Uncertainty Principle of the 2-D Affine Generalized Fractional Fourier Transform

Soo-Chang Pei and Jian-Jiun Ding†*
Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, 10617
E-mail: pei@cc.ee.ntu.edu.tw *, djj@cc.ee.ntu.edu.tw †, TEL: 886-2-33663700-321

***Abstract*— The uncertainty principles of the 1-D fractional Fourier transform and the 1-D linear canonical transform have been derived. We extend the previous works and discuss the uncertainty principle for the two-dimensional affine generalized Fourier transform (2-D AGFFT). We find that derived uncertainty principle of the 2-D AGFFT can also be used for determining the uncertainty principles of many 2-D operations, such as the 2-D fractional Fourier transform, the 2-D linear canonical transform, and the 2-D Fresnel transform. These uncertainty principles are useful for time-frequency analysis and signal analysis. Moreover, we find that the rotation and the chirp multiplication of the 2-D Gaussian function can satisfy the lower bound of the uncertainty principle of the 2-D AGFFT.**

I. INTRODUCTION

The well-known Heisenberg uncertainty principle states that, if $X(\omega)$ is the 1-D Fourier transform (FT) of $x(t)$

$$\textbf{FT:}\quad X(\omega)=FT\left[x(t)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt \tag{1}$$

and the 2nd moments of time and frequency are

$$\Delta_t^2=\int_{-\infty}^{\infty}t^2\left|x(t)\right|^2dt\Big/\int_{-\infty}^{\infty}\left|x(t)\right|^2dt, \tag{2}$$

$$\Delta_\omega^2=\int_{-\infty}^{\infty}\omega^2\left|X(\omega)\right|^2d\omega\Big/\int_{-\infty}^{\infty}\left|X(\omega)\right|^2d\omega, \tag{3}$$

when $\int_{-\infty}^{\infty}\left|x(t)\right|^2dt=1$, the following inequality is satisfied [1]

$$\Delta_t^2\Delta_\omega^2\geq\frac{1}{4}. \tag{4}$$

Then, in [2], the uncertainty principle was generalized into the case of the 1-D fractional Fourier transform (FRFT) [3]:

$$\textbf{FRFT:}\quad X_\alpha(u)=\sqrt{\frac{1-j\cot\alpha}{2\pi}}e^{ju^2\frac{\cot\alpha}{2}}\int_{-\infty}^{\infty}e^{-jut\csc\alpha}e^{jt^2\frac{\cot\alpha}{2}}x(t)\,dt. \tag{5}$$

If

$$\Delta_u^2=\int_{-\infty}^{\infty}u^2\left|X_\alpha(u)\right|^2du\Big/\int_{-\infty}^{\infty}\left|X_\alpha(u)\right|^2du, \tag{6}$$

then

$$\Delta_t^2\Delta_u^2\geq\frac{\sin^2\alpha}{4}. \tag{7}$$

Recently, the uncertainty principle was generalized into the case of the 1-D linear canonical transform (LCT) [4][5]. If

$$\textbf{LCT:}\quad X_{(a,b,c,d)}(u)=\sqrt{\frac{1}{j2\pi b}}e^{ju^2\frac{d}{2b}}\int_{-\infty}^{\infty}e^{-j\frac{ut}{b}}e^{jt^2\frac{a}{2b}}x(t)\,dt, \tag{8}$$

then

$$\Delta_t^2\Delta_u^2\geq\frac{b^2}{4}, \tag{9}$$

where

$$\Delta_u^2=\int_{-\infty}^{\infty}u^2\left|X_{(a,b,c,d)}(u)\right|^2du\Big/\int_{-\infty}^{\infty}\left|X_{(a,b,c,d)}(u)\right|^2du. \tag{10}$$

The uncertainty principle of the 1-D case has been discussed a lot. In this paper, we extend the previous works and derive the uncertainty principle for the two dimensional affine generalized fractional Fourier transform (2-D AGFFT). **The derived uncertainty principle is shown in Theorem 2**. As Heisenberg's uncertainty principle, the derived uncertainty principle will be useful in signal processing applications, such as time-frequency analysis, signal synthesis, communication, sampling theory, and filter design.

