On the Flexural Deflection of a Moderately Thick Plate

Part II. Solution of Equation for Deflection of the Plate

Satoru IGARASHI* Yasutaka SARUWATARI*
Éi Iti TAKIZAWA* Tohru NISHIMURA**

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Résumé

The equation of deflection for a moderately thick plate presented in our previous paper, is solved for two cases, namely, a) a simply supported rectangular plate under distributed pressure, and b) a simply supported circular plate under uniformly distributed pressure.

Results obtained here are used to calculate the maximum deflection of the plate, and are compared with the results hitherto obtained.

§ 1. Notations and Equation of Flexural Deflection of a Moderately Thick Plate

Notations

x_i : rectangular coordinates, $(i=1, 2, 3)$

ξ_i : components of displacement, $(i=1, 2, 3)$

$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i}+\frac{\partial \xi_i}{\partial x_j}\right)$: components of strain, $(i, j=1, 2, 3)$

$\varepsilon_{kk}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$,

$A_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}$: components of stress, $(i, j=1, 2, 3)$

with Lamé's constants λ and μ,

$(l, m)=l\lambda+m\mu$, $(l, m, n)=l\lambda^2+m\lambda\mu+n\mu^2$, etc.

h : thickness of plate,

$D=\frac{h^3(0, 1)(1, 1)}{3(1, 2)}$: flexural rigidity of plate, and

w_0 : deflection of plate, *i. e.* vertical displacement of the middle plane of plate.

We shall take x_1- and x_2-axes on the middle plane of the plate, x_3-axis being directed downwards.

Equation of Flexural Deflection of a Moderately Thick Plate

In the previous paper[1], the fundamental equations for deflection of a moderately thick plate were presented, and it was shown that one can obtain approximate

* Institute of Precision Mechanics, Faculty of Engineering, Hokkaidô University, Sapporo, JAPAN.

** Institute of Aeronautical and Space Science, Faculty of Engineering, Nagoya University, Nagoya, JAPAN.

equations with any desired accuracy after considering that the thickness of plate is small. We shall summarize the results here.

Expanding components of displacement ξ_i $(i=1, 2, 3)$ of the plate into power series in x_3:

$$\left.\begin{aligned}\xi_1 &= \sum_{k=0}^{\infty} u_{2k+1}\, x_3^{2k+1},\\ \xi_2 &= \sum_{k=0}^{\infty} v_{2k+1}\, x_3^{2k+1},\\ \xi_3 &= \sum_{k=0}^{\infty} w_{2k}\, x_3^{2k},\end{aligned}\right\} \tag{1-1}$$

and introducing (1-1) into equations of equilibrium of an elastic body:

$$0 = \frac{\partial A_{ij}}{\partial x_j} = (1, 1)\frac{\partial}{\partial x_i}\varepsilon_{kk} + (0, 1)\,\Delta_3 \xi_i, \quad (i=1, 2, 3) \tag{1-2}$$

with

$$\Delta_3 = \frac{\partial^2}{\partial x_k^2} = \Delta + \frac{\partial^2}{\partial x_3^2},$$

we obtain the following relations* among the coefficients of power series in eqs. (1-1):

$$\left.\begin{aligned}u_{2k+1} &= \frac{(-1)^k}{(2k+1)!}\left[\Delta^k u_1 + \frac{k(1, 1)}{(1, 2)}\Delta^{k-1}\frac{\partial}{\partial x_1}\{\Xi_1 - \Delta w_0\}\right], \quad (k=0, 1, 2, \cdots)\\ v_{2k+1} &= \frac{(-1)^k}{(2k+1)!}\left[\Delta^k v_1 + \frac{k(1, 1)}{(1, 2)}\Delta^{k-1}\frac{\partial}{\partial x_2}\{\Xi_1 - \Delta w_0\}\right], \quad (k=0, 1, 2, \cdots)\\ &\text{and}\\ w_{2k} &= \frac{(-1)^k}{(2k)!}\left[\Delta^k w_0 + \frac{k(1, 1)}{(1, 2)}\Delta^{k-1}\{\Xi_1 - \Delta w_0\}\right], \quad (k=0, 1, 2, \cdots)\end{aligned}\right\} \tag{1-3}$$

with

$$\Xi_1 = \frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2}. \tag{1-4}$$

