北海道大学工学部研究報告
第121号（昭和59年）

Bulletin of the Faculty of Engineering.
Hokkaido University. No. 121(1984)

Resonant Faraday Rotation and Holography for the Measurement of Spatial Magnetic Field and Atomic Density in a Plasma

Shun-ichi HIMENO*, Takeaki ENOTO**, Hitoshi MOCHIZUKI*,
Kei-ichi HIRANO***, and Yasutomo OZAWA****

(Received December 27, 1983)

Abstract

This paper describes a method of measuring spatial distributions of magnetic field by resonant Faraday rotation in combination with resonant holographic interferometry using a tunable dye laser.

The sensitivity and selectivity of these resonant methods are high in comparison with conventional Faraday rotation and holographic interferometry.

1. Introduction

The determination of spatial distributions of magnetic field is an important subject in the diagnostics of high temperature plasmas. At high energy densities, magnetic probes[1] not only perturb the plasma but are frequently destroyed by contact with it. The use of spectroscopic methods or scattering method[1] in plasmas is extremely difficult because of the large Doppler and Stark broadening of spectral lines or complicated devices. The measurement of the magnetic field by observation of the conventional Faraday rotation[1] of the plane of polarization of visible light by free electrons in applicable only for fairly dense plasmas in strong magnetic fields.

Here we examine theoretically the possibility of determining spatial distribution of magnetic field in a plasma from the resonant Faraday rotation by residual neutral atoms or specially introduced impurity atoms in the plasma. Resonant holographic interferometry is used to determine the density of the atoms in the plasma. If the wavelength of the light is adjacent to the wavelength of an atomic or ionic transition, the change in the polarization of light and the enhancement in the index of refraction is caused by a resonant interaction of light with atoms or ions. A tunable dye laser is the most successful candidate as a light source for resonant Faraday rotation and holographic interferametry.

*University of Electro-Communications
**Department of Nuclear Engineering, Hokkaido University
***Institute of Plasma Physics, Nagoya University
****Hokkaido Institute of Technology

2. Resonant holographic interferometry

The method of resonant holographic interferometry in based on obtaining interferograms in the light close to the absorption lines of one component of the plasma to be studied.

We shall first discuss about the classical theory of dispersion.
The light propagation in a absorbing and dispersive media, can be written with the help of complex refractive index[2)]

$$\tilde{n} = n(1 - i\varkappa) \qquad (2.1)$$

where n and $\varkappa$ are ordinary refractive indexes and absorption indexes, respectively. The light wave propagating in the z-direction with angular frequency $\omega = 2\pi\nu = 2\pi c/\lambda$ is written by

$$A = A_0 e^{i\omega(t-nZ/c)} = A_0 e^{-\omega n \varkappa z/c} e^{i\omega(t-nZ/c)} \qquad (2.2)$$

where A and A_0 characterize the electric or magnetic vector. The absorption coefficient per unit length is

$$k = \frac{2\omega_n \varkappa}{c} = \frac{4\pi n \varkappa}{\lambda} \qquad (2.3)$$

The complex refractive index n is connected with the polarization P in c.g.s. units

$$\tilde{n}^2 E = E + 4\pi p \qquad (2.4)$$

where $E = E_0 e^{i\omega t}$ is the electric field at the oscillators. The polarization is also written by

$$P = Nex \qquad (2.5)$$

where N is the particle number per unit volume (harmonic oscillators with eigen frequency $\omega = 2\pi\nu_0$) and x is the displacement of oscillators.

Let us consider the equation of motion for an electron of mass m

$$m\ddot{x} + m\omega_0^2 x + \gamma\dot{x} = eE \qquad (2.6)$$

where γ is the damping constant of the oscillators. If we make a periodic oscillation $x = x_0 e^{i\omega t}$, Eq. (2.6) becomes

$$(-\omega^2 + \omega_0^2 + i\omega\gamma)\, x = eE/m \qquad (2.7)$$

then Eq. (2.5) becames

$$P = \frac{Ne^2/m}{-\omega^2 + \omega_0^2 + i\omega\gamma} \qquad (2.8)$$

Therefore Eq. (2.4) is written as

$$\tilde{n}^2 - 1 = 4\pi Nex/E = \frac{4\pi Ne^2}{m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} \qquad (2.9)$$

