Generalized Sampling Theorem, Generalized Frequency, and Generalized Band-Limited Spectrum Function

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Abstract

Several remarks are made on the generalized sampling theorem presented by Takizawa and Isigaki. In relation to the generalized sampling theorem, the generalized frequency and the generalized band-limited spectrum function are discussed. The condition for establishing the generalized sampling theorem is stated in terms of the generalized frequency.

Zusammenfassung

Einige Erweiterungen des von Takizawa und Isigaki abgeleiteten verallgemeinerten Abtasttheorems werden in der vorliegenden Arbeit angegeben. Bezüglich dieses Theorems werden die verallgemeinerte Frequenz und die verallgemeinerte band-begrenzte Frequenzfunktion diskutiert. Die Bedingung für die Herleitung des verallgemeinerten Abtasttheorems wird in Abhängigkeit von der verallgemeinerten Frequenz angegeben.

§ 0. Preliminaries

In a previous paper[1] the author presented a generalized sampling theorem, by which a continuous function can be reconstructed from its sampled values and sampled derivatives. Here, in the present paper, the author makes several remarks on his generalized sampling theorem. He also introduces the notion of the generalized frequency and the generalized band-limited spectrum function, and discusses the condition for establishing the generalized sampling theorem in relation to the generalized frequency and the generalized band-limited spectrum function. The bound for truncation error of the sampling expansion is also estimated.

§ 1. Generalized Sampling Theorem

The generalized sampling theorem presented by Takizawa and Isigaki reads as follows:

Theorem 1–1. Generalized Sampling Theorem

An entire function $f(z)$ is expressed by:

$$f(z)=\sum_{n}\sum_{k=0}^{m_n}\sum_{j=0}^{m_n-k}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{k!}\cdot(z-z_n)^{j+k}\cdot\frac{g(z)}{(z-z_n)^{m_n+1}} \tag{1–1}$$

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$$= \sum_n \sum_{s=0}^{m_n} \sum_{j=0}^{s} \frac{f_n^{(j)}}{j!} \cdot \frac{H_n^{(s-j)}}{(s-j)!} \cdot (z-z_n)^s \cdot \frac{g(z)}{(z-z_n)^{m_n+1}} \qquad (1-1')$$

$$= \sum_n \sum_{s=0}^{m_n} \frac{(z-z_n)^s}{s!} \cdot \left[\frac{d^s}{dz^s}\left\{f(z) \cdot H(z)\right\}\right]_{z=z_n} \cdot \frac{g(z)}{(z-z_n)^{m_n+1}}, \qquad (1-1'')$$

in which the series is uniformly convergent in any finite closed domain of the complex z-plane, if the following conditions are satisfied:

(I) $f(z)$ and $g(z)$ are entire, (1–2)

(II) $g(z)$ has zeros of (m_n+1)-th order at points $z=z_n$ (n=integers), *i. e.*

$$g(z_n) = g'(z_n) = g''(z_n) = \cdots = g^{(m_n)}(z_n) = 0, \text{ and } g^{(m_n+1)}(z_n) \neq 0, \qquad (1-3)$$

for m_n=non-negative integers, which depend on n, and

$$\text{(III)} \quad \lim_{z\to\infty} \frac{f(z)}{g(z)} = 0. \qquad (1-4)$$

Here, for the sake of brevity, we have put*):

$$f_n^{(k)} = \left[d^k f(z)/dz^k\right]_{z=z_n}, \qquad (1-5)$$

$$g_n^{(k)} = \left[d^k g(z)/dz^k\right]_{z=z_n}, \qquad H_n^{(k)} = \left[d^k H(z)/dz^k\right]_{z=z_n}, \qquad (1-6)$$

and

$$h_n^{(k)} = \left[\frac{d^k}{dz^k} h(z)\right]_{z=z_n} = \left[\frac{d^k}{dz^k} \frac{g(z)}{(z-z_n)^{m_n+1}}\right]_{z=z_n}. \qquad (1-7)$$

The function:

$$h(z) \equiv 1/H(z) \equiv g(z)/(z-z_n)^{m_n+1}, \qquad (1-8)$$

is called a *generalized sampling function.* The summation over n in (1–1)∼(1–1″) covers the whole set of *sampling points* $z=z_n$ (n=integers).

The proof of Theorem 1–1 is straightforward. Under conditions (I) and (II), function $f(z)/g(z)$ is meromorphic in the complex plane. It has poles of (m_n+1)-th order at points $z=z_n$ (n=integers). By means of the Cauchy theorem, we express the function $f(z)/g(z)$ by a contour integration along a circle of radius R with its centre at the origin, the poles of the function $f(z)/g(z)$ being in the domain enclosed by the circle $|z|=R$. We tend the radius R to infinity, then the contour integral vanishes under condition (III), and we have merely to calculate the residues at the points $z=z_n$ (n=integers). Taking the sum of residues at $z=z_n$ (n=integers) and multiplying $g(z)$ to both sides of the expression thus obtained, we have Theorem 1–1.

The explicit expression of $H_n^{(k)}$ in (1–1)∼(1–1″) is as follows:

*) By $f^{(k)}(z)$, $H^{(k)}(z)$, and $h^{(k)}(z)$, we understand $d^k f(z)/dz^k$, $d^k H(z)/dz^k$, and $d^k h(z)/dz^k$, respectively. While, $g^{(0)}(z)$, $H^{(0)}(z)$, and $h^{(0)}(z)$, are respectively functions $g(z)$, $H(z)$, and $h(z)$ themselves.

$$H_n^{(k)} = \frac{(-1)^k}{h_n} \cdot \begin{vmatrix} \frac{h_n'}{h_n}, & 1, & 0, & & \\ \frac{h_n''}{h_n}, & {}_2C_1 \frac{h_n'}{h_n}, & 1, & 0, & \mathbf{0} \\ \vdots & \vdots & \vdots & & \\ \frac{h_n^{(k-1)}}{h_n}, & {}_{k-1}C_1 \frac{h_n^{(k-2)}}{h_n}, & {}_{k-1}C_2 \frac{h_n^{(k-3)}}{h_n}, & & , 1 \\ \frac{h_n^{(k)}}{h_n}, & {}_kC_1 \frac{h_n^{(k-1)}}{h_n}, & {}_kC_2 \frac{h_n^{(k-2)}}{h_n}, & \cdots, & {}_kC_{k-1} \frac{h_n'}{h_n} \end{vmatrix}, \quad (1\text{-}9)$$

for positive integers k, with

$${}_pC_q \frac{h_n^{(p-q)}}{h_n} = \frac{p!\,(m_n+1)!}{q!\,(m_n+1+p-q)!} \cdot \frac{g_n^{(m_n+1+p-q)}}{g_n^{(m_n+1)}}, \quad (p, q = 0, 1, 2, \cdots) \quad (1\text{-}10)$$

$$g_n^{(r)} = \Big[d^r g(z)/dz^r\Big]_{z=z_n}, \qquad (r = 0, 1, 2, \cdots) \quad (1\text{-}11)$$

and

$$h_n^{(r)} = \left[\frac{d^r}{dz^r} h(z)\right]_{z=z_n} = \left[\frac{d^r}{dz^r} \frac{g(z)}{(z-z_n)^{m_n+1}}\right]_{z=z_n}. \quad (r = 0, 1, 2, \cdots) \quad (1\text{-}12)$$

§ 2. Remarks on the Generalized Sampling Theorem

Here, we shall make several remarks with regard to the generalized sampling theorem given by Takizawa and Isigaki.