Boundary conditions at the surfaces of the plate read:

$$\left.\begin{aligned}\pm\frac{p}{2} = A_{33} &= (1, 0)\,\varepsilon_{kk} + 2(0, 1)\varepsilon_{33}, \quad \text{at} \quad x_3 = \pm\frac{h}{2}\\ 0 = A_{31} &= 2(0, 1)\,\varepsilon_{31}, \quad \text{at} \quad x_3 = \pm\frac{h}{2}\\ 0 = A_{32} &= 2(0, 1)\,\varepsilon_{32}, \quad \text{at} \quad x_3 = \pm\frac{h}{2}\end{aligned}\right\} \tag{1-5}$$

where $p = p(x_1, x_2)$ is the distributed external pressure over the upper surface of the plate. Introducing (1-1) with (1-3) into (1-5), we have the following equations:

* We understand that the zero-th power of the Laplacian operator is equal to unity, *i.e.* $\Delta^0 = 1$.

$$\frac{4}{3}(1,1)\,p=D\sum_{k=0}^{\infty}\frac{(-1)^k(2k+2)}{(2k+3)\,!}\left(\frac{h}{2}\right)^{2k}\Big[(2k+1,2k)\,\varDelta^{k+2}w_0-$$
$$-(2k+3,2k+4)\,\varDelta^{k+1}\varXi_1\Big], \qquad (1\text{-}6)$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)\,!}\left(\frac{h}{2}\right)^{2k}\Big[(2k+1,2k+2)\,\varDelta^k\varXi_1-(2k-1,2k-2)\,\varDelta^{k+1}w_0\Big]=0\,, \qquad (1\text{-}7)$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)\,!}\left(\frac{h}{2}\right)^{2k}\varDelta^k\varOmega_1=0\,, \qquad (1\text{-}8)$$

with flexural rigidity D of the plate of thickness h:

$$D=\frac{h^3(0,1)(1,1)}{3(1,2)}\,,$$

and

$$\varOmega_1=\frac{\partial u_1}{\partial x_2}-\frac{\partial v_1}{\partial x_1}\,. \qquad (1\text{-}9)$$

When the thickness of plate is small compared with its lateral dimensions, one can find:

$$\varOmega_1=\frac{\partial u_1}{\partial x_2}-\frac{\partial v_1}{\partial x_1}=0\,. \qquad (1\text{-}10)$$

Eq. (1-10) shows that there is a function ϕ such that:

$$u_1=\frac{\partial\phi}{\partial x_1}\,, \qquad \text{and} \qquad v_1=\frac{\partial\phi}{\partial x_2}\,, \qquad (1\text{-}11)$$

with

$$\varXi_1=\varDelta\phi\,. \qquad (1\text{-}12)$$

From (1-7)~(1-12), ϕ is expressed in the form:

$$\phi=-w_0-\frac{2(1,1)}{(1,2)}\left(\frac{h}{2}\right)^2\varDelta w_0-\frac{2(1,1)(4,5)}{3(1,2)^2}\left(\frac{h}{2}\right)^4\varDelta\varDelta w_0-$$
$$-\frac{2(1,1)(27,68,43)}{15(1,2)^3}\left(\frac{h}{2}\right)^6\varDelta\varDelta\varDelta w_0-\cdots. \qquad (1\text{-}13)$$

Eliminating $\varXi_1$ from (1-6) by means of (1-12) and (1-13), we obtain:

$$p=D\varDelta\varDelta\Big\{w_0+\frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2\varDelta w_0+\frac{(1479,3704,2332)}{840(1,2)^2}\left(\frac{h}{2}\right)^4\varDelta\varDelta w_0+$$
$$+\frac{(35969,135768,171420,72400)}{15120(1,2)^3}\left(\frac{h}{2}\right)^6\varDelta\varDelta\varDelta w_0+\cdots\Big\}. \qquad (1\text{-}14)$$

Retaining terms of $O(h^{2n})$ in the right-hand side of eqs. (1-13) and (1-14), we obtain the equation for deflection of a moderately thick plate in the n-th order approximation in our theory. We shall cite here merely the equations in the zero-th, the first, the second, and the third order approximations. For the sake of simplicity, we shall omit the subscript of w_0 hereafter, writing w instead of w_0.