The ordinary refractive index and absorption coefficient can be obtained by separating the real part and the imaginary part. From Eq.(2.1)

$$\tilde{n}^2 - 1 = n^2(1 - \varkappa^2) - 1 - i\, 2n^2\varkappa \qquad (2.10)$$

From Eq. (2.9) and (2.10) we have

$$n^2(1 - \varkappa^2) - 1 = \frac{4\pi Ne^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \qquad (2.11)$$

$$2n^2\varkappa=\frac{4\pi Ne^2}{m}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \quad (2.12)$$

Let us restrict the region of the eigenfrequency of the oscillators $\omega\approx\omega_0$ and write

$$\Delta\omega=\omega-\omega_0\ll\omega \text{ or } \omega_0 \quad (2.13)$$

we have the simple form from Eqs. (2.11) and (2.12):

$$n^2(1-\varkappa^2)-1=-\frac{\pi e^2N}{m\omega_0}\frac{2\Delta\omega}{(\Delta\omega)^2+(\gamma/2)^2} \quad (2.14)$$

$$2n^2\varkappa=\frac{\pi e^2N}{m\omega_0}\frac{\gamma}{(\Delta\omega)^2+(\gamma/2)^2} \quad (2.15)$$

In a fully thin gas $\varkappa\ll1$ and $n\approx1$. We can obtain refractive index n and absorption coefficient K directly from Eqs. (2.14) and (2.15):

$$n-1\approx-\frac{\pi e^2N}{2m\omega_0}\frac{2\Delta\omega}{(\Delta\omega)^2+(\gamma/2)^2} \quad (2.16)$$

$$K=\frac{2\omega n\varkappa}{c}\approx\frac{\pi e^2N}{mc}\frac{\gamma}{(\Delta\omega)^2+(\gamma/2)^2} \quad (2.17)$$

Near the absorption line the refractive index of atoms and ions is described in wavelength by

$$n-1=\frac{e^2}{4\pi mc^2}\lambda_0^2N\frac{\lambda-\lambda_0}{(\lambda-\lambda_0)^2+\left(\frac{\Delta\lambda_{\frac{1}{2}}}{2}\right)^2} \quad (2.18)$$

where λ_0 is the absorption wavelength, $\Delta\lambda_{\frac{1}{2}}$ is the full width at half maximum of the absorption line, λ is the wavelength of the diagnostic light, N is the atomic density at the absorbing level, and the oscillator strength f for the line is introduced.

The shape of the absorption line is described by

$$K=2\pi\frac{e^2}{4\pi mc^2}\lambda_0^2Nf\frac{\Delta\lambda_{\frac{1}{2}}}{(\lambda-\lambda_0)^2+\left(\frac{\Delta\lambda_{\frac{1}{2}}}{2}\right)^2} \quad (2.19)$$

Then the fringe shift F_r due to excited atoms is given by

$$F_r=\int\frac{n-1}{\lambda}dl$$

$$\approx\int\frac{e^2}{4\pi mc^2}\lambda_0^2Nf\frac{\lambda-\lambda_0}{(\lambda-\lambda_0)^2+\left(\frac{\Delta\lambda_{\frac{1}{2}}}{2}\right)^2}dl \quad (2.20)$$

where dl is a path length element. This means that by using a wavelength near the absorption line to obtain interferograms we can greatly increase the sensitivity of measuring the atom density.

Taking the fringe shifts due to free electrons into account and neglecting, we can write the corresponding total fringe shift F as

$$F=\frac{e^2\lambda_0^2f}{4\pi mc^2}\frac{1}{\lambda-\lambda_0}\int Ndl$$

$$-\frac{e^2}{2\pi mc^2}\lambda\int n_edl \quad (2.21)$$

where n_e is the electron density. We can distinguish between the two contributions to the fringe shifts if the refractive index is measured for two wavelengths.

The wavelength-dependence of the fringe shift near a spectral line is shown in Fig. 1. We can see from this figure that the fringe shift due to bound electrons is larger than the one due to free electrons and the fringe shift increases as the difference between the diagnostic line and the absorption line decreases.