1) As was seen in the proof mentioned above, we can replace conditions (I) and (II) by the following one :

(I′) The function $f(z)/g(z)$ is meromorphic, with poles of (m_n+1)-th order at points $z=z_n$ $(n=\text{integers})$. (2–1)

2) If we replace condition (III) by the following one :

$$(\text{III}') \quad \lim_{z\to\infty} \frac{f(z)}{g(z)} = K \quad (=\text{const}), \quad (2\text{-}2)$$

then the contour integral along the circle of infinite radius does not vanish but remains a finite value K. Thus, we have the following theorem :

Theorm 2–1

If conditions $\{(\text{I}), (\text{II}), (\text{III}')\}$ or $\{(\text{I}'), (\text{III}')\}$, are satisfied, we have

$$f(z) = \sum_n \sum_{s=0}^{m_n} \sum_{j=0}^{s} \frac{f_n^{(j)}}{j!} \cdot \frac{H_n^{(s-j)}}{(s-j)!} \cdot (z-z_n)^s \cdot \frac{g(z)}{(z-z_n)^{m_n+1}} + K \cdot g(z) \quad (2\text{-}3)$$

$$= \sum_n \sum_{s=0}^{m_n} \frac{(z-z_n)^s}{s!} \cdot \left[\frac{d^s}{dz^s}\Big\{f(z)\cdot H(z)\Big\}\right]_{z=z_n} \cdot \frac{g(z)}{(z-z_n)^{m_n+1}} + K \cdot g(z). \quad (2\text{-}3')$$

3) Condition (III′) can be further weakened, *i. e.,* we can replace condition (III′) by the following one :

$$\text{(III}'') \quad \lim_{C\to\infty}\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-z)\cdot g(\zeta)}d\zeta = K \quad (=\text{const}), \tag{2-4}$$

where C represents the contour of a closed curve, and $\lim_{C\to\infty}$ means to make the contour infinitely large. The left-hand side of (2–4) appears in calculating the value $f(z)/g(z)$ by the Cauchy theorem. Referring to the proof mentioned above, we see at once that the condition (III′) can be replaced by (III″). Hence, Theorem 2–1 holds under conditions {(I), (II), (III″)}, or {(I′), (III″)}.

4) If all the m_n's are equal to the same value m, then expressions (1–1)∼(1–1″), (2–3), and (2–3′), can be simplified.

5) For the case $m\geqq 0$ in (2–3) or (2–3′), we can take function $g(z)$ practically as

$$g(z)=\phi^{m+1}(z), \tag{2-5}$$

where $\phi(z)$ is an entire function with simple zeros at points $z=z_n$ (n=integers). Then the sampling formula (2–3) for an entire function $f(z)$ reduces to:

$$f(z)=\sum_n\sum_{s=0}^{m}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^s\cdot\frac{\phi^{m+1}(z)}{(z-z_n)^{m+1}}+K\cdot\phi^{m+1}(z), \tag{2-6}$$

where $H_n^{(k)}$'s are given by (1–9), with

$$h_n=g_n^{(m+1)}/(m+1)!=(\phi_n')^{m+1}, \tag{2-7}$$

and

$$\begin{aligned}{}_sC_r\cdot\frac{h_n^{(s-r)}}{h_n}&=\frac{s!\,(m+1)!}{r!\,(m+1+s-r)!}\cdot\frac{g_n^{(m+1+s-r)}}{g_n^{(m+1)}}=\frac{s!\,(m+1)!}{r!\,(\phi_n')^{m+1}}\cdot\\ &\cdot\sum_{\substack{p+q+u+\cdots=m+1\\ p+2q+3u+\cdots=m+1+s-r}}\frac{1}{p!\,q!\,u!\cdots}\cdot(\phi_n')^p\cdot\left(\frac{1}{2!}\phi_n''\right)^q\cdot\left(\frac{1}{3!}\phi_n'''\right)^u\cdots.\end{aligned} \tag{2-8}$$

6) Some examples of the sampling formula for small m.

a) If all the poles of the function $f(z)/g(z)$ are simple poles (*i.e.*, $m=0$) at points $z=z_n$ (n=integers), then from (2–3) or (2–3′), we have:

$$f(z)=\sum_n f_n\cdot H_n\cdot\frac{g(z)}{z-z_n}+K\cdot g(z)=\sum_n f(z_n)\cdot\frac{g(z)}{(z-z_n)\cdot g'(z_n)}+K\cdot g(z), \tag{2-9}$$

which, in case $K=0$, reduces to the formula suggested by van der Pol[2].

b) If all the poles of the function $f(z)/g(z)$ are of second order (*i.e.*, $m=1$) at points $z=z_n$ (n=integers), then expression (2–3) or (2–3′) reduces to:

$$\begin{aligned}f(z)&=\sum_n\left[f_n\cdot H_n+(z-z_n)\cdot\left\{f_n\cdot H_n'+f_n'\cdot H_n\right\}\right]\cdot\frac{g(z)}{(z-z_n)^2}+K\cdot g(z)\\ &=\sum_n\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-\frac{1}{3}f(z_n)\cdot\frac{g_n^{(3)}}{g_n^{(2)}}\right\}\right]\cdot\frac{2!\,g(z)}{(z-z_n)^2\cdot g_n^{(2)}}+K\cdot g(z).\end{aligned} \tag{2-10}$$

c) If all the poles of the function $f(z)/g(z)$ are of third order (*i. e.*, $m=2$) at points $z=z_n$ (n=integers), then from (2–3) or (2–3′) we have:

$$f(z)=\sum_n\left[f_n\cdot H_n+(z-z_n)\cdot\left\{f_n\cdot H_n'+f_n'\cdot H_n\right\}+\frac{1}{2}(z-z_n)^2\cdot\right.$$
$$\left.\cdot\left\{f_n\cdot H_n''+2f_n'\cdot H_n'+f_n''\cdot H_n\right\}\right]\cdot\frac{g(z)}{(z-z_n)^3}+K\cdot g(z)=$$
$$=\sum_n\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-\frac{1}{4}f(z_n)\cdot\frac{g_n^{(4)}}{g_n^{(3)}}\right\}+\frac{1}{2}(z-z_n)^2\cdot\right.$$
$$\left.\cdot\left\{f''(z_n)-\frac{1}{2}f'(z_n)\cdot\frac{g_n^{(4)}}{g_n^{(3)}}+\frac{1}{2}f(z_n)\cdot\left[\frac{1}{4}\left(\frac{g_n^{(4)}}{g_n^{(3)}}\right)^2-\frac{1}{5}\frac{g_n^{(5)}}{g_n^{(3)}}\right]\right\}\right]\cdot$$
$$\cdot\frac{3!\,g(z)}{(z-z_n)^3\cdot g_n^{(3)}}+K\cdot g(z)\,. \tag{2–11}$$

Detailed examples of the sampling formulae (1–1)∼(1–1″), (2–3)∼(2–3′) with (2–4), (2–6), and (2–9)∼(2–11), will be given in the following papers[9),10)].

7) Truncation error of the sampling expansion (2–3) or (2–3′) can be estimated in a simple manner (*cf.* § 6). For example, we shall consider the *bound for truncation error* of the sampling expansion (2–9) for real z and z_n (n=integers). Let $R_{N,M}(z)$ be a truncation error defined by:

$$R_{N,M}(z)\equiv f(z)-\sum_{n=-M}^{N}f_n\cdot\frac{g(z)}{(z-z_n)\cdot g_n'}-K\cdot g(z)\,,\quad(\text{for } z_{-M}<z<z_N) \tag{2–12}$$

with $f_n=f(z_n)$, $g_n'=[dg(z)/dz]_{z=z_n}$, and positive integers N and M. The sampling points z_n (n=integers) are arranged in ascending order of magnitude, in such a way that $\cdots<z_{-2}<z_{-1}<z_0<z_1<z_2<\cdots$, where positive values of z_n correspond to positive suffixes $n>0$, and negative values of z_n to negative suffixes $n<0$.

From the definition of $R_{N,M}(z)$ in (2–12), it is easily seen that

$$|R_{N,M}(z)|\leqslant|g(z)|\cdot\left[\frac{K_N\cdot A_N}{\sqrt{z_N-z}}+\frac{L_M\cdot B_M}{\sqrt{z-z_{-M}}}\right],\qquad(\text{for } z_{-M}<z<z_N) \tag{2–13}$$

where

$$K_N=\sqrt{\sum_{n>N}|f_n|^2}\,,\qquad L_M=\sqrt{\sum_{n<-M}|f_n|^2}\,,\qquad A_N=\max_{n>N}|g_n'|^{-1}\,,$$
$$\text{and } B_M=\max_{n<-M}|g_n'|^{-1}\,. \tag{2–14}$$

Especially, in case $g(z)=\sin(\beta z+\gamma)$ in (2–12), we have: $z_n=(n\pi-\gamma)/\beta$, $g_n'=(-1)^n\cdot\beta$, (n=integers)
and

$$|g(z)|\cdot A_N=|g(z)|\cdot B_M=|\sin(\beta z+\gamma)|/|\beta|\leqslant 1/|\beta|, \tag{2–15}$$

then expression (2–13) leads to:

$$|R_{N,M}(z)|\leqslant\frac{|\sin(\beta z+\gamma)|}{|\beta|}\cdot\left[\frac{K_N}{\sqrt{z_N-z}}+\frac{L_M}{\sqrt{z-z_{-M}}}\right]\leqslant$$
$$\leqslant\frac{|\sin(\beta z+\gamma)|}{\sqrt{|\beta|}}\cdot\left[\frac{K_N}{\sqrt{N\pi-(\beta z+\gamma)}}+\frac{L_M}{\sqrt{\beta z+\gamma+M\pi}}\right],\quad(\beta\neq 0) \tag{2–16}$$

which reduces to the inequality obtained by Jagerman[3)] for the case $M=N$.