A) Zero-th order approximation

$$\phi = -w\,, \tag{1-15}$$

and

$$p = D\Delta\Delta w\,. \tag{1-16}$$

B) First order approximation

$$\phi = -w - \frac{2(1,1)}{(1,2)}\left(\frac{h}{2}\right)^2 \Delta w\,, \tag{1-17}$$

and

$$p = D\Delta\Delta\left\{w + \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta w\right\}. \tag{1-18}$$

C) Second order approximation

$$\phi = -w - \frac{2(1,1)}{(1,2)}\left(\frac{h}{2}\right)^2 \Delta w - \frac{2(1,1)(4,5)}{3(1,2)^2}\left(\frac{h}{2}\right)^4 \Delta\Delta w\,, \tag{1-19}$$

and

$$p = D\Delta\Delta\left\{w + \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta w + \frac{(1479,3704,2332)}{840(1,2)^2}\left(\frac{h}{2}\right)^4 \Delta\Delta w\right\}. \tag{1-20}$$

D) Third order approximation

$$\phi = -w - \frac{2(1,1)}{(1,2)}\left(\frac{h}{2}\right)^2 \Delta w - \frac{2(1,1)(4,5)}{3(1,2)^2}\left(\frac{h}{2}\right)^4 \Delta\Delta w - \\ - \frac{2(1,1)(27,68,43)}{15(1,2)^3}\left(\frac{h}{2}\right)^6 \Delta\Delta\Delta w\,, \tag{1-21}$$

and

$$p = D\Delta\Delta\left\{w + \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta w + \frac{(1479,3704,2332)}{840(1,2)^2}\left(\frac{h}{2}\right)^4 \Delta\Delta w + \right. \\ \left. + \frac{(35969,135768,171420,72400)}{15120(1,2)^3}\left(\frac{h}{2}\right)^6 \Delta\Delta\Delta w\right\}. \tag{1-22}$$

§ 2. Solution of Equation of Deflection for a Simply Supported Rectangular Thick Plate

Eqs. (1-6)~(1-8) as well as equations in any order approximation can be solved for a simply supported rectangular plate under distributed pressure.

Let the rectangular plate occupy the region: $-a/2 \leqslant x_1 \leqslant a/2$ and $-b/2 \leqslant x_2 \leqslant b/2$, and be simply supported at $x_1 = \pm a/2$ and $x_2 = \pm b/2$. We shall take the boundary conditions in this case as follows:

$$\xi_2 = \xi_3 = 0, \qquad \text{and} \quad A_{11} = 0, \qquad \text{at} \quad x_1 = \pm a/2 \tag{2-1}$$

and

$$\xi_1 = \xi_3 = 0, \qquad \text{and} \quad A_{22} = 0. \qquad \text{at} \quad x_2 = \pm b/2 \tag{2-2}$$

Under the boundary conditions (2-1) and (2-2), solution of eqs. (1-6)～(1-8) can be expressed by:

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \cos \frac{(2m-1)\pi}{a} x_1 \cdot \cos \frac{(2n-1)\pi}{b} x_2, \tag{2-3}$$

$$\Omega_1 = 0, \tag{2-4}$$

and

$$\phi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Phi_{mn} \cos \frac{(2m-1)\pi}{a} x_1 \cdot \cos \frac{(2n-1)\pi}{b} x_2, \tag{2-5}$$

with (1-11) and (1-12).

The distributed pressure $p = p(x_1, x_2)$ is expressed in a double Fourier series:

$$p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \cos \frac{(2m-1)\pi}{a} x_1 \cdot \cos \frac{(2n-1)\pi}{b} x_2, \tag{2-6}$$

with

$$p_{mn} = \frac{4}{ab} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} p(x_1, x_2) \cos \frac{(2m-1)\pi}{a} x_1 \cdot \cos \frac{(2n-1)\pi}{b} x_2 dx_1 dx_2. \quad (m, n = 1, 2, 3, \cdots) \tag{2-7}$$

When the pressure is uniformly distributed, *i. e.* $p = \text{const}$ $(= p_0)$, expression (2-7) turns simply to be:

$$p_{mn} = \frac{2^4}{\pi^2} \frac{(-1)^{m+n}}{(2m-1)(2n-1)} p_0. \qquad (m, n = 1, 2, 3, \cdots) \tag{2-8}$$

