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On a Solution of Non-linear Differential Equation

$$\frac{\partial u}{\partial t}-\alpha u^2\frac{\partial u}{\partial x}+\gamma\frac{\partial^5 u}{\partial x^5}=0$$

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Résumé

Non-linear partial differential equation:

$$\frac{\partial u}{\partial t}-\alpha u^2\frac{\partial u}{\partial x}+\gamma\frac{\partial^5 u}{\partial x^5}=0,$$

with $\alpha\gamma>0$, has a solution: $u(x,t)=A\cdot\wp(b(x+vt))$, where $\wp(z)$ is Weierstraß' $\wp$-function. $A^2=360\,\gamma b^4/\alpha$, b, and v are constants.

The Weierstraßian $\wp$-function multiplied by the squared Jacobian *sn*-function is found to be a solution of the Korteweg-de Vries equation.

The general evolution equation can be written as:

$$\frac{\partial}{\partial t}u=-\frac{\partial}{\partial x}F(t,x,u,u_x,u_{xx},\cdots), \tag{1}$$

with the conserved density $u=u(x,t)$ and flux F. The subscript denotes differentiation.

If we take:

$$F=\frac{\delta I}{\delta u}, \tag{2}$$

and

$$I=\int_D\left\{\frac{\varepsilon}{2}u^2+\frac{\zeta}{6}u^3+\frac{\alpha}{12}u^4+\frac{\beta}{2}(u_x)^2+\frac{\gamma}{2}(u_{xx})^2\right\}dx, \tag{3}$$

where $\delta I/\delta u$ means the functional derivative of I with regard to u, and α, β, γ, ε, and ζ are constants.

Assuming that all the functions u, u_x, u_{xx}, and u_{xxx}, vanish at the upper and the lower boundaries of domain D, we have

$$\frac{\delta I}{\delta u}=\varepsilon u+\frac{\zeta}{2}u^2+\frac{\alpha}{3}u^3-\beta u_{xx}+\gamma u_{xxxx}. \tag{4}$$

From (1), (2), and (4), we obtain

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$$u_t+\varepsilon u_x+\zeta uu_x+\alpha u^2u_x-\beta u_{xxx}+\gamma u_{xxxxx}=0\,. \tag{5}$$

While, the solution of the higher order Korteweg - de Vries equation:

$$u_t-\alpha u^n u_x+\eta(uu_{xx})_x+\eta' u_x u_{xx}-\beta u_{xxx}+\gamma u_{xxxxx}=0\,, \tag{6}$$

was discussed by several authors. *e.g.* For $n=2$, $\alpha=-45$, $\eta=15$, $\eta'=\beta=0$, and $\gamma=1$, soliton solutions of eq. (6) was found by Hirota's method[1,2]. For $n=2$, $\alpha=-30$, $\eta=\eta'=10$, $\beta=0$, $\gamma=1$, eq. (6) has also soliton solutions[2,3].

An oscillatory solitary wave was observed[4] in a non-linear electric circuit for $n=1$, $\alpha=-1$, $\eta=\eta'=\beta=0$, and $\gamma=-1$ in (6), with reference to the computation[5], although the exact soliton solution for this equation seems to have not yet been found.

In the present paper we shall consider eq. (5) with $\alpha=-\alpha$, $\varepsilon=\zeta=\beta=0$, or eq. (6) with $n=2$ and $\eta=\eta'=\beta=0$, *i.e.*

$$u_t-\alpha u^2u_x+\gamma u_{xxxxx}=0\,. \qquad (\alpha\gamma>0) \tag{7}$$

Assuming a travelling solution with velocity v:

$$u(x,t)=u(x+vt)\equiv u(\xi)\,, \tag{8}$$

with

$$\xi=x+vt\,, \qquad (v=\text{const}) \tag{9}$$

and putting (8) into eq. (7), we obtain

$$vu_\xi-\alpha u^2u_\xi+\gamma u_{\xi\xi\xi\xi\xi}=0\,, \tag{10}$$

which can be integrated to give

$$vu-\frac{\alpha}{3}u^3+\gamma u_{\xi\xi\xi\xi}=-C\,, \tag{11}$$

with an integration constant C.