Moreover, since many 2-D operations are the special cases of the 2-D AGFFT (such as the 2-D FRFT and the 2-D Fresnel transform), we can use the derived uncertainty principle to find the uncertainty principles for these operations.

II. TWO-DIMENSIONAL AFFINE GENERALIZED FRACTIONAL FOURIER TRANSFORM

The two-dimensional affine generalized fractional Fourier transform **(2-D AGFFT)** is defined as [6][7]

$$G_{(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})}(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_{(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})}(u,v,x,y)\cdot g(x,y)\cdot dxdy, \tag{11}$$

where

$$\mathbf{A}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}d_{11}&d_{12}\\d_{21}&d_{22}\end{bmatrix} \tag{12}$$

represents the 16 parameters of 2-D AGFFT, and

$$K_{(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})}(u,v,x,y)=\frac{1}{2\pi\sqrt{-\det(\mathbf{B})}}e^{\frac{j}{2\det(\mathbf{B})}\left(k_1\cdot u^2+k_2\cdot u\cdot v+k_3\cdot v^2\right)}$$
$$e^{\frac{j}{\det(\mathbf{B})}\left((-b_{22}u+b_{12}v)x+(b_{21}u-b_{11}v)y\right)}\;e^{\frac{j}{2\det(\mathbf{B})}\left(p_1\cdot x^2+p_2\cdot x\cdot y+p_3\cdot y^2\right)}, \tag{13}$$

where $k_1=d_{11}b_{22}-d_{12}b_{21}$, $k_2=2(-d_{11}b_{12}+d_{12}b_{11})$,
$k_3=-d_{21}b_{12}+d_{22}b_{11}$, $p_1=a_{11}b_{22}-a_{21}b_{12}$,
$p_2=2(a_{12}b_{22}-a_{22}b_{12})$, $p_3=-a_{12}b_{21}+a_{22}b_{11}$. (14)

Moreover, the following constraints should be satisfied [6][7]:

$$\mathbf{A}^\mathbf{T}\mathbf{C}=\mathbf{C}^\mathbf{T}\mathbf{A},\quad \mathbf{B}^\mathbf{T}\mathbf{D}=\mathbf{D}^\mathbf{T}\mathbf{B},\quad \mathbf{A}^\mathbf{T}\mathbf{D}-\mathbf{C}^\mathbf{T}\mathbf{B}=\mathbf{I}. \tag{15}$$

The 2-D AGFFT is useful for filter design, signal analysis, data compression, communication, optics, and image processing [6]. It is a generalization of many 2-D operations. For example, the 2-D FT is a special case of the AGFFT where

$b_{11}=b_{22}=1$, $c_{11}=c_{22}=-1$, $a_{11}=a_{12}=a_{21}=a_{22}=0$,
$b_{12}=b_{21}=c_{12}=c_{21}=d_{11}=d_{12}=d_{21}=d_{22}=0$. (16)

The 2-D fractional Fourier transform (2-D FRFT) [3] is:

2-D FRFT: $G_{\alpha,\beta}(u,v)=\frac{\sqrt{(1-j\cot\alpha)(1-j\cot\beta)}}{2\pi}e^{\frac{j}{2}(u^2\cot\alpha+v^2\cot\beta)}\times$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j(ux\csc\alpha+vy\csc\beta)}e^{\frac{j}{2}(x^2\cot\alpha+y^2\cot\beta)}g(x,y)dxdy\ . \quad (17)$$

It is a special case of the 2-D AGFFT where

$a_{11}=d_{11}=\cos\alpha$, $b_{11}=-c_{11}=\sin\alpha$, $a_{22}=d_{22}=\cos\beta$, $b_{22}=-c_{22}=\sin\beta$,

$a_{21}=a_{12}=b_{12}=b_{21}=c_{12}=c_{21}=d_{12}=d_{21}=0$. (18)

The 2-D linear canonical transform (LCT) is defined as

2-D LCT: $G_{(a,b,c,d,a_1,b_1,c_1,d_1)}(u,v)=\frac{1}{2\pi\sqrt{-bb_1}}e^{j(\frac{d}{2b}u^2+\frac{d_1}{2b_1}v^2)}\times$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j(\frac{ux}{b}+\frac{vy}{b_1})}e^{j(\frac{a}{2b}x^2+\frac{a_1}{2b_1}y^2)}g(x,y)dxdy\ . \quad (19)$$