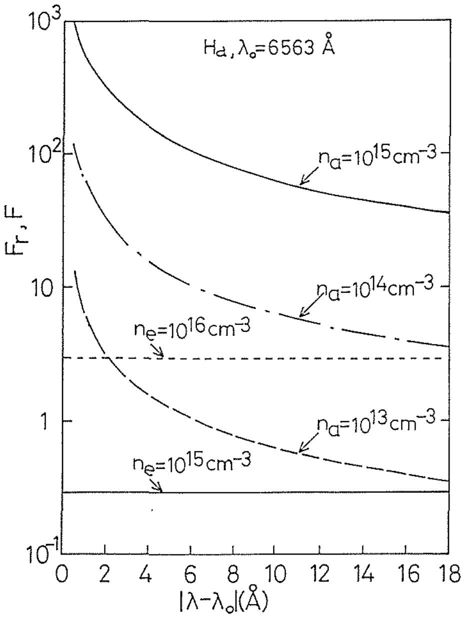

Fig.1 The fringe shift F_r due to bound electrons and the fringe shift F_c due to free electrons as a function of $\lambda-\lambda_0$ for $\Delta\lambda_{\frac{1}{2}} \ll \lambda-\lambda_0$
The lines indicated by n_a are for F_r and those by n_e are for F_c.
$l=10^2$cm, $\lambda_0=6563$Å(H_a):

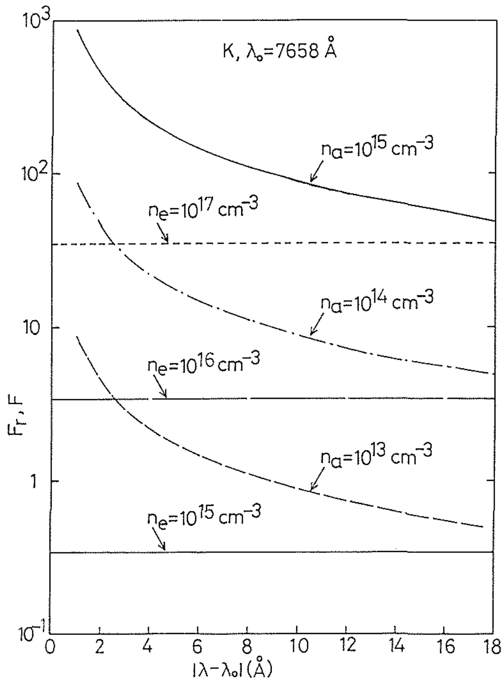

Fig.2 F_r and F_c as a function of $\lambda-\lambda_0$ for $\Delta\lambda_{\frac{1}{2}} \ll \lambda-\lambda_0$,
$l=10^2$cm, $\lambda_0=7658$ Å(k).

3. Resonant Faraday rotation

Taking the same procedures in resonant holographic interferometry,we can derive the Faraday rotation formula. When a linearly polarized light close to the absorption lines of the plasma is transmitted in the z-direction parallel to an applied magnetic field B, an enhanced rotation in the plane of polarization is observed.

The equation of motion for an electron of mass m, bound with a force constant m ω_0^2 driven by an electric field $\mathbf{E}$ and a static magnetic field $\mathbf{B}$, includes a damping term γ,

$$m\ddot{\mathbf{r}}+m\omega_0^2\mathbf{r}+m\gamma\dot{\mathbf{r}}=e(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}) \qquad (3.1)$$

where e is the electron charge and $\mathbf{r} = \mathbf{r}_0 e^{i\omega t}$ is the position vector.