Introducing (2-3)～(2-6) with (1-12) into (1-6)～(1-8), we obtain:

$$W_{mn} = \frac{1}{6D} \frac{p_{mn}}{\gamma_{mn}^4} (\gamma_{mn} h)^3 \frac{\frac{(1,1)}{(1,2)} \left(\gamma_{mn} \frac{h}{2}\right) \sinh\left(\gamma_{mn} \frac{h}{2}\right) + \cosh\left(\gamma_{mn} \frac{h}{2}\right)}{\sinh(\gamma_{mn} h) - \gamma_{mn} h}, \quad (m, n = 1, 2, 3, \cdots) \tag{2-9}$$

$$\Phi_{mn} = \frac{1}{6D} \frac{p_{mn}}{\gamma_{mn}^4} (\gamma_{mn} h)^3 \frac{\frac{(1,1)}{(1,2)} \left(\gamma_{mn} \frac{h}{2}\right) \sinh\left(\gamma_{mn} \frac{h}{2}\right) - \cosh\left(\gamma_{mn} \frac{h}{2}\right)}{\sinh(\gamma_{mn} h) - \gamma_{mn} h}, \quad (m, n = 1, 2, 3, \cdots) \tag{2-10}$$

with

$$\gamma_{mn}^2 = \left[\frac{(2m-1)\pi}{a}\right]^2 + \left[\frac{(2n-1)\pi}{b}\right]^2. \qquad (m, n = 1, 2, 3, \cdots) \tag{2-11}$$

By means of (2-3) and (2-9), we have the expression for w, which is found to be identical with the result obtained by Iyenger et al.[2] They obtained the solution of the equation derived from Vlasov's method[3].

If we express the right-hand side of eq. (2-9) into power series in h:

$$W_{mn}=\frac{1}{D}\frac{p_{mn}}{\gamma_{mn}{}^4}\left[1+\frac{(13,16)}{10(1,2)}\left(\gamma_{mn}\frac{h}{2}\right)^2-\frac{(297,454)}{4200(1,2)}\left(\gamma_{mn}\frac{h}{2}\right)^4+\cdots\right],$$
$$(m,n=1,2,3,\cdots) \qquad (2\text{-}12)$$

and truncate the series at the terms of $O(h^{2n})$, we can obtain the solution of equation in the n-th order approximation.

§ 3. Solution of Equation of Deflection for a Simply Supported Circular Thick Plate under Uniform Pressure

Let us take cylindrical coordinates $r=\sqrt{x_1^2+x_2^2}$, $\theta=\text{arctg}\,(x_2/x_1)$, and $z=x_3$. Then, Laplacian operator $\varDelta$ reads:

$$\varDelta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

which appears in eqs. (1-6)~(1-8), (1-16), (1-18), (1-20), etc.

Components of displacement and components of stress are to be written also in cylindrical coordinates:

$$\xi_r=\xi_1\cos\theta+\xi_2\sin\theta, \qquad \xi_\theta=-\xi_1\sin\theta+\xi_2\cos\theta, \qquad \xi_z=\xi_3,$$
$$A_{rr}=\lambda\varepsilon_{kk}+2\mu(\partial\xi_r/\partial r), \quad \text{etc.}$$

with

$$\varepsilon_{kk}=\frac{1}{r}\frac{\partial(r\xi_r)}{\partial r}+\frac{1}{r}\frac{\partial\xi_\theta}{\partial\theta}+\frac{\partial\xi_z}{\partial z}.$$

We shall solve eqs. (1-16), (1-18), and (1-20) for a simply supported circular plate under *uniform* pressure p_0 over the surface of the plate. Accordingly we have:

$$\xi_\theta=0, \qquad \frac{\partial}{\partial\theta}=0, \qquad \text{and} \quad \varDelta=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right).$$

Let the circular plate occupy the region: $0\leqslant r\leqslant a$, and be simply supported at $r=a$. As for the boundary conditions, we shall take:

$$\xi_z=0, \qquad \text{and} \quad M_r\equiv\int_{-\frac{h}{2}}^{\frac{h}{2}}A_{rr}z\,dz=0, \qquad \text{at} \quad r=a \qquad (3\text{-}1)$$

and let us start from the zero-th order approximation.