It is a special case of the 2-D AGFFT where

$a_{11}=a$, $a_{22}=a_1$, $b_{11}=b$, $b_{22}=b_1$, $c_{11}=c$, $c_{22}=c_1$, $d_{11}=d$, $d_{22}=d_1$,

$a_{21}=a_{12}=b_{12}=b_{21}=c_{12}=c_{21}=d_{12}=d_{21}=0$. (20)

The **2-D Fresnel transform** is:

$$G_{(z,z_1)}(u,v)=-i\frac{e^{jkz}}{\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j\frac{\pi}{\lambda z}\left[(u-x)^2+(v-y)^2\right]}g(x,y)dxdy\ . \quad (21)$$

It describes the light propagation in the free space. If the constant phase is ignored, the 2-D Fresnel transform can be viewed as the special case of the AGFFT where

$a_{11}=a_{22}=d_{11}=d_{22}=1$, $b_{11}=b_{22}=\lambda z/2\pi$, (22)

$a_{21}=a_{12}=b_{12}=b_{21}=c_{11}=c_{12}=c_{21}=c_{22}=d_{12}=d_{21}=0$. (23)

III. UNCERTAINTY PRINCIPLE OF THE 2-D AGFFT

As the 1-D case, in this paper, we always suppose that the signal $g(x, y)$ is normalized

$$\int_{-\infty}^{\infty}|g(x,y)|^2\,dxdy=1\ . \quad (24)$$

We will try to find the lower bound of $\Delta_{x,y}^2\Delta_{u,v}^2$, where

$$\Delta_{x,y}^2=\int_{-\infty}^{\infty}\left(x^2+y^2\right)|g(x,y)|^2\,dxdy\ , \quad (25)$$

$$\Delta_{u,v}^2=\int_{-\infty}^{\infty}\left(u^2+v^2\right)\left|G_{(A,B,C,D)}(u,v)\right|^2 dudv\ . \quad (26)$$

and $G_{(A,B,C,D)}(u, v)$ is the 2-D AGFFT (defined in (11)-(15)) of $g(x, y)$. The formula of the 2-D AGFFT is very complicated. It has 16 parameters. We should use some ways to simplify the derivation of the uncertainty principle.

[Lemma 1] First, note that, if

$$g_0(x,y)=e^{\frac{j}{2\det(\mathbf{B})}\left(p_1\cdot x^2+p_2\cdot x\cdot y+p_3\cdot y^2\right)}g(x,y), \quad (27)$$

$$H(u,v)=e^{\frac{-j}{2\det(\mathbf{B})}\left(k_1\cdot u^2+k_2\cdot u\cdot v+k_3\cdot v^2\right)}G_{(A,B,C,D)}(u,v), \quad (28)$$

then, since $|g_0(x, y)| = |g(x, y)|$ and $|H(u, v)| = |G_{(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})}(u, v)|$,

$$\Delta_{x,y}^2=\int_{-\infty}^{\infty}\left(x^2+y^2\right)|g_0(x,y)|^2\,dxdy\ , \quad (29)$$

$$\Delta_{u,v}^2=\int_{-\infty}^{\infty}\left(u^2+v^2\right)|H(u,v)|^2\,dudv\ . \quad (30)$$

Note that

$$H(u,v)=\frac{1}{2\pi\sqrt{-\det(\mathbf{B})}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{j}{\det(\mathbf{B})}((-b_{22}u+b_{12}v)x+(b_{21}u-b_{11}v)y)}g_0(x,y)dxdy\ . \quad (31)$$

[Lemma 2] Moreover, the rotation operation does not affect the 2nd order moment. That is, if

$$g_1(x,y)=g_0(x\cos\theta+y\sin\theta,-x\sin\theta+y\cos\theta)\ , \quad (32)$$

$$H_1(u,v)=H(u\cos\phi+v\sin\phi,-u\sin\phi+v\cos\phi)\ , \quad (33)$$

then

$$\int_{-\infty}^{\infty}\left(x^2+y^2\right)|g_1(x,y)|^2\,dxdy=\int_{-\infty}^{\infty}\left(x^2+y^2\right)|g_0(x,y)|^2\,dxdy\ , \quad (34)$$

$$\int_{-\infty}^{\infty}\left(u^2+v^2\right)|H_1(u,v)|^2\,dudv=\int_{-\infty}^{\infty}\left(u^2+v^2\right)|H(u,v)|^2\,dudv\ . \quad (35)$$