Let us introduce the following variables corresponding to right and left circular polarization[2)]

$$\begin{aligned} \mathbf{r}_\pm &= x \pm iy \\ \mathbf{E}_\pm &= E_x \pm iE_y \\ \mathbf{P}_\pm &= P_x \pm iP_y \end{aligned} \quad (3.2)$$

We have now

$$(-\omega^2 + \omega_0^2 \pm \frac{eB}{m}\omega + i\gamma\omega)\mathbf{r}_\pm = e\,\mathbf{E}_\pm/m \quad (3.3)$$

and theretore

$$\begin{aligned} \mathbf{P}_\pm &= Ne\,\mathbf{r}_\pm \\ &= \frac{Ne^2}{m}\frac{\mathbf{E}_\pm}{-\omega^2 + \omega_0^2 + i\gamma\omega \pm \frac{eB}{mc}\omega} \end{aligned} \quad (3.4)$$

The complex refractive indices considering oscillator strength f are given by

$$\tilde{n}_\pm^2 - 1 = \frac{4\pi Nfe^2}{m}\frac{1}{\omega_0^2 - \omega^2 + \omega(i\gamma \pm eB/\omega)} \quad (3.5)$$

If we introduce the mean index of refraction $n = (n_+ - n_-)/2$ and neglect the γterm, the difference of two indices is

$$\begin{aligned} n_+ - n_- &= \frac{n_+^2 - n_-^2}{2n} \\ &= -\frac{4\pi Nfe^2}{m}\frac{eB}{ncm}\frac{\omega}{(\omega_0^2 - \omega^2)^2} \end{aligned} \quad (3.6)$$

Thus the Faraday rotation θ_r due to bound electrons can be measured by

$$\begin{aligned} \theta_r &= -\int \frac{1}{2}k(n_+ - n_-)dl \\ &= \int \frac{4\pi Nfe^2}{m}\frac{eB}{2nc^2m}\frac{\omega^2}{(\omega_0^2 - \omega^2)^2}dl \end{aligned} \quad (3.7)$$

Taking the free electron contribution into account and replacing ω with λ, the total rotation angle θ can be written as

$$\begin{aligned} \theta &= \frac{e^3 f}{2\pi m^2 c^4}\frac{\lambda^6}{(\lambda^2 - \lambda_0^2)^2}\int Bn_e dl \\ &\quad + \frac{e^3}{2\pi m^2 c^3}\lambda^3 \int Bn_e dl \end{aligned} \quad (3.8)$$

where N and n_e are excited the atom density and electron density, respectively.

Figure 3 shows that the rotation due to bound electrons is much larger than the one due to free electrons and the former increases as the diagnostic line approaches the absorption line.

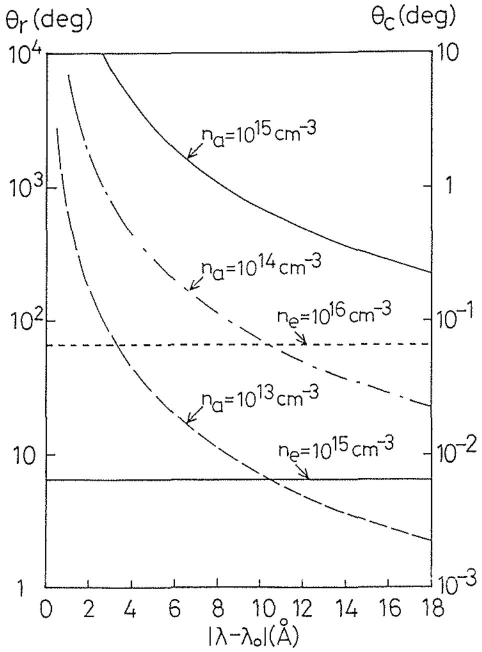

Fig.3 The resonant Faraday rotation θ_r due to bound electrons and classical Faraday rotation θ_c due to free electrons as a function of $\lambda-\lambda_0$
The lines indicated by n_a are for θ_r and those by n_e are for θ_c.
$B=10^4$ G, $l=10^2$cm, $\lambda_0=6563$ Å(H_a).