A) *Zero-th Order Approximation*

The solution $w^{(0)}$ of equation for deflection in the zero-th order approximation (1-16), is expressed by a particular solution of (1-16), say $w_{\text{I}}^{(0)}$, plus a biharmonic function $w_{\text{II}}^{(0)}$, namely

$$w^{(0)} = w_{\text{I}}^{(0)} + w_{\text{II}}^{(0)} , \tag{3-2}$$

with

$$D\Delta\Delta w_{\text{I}}^{(0)} = p_0 , \tag{3-3}$$

and

$$\Delta\Delta w_{\text{II}}^{(0)} = 0 . \tag{3-4}$$

The solution of eq. (3-3) is given by:

$$w_{\text{I}}^{(0)} = \frac{1}{64}\frac{p_0}{D} r^4 . \tag{3-5}$$

While, the solution of eq. (3-4), which is finite at $r=0$, reads

$$w_{\text{II}}^{(0)} = C_1 r^2 + C_0 , \tag{3-6}$$

with constants C_1 and C_0. Determining C_1 and C_0, so as to satisfy the boundary conditions (3-1) for $w^{(0)}$, we obtain the solution of eq. (1-16):

$$w^{(0)} = w_{\text{I}}^{(0)} + w_{\text{II}}^{(0)} = \frac{1}{64}\frac{p_0}{D}(a^2 - r^2)\left\{\frac{(11, 10)}{(3, 2)} a^2 - r^2\right\} . \tag{3-7}$$

Expression (3-7) is nothing but the usual solution for a circular thin plate.

B) First Order Approximation

The solution $w^{(1)}$ of eq. (1-18) for deflection in the first order approximation is the sum of a particular solution of (1-18), say $w_{\text{I}}^{(1)}$, and the solution $w_2^{(1)}$ of the homogeneous equation for eq. (1-18), namely

$$w^{(1)} = w_{\text{I}}^{(1)} + w_2^{(1)} , \tag{3-8}$$

with

$$D\Delta\Delta\left\{w_{\text{I}}^{(1)} + \frac{(13, 16)}{10(1, 2)}\left(\frac{h}{2}\right)^2 \Delta w_{\text{I}}^{(1)}\right\} = p_0 , \tag{3-9}$$

and

$$\Delta\Delta\left\{1 + \frac{(13, 16)}{10(1, 2)}\left(\frac{h}{2}\right)^2 \Delta\right\} w_2^{(1)} = \left\{1 + \frac{(13, 16)}{10(1, 2)}\left(\frac{h}{2}\right)^2 \Delta\right\} \Delta\Delta w_2^{(1)} = 0 . \tag{3-10}$$

The solution $w_2^{(1)}$ of eq. (3-10) is decomposed into two parts, namely

$$w_2^{(1)} = w_{\text{II}}^{(1)} + w_{\text{III}}^{(1)} , \tag{3-11}$$

where

$$\Delta\Delta w_{\text{II}}^{(1)} = 0 , \tag{3-12}$$

and

$$\Delta w_{\text{III}}^{(1)} + \alpha^2 w_{\text{III}}^{(1)} = 0 , \tag{3-13}$$

with

$$\alpha^2=\frac{10(1,2)}{(13,16)}\left(\frac{2}{h}\right)^2.$$

The solutions of eqs. (3-9) and (3-12) are:

$$w_{\mathrm{I}}^{(1)}=\frac{1}{64}\frac{p_0}{D}r^4, \tag{3-14}$$

and

$$w_{\mathrm{II}}^{(1)}=C_1r^2+C_0, \tag{3-15}$$

with constants C_1 and C_0. While, the solution of eq. (3-13) is expressed by a Bessel function of order zero and is written as:

$$w_{\mathrm{III}}^{(1)}=AJ_0(\alpha r), \tag{3-16}$$

with a constant A.