8) The generalization of the sampling theorem 2–1 to the case of many variables can be made straightforward. A similar analysis as was stated in the proof mentioned above, leads to the sampling formula for many variables. For example, we have:

Theorem 2–2

An entire function $f(z, w)$ with respect to two complex variables z and w, can be expressed in the following form:

$$f(z, w)=\sum_n \sum_r \sum_{s=0}^{m_n} \sum_{u=0}^{M_r} \sum_{j=0}^{s} \sum_{k=0}^{u} \frac{f_{n,r}^{(j,k)}}{j!\,k!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot\frac{G_r^{(u-k)}}{(u-k)!}\cdot(z-z_n)^s\cdot$$
$$\cdot(w-w_r)^u\cdot\frac{g_1(z)\cdot g_2(w)}{(z-z_n)^{m_n+1}\cdot(w-w_r)^{M_r+1}}+\sum_n \sum_{s=0}^{m_n}\sum_{j=0}^{s}\frac{A_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot$$
$$\cdot(z-z_n)^s\cdot\frac{g_1(z)\cdot g_2(w)}{(z-z_n)^{m_n+1}}+\sum_r\sum_{u=0}^{M_r}\sum_{k=0}^{u}\frac{B_r^{(k)}}{k!}\cdot\frac{G_r^{(u-k)}}{(u-k)!}\cdot(w-w_r)^u\cdot$$
$$\cdot\frac{g_1(z)\cdot g_2(w)}{(w-w_r)^{M_r+1}}+K_0\cdot g_1(z)\cdot g_2(w) \tag{2–17}$$

$$=\sum_n\sum_r\sum_{s=0}^{m_n}\sum_{u=0}^{M_r}\frac{(z-z_n)^s\cdot(w-w_r)^u}{s!\,u!}\cdot$$
$$\cdot\left[\frac{\partial^{s+u}}{\partial z^s\partial w^u}\left\{f(z, w)\cdot H(z)\cdot G(w)\right\}\right]_{z=z_n,\ w=w_r}\cdot\frac{g_1(z)\cdot g_2(w)}{(z-z_n)^{m_n+1}\cdot(w-w_r)^{M_r+1}}+$$
$$+\sum_n\sum_{s=0}^{m_n}\sum_{j=0}^{s}\frac{A_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^s\cdot\frac{g_1(z)\cdot g_2(w)}{(z-z_n)^{m_n+1}}+$$
$$+\sum_r\sum_{u=0}^{M_r}\sum_{k=0}^{u}\frac{B_r^{(k)}}{k!}\cdot\frac{G_r^{(u-k)}}{(u-k)!}\cdot(w-w_r)^u\cdot\frac{g_1(z)\cdot g_2(w)}{(w-w_r)^{M_r+1}}+$$
$$+K_0\cdot g_1(z)\cdot g_2(w), \tag{2–17'}$$

if the following conditions (IV), (V), and (VI) are satisfied:

(IV) Function $f(z, w)$ is entire with regard to z and w, respectively. (2–18)

(V) $g_1(z)$ and $g_2(w)$ are entire, and $g_1(z)$ has zeros of (m_n+1)-th order at points $z=z_n$ (n=integers), and $g_2(w)$ has zeros of (M_r+1)-th order at points $w=w_r$ (r=integers). (2–19)

(VI)
$$\lim_{c_2\to\infty}\frac{1}{2\pi i}\int_{c_2}\frac{\left[\frac{\partial^j}{\partial\zeta^j}f(\zeta, \eta)\right]_{\zeta=z_n}}{(\eta-w)\cdot g_2(\eta)}d\eta=A_n^{(j)},$$
$$\lim_{c_1\to\infty}\frac{1}{2\pi i}\int_{c_1}\frac{\left[\frac{\partial^k}{\partial\eta^k}f(\zeta, \eta)\right]_{\eta=w_r}}{(\zeta-z)\cdot g_1(\zeta)}d\zeta=B_r^{(k)},$$

and

$$\lim_{\substack{c_1\to\infty\\ c_2\to\infty}}\left(\frac{1}{2\pi i}\right)^2\cdot\int_{c_1}\int_{c_2}\frac{f(\zeta, \eta)}{(\zeta-z)\cdot(\eta-w)\cdot g_1(\zeta)\cdot g_2(\eta)}d\zeta d\eta=K_0\,(=\text{const}), \tag{2–20}$$

where $\mathfrak{c}_1$ and $\mathfrak{c}_2$ represent contours of closed curves in ζ- and η-planes, respectively, and $\lim_{\mathfrak{c}_1\to\infty}$ and $\lim_{\mathfrak{c}_2\to\infty}$ mean to make the contours infinitely large.

In expression (2–17), we have put:

$$f_{n,r}^{(j,k)}=\left[\frac{\partial^{j+k}}{\partial z^j\partial w^k}f(z,w)\right]_{z=z_n,\ w=w_r}, \tag{2–21}$$

$$\left.\begin{aligned} &h(z)\equiv 1/H(z)\equiv g_1(z)/(z-z_n)^{m_n+1},\\ &p(w)\equiv 1/G(w)\equiv g_2(w)/(w-w_r)^{M_r+1},\\ &H_n^{(j)}=\left[\partial^j H(z)/\partial z^j\right]_{z=z_n},\qquad G_r^{(k)}=\left[\partial^k G(w)/\partial w^k\right]_{w=w_r}. \end{aligned}\right\} \tag{2–22}$$

The summation over n and r in (2–17) and (2–17′), means to cover all the points $z=z_n$ ($n=$integers) and $w=w_r$ ($r=$integers), respectively in z- and w-planes. The points $z=z_n$ ($n=$integers) are called the *sampling points in z-plane*, and the points $w=w_r$ ($r=$integers) the *sampling points in w-plane.* Functions $h(z)=g_1(z)/(z-$ $-z_n)^{m_n+1}$ and $p(w)=g_2(w)/(w-w_r)^{M_r+1}$ shall be called the *sampling functions*, respectively in z- and w-planes.

§ 3. Generalized Frequency and Generalized Spectrum Function

We shall take the integral transform $F(s)$ of a function $f(z)$:

$$F(s)=\int_A K(s,z)\cdot f(z)\cdot dz, \tag{3–1}$$

with an integral kernel $K(s,z)$ and the domain of integration A. The inverse transform, if it exists, shall be written by:

$$f(z)=\int_B \widetilde{K}(z,s)\cdot F(s)\cdot ds, \tag{3–2}$$

with the integral kernel $\widetilde{K}(z,s)$ and the domain of integration B.

When we take in (3–1) and (3–2): $f(z)\in L^2(-\infty,+\infty)$, $F(s)\in L^2(-\infty,+\infty)$, $K(s,z)=\frac{1}{\sqrt{2\pi}}\cdot\exp[-isz]$, $\widetilde{K}(z,s)=\frac{1}{\sqrt{2\pi}}\cdot\exp[+izs]$, $A=(-\infty,+\infty)$, $B=(-\infty,$ $+\infty)$, then we have the *Fourier transform* and its inverse:

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp[-isz]\cdot f(z)\cdot dz, \tag{3–3}$$

and

$$f(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp[+izs]\cdot F(s)\cdot ds. \tag{3–4}$$

The function $F(s)$ in (3–3) is called the *Fourier spectrum function* of $f(z)$, and the variable s in (3–3) the *frequency* of $f(z)$.