Substituting (32) and (33) into (31), we obtain

$$H_1(u,v)=\frac{1}{2\pi\sqrt{-\det(\mathbf{B})}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j((\eta_1u+\eta_2v)x+(\eta_3u+\eta_4v)y)}g_1(x,y)dxdy\ , \quad (36)$$

where η_1, η_2, η_3, and η_4 can be calculated from:

$$\begin{bmatrix}\eta_1 & \eta_2\\ \eta_3 & \eta_4\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\frac{b_{22}}{\det(\mathbf{B})} & \frac{-b_{12}}{\det(\mathbf{B})}\\ \frac{-b_{21}}{\det(\mathbf{B})} & \frac{b_{11}}{\det(\mathbf{B})}\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}. \quad (37)$$

Note that, if η_2 and η_3 are zero, the relation between $H_1(u, v)$ and $g_1(x, y)$ in (36) will be simplified into the 2-D scaled FT. The uncertainty principle of the 2-D scaled FT is easier to find. To make $\eta_2 = \eta_3 = 0$, θ and ϕ should satisfy

$$\begin{bmatrix}\frac{b_{22}}{\det(\mathbf{B})} & \frac{-b_{12}}{\det(\mathbf{B})}\\ \frac{-b_{21}}{\det(\mathbf{B})} & \frac{b_{11}}{\det(\mathbf{B})}\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\eta_1 & 0\\ 0 & \eta_4\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}, \quad (38)$$

$$b_{11}+b_{22}=(\eta_1^{-1}+\eta_4^{-1})\cos(\phi-\theta)\ ,\ b_{11}-b_{22}=(\eta_1^{-1}-\eta_4^{-1})\cos(\phi+\theta)\ ,$$
$$b_{12}+b_{21}=(\eta_4^{-1}-\eta_1^{-1})\sin(\phi+\theta)\ ,\ b_{12}-b_{21}=(\eta_4^{-1}+\eta_1^{-1})\sin(\phi-\theta)\ . \quad (39)$$

Therefore,

$$\sqrt{(b_{11}+b_{22})^2+(b_{12}-b_{21})^2}=\left|\eta_1^{-1}+\eta_4^{-1}\right|, \quad (40)$$

$$\sqrt{(b_{11}-b_{22})^2+(b_{21}+b_{21})^2}=\left|\eta_1^{-1}-\eta_4^{-1}\right|. \quad (41)$$

Thus, we can choose

$$\eta_1=2/(\sqrt{(b_{11}+b_{22})^2+(b_{12}-b_{21})^2}+\sqrt{(b_{11}-b_{22})^2+(b_{12}+b_{21})^2})$$
$$\eta_4=2/(\sqrt{(b_{11}+b_{22})^2+(b_{12}-b_{21})^2}-\sqrt{(b_{11}-b_{22})^2+(b_{12}+b_{21})^2})$$

if $(b_{11}+b_{22})^2 + (b_{12}-b_{21})^2 > (b_{11}-b_{22})^2 + (b_{12}+b_{21})^2$ (42)

and

$$\eta_1=2/(\sqrt{(b_{11}+b_{22})^2+(b_{12}-b_{21})^2}-\sqrt{(b_{11}-b_{22})^2+(b_{12}+b_{21})^2})$$
$$\eta_4=2/(\sqrt{(b_{11}+b_{22})^2+(b_{12}-b_{21})^2}+\sqrt{(b_{11}-b_{22})^2+(b_{12}+b_{21})^2})$$

if $(b_{11}+b_{22})^2 + (b_{12}-b_{21})^2 < (b_{11}-b_{22})^2 + (b_{12}+b_{21})^2$. (43)

