-1} &= R_{1-2k}^T A R_{2k-1} = D[R_{3-2k}]^T A R_{2k-3} B^{-1} G (B^{(k)})^{-1} B \\ &= \{D[R_{3-2k}^T A R_{2k-3}] - R_{3-2k}^T D[A R_{2k-3}]\} B^{-1} G (B^{(k)})^{-1} B \\ &= -R_{3-2k}^T A R_{2k-5} B^{-1} G (B^{(k-1)})^{-1} B B^{-1} B^{(k-1)} (B^{(k)})^{-1} B \\ &= -R_{3-2k}^T A R_{2k-3} B^{-1} B^{(k-1)} (B^{(k)})^{-1} B \\ &= (-1)^{k+1} B^T (B^{(k-1)})^{-1} B B^{-1} B^{(k-1)} (B^{(k)})^{-1} B \\ &= (-1)^{k+1} B^T (B^{(k)})^{-1} B. \end{aligned}$$

Case 1.2. Next assume that $m = 2k - 1$ with $k \leq 0$. Recall that

$$(B^{(1-k)})^T = -kB + (1-k) B^T = -B^{(k)}.$$

Then

$$\begin{aligned} R_{1-2k}^T A R_{2k-1} &= (R_{2k-1}^T A R_{1-2k})^T = ((-1)^k B^T (B^{(1-k)})^{-1} B)^T \\ &= (-1)^{k+1} B^T (B^{(k)})^{-1} B. \end{aligned}$$

(2) Apply (1), Proposition 2.6 (2) and (3) to compute

$$\begin{aligned} R_{-2k}^T A R_{2k} &= J(\mathbf{P})^{-T} (B^{(1-k)})^T B^{-T} R_{-2k+1}^T A R_{2k-1} B^{-1} J(\mathbf{P})^T A \\ &= J(\mathbf{P})^{-T} (B^{(1-k)})^T B^{-T} Y_{2k-1} B^{-1} J(\mathbf{P})^T A = (-1)^k A. \quad \square \end{aligned}$$

Remark. Corollaries 3.4 and 3.5 show that the definitions of $D(\mathcal{A}, m)$ and $\Omega(\mathcal{A}, m)$ for $m \in \mathbb{Z}_{<0}$ are equivalent to those of $D\Omega(\mathcal{A}, m)$ and $\Omega D(\mathcal{A}, m)$ in [1].

Consider the T-linear connection (covariant derivative)

$$\nabla_D : \mathrm{Der}_F \to \mathrm{Der}_F$$

characterized by $\nabla_D(fX) = (Df)X + f(\nabla_D X)$ and $\nabla_D(\partial_{x_j}) = 0$ for $f \in F$, $X \in \mathrm{Der}_F$ and $1 \leq j \leq \ell$. Then it is easy to see the diagram

$$\begin{array}{ccc} \Omega_F & \xrightarrow{\nabla_D} & \Omega_F \\ {\scriptstyle I^*}\downarrow & & {\scriptstyle I^*}\downarrow \\ \mathrm{Der}_F & \xrightarrow{\nabla_D} & \mathrm{Der}_F \end{array}$$

is commutative. In fact

$$\begin{aligned}\nabla_D \circ I^* \left(\sum_{j=1}^{\ell} f_j \, dx_j \right) &= \nabla_D \left[\sum_{j=1}^{\ell} \left(\sum_{i=1}^{\ell} I^*(dx_i, dx_j) f_i \right) \partial_{x_j} \right] \\ &= \sum_{j=1}^{\ell} \left(\sum_{i=1}^{\ell} I^*(dx_i, dx_j) D(f_i) \right) \partial_{x_j} \\ &= I^* \left(\sum_{j=1}^{\ell} D(f_j) \, dx_j \right) = I^* \circ \nabla_D \left(\sum_{j=1}^{\ell} f_j \, dx_j \right).\end{aligned}$$

Define $\mathcal{C}_k := I^*(\mathcal{B}_{k-1}) = \{\eta_1^{(2k-1)}, \eta_2^{(2k-1)}, \dots, \eta_\ell^{(2k-1)}\}$ for each $k \in \mathbb{Z}$. The following Theorems 3.7 and 3.9 can be proved by translating Theorems 1.1 and 1.2 through ∇_D.

Theorem 3.7

(1) The R-module $D(\mathcal{A}, 2k-1)^W$ is free with a basis $\mathcal{C}_k$ for $k \in \mathbb{Z}$.