Corresponding to the Fourier spectrum and the frequency, we shall call the function $F(s)$ in (3–1) a *generalized spectrum function*[8] of $f(z)$ with regard to the transform (3–1), while the variable s in (3–1) shall be called a *generalized frequency*[8] of $f(z)$ with regard to the transform (3–1).

For example, if an entire function $f(z)$ satisfies the following condition:

$$\sqrt{z}\cdot f(z)\in L^1(0,\ +\infty), \tag{3-5}$$

then we have an integral transform and its inverse transform[4]:

$$F(s)=\int_0^{+\infty} z\cdot J_\nu(sz)\cdot f(z)\cdot dz\,, \tag{3-6}$$

and

$$f(z)=\int_0^{+\infty} s\cdot J_\nu(zs)\cdot F(s)\cdot ds\,, \tag{3-7}$$

with Bessel function $J_\nu(z)$ of order $\nu\geqslant -1/2$. In this case, we call $F(s)$ a *generalized spectrum function* of $f(z)$ with regard to the Hankel transform (3–6), and the variable s a *generalized frequency* of $f(z)$ with regard to the Hankel transform (3–6).

Another example of a generalized spectrum $F(s)$ and a generalized frequency s is as follows[5]:

$$F(s)=\int_p^{+\infty} z\cdot T_\nu(\alpha z,\ sz)\cdot f(z)\cdot dz\,, \tag{3-8}$$

and

$$B_\nu(z)\cdot f(z)=\int_\alpha^{+\infty} s\cdot T_\nu(\alpha z,\ zs)\cdot F(s)\cdot ds\,, \tag{3-9}$$

with

$$B_\nu(z)=J_\nu^2(z)+Y_\nu^2(z)\,,$$

where

$$T_\mu(x,\ z)=Y_\nu(x)\cdot J_\mu(z)-J_\nu(x)\cdot Y_\mu(z)\,, \tag{3-10}$$

with Bessel function $J_\mu(z)$ and Neumann function $Y_\mu(z)$, and $z\cdot f(z)\in L^1(p,\ +\infty)$, *i. e.*, the integral: $\int_p^{+\infty} z\cdot f(z)\cdot dz<+\infty$, is absolutely convergent, with $\alpha\geqslant p>0$, and $\nu\geqslant -1/2$.

§ 4. Band-limited Function and Fourier Spectrum Function

A class of functions which is of particular interest is the class of *band-limited functions* defined as follows:

A function $f(z)$ is called *band-limited*, if there exists a constant $W>0$ and a frequency function $G(\omega)$ of *bounded variation* over the interval $(-W,\ W)$, so that

$$f(z)=\frac{1}{\sqrt{2\pi}}\int_{-W}^{+W}\exp\,[i\omega z]\cdot dG(\omega)\,. \tag{4-1}$$

The function $G(\omega)$ is called the *Fourier-Stieltjes spectrum function* of $f(z)$, and W the *cut-off frequency*, or the *maximum frequency*. In case $G(\omega)$ is absolutely continuous over the interval $(-W, W)$, then (4–1) can be written as:

$$f(z)=\frac{1}{\sqrt{2\pi}}\int_{-W}^{+W}\exp[i\omega z]\cdot G'(\omega)\cdot d\omega, \tag{4–2}$$

and $G'(\omega)$ is called the *Fourier spectrum function* of $f(z)$ with *maximum frequency* W (*cf.* (3–4)). Clearly *a band-limited function is entire.*

It is convenient to introduce a function $\Xi(y)$ defined by:

$$\Xi(y)=\max_{-\infty<x<+\infty}|f(z)|, \qquad (\text{with } z=x+iy) \tag{4–3}$$

for any entire function $f(z)$.

From (4–1) we have

$$|f(z)|\leqslant\frac{1}{\sqrt{2\pi}}\int_{-W}^{+W}|\exp[i\omega x-\omega y]|\cdot dV(\omega)\leqslant\frac{1}{\sqrt{2\pi}}\int_{-W}^{+W}|\exp[i\omega x]|\cdot$$
$$\cdot|\exp[-\omega y]|\cdot dV(\omega)\leqslant\frac{1}{\sqrt{2\pi}}\int_{-W}^{+W}|\exp[-\omega y]|\cdot dV(\omega), \tag{4–4}$$

where $V(\omega)$ is the variation of $G(\omega)$ over the interval $(-W, \omega)$. Let us take:

$$V=\int_{-W}^{+W}dV(\omega), \tag{4–5}$$

i. e., V is the total variation of $G(\omega)$ over the interval $(-W, W)$, then from (4–4) and (4–5), we obtain:

$$|f(z)|\leqslant\frac{V}{\sqrt{2\pi}}\cdot\exp[W\cdot|y|]. \tag{4–6}$$

Thus we may define the function $\Xi(y)$ in (4–3) as

$$\Xi(y)=\frac{V}{\sqrt{2\pi}}\cdot\exp[W\cdot|y|], \tag{4–7}$$

and obtain:

$$\exp[-|\beta|\cdot|y|]\cdot\Xi(y)=\frac{V}{\sqrt{2\pi}}\cdot\exp[(W-|\beta|)\cdot|y|]. \tag{4–8}$$

When $|y|\to+\infty$, the expression (4–8) approaches zero, provided that $W-|\beta|<0$, *i. e.*

$$1/|\beta|<1/W. \tag{4–9}$$

From (4–6)∼(4–8), the left-hand side of condition (III) in (1–4) with $g(z)=$ $=\sin(\beta z+\gamma)$, is seen to be**):

$$\lim_{z\to\infty}\left|\frac{f(z)}{\sin(\beta z+\gamma)}\right|\leqslant\lim_{|y|\to+\infty}4\cdot\exp[-|\beta|\cdot|y|]\cdot\Xi(y). \qquad (\beta\neq0) \tag{4–10}$$

**) $1/|\sin^{m+1}(\beta z+\gamma)|=1/[\sqrt{\sin^2(\beta x+\gamma)+\sinh^2(\beta y)}]^{m+1}\leqslant1/|\sinh(\beta y)|^{m+1}\leqslant4^{m+1}\cdot\exp[-(m+1)\cdot|\beta|\cdot|y|]$, for large y (*i. e.*, for $|\beta|\cdot|y|\geqslant\log\sqrt{2}$), with $(m+1)$ positive integers, β and γ real, and $z=x+iy$. We shall call the inequality in the first line (4–10′).

Referring to (4–6)∼(4–10), we have the following expression:

$$\lim_{z\to\infty}\left|\frac{f(z)}{\sin(\beta z+\gamma)}\right|=0, \tag{4–11}$$

i. e., condition (III) in (1–4) holds, if function $f(z)$ is band-limited with maximum frequency W, and if condition:

$$1/|\beta|<1/W, \tag{4–12}$$

is satisfied. Hence, from (1–1) or (2–9) we obtain:

Theorem 4–1

If a function $f(z)$ is band-limited with maximum frequency W, then $f(z)$ is representable by:

$$f(z)=\sum_{n=-\infty}^{+\infty}f_n\cdot\frac{\sin(\beta z+\gamma-n\pi)}{\beta z+\gamma-n\pi}, \qquad (\beta\neq 0) \tag{4–13}$$

with $f_n=f(z_n)$, and $z_n=(n\pi-\gamma)/\beta$ (n=integers), provided that the *sampling frequency****) $|\beta|$ satisfies the following condition:

$$1/|\beta|<1/W. \tag{4–14}$$

Referring to the proof above, and taking $g(z)=\sin^{m+1}(\beta z+\gamma)$ in (1–1) or (2–3), we obtain the following theorems in a similar manner.

Theorem 4–2

An entire function $f(z)$ is given by:

$$f(z)=\sum_{n=-\infty}^{+\infty}\sum_{s=0}^{m}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^s\cdot\frac{\sin^{m+1}(\beta z+\gamma)}{(z-z_n)^{m+1}}, \ (\beta\neq 0) \tag{4–15}$$

provided that condition:

$$\lim_{|y|\to+\infty}\exp[-(m+1)\cdot|\beta|\cdot|y|]\cdot\Xi(y)=0, \ \text{(for positive integer } (m+1)) \tag{4–16}$$

is satisfied. Here we put $z_n=(n\pi-\gamma)/\beta$, and $f_n^{(j)}=f^{(j)}(z_n)$. The values $H_n^{(s-j)}$ are given by (1–9), (2–5), (2–7), and (2–8), with $g(z)=\sin^{m+1}(\beta z+\gamma)$.