4. Quantum theory of dispersion without magnetic field

Let us suppose that each atom has one electron with a bound state φ_m (r) initially which is excited to φ_n (r) by the external electric field

$$E(t)=E_x\cos\omega t \qquad (4.1)$$

Electron wave function is described by

$$\varphi(r,t)=\zeta_m e^{-iE_m t/\hbar}+\sum_n c_n\varphi_n e^{-iE_n t/\hbar} \qquad (4.2)$$

Here we apply a time-dependent perturbation theory. If we put Eq. (4.2) into time-dependent Schrödinger equation the coefficients in the excited states satisfy the next differential equation

$$i\hbar\frac{dc_n}{dt}=\int\varphi_n^*(e\mathbf{E}(\mathbf{t})\cdot\mathbf{r})\varphi_m e^{i(E_n-E_m)t/\hbar}\mathbf{dr} \qquad (4.3)$$

The solution[3] is

$$\begin{aligned}C_n&=\frac{1}{2i\hbar}\int eE_xX_{nm}(e^{i\omega t}+e^{-i\omega t})e^{i(E_n-E_m)t/\hbar}dt\\&=\frac{1}{2}eE_xX_{nm}\left[\frac{1-e^{i(\hbar\omega+E_n-E_m)t/\hbar}}{\hbar\omega+(E_n-E_m)}\right.\\&\quad\left.-\frac{1-e^{-i(\hbar\omega-(E_n-E_m))\,t/\hbar}}{\hbar\omega-(E_n-E_m)}\right]\end{aligned} \qquad (4.4)$$

where

$$X_{nm}=\int\varphi_n^*\times\varphi_m dx \qquad (4.5)$$

which is the matrix element of the electron dipole moment in the direction of the electric field vector, between states φ_m and φ_n.

From Eqs. (4.4) and (4.5) we have

$$<ex(t)> = \int \varphi_n^*(\mathbf{r},t)\, ex\, \varphi_m(\mathbf{r},t)\, \mathbf{d\,r}$$

$$= \sum_{\mu} [ex_{mn} C_n(t) e^{-i\omega t} + ex_{nm} C_n^*(t) e^{i\omega t}]$$

$$= E_x \sum_n \frac{e^2|X_{mn}|^2}{2\hbar} \left[\frac{1}{\omega_n - \omega} + \frac{1}{\omega_n + \omega}\right] (e^{i\omega t} + e^{-i\omega t}) \qquad (4.6)$$

Hence atomic polarization $\alpha(=P/NE)$ is given by

$$\alpha = \sum_n \frac{e^2|X_{mn}|^2}{\hbar} \frac{2\omega_n}{\omega_n^2 - \omega^2} \qquad (4.7)$$

Here oscillator strength of the n-th transition is defined by

$$f_n = \frac{2m}{\hbar^2} \hbar \omega_n |X_{mn}|^2$$

$$\sum_n f_n = 1 \qquad (4.8)$$

Then we have

$$\alpha(\omega) = \frac{e^2}{m} \sum_n \frac{f_n}{\omega_n^2 - \omega^2} \qquad (4.9)$$

If N is the number of atoms per unit volume, dielectric constant is given by

$$\Sigma(\omega) = 1 + 4\pi N \alpha(\omega)$$

$$= 1 + \frac{4\pi N e^2}{m} \sum_n \frac{f_n}{\omega_n^2 - \omega^2} \qquad (4.10)$$

which is consistent with Eq. (2.16) neglecting the γ term.

5. Quantum theory of Faraday rotation

We now consider the Faraday rotation due to electrons bound in atoms. We neglect the spin-orbit and Doppler effect and assume that the electron may be described by a hamiltonian of hydrogen form in c.g.s. units,

$$H = \frac{1}{2m}(\mathbf{p} - \frac{e\mathbf{A}}{c})^2 - \frac{e^2}{\varepsilon r} \qquad (5.1)$$

where ε is the static dielectric constant, m is the electron mass, $\mathbf{P} = -i\hbar$ grad. For the external magnetic field $\mathbf{B}$ along the z-direction we may choose the vector potential $\mathbf{A} = (\mathbf{B} \times \mathbf{r})/2$. Then Eq. (5.1) becomes

$$H = \frac{1}{2m}\left[(P_x + \frac{eBy}{2c})^2 + (P_y - \frac{eBx}{2c})^2 + P_z^2\right] - \frac{e^2}{\varepsilon r} \qquad (5.2)$$

The eigenstates and eigenvalues of Eq. (5.2) can be written in pairs as follows[4)]

$$\Psi_{n,m}(r,\theta,\phi) = \Phi_{n,|m|}(r,\theta) e^{im\phi}$$

$$E_{n,m} = E_{n,|m|} + \frac{1}{2}\hbar\omega_c m \qquad (5.3)$$

Where $m = 0, \pm 1, \pm 2, ..$ and $\omega_c = eB/mc$ is the cyclotron frequency of electrons. The transitions from the ground state indicated by the right- and left-handed wave correspond to the states (n, m=1) and (n, m=−1), respectively.