(2) The T-module $D(\mathcal{A}, 2k-1)^W$ is free with a basis $\bigcup_{p \geq k} \mathcal{C}_p$ for $k \in \mathbb{Z}$.

(3) $\mathcal{C} := \bigcup_{k \in \mathbb{Z}} \mathcal{C}_k$ is a basis for $D(\mathcal{A}, -\infty)^W$ as a T-module.

Definition 3.8

Define

$$\mathcal{G}_k := I^*(\mathcal{F}_{k-1}), \quad \mathcal{H}^{(k)} := I^*(\mathcal{J}^{(k-1)}) \quad (k \in \mathbb{Z}, \ 1 \leq j \leq \ell).$$

Then

$$\mathcal{G}_k = \bigoplus_{1 \leq j \leq \ell} T \, \eta_j^{(2k-1)}, \quad \mathcal{H}^{(k)} = \bigoplus_{p \geq k} \mathcal{G}_p.$$

The ∇_D induces T-isomorphisms

$$\nabla_D : \mathcal{G}_{k+1} \xrightarrow{\sim} \mathcal{G}_k, \quad \nabla_D : D(\mathcal{A}, 2k+1)^W \xrightarrow{\sim} D(\mathcal{A}, 2k-1)^W.$$

In particular,

$$\mathcal{G}_0 = \bigoplus_{j=1}^{\ell} T \, \partial_{P_j}, \quad \text{and} \quad \mathcal{H}^{(0)} = \bigoplus_{j=1}^{\ell} R \, \partial_{P_j} = \mathrm{Der}_R \, .$$

Theorem 3.9

(1) The ∇_D induces a T-linear automorphism $\nabla_D : D(\mathcal{A}, -\infty)^W \xrightarrow{\sim} D(\mathcal{A}, -\infty)^W$.

(2) $D(\mathcal{A}, -\infty)^W = \bigoplus_{k \in \mathbb{Z}} \mathcal{G}_k$.

(3) $D(\mathcal{A}, 2k-1)^W = \mathcal{H}^{(k)} = \bigoplus_{p \geq k} \mathcal{G}_p$. $(k \in \mathbb{Z})$.

Remark. The construction of a basis $\eta_1^{(1)}, \ldots, \eta_\ell^{(1)}$ for $D(\mathcal{A}, 1)$ is due to K. Saito [6]. A basis for $D(\mathcal{A}, 2)$ was constructed in [10]. In [11] $D(\mathcal{A}, m)$ was found to be a free S-module for all $m \geq 0$ whenever $\mathcal{A}$ is a Coxeter arrangement. Note that it is re-proved in Theorem 3.3 in this article. In [8] K. Saito called the decreasing filtration $\mathrm{Der}_R = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \ldots$ and the decomposition $\mathrm{Der}_R = D(\mathcal{A}, -1)^W = \mathcal{H}^{(0)} = \bigoplus_{p \geq 0} \mathcal{G}_p$ the Hodge filtration and the Hodge decomposition respectively. They are essential to define the flat structure (or equivalently the Frobenius manifold structure in topological field theory) on the orbit space V/W. Note that Theorem 3.9 (3), when $k \geq 0$, is the main theorem of [13].

4 Relation among bases for logarithmic forms and derivations

In the previous section we constructed a basis $\{\omega_j^{(m)}\}$ for $\Omega(\mathcal{A}, m)$ and a basis $\{\eta_j^{(m)}\}$ for $D(\mathcal{A}, m)$ for $m \in \mathbb{Z}$. In this section we briefly describe their relations to other bases constructed in the earlier works [11], [15], and [2]. In [11], the following bases for $D(\mathcal{A}, 2k+1)$ and $D(\mathcal{A}, 2k)$ are given:

$$[\xi_1^{(2k+1)}, \ldots, \xi_\ell^{(2k+1)}] := [\partial_{x_1}, \ldots, \partial_{x_\ell}] A J(D^k[\mathbf{x}])^{-1} J(\mathbf{P}),$$
$$[\xi_1^{(2k)}, \ldots, \xi_\ell^{(2k)}] := [\partial_{x_1}, \ldots, \partial_{x_\ell}] A J(D^k[\mathbf{x}])^{-1}.$$

The two bases $\{\eta_j^{(m)}\}$ and $\{\xi_j^{(m)}\}$ are related as follows:

Proposition 4.1
For $k \in \mathbb{Z}_{\geq 0}$,

$$[\xi_1^{(2k+1)}, \ldots, \xi_\ell^{(2k+1)}] = (-1)^k [\eta_1^{(2k+1)}, \ldots, \eta_\ell^{(2k+1)}] B^{-1} B^{(k+1)},$$
$$[\xi_1^{(2k)}, \ldots, \xi_\ell^{(2k)}] = (-1)^k [\eta_1^{(2k)}, \ldots, \eta_\ell^{(2k)}].$$

Proof. The second formula is immediate from Definition 2.5. The following computation proves the first formula:

$$\begin{aligned} J(D^k[\mathbf{x}])^{-1} J(\mathbf{P}) &= (-1)^{k+1} R_{2k+1} D[J(\mathbf{P})]^{-1} J(D^{k+1}[\mathbf{x}]) J(D^k[\mathbf{x}])^{-1} J(\mathbf{P}) \\ &= (-1)^k R_{2k+1} D[J(\mathbf{P})]^{-1} A^{-1} J(\mathbf{P})^{-T} B^{(k+1)} \\ &= (-1)^k R_{2k+1} B^{-1} B^{(k+1)}. \ \square \end{aligned}$$

In [15], the following bases are given:

$$[\nabla_{I^*(dP_1)}\nabla_D^{-k}\theta_E,\ldots,\nabla_{I^*(dP_\ell)}\nabla_D^{-k}\theta_E] \quad \text{for} \quad D(\mathcal{A},2k+1),$$
$$[\nabla_{\partial_{x_1}}\nabla_D^{-k}\theta_E,\ldots,\nabla_{\partial_{x_\ell}}\nabla_D^{-k}\theta_E] \quad \text{for} \quad D(\mathcal{A},2k).$$

Here θ_E is the Euler derivation. Their relations to $\{\eta_j^{(m)}\}$ are given as follows:

Proposition 4.2
Let $k \in \mathbb{Z}_{\geq 0}$. Then

$$[\nabla_{I^*(dP_1)}\nabla_D^{-k}\theta_E,\ldots,\nabla_{I^*(dP_\ell)}\nabla_D^{-k}\theta_E] = [\eta_1^{(2k+1)},\ldots,\eta_\ell^{(2k+1)}]B^{-1}B^{(k+1)},$$
$$[\nabla_{\partial_{x_1}}\nabla_D^{-k}\theta_E,\ldots,\nabla_{\partial_{x_\ell}}\nabla_D^{-k}\theta_E] = [\eta_1^{(2k)},\ldots,\eta_\ell^{(2k)}]A^{-1}.$$

Proof. By [12, Theorem 1.2.] and [14] one has

$$[\nabla_{I^*(dP_1)}\nabla_D^{-k}\theta_E,\ldots,\nabla_{I^*(dP_\ell)}\nabla_D^{-k}\theta_E] = (-1)^k[\xi_1^{(2k+1)},\ldots,\xi_\ell^{(2k+1)}].$$

Combining with Proposition 4.1, we have the first relation. For the second one, compute

$$\begin{aligned}[\nabla_{\partial_{x_1}}\nabla_D^{-k}\theta_E,\ldots,\nabla_{\partial_{x_\ell}}\nabla_D^{-k}\theta_E]AJ(\mathbf{P}) &= [\nabla_{I^*(dP_1)}\nabla_D^{-k}\theta_E,\ldots,\nabla_{I^*(dP_\ell)}\nabla_D^{-k}\theta_E] \\ &= [\eta_1^{(2k+1)},\ldots,\eta_\ell^{(2k+1)}]B^{-1}B^{(k+1)} \\ &= [\eta_1^{(2k)},\ldots,\eta_\ell^{(2k)}]J(\mathbf{P})\end{aligned}$$

by Proposition 2.6 (3). □

Next let us review the bases for $\Omega(\mathcal{A},m)$ described in [2, Theorem 6]: Let $k \in \mathbb{Z}_{\geq 0}$ and P_1 the smallest degree basic invariant. Then

$$\{\nabla_{\partial_{P_1}}\nabla_D^k dP_1,\ldots,\nabla_{\partial_{P_\ell}}\nabla_D^k dP_1\}$$

forms a basis for $\Omega(\mathcal{A},2k+1)$ and

$$\{\nabla_{\partial_{x_1}}\nabla_D^k dP_1,\ldots,\nabla_{\partial_{x_\ell}}\nabla_D^k dP_1\}$$

forms a basis for $\Omega(\mathcal{A},2k)$.