Theorem 4–3

A band-limited function $f(z)$ with maximum frequency W is representable by (4–15), if condition:

$$1/|\beta|<(m+1)/W, \qquad \text{(for positive integer } (m+1)) \tag{4–17}$$

is satisfied.

Expression (4–15) is somewhat different from the expression obtained by Linden and Abramson[6].

***) We shall call the distance between successive sampling points, *i. e.*, $|z_{n+1}-z_n|=\pi/|\beta|$, a *sampling interval*, and value $|\beta|=\pi/|z_{n+1}-z_n|$ a *sampling frequency*.

Let $z \cdot f(z)$ be band-limited with maximum frequency W and $\lim_{z\to 0} z \cdot f(z)=0$, then $f(z)$ is entire. Referring to inequality (4–6), we have

$$|f(z)| \leqslant \frac{V}{\sqrt{2\pi}\cdot|z|}\cdot \exp[W\cdot|y|] \leqslant \frac{V}{\sqrt{2\pi}\cdot|y|}\cdot \exp[W\cdot|y|]. \tag{4–18}$$

Accordingly**), from (4–10′) we obtain:

$$\left|\frac{f(z)}{\sin^{m+1}(\beta z+\gamma)}\right| \leqslant \frac{4^{m+1}\cdot V}{\sqrt{2\pi}\,|y|}\cdot \exp[(W-(m+1)\cdot|\beta|)\cdot|y|], \tag{4–19}$$

with V the total variation of Fourier-Stieltjes spectrum function of $z \cdot f(z)$ over $(-W, W)$. We refer to expressions (4–18) and (4–19), and obtain the following theorem:

Theorem 4–4

If $z \cdot f(z)$ is band-limited with maximum frequency W and $\lim_{z\to 0} z \cdot f(z)=0$, then $f(z)$ is representable by (4–15) provided that $1/|\beta|<(m+1)/W$.

Let $f(z)$ be band-limited over the interval $(-W, W)$ with a Fourier spectrum function $G(\omega)$ of bounded variation, *i. e.*

$$f(z)=\frac{1}{\sqrt{2\pi}}\int_{-W}^{+W} \exp[i\omega z]\cdot G(\omega)\cdot d\omega. \tag{4–20}$$

Define $G(\omega)$ to be zero outside the interval $(-W, W)$, then

$$f(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp[i\omega z]\cdot G(\omega)\cdot d\omega. \tag{4–21}$$

Integration by parts shows that

$$z\cdot f(z)=\frac{i}{\sqrt{2\pi}}\int_{-W}^{+W} \exp[i\omega z]\cdot dG(\omega), \tag{4–22}$$

i. e., $z \cdot f(z)$ is band-limited. Also, from (4–20), we can see that $\lim_{z\to 0} z \cdot f(z)=0$. Hence, the function $f(z)$ is entire. Accordingly we have:

Theorem 4–5

If $f(z)$ is band-limited with maximum frequency W and its Fourier spectrum function is of bounded variation over the interval $(-W, W)$, then $f(z)$ is representable by (4–15), provided that $1/|\beta|<(m+1)/W$.

Theorems 4–2∼4–5 for the case $m \leqslant 1$ and Theorem 4–1, were given by Jagerman and Fogel[7]. Their theorems assert that the use of both f_n and f'_n allows exact representation using a sampling domain twice that requires for representation of the same function when using f_n alone. While the present theorems 4–2∼4–5 give an interpolation series applicable to the general case where the values of function f_n and derivatives of higher order $f_n^{(j)}$ $(j=1, 2, 3, \cdots, m)$ are available.

§ 5. Hankel Transform and Generalized Frequency

A similar analysis can be carried out, when we take $g(z)=J_\mu(\beta z)$ in (1–1), where $J_\mu(z)$ is Bessel function of integral order μ. In this case, we have:

$$f(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} f(z_n)\cdot\frac{1}{J'_\mu(\beta z_n)}\cdot\frac{J_\mu(\beta z)}{\beta(z-z_n)}+E(z), \qquad (\beta\neq 0) \tag{5–1}$$

$$E(z)=\sum_{s=0}^{|\mu|-1}\sum_{j=0}^{s}\frac{f_0^{(j)}}{j!}\cdot\frac{H_0^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{J_\mu(\beta z)}{z^{|\mu|}}, \qquad (\beta\neq 0) \tag{5–1′}$$

with

$$J_\mu(\beta z_n)=0, \qquad (\mu=\text{integer},\ n=\text{integers}) \tag{5–1″}$$

where $f_0^{(j)}=f^{(j)}(z_0)=f^{(j)}(0)$, and $H_0^{(s-j)}=H^{(s-j)}(z_0)=H^{(s-j)}(0)$ is given by (1–9), with $g(z)=J_\mu(\beta z)$. In case $\mu=0$, $E(z)$ reduces to a null function.

We shall consider the case, where function $\sqrt{z}\cdot f(z)$ is expressed by the *Hankel-Stieltjes transform* (3–7) with a *generalized maximum frequency* M:

$$\sqrt{z}\cdot f(z)=\int_0^M s\cdot J_\nu(zs)\cdot dF(s). \qquad \left(\nu\geqslant -\frac{1}{2}\right) \tag{5–2}$$

If function $F(s)$ is absolutely continuous over the interval $(0, M)$, we have:

$$\sqrt{z}\cdot f(z)=\int_0^M s\cdot J_\nu(zs)\cdot F'(s)\cdot ds. \qquad \left(\nu\geqslant -\frac{1}{2}\right) \tag{5–3}$$

In expression (5–3), function $F'(s)$ is assumed to be zero outside the interval $(0, M)$, *i.e.*, $F'(s)=0$ for $s\geqslant M$. The function $F'(s)$ shall be called the *Hankel spectrum* of $f(z)$, and M a *generalized maximum frequency* with regard to the Hankel transform (3–7).

Let us take $U(s)$ the variation of $F(s)$ over the interval $(0, s)$ and U the total variation of $F(s)$ over the interval $(0, M)$, *i.e.*

$$U=\int_0^M dU(s). \tag{5–4}$$

Then, we have:

$$|\sqrt{z}|\cdot|f(z)|\leqslant\int_0^M |s\cdot J_\nu(zs)|\cdot dF(s)\leqslant M\cdot U\cdot\exp\,[M\cdot|y|],$$

$$\text{(for any real } x \text{ and an integer } \nu) \tag{5–5}$$

with $z=x+iy$. Inequality (5–5) is obtained from the following expressions:

$$|J_\nu(zs)|=|J_\nu(xs+iys)|\leqslant\sum_{r=-\infty}^{+\infty}|J_{\nu-r}(xs)|\cdot|J_r(iys)|\leqslant\sum_{r=-\infty}^{+\infty}|i^r\cdot I_r(ys)|\leqslant$$

$$\leqslant\sum_{r=-\infty}^{+\infty}I_r(|y|\cdot s)=\exp\,[|y|\cdot s],$$

$$\text{(for any real } x \text{ and an integer } \nu) \tag{5–6}$$

by means of the addition theorem of Bessel functions, and because of $|J_r(x)|\leqslant 1$, for any real x and integer r. Modified Bessel function $I_r(y)$ of non-negative integral order r is positive and monotonously increasing function of increasing positive y, so we obtain, from (5–4), (5–5), and (5–6), the following inequality:

$$|\sqrt{z}|\cdot|f(z)|\leqslant\int_0^M|s\cdot J_\nu(zs)|\cdot dF(s)\leqslant\int_0^M s\cdot\exp[|y|\cdot s]\cdot dF(s)\leqslant$$
$$\leqslant M\cdot U\cdot\exp[M\cdot|y|]\,. \qquad \text{(for any integer } \nu) \qquad (5\text{–}5')$$