Then, from (39),

$$\phi=(\psi_1+\psi_2)/2, \qquad \theta=(\psi_1-\psi_2)/2, \quad (44)$$

where $\psi_1=\cos^{-1}\frac{b_{11}-b_{22}}{\eta_1^{-1}-\eta_4^{-1}}=\sin^{-1}\frac{b_{12}+b_{21}}{\eta_1^{-1}-\eta_4^{-1}}$,

$$\psi_2=\cos^{-1}\frac{b_{11}+b_{22}}{\eta_1^{-1}+\eta_4^{-1}}=\sin^{-1}\frac{b_{12}-b_{21}}{\eta_1^{-1}+\eta_4^{-1}}\ . \quad (45)$$

If we choose η_1, η_2, η_3, η_4, ϕ, and θ as (42) (or (43)) and (44), $\eta_2 = \eta_3 = 0$ and the relation between $H_1(u, v)$ and $g_1(x, y)$ in (36) becomes the 2-D scaled FT.

$$H_1(u,v) = \frac{\sqrt{-\eta_1\eta_4}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j(\eta_1 ux+\eta_4 vy)} g_1(x,y)\,dxdy. \qquad (46)$$

[Theorem 1] For the 2-D **scaled** Fourier transform:

$$G_{SF}(f,h) = \frac{\sqrt{-\sigma_1\sigma_2}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j(\sigma_1 fx+\sigma_2 hy)} g(x,y)\,dxdy. \qquad (47)$$

If $\Delta_{x,y}^2 = \int_{-\infty}^{\infty}(x^2+y^2)|g(x,y)|^2 dxdy$,

$$\Delta_{SF}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(f^2+h^2)|G_{SF}(f,h)|^2 dfdh, \qquad (48),$$

then

$$\Delta_{x,y}^2\Delta_{SF}^2 \ge \frac{1}{4}(|\sigma_1^{-1}|+|\sigma_2^{-1}|)^2. \qquad (49)$$

(Proof): Since $G_{SF}(f,h) = \sqrt{\sigma_1\sigma_2}G(\sigma_1 f,\sigma_2 h)$, where $G(f, h)$ is the FT of $g(x, y)$, if we set $f_1 = \sigma_1 f$ and $h_1 = \sigma_2 h$, then $dfdh = df_1dh_1/|\sigma_1\sigma_2|$ and (47) becomes

$$\Delta_{SF}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(p^2f_1^2+q^2h_1^2)|G(f_1,h_1)|^2 df_1dh_1. \qquad (50)$$

where $p = 1/\sigma_1$ and $q = 1/\sigma_2$. Then since

$$\begin{aligned}&(p^2f^2+q^2h^2)|G(f,h)|^2\\ &= (-|p|f+j|q|h)G(f,h)(-|p|f-j|q|h)G^*(f,h),\end{aligned} \qquad (51)$$

$$IFT\left[(-|p|f+j|q|)G(f,h)\right] = [j|p|\frac{\partial}{\partial x}+|q|\frac{\partial}{\partial y}]g(x,y), \quad (52)$$

from Parseval's Theorem of the 2-D FT:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(x,y)|^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|G(f,h)|^2 dfdh, \qquad (53)$$

if $G(f, h) = FT[g(x, y)]$, (51) can be rewritten as

$$\Delta_{SF}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[j|p|\frac{\partial}{\partial x}+|q|\frac{\partial}{\partial y}]g(x,y)[-j|p|\frac{\partial}{\partial x}+|q|\frac{\partial}{\partial y}]\,g^*(x,y)\,dxdy. \qquad (54)$$

Furthermore, in (48),

$$(x^2+y^2)|g(x,y)|^2 = (jx+y)g(x,y)(-jx+y)g^*(x,y). \qquad (55)$$

Therefore,

$$\Delta_{x,y}^2\Delta_{SF}^2 = \|(jx+y)g(x,y)\|^2\,\|[j|p|\frac{\partial}{\partial x}+|q|\frac{\partial}{\partial y}]g(x,y)\|^2, \quad (56)$$

Then, from Cauchy-Schwartz inequality,

$$\|f(x,y)\|^2\|g(x,y)\|^2 \ge |\langle f(x,y),g(x,y)\rangle|^2, \qquad (57)$$

$$\|f(x,y)\|^2\|g(x,y)\|^2 \ge [|\langle f(x,y),g(x,y)\rangle|^2+|\langle f^*(x,y),g^*(x,y)\rangle|^2]/2, \qquad (58)$$