A solution of eq. (11) can be obtained as

$$u(\xi)=A\cdot\wp(b\xi)\,, \qquad (A:\ \text{real positive}) \tag{12}$$

with a real constant b, and $A^2=360\,\gamma b^4/\alpha$. $\wp(z)=\wp(z|2\omega_1,\ 2\omega_3)$ is the Weierstraßian $\wp$-function with fundamental periods $2\omega_1$ and $2\omega_3$:

$$2\omega_1=2\int_{e_1}^{+\infty}\frac{dz}{\sqrt{4z^3-g_2z-g_3}}\,, \tag{13}$$

and

$$2\omega_3=2i\int_{-\infty}^{e_3}\frac{dz}{\sqrt{-(4z^3-g_2z-g_3)}}\,. \tag{14}$$

$2\omega_1$ is real and $2\omega_3$ is purely imaginary, and $e_i=\wp(\omega_i)$ $(i=1,2,3)$, are all real quantities, for

$$g_2^3-27g_3^2>0\,. \tag{15}$$

We put here $\omega_2=\omega_1+\omega_3$. e_i's satisfy the following cubic equation:

$$4z^3-g_2z-g_3 \equiv 4(z-e_1)(z-e_2)(z-e_3)=0\,, \tag{16}$$

with $e_3<e_2<e_1$.
By means of relations:

$$e_1+e_2+e_3=0\,, \tag{17}$$

$$g_2=-4(e_1e_2+e_2e_3+e_3e_1)=\frac{v}{18\gamma b^4}=20\frac{v}{\alpha A^2}\,, \tag{18}$$

and

$$g_3=4e_1e_2e_3=\frac{C}{12A\gamma b^4}=30\frac{C}{\alpha A^3}\,, \tag{19}$$

eq. (16) is written as:

$$4z^3-20\frac{v}{\alpha A^2}z-30\frac{C}{\alpha A^3}=0\,. \tag{20}$$

Function $\wp(z)$ is an even function of z and has a pole of order 2 in any primitive period-parallelogram on the complex z-plane. So, the solution (12) is a real solution $A\cdot\wp(b\xi)$ with $v<0$, travelling towards $+x$-direction for $\alpha<0$ and $\gamma<0$, while it is also a real solution with $v>0$, travelling towards $-x$-direction for $\alpha>0$ and $\gamma>0$.

The function $\wp(z)$ is expressed as

$$\wp(z)=e_3+\frac{e_1-e_3}{sn^2(z\sqrt{e_1-e_3}\,,k)}\,, \tag{21}$$

with the Jacobian elliptic function $sn(z,k)$ of modulus k:

$$k=\sqrt{\frac{e_2-e_3}{e_1-e_3}}\,, \tag{22}$$

and we can find that the function $\phi(\xi)$ defined by:

$$\begin{aligned}\phi(\xi)&=A\cdot\wp(b\xi)\cdot sn^2\left(b\xi\sqrt{e_1-e_3}\,,k\right)\\&=A\left\{(e_1-e_3)+e_3\cdot sn^2\left(b\xi\sqrt{e_1-e_3}\,,k\right)\right\},\end{aligned} \tag{23}$$

satisfies the following Korteweg - de Vries equation:

$$\frac{\partial\phi}{\partial t}-\alpha_0\phi\frac{\partial\phi}{\partial x}+\gamma_0\frac{\partial^3\phi}{\partial x^3}=0\,, \tag{24}$$

with

$$\alpha_0=\frac{v(e_2-e_3)}{A(e_3^2+e_1e_2)}\,, \tag{25}$$

and

$$\gamma_0=\frac{ve_3}{12b^2(e_3^2+e_1e_2)}\,. \tag{26}$$

A) If we tend $k \to 0$, then we have:

$$2\omega_1 \longrightarrow \pi/\sqrt{-3e_3}\,,\ 2\omega_3 \longrightarrow i\infty\,, \tag{27}$$

$$\left.\begin{aligned} &e_1 \longrightarrow \frac{2}{A}\sqrt{\frac{5v}{3\alpha}}\,, \\ &e_2 = e_3 \longrightarrow -\frac{1}{A}\sqrt{\frac{5v}{3\alpha}}\,, \end{aligned}\right\} \tag{28}$$

and the integration constant C should be chosen to be:

$$C = \frac{4v}{9}\sqrt{\frac{5v}{3\alpha}}\,, \tag{29}$$

so that eq. (7) could have a real solution. And the solution (12) takes the form:

$$u(x,t) = \sqrt{\frac{5v}{3\alpha}} \cdot \left\{3\,\mathrm{cosec}^2\left(b\xi\sqrt{\frac{1}{A}\sqrt{\frac{15v}{\alpha}}}\right) - 1\right\}. \tag{30}$$