The left-hand side of condition (III) in (1–4) for $g(z)=J_\mu(\beta z)$, is written, by means of (5–5'), as follows:

$$\lim_{z\to\infty}\left|\frac{f(z)}{g(z)}\right|\leqslant\lim_{z\to\infty}M\cdot U\cdot\left|\frac{\exp[M\cdot|y|]}{\sqrt{z}\cdot J_\mu(\beta z)}\right|\leqslant\lim_{|y|\to+\infty}M\cdot U\cdot\sqrt{\frac{|\beta|\cdot\pi}{2}}\cdot\exp[(M-|\beta|)\cdot|y|]\,, \qquad (5\text{–}7)$$

with $z=x+iy$. Here, Bessel function $J_\mu(\beta z)$ for large argument was expanded in the asymptotic expansion: $J_\mu(\beta z)\sim\sqrt{2/(\pi\beta z)}\cdot\cos[\beta z-(2\mu+1)\pi/4]-\cdots$. Accordingly, condition (III) is satisfied, provided that $M<|\beta|$, *i. e.*

$$1/|\beta|<1/M\,. \qquad (5\text{–}8)$$

For $g(z)=J_\mu^{m+1}(\beta z)$, we have a sampling formula (1–1)$\sim$(1–1''), and by a similar analysis as in (5–7) and (5–8), we can find that condition (III) is satisfied, provided that $z^{(m+1)/2}\cdot f(z)$ is expressed by the Hankel transform (3–7) and that $M<$ $<(m+1)\cdot|\beta|$, *i. e.*

$$1/|\beta|<(m+1)/M\,. \qquad \text{(for positive integer } (m+1)) \qquad (5\text{–}9)$$

Hence, we obtain the following theorems:

Theorem 5–1

If $\sqrt{z}\cdot f(z)$ is entire and band-limited in the sense of the generalized maximum frequency M with regard to the Hankel transform (3–7), then $f(z)$ is representable by the interpolation series (5–1), provided that

$$1/|\beta|<1/M\,. \qquad (5\text{–}10)$$

Theorem 5–2

If $z^{(m+1)/2}\cdot f(z)$ is entire and band-limited in the sense of the generalized maximum frequency M with regard to the Hankel transform (3–7), then $f(z)$ if representable by the interpolation series:

$$f(z)=\sum_{\substack{n=-\infty\\n\neq0}}^{+\infty}\sum_{s=0}^{m}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^s\cdot\frac{J_\mu^{m+1}(\beta z)}{(z-z_n)^{m+1}}+G(z),\ (\beta\neq0) \qquad (5\text{–}11)$$

provided that $1/|\beta|<(m+1)/M$, with

$$G(z)=\sum_{s=0}^{|\mu|\cdot(m+1)-1}\sum_{j=0}^{s}\frac{f_0^{(j)}}{j!}\cdot\frac{H_0^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{J_\mu^{m+1}(\beta z)}{z^{|\mu|\cdot(m+1)}}\,, \qquad (\beta\neq0) \qquad (5\text{–}11')$$

where $J_\mu(\beta z_n)=0$ (μ=integer, n=integers), $f_n^{(j)}=f^{(j)}(z_n)$, $f_0^{(j)}=f^{(j)}(z_0)=f^{(j)}(0)$, and $H_n^{(s-j)}$ are given by (1–9), with $g(z)=J_\mu^{m+1}(\beta z)$.

Here we shall note that Bessel function $J_\mu(z)$ has a zero of $|\mu|$-th order, and $J_\mu^{m+1}(z)$ has a zero of $|\mu|\cdot(m+1)$-th order, respectively at the origin. In case $\mu=0$, $G(z)$ reduces to a null function.

§ 6. Truncation Error of the Sampling Expansion

The truncation error of the sampling expansion (2–3)$\sim$(2–3′) can be easily estimated. For the sake of simplicity, we shall concern ourselves merely with cases of $m_n=0$, 1, and 2.

For the sake of convenience in printing, all the formulae and expressions in § 6 shall be placed in the Appendix. This was done, in accordance with the suggestions by the editorial committee of this Memoirs.

a) As for the case $m_n=0$, we shall consider the *bound for truncation error* of the sampling expansion (2–9) for real z and z_n (n=integers). Let $R_{N,M}(z)$ be a truncation error defined by (6–1), with $f_n=f(z_n)$, $g'_n=g'(z_n)$, and positive integers N and M. The sampling points z_n (n=integers) are arranged in ascending order of magnitude in such a way that $\cdots<z_{-2}<z_{-1}<z_0<z_1<z_2\cdots$, where positive values of z_n correspond to positive suffixes $n>0$, and negative values of z_n to negative suffixes $n<0$.

From the Cauchy inequality applied to (6–1), we have (6–2). Accordingly, we obtain (6–3) with (6–4), K_N and L_M being assumed to be bounded. Function $1/(z-z_n)^2$ is a monotonic decreasing function of z_n for increasing $z_n>z$, hence we obtain (6–5). Similarly we have (6–6).

From (6–3), (6–5), and (6–6), we obtain the bound for truncation error (6–1) as in (6–7).

In case $g(z)=\sin(\beta z+\gamma)$ with $\beta\neq 0$, inequality (6–7) reduces to (6–8), making use of the expressions (6–9) and (6–10), with $z_N=(N\pi-\gamma)/\beta$ and $z_{-M}=(-M\pi-\gamma)/\beta$.

If we take $|\beta z+\gamma|\leqslant S\pi$ with $S<N$ and $S<M$, inequality (6–8) implies (6–11), because of the relations: $N\pi-(\beta z+\gamma)\geqslant N\pi-S\pi$, and $\beta z+\gamma+M\pi\geqslant M\pi-S\pi$.

Inequalities (6–8) and (6–11) reduce to the inequalities obtained by Jagerman[3] for the case $\gamma=0$ and $M=N$. In case $S=1/2$, (6–11) is more simplified as was shown by him.

b) As for the case $m_n=1$, we shall consider the bound for truncation error of the sampling expansion (2–10) for real z and z_n (n=integers). Let $R_{N,M}(z)$ be a truncation error defined by (6–12), with $f_n^{(k)}=f^{(k)}(z_n)$, $g_n^{(k)}=g^{(k)}(z_n)$, and positive integers N and M. The sampling points z_n (n=integers), *i.e.*, zeros of second order of $g(z)$, are arranged in ascending order of magnitude with increasing n, in such a way that $\cdots z_{-2}<z_{-1}<z_0<z_1<z_2<\cdots$, where positive values of z_n correspond to positive suffixes $n>0$, and negative values of z_n to negative suffixes $n<0$. We shall write $z_0=0$, either z_0 is a zero of $g(z)$ or not.

From the Cauchy inequality applied to (6–12), we obtain (6–13). Inequality (6–13) implies (6–14), with (6–15) and (6–16).

Functions $1/(z-z_n)^2$ and $1/(z-z_n)^4$ are monotonic decreasing functions of z_n for increasing $z_n>z$, hence we have (6–17) and (6–18). Similarly we obtain (6–19) and (6–20).

From (6–14) and (6–17)∼(6–20), we have the bound for truncation error (6–12) as in (6–21).

In case $g(z)=\sin^2(\beta z+\gamma)$ with $\beta\neq 0$, inequality (6–21) reduces to (6–22), making use of expressions (6–23) and (6–24), with $z_N=(N\pi-\gamma)/\beta$, and $z_{-M}=(-M\pi-\gamma)/\beta$.

If we take $|\beta z+\gamma|\leqslant S\pi$ with $S<N$ and $S<M$, inequality (6–22) reduces to (6–25), because of the relations: $N\pi-(\beta z+\gamma)\geqslant N\pi-S\pi$, and $\beta z+\gamma+M\pi\geqslant M\pi-S\pi$.

In case $N=M$ in (6–25), we have (6–26).

Inequalities (6–21), (6–22), and (6–25), are of some interest, if we compare them with expressions (6–7), (6–8), and (6–11). The latters express the bound for truncation error of the sampling expansion, making use of zero-th order derivatives of the sampled function. While, inequalities (6–21), (6–22) and (6–25), are available for the case of sampling expansion, which takes into account the sampled first order derivatives of an entire function.

c) A similar analysis leads to the expression for estimating the bound for truncation error of the sampling expansion (2–3)∼(2–3′) with $m_n\geqslant 2$.

As an example of the expressions for truncation error, let us consider the bound for truncation error of the sampling expansion (2–11) for real z and z_n (n=integers). Let $R_{N,M}(z)$ be a truncation error defined by (6–27), with $f_n^{(k)}=f^{(k)}(z_n)$ and $g_n^{(k)}=g^{(k)}(z_n)$.