(56) can be rewritten as:

$$\begin{aligned}&\Delta_{x,y}^2\Delta_{SF}^2 \ge\\ &\frac{1}{2}\left|\left\langle jxg,j|p|\frac{\partial}{\partial x}g\right\rangle+\left\langle yg,j|p|\frac{\partial}{\partial x}g\right\rangle+\left\langle jxg,|q|\frac{\partial}{\partial y}g\right\rangle+\left\langle yg,|q|\frac{\partial}{\partial y}g\right\rangle\right|^2+\\ &\frac{1}{2}\left|\left\langle j|p|\frac{\partial}{\partial x}g,jxg\right\rangle+\left\langle j|p|\frac{\partial}{\partial x}g,yg\right\rangle+\left\langle |q|\frac{\partial}{\partial y}g,jxg\right\rangle+\left\langle |q|\frac{\partial}{\partial y}g,yg\right\rangle\right|^2.\end{aligned} \quad (59)$$

Note that (56) can also be expressed as

$$\Delta_{x,y}^2\Delta_{SF}^2 = \|(jx-y)g(x,y)\|^2\,\|[j|p|\frac{\partial}{\partial x}-|q|\frac{\partial}{\partial y}]g(x,y)\|^2. \quad (60)$$

From the similar process, we obtain

$$\begin{aligned}&\Delta_{x,y}^2\Delta_{SF}^2 \ge\\ &\frac{1}{2}\left|\left\langle jxg,j|p|\frac{\partial}{\partial x}g\right\rangle-\left\langle yg,j|p|\frac{\partial}{\partial x}g\right\rangle-\left\langle jxg,|q|\frac{\partial}{\partial y}g\right\rangle+\left\langle yg,|q|\frac{\partial}{\partial y}g\right\rangle\right|^2+\\ &\frac{1}{2}\left|\left\langle j|p|\frac{\partial}{\partial x}g,jxg\right\rangle-\left\langle j|p|\frac{\partial}{\partial x}g,yg\right\rangle-\left\langle |q|\frac{\partial}{\partial y}g,jxg\right\rangle+\left\langle |q|\frac{\partial}{\partial y}g,yg\right\rangle\right|^2.\end{aligned} \quad (61)$$

Adding (61) by (63) and using the fact that

$$|a|^2+|b|^2+|c|^2+|d|^2 \ge 4\left|\frac{a+b+c+d}{4}\right|^2, \qquad (62)$$

we obtain

$$\begin{aligned}\Delta_{x,y}^2\Delta_{SF}^2 \ge \frac{1}{4}\Bigg|&\left\langle jxg,j|p|\frac{\partial}{\partial x}g\right\rangle+\left\langle j|p|\frac{\partial}{\partial x}g,jxg\right\rangle+\\ &\left\langle yg,|q|\frac{\partial}{\partial y}g\right\rangle+\left\langle |q|\frac{\partial}{\partial y}g,yg\right\rangle\Bigg|^2.\end{aligned} \qquad (63)$$

Then,

$$\begin{aligned}&\left\langle jxg,j|p|\frac{\partial}{\partial x}g\right\rangle+\left\langle j|p|\frac{\partial}{\partial x}g,jxg\right\rangle\\ &= |p|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\frac{\partial}{\partial x}\left[g(x,y)g^*(x,y)\right]dxdy\\ &= |p|\int_{-\infty}^{\infty}\left[xg(x,y)g^*(x,y)\Big|_{x=-\infty}^{x=\infty}-\int_{-\infty}^{\infty}g(x,y)g^*(x,y)dx\right]dy\\ &= -|p|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(x,y)|^2 dxdy = -|p|.\end{aligned} \qquad (64)$$

Similarly,

$$\left\langle yg,|q|\frac{\partial}{\partial y}g\right\rangle+\left\langle |q|\frac{\partial}{\partial y}g,yg\right\rangle = -|q|. \qquad (65)$$

Therefore,

$$\Delta_{x,y}^2\Delta_{pf,qh}^2 \ge \frac{1}{4}(|p|+|q|)^2 = \frac{1}{4}(|\sigma_1^{-1}|+|\sigma_2^{-1}|)^2. \qquad \#$$