Accordingly, the solution of eq. (1-18) under the boundary conditions (3-1) can be written by:

$$w^{(1)}=w_{\mathrm{I}}^{(1)}+w_{\mathrm{II}}^{(1)}+w_{\mathrm{III}}^{(1)}=\frac{1}{64}\frac{p_0}{D}r^4+C_1r^2+C_0+AJ_0(\alpha r), \tag{3-17}$$

with constants A, C_1, and C_0, expressed as follows:

$$A=-\frac{(13,16)}{80(3,2)}a^4\frac{p_0}{D}\left(\frac{h}{2a}\right)^2\frac{1+\dfrac{4(35,66,32)}{5(1,2)^2}\left(\dfrac{h}{2a}\right)^2}{J_0(\alpha a)-\dfrac{4(1,0)}{5(3,2)}\alpha a\left(\dfrac{h}{2a}\right)^2J_1(\alpha a)}, \tag{3-18}$$

$$C_1=-\frac{1}{16}\frac{p_0}{D}a^2+\frac{1}{4}\alpha^2AJ_0(\alpha a), \tag{3-19}$$

and

$$C_0=\frac{3}{64}\frac{p_0}{D}a^4-\left(1+\frac{\alpha^2a^2}{4}\right)AJ_0(\alpha a). \tag{3-20}$$

Expression (3-17) obtained here has a somewhat different feature from a solution given by Love[4]:

$$w=\frac{1}{64}\frac{p_0}{D}(a^2-r^2)\left\{\frac{(11,10)}{(3,2)}a^2-r^2+\frac{8(35,66,32)}{5(1,2)(3,2)}\left(\frac{h}{2}\right)^2\right\}, \tag{3-21}$$

where he took another boundary condition. Eq. (3-21), however, has a similar expression to (3-7) and also contains a term of $O(h^2)$.

C) Second Order Approximation

In a similar manner, the solution $w^{(2)}$ of eq. (1-20) in the second order approximation can be decomposed into three terms, namely

$$w^{(2)}=w_{\mathrm{I}}^{(2)}+w_{\mathrm{II}}^{(2)}+w_{\mathrm{III}}^{(2)}, \tag{3-22}$$

with

$$D\Delta\Delta\left\{w_{\mathrm{I}}^{(2)}+\frac{2\alpha_2^2}{\beta_2^4}\Delta w_{\mathrm{I}}^{(2)}+\frac{1}{\beta_2^4}\Delta\Delta w_{\mathrm{I}}^{(2)}\right\}=p_0, \tag{3-23}$$

$$\Delta\Delta w_{\mathrm{II}}^{(2)} = 0 \,, \tag{3-24}$$

and

$$\Delta\Delta w_{\mathrm{III}}^{(2)} + 2\alpha_2^2 \Delta w_{\mathrm{III}}^{(2)} + \beta_2^4 w_{\mathrm{III}}^{(2)} = 0 \,, \tag{3-25}$$

with

$$\alpha_2{}^2 = \frac{42(1,2)(13,16)}{(1479,3704,2332)}\left(\frac{2}{h}\right)^2, \qquad \text{and} \quad \beta_2{}^4 = \frac{840(1,2)^2}{(1479,3704,2332)}\left(\frac{2}{h}\right)^4.$$

The solutions of eqs. (3-23) and (3-24) are given as :

$$w_{\mathrm{I}}^{(2)} = \frac{1}{64}\frac{p_0}{D} r^4 \,, \tag{3-26}$$

and

$$w_{\mathrm{II}}^{(2)} = C_1 r^2 + C_0 \,, \tag{3-27}$$

with constants C_1 and C_0. While, the finite solution of (3-25) at $r=0$ reads :

$$w_{\mathrm{III}}^{(2)} = A_1 \,\mathfrak{Re}\, I_0(\kappa r) + B_1 \,\mathfrak{Im}\, I_0(\kappa r) \,, \tag{3-28}$$

where $I_0(\kappa r)$ is a modified Bessel function of order zero, with constants A_1, and B_1. We wrote one of the roots of the equation :

$$\kappa^4 + 2\alpha_2{}^2 \kappa^2 + \beta_2{}^4 = 0 \,, \tag{3-29}$$

to be $\kappa = \zeta + i\eta$ $(\zeta \geqq 0$ and $\eta \geqq 0)$,
with

$$\zeta = \sqrt{\frac{\beta_2{}^2 - \alpha_2{}^2}{2}} \,, \qquad \text{and} \quad \eta = \sqrt{\frac{\beta_2{}^2 + \alpha_2{}^2}{2}} \,. \tag{3-30}$$