Similar calculations as in (6–12)∼(6–20) lead to inequality (6–28) with (6–29) ∼(6–32), to express the bound for truncation error (6–27).

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Appendix

$$R_{N,M}(z)\equiv f(z)-\sum_{n=-M}^{N} f_n\cdot\frac{g(z)}{(z-z_n)\cdot g'_n}-K\cdot g(z)$$

$$=\sum_{n>N} f_n\cdot\frac{g(z)}{(z-z_n)\cdot g'_n}+\sum_{n<-M} f_n\cdot\frac{g(z)}{(z-z_n)\cdot g'_n}.\quad (\text{for } z_{-M}<z<z_N)\qquad (6\text{-}1)$$

$$\frac{|R_{N,M}(z)|}{|g(z)|}\leqslant\left[\sum_{n>N}|f_n|^2\cdot\sum_{n>N}\left|\frac{1}{(z-z_n)\cdot g'_n}\right|^2\right]^{1/2}+\left[\sum_{n<-M}|f_n|^2\cdot\sum_{n<-M}\left|\frac{1}{(z-z_n)\cdot g'_n}\right|^2\right]^{1/2},\qquad (6\text{-}2)$$

$$\frac{|R_{N,M}|}{|g(z)|}\leqslant K_N\cdot A_N\cdot\left[\sum_{n>N}\frac{1}{(z-z_n)^2}\right]^{1/2}+L_M\cdot B_M\cdot\left[\sum_{n<-M}\frac{1}{(z-z_n)^2}\right]^{1/2},\qquad (6\text{-}3)$$

$$K_N=\sqrt{\sum_{n>N}|f_n|^2},\quad L_M=\sqrt{\sum_{n<-M}|f_n|^2},\quad A_N=\max_{n>N}|g'_n|^{-1},\ \text{and}\ B_M=\max_{n<-M}|g'_n|^{-1}.\qquad (6\text{-}4)$$

$$\sum_{n>N}\frac{1}{(z-z_n)^2}<\int_{z_N}^{+\infty}\frac{du}{(z-u)^2}=\frac{1}{z_N-z},\qquad (6\text{-}5)$$

$$\sum_{n<-M}\frac{1}{(z-z_n)^2}<\int_{-\infty}^{z_{-M}}\frac{du}{(z-u)^2}=\frac{1}{z-z_{-M}},\qquad (6\text{-}6)$$

$$|R_{N,M}(z)|\leqslant|g(z)|\cdot\left[\frac{K_N\cdot A_N}{\sqrt{z_N-z}}+\frac{L_M\cdot B_M}{\sqrt{z-z_{-M}}}\right].\quad (\text{for } z_{-M}<z<z_N)\qquad (6\text{-}7)$$

$$|R_{N,M}(z)|\leqslant\frac{1}{\sqrt{|\beta|}}\cdot|\sin(\beta z+\gamma)|\cdot\left[\frac{K_N}{\sqrt{N\pi-(\beta z+\gamma)}}+\frac{L_M}{\sqrt{\beta z+\gamma+M\pi}}\right].$$

$$(\beta\neq0,\ -M\pi<\beta z+\gamma<N\pi)\qquad (6\text{-}8)$$

$$z_n=(n\pi-\gamma)/\beta,\qquad g'_n=(-1)^n\cdot\beta,\qquad (\beta\neq0)\qquad (6\text{-}9)$$

$$A_N=B_M=\frac{1}{|\beta|}.\qquad (6\text{-}10)$$

$$|R_{N,M}(z)|\leqslant\frac{1}{\sqrt{|\beta|}}\cdot|\sin(\beta z+\gamma)|\cdot\left[\frac{K_N}{\sqrt{N\pi-S\pi}}+\frac{L_M}{\sqrt{M\pi-S\pi}}\right].\qquad (\beta\neq0)\qquad (6\text{-}11)$$

$$R_{N,M}(z)\equiv f(z)-\sum_{n=-M}^{N}\left[f_n+(z-z_n)\cdot\left\{f'_n-\frac{1}{3}f_n\cdot\frac{g_n^{(3)}}{g_n^{(2)}}\right\}\right]\cdot\frac{2!\,g(z)}{(z-z_n)^2\cdot g_n^{(2)}}-K\cdot g(z)=$$

$$=\sum_{n>N}\left[f_n+(z-z_n)\cdot\left\{f'_n-\frac{1}{3}f_n\cdot\frac{g_n^{(3)}}{g_n^{(2)}}\right\}\right]\cdot\frac{2!\,g(z)}{(z-z_n)^2\cdot g_n^{(2)}}+\sum_{n<-M}\left[f_n+(z-z_n)\cdot\right.$$

$$\left.\cdot\left\{f'_n-\frac{1}{3}f_n\cdot\frac{g_n^{(3)}}{g_n^{(2)}}\right\}\right]\cdot\frac{2!\,g(z)}{(z-z_n)^2\cdot g_n^{(2)}}.\quad (\text{for } z_{-M}<z<z_N)\qquad (6\text{-}12)$$

$$\frac{|R_{N,M}(z)|}{|g(z)|}\leqslant\left[\sum_{n>N}|f_n|^2\cdot\sum_{n>N}\left|\frac{2!}{(z-z_n)^2\cdot g_n^{(2)}}\right|^2\right]^{1/2}+\left[\sum_{n>N}|f'_n|^2\cdot\sum_{n>N}\left|\frac{2!}{(z-z_n)\cdot g_n^{(2)}}\right|^2\right]^{1/2}+$$

$$+\left[\sum_{n>N}|f_n|^2\cdot\sum_{n>N}\left|\frac{2!\,g_n^{(3)}}{3(z-z_n)\cdot(g_n^{(2)})^2}\right|^2\right]^{1/2}+\left[\sum_{n<-M}|f_n|^2\cdot\right.$$

$$\left.\cdot\sum_{n<-M}\left|\frac{2!}{(z-z_n)^2\cdot g_n^{(2)}}\right|^2\right]^{1/2}+\left[\sum_{n<-M}|f'_n|^2\cdot\sum_{n<-M}\left|\frac{2!}{(z-z_n)\cdot g_n^{(2)}}\right|^2\right]^{1/2}+$$

$$+\left[\sum_{n<-M}|f_n|^2\cdot\sum_{n<-M}\left|\frac{2!\,g_n^{(3)}}{3(z-z_n)\cdot(g_n^{(2)})^2}\right|^2\right]^{1/2},\qquad (6\text{-}13)$$

$$\frac{|R_{N,M}(z)|}{|g(z)|}\leqslant K_N\cdot C_N\cdot\left[\sum_{n>N}\frac{1}{(z-z_n)^4}\right]^{1/2}+C_N\cdot(K'_N+K_N\cdot P_N)\cdot\left[\sum_{n>N}\frac{1}{(z-z_n)^2}\right]^{1/2}+$$

$$+L_M\cdot D_M\cdot\left[\sum_{n<-M}\frac{1}{(z-z_n)^4}\right]^{1/2}+D_M\cdot(L'_M+L_M\cdot Q_M)\cdot\left[\sum_{n<-M}\frac{1}{(z-z_n)^2}\right]^{1/2},\qquad (6\text{-}14)$$

$$K_N=\sqrt{\sum_{n>N}|f_n|^2},\quad K'_N=\sqrt{\sum_{n>N}|f'_n|^2},\quad L_M=\sqrt{\sum_{n<-M}|f_n|^2},\quad L'_M=\sqrt{\sum_{n<-M}|f'_n|^2},\qquad (6\text{-}15)$$

$$\left.\begin{aligned}C_N&=\max_{n>N}2!/|g_n^{(2)}|, & D_M&=\max_{n<-M}2!/|g_n^{(2)}|,\\ P_N&=\max_{n>N}(1/3)\cdot|g_n^{(3)}|/|g_n^{(2)}|, & Q_M&=\max_{n<-M}(1/3)\cdot|g_n^{(3)}|/|g_n^{(2)}|.\end{aligned}\right\}\qquad (6\text{-}16)$$

$$\sum_{n>N}\frac{1}{(z-z_n)^2}<\int_{z_N}^{+\infty}\frac{du}{(z-u)^2}=\frac{1}{z_N-z},\qquad (6\text{-}17)$$