Proposition 4.3
Let $k \geq 0$. Then

$$[\nabla_{\partial_{P_1}}\nabla_D^k dP_1,\ldots,\nabla_{\partial_{P_\ell}}\nabla_D^k dP_1] = [\omega_1^{(-2k-1)},\ldots,\omega_\ell^{(-2k-1)}]B^{-1},$$
$$[\nabla_{\partial_{x_1}}\nabla_D^k dP_1,\ldots,\nabla_{\partial_{x_\ell}}\nabla_D^k dP_1] = [\omega_1^{(-2k)},\ldots,\omega_\ell^{(-2k)}]A^{-1}.$$

Proof. First, note that $[\nabla_D, \nabla_{\partial_{P_i}}]$ is W-invariant, hence in Der_R. Since the smallest degree of derivations in Der_R is $\deg \partial_{P_\ell}$, it follows that $[\nabla_D, \nabla_{\partial_{P_i}}] = 0$. In other words, $\nabla_{\partial_{P_i}}$ and $\nabla_{\partial_{P_\ell}} = \nabla_D$ commute for all i. Hence

$$[\nabla_{\partial_{P_1}} \nabla_D^k dP_1, \ldots, \nabla_{\partial_{P_\ell}} \nabla_D^k dP_1] = \nabla_D^k [\nabla_{\partial_{P_1}} dP_1, \ldots, \nabla_{\partial_{P_\ell}} dP_1].$$

Our proof is an induction on k. First assume that $k = 0$. Choose

$$P_1 = \frac{1}{2}[x_1, \ldots, x_\ell] A^{-1} [x_1, \ldots, x_\ell]^T,$$

and

$$dP_1 = [dx_1, \ldots, dx_\ell] A^{-1} [x_1, \ldots, x_\ell]^T.$$

Compute

$$\begin{aligned}
[\nabla_{\partial_{P_1}} dP_1, \ldots, \nabla_{\partial_{P_\ell}} dP_1] B &= [\nabla_{\partial_{x_1}} dP_1, \ldots, \nabla_{\partial_{x_\ell}} dP_1] J(\mathbf{P})^{-T} B \\
&= [dx_1, \ldots, dx_\ell] A^{-1} J(\mathbf{P})^{-T} B \\
&= [dx_1, \ldots, dx_\ell] D[J(\mathbf{P})] = [\omega_1^{(-1)}, \ldots, \omega_\ell^{(-1)}].
\end{aligned}$$

For $k > 0$, apply ∇_D^k and use the commutativity. Then we have the first relation. For the second relation use Proposition 2.6 (2) to compute:

$$\begin{aligned}
[\nabla_{\partial_{x_1}} \nabla_D^k dP_1, \ldots, \nabla_{\partial_{x_\ell}} \nabla_D^k dP_1] &= [\nabla_{\partial_{P_1}} \nabla_D^k dP_1, \ldots, \nabla_{\partial_{P_\ell}} \nabla_D^k dP_1] J(\mathbf{P})^T \\
&= [\omega_1^{(-2k-1)}, \ldots, \omega_\ell^{(-2k-1)}] B^{-1} J(\mathbf{P})^T \\
&= [dx_1, \ldots, dx_\ell] R_{-2k-1} B^{-1} J(\mathbf{P})^T \\
&= [dx_1, \ldots, dx_\ell] R_{-2k} A^{-1} \\
&= [\omega_1^{(-2k)}, \ldots, \omega_\ell^{(-2k)}] A^{-1}.
\end{aligned}$$

□

Remark. If $k < 0$ in Propositions 4.2 and 4.3, then the derivations and 1-forms in the left hand sides are proved to form bases for the logarithmic modules $D\Omega(\mathcal{A}, 2k+1), D\Omega(\mathcal{A}, 2k), \Omega D(\mathcal{A}, 2k+1)$ and $\Omega D(\mathcal{A}, 2k)$ in [1]. By using the same arguments in the proofs above, we can show that Propositions 4.2 and 4.3 hold true for all integers k in the logarithmic modules $D\Omega(\mathcal{A}, \mathbf{m})$ and $\Omega D(\mathcal{A}, \mathbf{m})$ with $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$.

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