Lemmas 1 and 2 and Theorem 1 much simplify the derivation of the uncertainty principle. From (29), (30), (34), (35), (46), and (49),

$$\Delta_{x,y}^2\Delta_{u,v}^2 \ge \frac{1}{4}(|\eta_1^{-1}|+|\eta_4^{-1}|)^2. \qquad (66)$$

Moreover, from (42) and (43)

$$\begin{aligned}&|\eta_1^{-1}|+|\eta_4^{-1}| =\\ &\max\left(\sqrt{(b_{11}+b_{22})^2+(b_{12}-b_{21})^2},\sqrt{(b_{11}-b_{22})^2+(b_{12}+b_{21})^2}\right).\end{aligned} \quad (67)$$

Thus, we obtain:

[Theorem 2] Uncertainty Principle of the 2-D AGFFT:

If $\Delta^2_{x,y}$ and $\Delta^2_{u,v}$ are the 2nd order moments of $g(x, y)$ and the 2-D AGFFT of $g(x, y)$, as in (25) and (26), respectively, then

$$\begin{aligned}&\Delta_{x,y}^2\Delta_{u,v}^2 \ge\\ &\frac{1}{4}\max\left((b_{11}+b_{22})^2+(b_{12}-b_{21})^2,(b_{11}-b_{22})^2+(b_{12}+b_{21})^2\right).\end{aligned} \quad (68)$$

IV. THE RELATED PRINCIPLES

[Remark 1] More generally, if $G_{(A_1,B_1,C_1,D_1)}(u,v)$ and $G_{(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})}(u, v)$ are the 2-D AGFFTs of $g(x, y)$ with parameters $\{\mathbf{A_1}, \mathbf{B_1}, \mathbf{C_1}, \mathbf{D_1}\}$ and $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$, respectively, and

$$\Delta_{u_1,v_1}^2 = \int_{-\infty}^{\infty}\left(u_1^2+v_1^2\right)\left|G_{(A_1,B_1,C_1,D_1)}(u,v)\right|^2 dudv,$$

$$\Delta_{u,v}^2 = \int_{-\infty}^{\infty}\left(u^2+v^2\right)\left|G_{(A,B,C,D)}(u,v)\right|^2 dudv, \tag{69}$$

then

$$\Delta_{x,y}^2\Delta_{u,v}^2 \ge \frac{1}{4}\max\Big((q_{11}+q_{22})^2+(q_{12}-q_{21})^2, (q_{11}-q_{22})^2+(q_{12}+q_{21})^2\Big), \tag{70}$$

where $\begin{bmatrix}\mathbf{P} & \mathbf{Q}\\ \mathbf{R} & \mathbf{S}\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{A_1} & \mathbf{B_1}\\ \mathbf{C_1} & \mathbf{D_1}\end{bmatrix}^{-1}$, $\mathbf{Q} = \begin{bmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{bmatrix}$. (71)

This can be proven from the fact that $G_{(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})}(u, v)$ is the 2-D AGFFT of $G_{(A_1,B_1,C_1,D_1)}(u,v)$ with parameters $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}\}$.

[Theorem 3] It is known that, for the 1-D FT, the 1-D Gaussian function can satisfy the lower bound of inequality of the uncertainty principle [1]. For the 2-D AGFFT, **the chirp multiplication and rotation of the 2-D Gaussian** will satisfy the lower bound of inequality of the uncertainty principle. If

$$g(x,y) = \sqrt{\pi^{-1}}\, e^{\frac{-j}{2\det(\mathbf{B})}\left(p_1\cdot x^2+p_2\cdot x\cdot y+p_3\cdot y^2\right)} \times e^{-\frac{1}{2}\left((x\cos\theta-y\sin\theta)^2|\eta_1|+(x\sin\theta+y\cos\theta)^2|\eta_4|\right)}, \tag{72}$$

where p_1, p_2, and p_3 are defined in (14), η_1 and η_4 are calculated from (42) or (43), and θ is determined from (44), then the 2-D AGFFT of $g(x, y)$ is