$$\sum_{n>N}\frac{1}{(z-z_n)^4}<\int_{z_N}^{+\infty}\frac{du}{(z-u)^4}=\frac{1}{3(z_N-z)^3},\qquad (6\text{-}18)$$

$$\sum_{n<-M}\frac{1}{(z-z_n)^2}<\int_{-\infty}^{z_{-M}}\frac{du}{(z-u)^2}=\frac{1}{z-z_{-M}},\qquad (6\text{-}19)$$

$$\sum_{n<-M}\frac{1}{(z-z_n)^4}<\int_{-\infty}^{z_{-M}}\frac{du}{(z-u)^4}=\frac{1}{3(z-z_{-M})^3}.\qquad (6\text{-}20)$$

$$|R_{N,M}(z)|\leqslant|g(z)|\cdot\left[\frac{K_N\cdot C_N}{\sqrt{3(z_N-z)^3}}+\frac{C_N\cdot(K'_N+K_N\cdot P_N)}{\sqrt{z_N-z}}+\right.$$

$$\left.+\frac{L_M\cdot D_M}{\sqrt{3(z-z_{-M})^3}}+\frac{D_M\cdot(L'_M+L_M\cdot Q_M)}{\sqrt{z-z_{-M}}}\right].\qquad (6\text{-}21)$$

$$|R_{N,M}(z)|\leqslant|\sin(\beta z+\gamma)|^2\cdot\left[\frac{K_N}{\sqrt{3|\beta|\cdot(N\pi-\beta z-\gamma)^3}}+\frac{K'_N}{\sqrt{|\beta|^3\cdot(N\pi-\beta z-\gamma)}}+\right.$$

$$\left.+\frac{L_M}{\sqrt{3|\beta|\cdot(\beta z+\gamma+M\pi)^3}}+\frac{L'_M}{\sqrt{|\beta|^3\cdot(\beta z+\gamma+M\pi)}}\right].\qquad (\beta\neq0)\qquad (6\text{-}22)$$

$$\left.\begin{aligned}&z_n=(n\pi-\gamma)/\beta, && (n=\text{integers})\\ &g_n^{(2)}=g^{(2)}(z_n)=2\beta^2, && C_N=D_M=1/\beta^2,\end{aligned}\right\}\qquad (6\text{-}23)$$

$$g_n^{(3)}=0,\qquad P_N=Q_M=0.\qquad (6\text{-}24)$$

$$|R_{N,M}(z)|\leqslant|\sin(\beta z+\gamma)|^2\cdot\left[\frac{K_N}{\sqrt{3|\beta|\cdot(N\pi-S\pi)^3}}+\frac{K'_N}{\sqrt{|\beta|^3\cdot(N\pi-S\pi)}}+\right.$$

$$\left.+\frac{L_M}{\sqrt{3|\beta|\cdot(M\pi-S\pi)^3}}+\frac{L'_M}{\sqrt{|\beta|^3\cdot(M\pi-S\pi)}}\right].\qquad (\beta\neq0)\qquad (6\text{-}25)$$

$$|R_{N,N}(z)|\leqslant|\sin(\beta z+\gamma)|^2\cdot\left[\frac{K_N+L_N}{\sqrt{3|\beta|\cdot(N\pi-S\pi)^3}}+\frac{K'_N+L'_N}{\sqrt{|\beta|^3\cdot(N\pi-S\pi)}}\right].\qquad (\beta\neq0)\qquad (6\text{-}26)$$

$$R_{N,M}(z)\equiv f(z)-\sum_{n=-M}^{N}\left[f_n+(z-z_n)\cdot\left\{f'_n-\frac{1}{4}f_n\cdot\frac{g_n^{(4)}}{g_n^{(3)}}\right\}+\frac{1}{2}(z-z_n)^2\cdot\right.$$

$$\left.\cdot\left\{f''_n-\frac{1}{2}f'_n\cdot\frac{g_n^{(4)}}{g_n^{(3)}}+\frac{1}{2}f_n\cdot\left[\frac{1}{4}\left(\frac{g_n^{(4)}}{g_n^{(3)}}\right)^2-\frac{1}{5}\cdot\frac{g_n^{(5)}}{g_n^{(3)}}\right]\right\}\right]\cdot\frac{3!\,g(z)}{(z-z_n)^3\cdot g_n^{(3)}}-K\cdot g(z)=$$

$$=\sum_{n>N}\left[f_n+(z-z_n)\cdot\left\{f'_n-\frac{1}{4}f_n\cdot\frac{g_n^{(4)}}{g_n^{(3)}}\right\}+\frac{1}{2}(z-z_n)^2\cdot\right.$$

$$\left.\cdot\left\{f''_n-\frac{1}{2}f'_n\cdot\frac{g_n^{(4)}}{g_n^{(3)}}+\frac{1}{2}f_n\cdot\left[\frac{1}{4}\left(\frac{g_n^{(4)}}{g_n^{(3)}}\right)^2-\frac{1}{5}\cdot\frac{g_n^{(5)}}{g_n^{(3)}}\right]\right\}\right]\cdot\frac{3!\,g(z)}{(z-z_n)^3\cdot g_n^{(3)}}+$$

$$+\sum_{n<-M}\left[f_n+(z-z_n)\cdot\left\{f'_n-\frac{1}{4}f_n\cdot\frac{g_n^{(4)}}{g_n^{(3)}}\right\}+\frac{1}{2}(z-z_n)^2\cdot\left\{f''_n-\frac{1}{2}f'_n\cdot\frac{g_n^{(4)}}{g_n^{(3)}}+\right.\right.$$

$$\left.\left.+\frac{1}{2}f_n\cdot\left[\frac{1}{4}\left(\frac{g_n^{(4)}}{g_n^{(3)}}\right)^2-\frac{1}{5}\cdot\frac{g_n^{(5)}}{g_n^{(3)}}\right]\right\}\right]\cdot\frac{3!\,g(z)}{(z-z_n)^3\cdot g_n^{(3)}}.\quad (\text{for } z_{-M}<z<z_N)\qquad (6\text{-}27)$$

$$|R_{N,M}(z)|\leqslant|g(z)|\cdot\left[\frac{K_N\cdot E_N}{\sqrt{5(z_N-z)^5}}+\frac{E_N\cdot(K'_N+K_N\cdot S_N)}{\sqrt{3(z_N-z)^3}}+\right.$$

$$+\frac{E_N\cdot\{(1/2)\cdot K''_N+K'_N\cdot S_N+K_N\cdot(S_N^2+U_N)\}}{\sqrt{z_N-z}}+\frac{L_M\cdot F_M}{\sqrt{5(z-z_{-M})^5}}+$$

$$\left.+\frac{F_M\cdot(L'_M+L_M\cdot T_M)}{\sqrt{3(z-z_{-M})^3}}+\frac{F_M\cdot\{(1/2)\cdot L''_M+L'_M\cdot T_M+L_M\cdot(T_M^2+V_M)\}}{\sqrt{z-z_{-M}}}\right].\qquad (6\text{-}28)$$

$$\left.\begin{aligned}K_N&=\sqrt{\sum_{n>N}|f_n|^2}, & K'_N&=\sqrt{\sum_{n>N}|f'_n|^2}, & K''_N&=\sqrt{\sum_{n>N}|f''_n|^2},\\ L_M&=\sqrt{\sum_{n<-M}|f_n|_2}, & L'_M&=\sqrt{\sum_{n<-M}|f'_n|^2}, & L''_M&=\sqrt{\sum_{n<-M}|f''_n|^2},\end{aligned}\right\}\qquad (6\text{-}29)$$

$$E_N=\max_{n>N}3!/|g_n^{(3)}|,\qquad F_M=\max_{n<-M}3!/|g_n^{(3)}|,\qquad (6\text{-}30)$$

$$S_N=\max_{n>N}(3!/4!)\cdot|g_n^{(4)}|/|g_n^{(3)}|,\qquad T_M=\max_{n<-M}(3!/4!)\cdot|g_n^{(4)}|/|g_n^{(3)}|,\qquad (6\text{-}31)$$

$$U_N=\max_{n>N}(3!/5!)\cdot|g_n^{(5)}|/|g_n^{(3)}|,\qquad V_M=\max_{n<-M}(3!/5!)\cdot|g_n^{(5)}|/|g_n^{(3)}|.\qquad (6\text{-}32)$$