The solution $w^{(2)}$ of eq. (1-20) under the boundary conditions (3-1) can be written as :

$$\begin{aligned} w^{(2)} &= w_{\mathrm{I}}^{(2)} + w_{\mathrm{II}}^{(2)} + w_{\mathrm{III}}^{(2)} = \\ &= \frac{1}{64}\frac{p_0}{D} r^4 + C_1 r^2 + C_0 + A_1 \,\mathfrak{Re}\, I_0(\kappa r) + B_1 \,\mathfrak{Im}\, I_0(\kappa r) \,, \end{aligned} \tag{3-31}$$

with

$$A_1 = \frac{p_0}{D} a^4 \frac{\mathfrak{Im}\, L(\kappa a) - \left\{\frac{1}{8} + \frac{(17,16)}{10(1,2)}\left(\frac{h}{2a}\right)^2\right\} \mathfrak{Im}\,\{(\kappa a)^4 I_0(\kappa a)\}}{\mathfrak{Re}\, L(\kappa a)\, \mathfrak{Im}\,\{(\kappa a)^4 I_0(\kappa a)\} - \mathfrak{Im}\, L(\kappa a)\, \mathfrak{Re}\,\{(\kappa a)^4 I_0(\kappa a)\}} \,, \tag{3-32}$$

$$B_1 = -\frac{p_0}{D} a^4 \frac{\mathfrak{Re}\, L(\kappa a) - \left\{\frac{1}{8} + \frac{(17,16)}{10(1,2)}\left(\frac{h}{2a}\right)^2\right\} \mathfrak{Re}\,\{(\kappa a)^4 I_0(\kappa a)\}}{\mathfrak{Re}\, L(\kappa a)\, \mathfrak{Im}\,\{(\kappa a)^4 I_0(\kappa a)\} - \mathfrak{Im}\, L(\kappa a)\, \mathfrak{Re}\,\{(\kappa a)^4 I_0(\kappa a)\}} \,, \tag{3-33}$$

$$C_1 = -\frac{1}{16}\frac{p_0}{D} a^2 - \frac{1}{4a^2}\Big[A_1 \,\mathfrak{Re}\big\{(\kappa a)^2 I_0(\kappa a)\big\} + B_1 \,\mathfrak{Im}\big\{(\kappa a)^2 I_0(\kappa a)\big\}\Big], \tag{3-34}$$

and

$$C_0 = -\frac{1}{64}\frac{p_0}{D}a^4 - C_1 a^2 - A_1 \Re e\, I_0(\kappa a) - B_1 \Im m\, I_0(\kappa a)\,, \tag{3-35}$$

where

$$L(\kappa a) = (\kappa a)^2 I_0(\kappa a) - 2(\kappa a)\, I_1(\kappa a) + \frac{(17,16)}{5(1,2)}\left(\frac{h}{2a}\right)^2 \left\{(\kappa a)^4 I_0(\kappa a) - (\kappa a)^3 I_1(\kappa a)\right\} + \\ + \frac{(1479, 3704, 2332)}{420(1,2)^2}\left(\frac{h}{2a}\right)^4 \left\{\frac{2(1,1)}{(1,2)}(\kappa a)^6 I_0(\kappa a) - \\ - \frac{(1919, 4248, 2332)}{(1479, 3704, 2332)}(\kappa a)^5 I_1(\kappa a)\right\}. \tag{3-36}$$

§ 4. Numerical Example

As for numerical examples, we shall consider the maximum deflection of a plate under uniform pressure calculated from solutions of several approximate equations obtained in this paper.

A) Rectangular Plate

Uniform pressure $p = p_0$ ($=$const) is expressed in a double Fourier series in (2-7), giving

$$p_{mn} = \frac{2^4}{\pi^2}\frac{(-1)^{m+n}}{(2m-1)(2n-1)}p_0. \quad (m, n = 1, 2, 3, \cdots) \tag{4-1}$$

The maximum deflection $w_{\max}$ of a rectangular plate is obtained from (2-3) with $x_1 = x_2 = 0$, *i. e.*

$$w_{\max} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\,. \tag{4-2}$$

W_{mn} are given in eq. (2-9) for the exact solution of the plate and in eq. (2-12) for approximate solutions.