$$G_{(A,B,C,D)}(u,v) = \sqrt{(-\pi)^{-1}}\, e^{\frac{j}{2\det(\mathbf{B})}\left(k_1\cdot u^2+k_2\cdot u\cdot v+k_3\cdot v^2\right)} \times e^{-\frac{1}{2}\left((u\cos\phi+v\sin\phi)^2|\eta_1|+(-u\sin\phi+v\cos\phi)^2|\eta_4|\right)}, \tag{73}$$

where ϕ is calculated from (44). Then,

$$\Delta_{x,y}^2 = \Delta_{u,v}^2 = \left(\sqrt{1/|\eta_1|}+\sqrt{1/|\eta_4|}\right)/2, \tag{74}$$

$$\Delta_{x,y}^2\Delta_{u,v}^2 = \frac{1}{4}\max\Big((b_{11}+b_{22})^2+(b_{12}-b_{21})^2, (b_{11}-b_{22})^2+(b_{12}+b_{21})^2\Big). \tag{75}$$

Thus, the function in (74) satisfies the **lower bound** of inequality of the uncertainty principle for the 2-D AGFFT.

V. SOME IMPORTANT SPECIAL CASES

Since the 2-D AGFFT is the generalization of many operations, we can use (68) to find the uncertainty principle of these operations.

[Corollary 1] Uncertainty Principle of the 2-D FRFT:

From (68) and (18), if

$$\Delta_{u,v}^2 = \int_{-\infty}^{\infty}\left(u^2+v^2\right)\left|G_{\alpha,\beta}(u,v)\right|^2 dudv, \tag{76}$$

where $G_{\alpha,\beta}(u, v)$ is the 2-D FRFT of $g(x, y)$, as in (7), then

$$\Delta_{x,y}^2\Delta_{u,v}^2 \ge \frac{1}{4}\left(|\sin\alpha|+|\sin\beta|\right)^2. \tag{77}$$

Moreover, when

$$g(x,y) = \sqrt{\pi^{-1}}\, e^{\frac{-j}{2}\left(x^2\cot\alpha+p_3\cdot y^2\cot\beta\right)} e^{-\frac{1}{2}\left(x^2|\csc\alpha|+y^2|\csc\beta|\right)}, \tag{78}$$

the equality that $\Delta_{x,y}^2\Delta_{u,v}^2 = \left(|\sin\alpha|+|\sin\beta|\right)^2/4$ is satisfied.

[Corollary 2] Uncertainty Principle of the 2-D LCT:

From (68) and (20), if

$$\Delta_{u,v}^2 = \int_{-\infty}^{\infty}\left(u^2+v^2\right)\left|G_{(a,b,c,d,a_1,b_1,c_1,d_1)}(u,v)\right|^2 dudv, \tag{79}$$

then

$$\Delta_{x,y}^2\Delta_{u,v}^2 \ge \frac{1}{4}\left(|b|+|b_1|\right)^2. \tag{80}$$

More, the equality is satisfied when

$$g(x,y) = \sqrt{\pi^{-1}}\, e^{-j\left(\frac{a}{2b}x^2+\frac{a_1}{2b_1}y^2\right)} e^{-\frac{1}{2}\left(x^2/|b|+y^2/|b_1|\right)}, \tag{81}$$

[Corollary 3] Uncertainty Principle of the 2-D Fresnel Transform:

$$\Delta_{x,y}^2\Delta_{u,v}^2 \ge \frac{\lambda^2 z^2}{4\pi^2}. \tag{82}$$

The equality that $\Delta_{x,y}^2\Delta_{u,v}^2 = \lambda^2 z^2/4\pi^2$ is satisfied when

$$g(x,y) = \sqrt{\pi^{-1}}\, e^{-j\frac{\pi}{\lambda z}\left(x^2+y^2\right)} e^{-\frac{\pi}{\lambda z}\left(x^2+y^2\right)}. \tag{83}$$

VI. CONCLUSIONS

In this paper, we derived the uncertainty principle of the 2-D AGFFT (See Theorem 2.) We also showed that the lower bound can be achieved by the 2-D Gaussian function with rotation and chirp multiplication (See Theorem 3.) The uncertainty principle of the 2-D AGFFT would be very useful in time-frequency analysis, developing sampling theory in 2-D case, filter design, signal synthesis, and optics.

VII. REFERENCES

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