The methods of time-dependent perturbation theory can be used to obtain an expression for the dielectric constant tensor[3,5] in the same manner as in section 3:

$$\varepsilon_{ij}=\delta_{ij}-\frac{4\pi e^2}{\hbar\omega}\sum_k\sum_{k'}\frac{1}{\omega_{k'k}}\left[\frac{(v_i)_{kk'}(v_j)_{k'k}}{\omega+\omega_{k'k}}-\frac{(v_j)_{kk'}(v_i)_{k'k}}{\omega_{k'k}-\omega}\right]$$
$$-i\frac{4\pi^2e^2}{\hbar\omega}\sum_k\sum_{k'}\frac{1}{\omega_{k'k}}\Big[(v_i)_{kk'}(v_j)_{k'k}\delta(\omega+\omega_{k'k})$$
$$+(v_j)_{kk'}(v_i)_{k'k}\delta(\omega_{k'k}-\omega)\Big] \qquad (5.4)$$

Here the frequency $\omega_{k'k}$ are such that $\hbar\omega_{k'k}$ is the energy reguired to excite an electron from state k to k′. The velocity operator is related to the position operator by

$$(v)_{k'k}=i\omega_{k'k}(r)_{k'k}. \qquad (5.5)$$

At frequencies where there is hardly any absorption, a rotation of the plane of polarization θ_r per unit path length is given by

$$\theta_r=-\frac{\omega}{2c}(n_+-n_-)=-\frac{\omega}{2c}\frac{n_+^2-n_-^2}{2n}$$
$$=-\frac{\omega}{4nc}(\varepsilon_+-\varepsilon_-)=i\frac{\omega\varepsilon_{xy}}{2nc} \qquad (5.6)$$

Here $n=(n_+-n_-)/2$ may be approxinmated to the refractive index at zero field and ε_{xy} is the xy component of the imaginary part of dielectric constant tensor ε_{ij}.

If we calculate the component ε_{xy} neglecting the δ-function term in Eq. (5.4) we have the Faraday rotation from Eq. (5.6) as

$$\theta_r=-v\frac{2\pi e^2}{nc\hbar}\sum_k\sum_{k'}\frac{(v_x)_{kk'}(v_y)_{k'k}-(v_y)_{kk'}(v_x)_{k'k}}{\omega_{k'k}^2-\omega^2} \qquad (5.7)$$

Equation (5.7) can be written in terms of velocity matrix elements for the right and left circularly polarized wave as

$$\theta_r=-\frac{\pi e^2}{nc\hbar}\sum_k\sum_{k'}\frac{v_{k'k}(+)v_{k'k}^*(+)-v_{k'k}(-)v_{k'k}^*(-)}{\omega_{k'k}^2-\omega^2} \qquad (5.8)$$

in which $v(\pm)=v_x\pm iv_y$.

The definition of oscillator strength for absorption of the right- and left-handed polarization is

$$f_{k'k}(\pm)=mv_{k'k}^*(\pm)v_{k'k}(\pm)/\hbar\omega_{k'k} \qquad (5.9)$$

Then Eq. (5.8) becomes

$$\theta_r=-\frac{1}{2}\frac{2\pi e^2}{ncm}\sum_k\sum_{k'}\frac{\omega_{k'k}(f_{k'k}(+)-f_{k'k}(-1))}{\omega_{k'k}^2-\omega^2} \qquad (5.10)$$