While, function $\phi(\xi)$ defined in (23) reads:

$$\phi(\xi) = \sqrt{\frac{5v}{3\alpha}} \cdot \left\{3 - \sin^2\left(b\xi\sqrt{\frac{1}{A}\sqrt{\frac{15v}{\alpha}}}\right)\right\}. \tag{31}$$

B) If we take $k \to 1$, then from (13), (14), (17), (18), (19), and (22), we have:

$$2\omega_1 \longrightarrow +\infty,\ 2\omega_3 \longrightarrow i\pi/\sqrt{3e_1}\,, \tag{32}$$

$$\left.\begin{aligned} &e_1 = e_2 \longrightarrow \frac{1}{A}\sqrt{\frac{5v}{3\alpha}}\,, \\ &e_3 \longrightarrow -\frac{2}{A}\sqrt{\frac{5v}{3\alpha}}\,, \end{aligned}\right\} \tag{33}$$

and the integration constant C should be chosen to be:

$$C = -\frac{4v}{9}\sqrt{\frac{5v}{3\alpha}}\,. \tag{34}$$

Solution (12) turns to be:

$$u(x,t) = \sqrt{\frac{5v}{3\alpha}} \cdot \left\{3\coth^2\left(b\xi\sqrt{\frac{1}{A}\sqrt{\frac{15v}{\alpha}}}\right) - 2\right\}, \tag{35}$$

which gives a solitary pulse at $\xi = x + vt = 0$. While, function $\phi(\xi)$ for $k \to 1$ reads:

$$\phi(\xi) = \sqrt{\frac{5v}{3\alpha}} \cdot \left\{3 - 2\tanh^2\left(b\xi\sqrt{\frac{1}{A}\sqrt{\frac{15v}{\alpha}}}\right)\right\}. \tag{36}$$

C) If $0 < k < 1$, function $\wp(b\xi)$ is real for the values of C, which lies in the region:

$$-\frac{4}{9} \cdot |v| \cdot \sqrt{\frac{5v}{3\alpha}} < C < \frac{4}{9} \cdot |v| \cdot \sqrt{\frac{5v}{3\alpha}}\,. \tag{37}$$

When we take $C=0$, then eq. (11) reads:

$$vu-\frac{\alpha}{3}u^3+\gamma u_{\xi\xi\xi\xi}=0\,, \tag{38}$$

and eq. (20) becomes to be

$$z\left(z^2-5\frac{v}{\alpha A^2}\right)=0\,. \tag{39}$$

Then, solution (12) can be reduced to:

$$u(x,t)=\sqrt{\frac{5v}{\alpha}}\cdot\left\{\frac{2}{sn^2\left(b\xi\sqrt{\frac{2}{A}\sqrt{\frac{5v}{\alpha}}}\,,\sqrt{\frac{1}{2}}\right)}-1\right\}, \tag{40}$$

with

$$\left.\begin{aligned} &g_3=4e_1e_2e_3=0\,,\\ \text{and}\quad &g_2=\frac{v}{18\gamma b^4}=\frac{20v}{\alpha A^2}>0\,, \end{aligned}\right\} \tag{41}$$

i. e.

$$\left.\begin{aligned} &e_2=0\,,\\ &e_1=-e_3=\frac{1}{A}\sqrt{\frac{5v}{\alpha}}\,, \end{aligned}\right\} \tag{42}$$

and

$$k=\sqrt{\frac{-e_3}{e_1-e_3}}=\sqrt{\frac{1}{2}}\,. \tag{43}$$

While, function $\phi(\xi)$ for $C=0$, reads:

$$\phi(\xi)=\sqrt{\frac{5v}{\alpha}}\cdot\left\{2-sn^2\left(b\xi\sqrt{\frac{2}{A}\sqrt{\frac{5v}{\alpha}}}\,,\sqrt{\frac{1}{2}}\right)\right\}. \tag{44}$$

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