For a square plate, *i. e.* $a = b$, we calculate $w_{\max}$ for various values of h/a,

TABLE 1. Comparison of the maximum deflections of a simply supported square plate ($a = b$) under uniform pressure.
($\lambda/\mu = 3/2$, and Poisson's ratio $= 0.3$)

h/a	$w_{\max}/w_e$ (w_e: exact solution)		
	0-th order approx. (thin plate)	1st order approx.	Reissner[5)]
0.05	0.989	1.000	1.000
0.10	0.956	1.000	0.998
0.15	0.907	1.000	0.996
0.20	0.846	1.001	0.995
0.25	0.779	1.003	0.993
0.30	0.711	1.005	0.993

after truncating series (4-2) at appropriate terms ($m, n \simeq 5 \sim 6$). In Table 1, the values w_{max}/w_e are shown in the zero-th and the first order approximations in our theory and are compared with the values calculated from Reissner's theory[5]. w_e means the exact solution given by (4-2) with (2-9). The ratio λ/μ is taken to be 3/2, with Poisson's ratio $(\lambda/2)/(\lambda+\mu)=0.3$.

In the zero-th order approximation (*i.e.* thin plate), the relative error $|1-(w_{max}/w_e)|$ is comparatively small (*e.g.* less than 5% for $h/a=0.1$), while, for in creasing thickness, the error increases from *ca.* 9% (for $h/a=0.15$) to *ca.* 30% (for $h/a=0.3$). In the first order approximation in our theory, the values of w_{max} agree very well with the exact solution, *e.g.* the error is 0.5% for $h/a=0.3$. In the second order approximation, the values of w_{max} coincides very well with the exact values (*e.g.* the error is less than 0.1% for $h/a \leqslant 0.4$).

B) Circular Plate

The maximum deflection w_{max} of a circular plate under uniform pressure is obtained from eqs. (3-7), (3-17), and (3-31), with $r=0$.

In the zero-th order approximation, we have:

$$w_{max} = \frac{(11, 10)}{64(3, 2)} \frac{p_0}{D} a^4, \tag{4-3}$$

while the first order approximation gives:

$$w_{max} = \frac{3}{64} \frac{p_0}{D} a^4 - \left(1 + \frac{\alpha^2 a^2}{4}\right) A J_0(\alpha a) + A, \tag{4-4}$$

with (3-18). The maximum deflection in the second order approximation reads:

$$w_{max} = \frac{3}{64} \frac{p_0}{D} a^4 + A_1 + \left[\frac{1}{4} \Re\mathrm{e}\left\{(\kappa a)^2 I_0(\kappa a)\right\} - \Re\mathrm{e}\, I_0(\kappa a)\right] A_1 + \\ + \left[\frac{1}{4} \Im\mathrm{m}\left\{(\kappa a)^2 I_0(\kappa a)\right\} - \Im\mathrm{m}\, I_0(\kappa a)\right] B_1, \tag{4-5}$$

with (3-32) and (3-33).

Numerical results calculated from (4-3)~(4-5) and from Love's solution are compared in Table 2. In this case, the exact solution can not be obtained, and numerical values of w_{max} are shown as a ratio of w_{max} to $w^{(0)}$, for values of $h/2a$. $w^{(0)}$ means the values in the zero-th order approximation (thin plate). The ratio λ/μ is taken to be 3/2 and Poison's ratio 0.3.

For the plate of small thickness, the values $w_{max}/w^{(0)}$ in the first and the second order approximations and also in the theory of Love, do not differ very much. For example, the values of the difference $|1-(w_{max}/w^{(0)})|$ are less than 10% for $h/2a \leqslant 0.1$. The value increases with increasing thickness of the plate.

Although the features of (3-31) in our second order approximation and (3-21) given by Love are quite different from each other, the numerical values given in our second order approximation (4-5) show a very good agreement with those obtained by (3-21) for all the values of $h/2a$. The maximum difference between them is merely less than 7%.

TABLE 2. Comparison of the maximum deflections of a simply supported circular plate under uniform pressure. ($\lambda/\mu=3/2$, and Poisson's ratio$=0.3$)

$h/2a$	$w_{max}/w^{(0)}$ ($w^{(0)}$: 0-th order approximation)		
	1st order approx.	2nd order approx.	Love[4]
0.05	0.997	1.012	1.009
0.10	1.090	1.047	1.036
0.15	1.016	1.111	1.081
0.20	1.441	1.221	1.145
0.25	1.462	1.314	1.226
0.30	1.683	1.337	1.326

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