Equation (5.10) gives the rotation θ_r per bound electron as

$$\theta_r=-\frac{1}{2}\frac{2\pi e^2}{ncm}\sum_n\left[\frac{\omega_n^+f_n^+}{(\omega_n^+)^2-\omega^2}-\frac{\omega_n^-f_n^-}{(\omega_n^-)^2-\omega^2}\right] \qquad (5.11)$$

in which the transition frequencies are, by Eq. (5.3)

$$\omega_n^{\mp}=\omega_n\pm\frac{1}{2}\omega_c \qquad (5.12)$$

where $\hbar\omega_n = E_{n,1}$. From Eq. (5.12) and the definition of oscillator strengths, we see that

$$\frac{f_n^+}{(1+\omega_c/2\omega_n)} = \frac{f_n^-}{(1-\omega_c/2\omega_n)} \equiv f_n \qquad (5.13)$$

then Eq. (5.11) becomes

$$\theta_r = -\frac{1}{2}\frac{2\pi e^2}{ncm}\sum_n\left[\frac{(\omega_n+\frac{1}{2}\omega_c)^2}{(\omega_n+\frac{1}{2}\omega_c)^2-\omega^2} - \frac{(\omega_n-\frac{1}{2}\omega_c)^2}{(\omega_n-\frac{1}{2}\omega_c)^2-\omega^2}\right]\frac{f_n}{\omega_n} \qquad (5.14)$$

Equation (5.14) becomes, to the first order in B

$$\theta_r = \frac{1}{2}\frac{4\pi e^2}{ncm}\sum_M \frac{\omega_c\omega^2 f_n}{(\omega_n^2-\omega^2)^2} \qquad (5.15)$$

per unit length and bound electron, which is consistent with Eq. (3.7).
Here f_n is the oscillator strength in the zero field.

6. Experimental apparatus proposed

The experimental apparatus proposed in the investigation of resonant Faraday rotation and holographic interferometry is shown in Fig. 4.

The tunable dye laser DL is used as a light source in this experiment.

Holograms are recorded on Kodak type 649F holographic plates HP, with a plasma and a non-plasma exposure superimposed.

A small angular shift of the reference beam is made by the mirror M_3, between exposures to provide a background fringe pattern.

On the other hand, the light from the source is linearly polarized by the Nichol prism

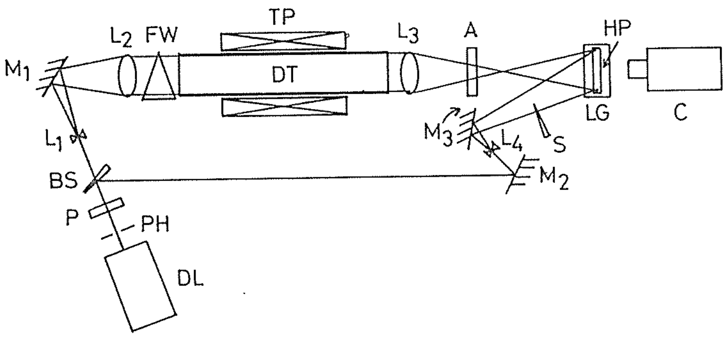

Fig.4 Proposed arrangement of the experimental apparatus for resonant Faraday rotation and holographic interferometry of a θ-pinch plasma.
TP-θ-pinch apparatus; DT-discharge tube; DL-dye laser; PH-pinhole; P-poralizer; BS-beam splitter; $L_1 \sim L_3$-Lens; $M_1 \sim M_2$-fixed mirror; M_3-rotating mirror; FW-Faraday wedge; A-analizer; HP-holographic plate; LG-liquid gate; C-image converter camera; S-shield plate.

P and then traverses the discharge tube DT before passing through the analyzing Nichol A to the holographic plate.

The Faraday wedge FW made of a high Verdet-constant material is inserted into the object beam in order to produce fractional background fringes in the Faraday rotation photographs.

The plates are developed in Kodak D-19 developer, stopped, fixed, washed, and bleached in the liquid gate LG for a rapid in situ processing in real-time holographic interferometry.

With a high speed image converter camera C, holographic interferograms and Faraday rotation photographs of a θ-pinch plasma are obtained.

Acknowledgement

We wish to thank Dr. M. Otsuka of Nagoya Univ., and also Dr. K. Murata and Dr. T. Asakura of Hokkaido Univ. for their many helpful discussions and valuable advice.

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