New Formulae of the Generalized Sampling Theorem II

—Formulae making use of the Sampled First and Higher Order Derivatives—

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Abstract

New sampling formulae are obtained, based on the generalized sampling theorem presented by Takizawa and Isigaki. The examples stated here include the sampling formulae, by which one can reconstruct a continuous function from its sampled values and sampled higher order derivatives. Sampling formulae presented here can be effectively applied as interpolation or extrapolation formulae.

Zusammenfassung

Neue Abtastformeln werden auf der Grundlage des von Takizawa und Isigaki verallgemeinerten Abtasttheorems angegeben. Die in der vorliegenden Arbeit aufgeführten Beispiele enthalten solche Abtastformeln, mit denen man eine kontinuierliche Funktion durch ihre abgetasteten Werte und abgetasteten Ableitungen höherer Ordnung wieder konstruieren kann. Die hier angegebenen Formeln können als Interpolations- oder Extrapolationsformeln erfolgreich angewandt werden.

Preliminaries

In previous papers[1)~3)], the authors presented several examples of the generalized sampling theorem, by which a continuous function is reconstructed from sampled values and sampled derivatives. In the preceding paper[3)], the authors' emphasis was laid especially on cases of sampling formulae, which make use of zero-th order derivatives of the sampled function (*i.e.*, the values of the sampled function itself).

Here, in the present paper, the authors concern themselves mainly with the examples of sampling formulae, which contain the sampled values and sampled higher order derivatives of an entire function, being based on the generalized sampling theorem presented by Takizawa and Isigaki[1)].

The expressions in the references 1), 2), and 3), shall be quoted with authors' initials, *e.g.*, (TI–2–1), (T–1–1), (TH–1–1) etc. We shall begin this paper with §2, which is a continuation of §1 of the authors' previous paper[3)] entitled "New Formulae of the Generalized Sampling Theorem I".

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In §2 the authors present several examples of sampling formulae which make use of first order derivatives of the sampled function, while §3 is treated by Horng, giving examples of sampling formulae which take the sampled second order derivatives into account.

§2. Examples of Sampling Formulae making use of the First Order Derivatives of Sampled Function

We shall refer to expressions (TI-2-1)~(TI-2-1′) or (T-1-1)~(T-1-1″) mainly for the case $m_n=1$, which corresponds to the case for the sampling formula making use of the sampled first order derivatives of an entire function $f(z)$.

In case $m=1$ in (TI-2-27) or (T-2-3), we obtain the sampling formula (TI-2-23) or (T-2-10), which reads

$$f(z)=\sum_n\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-\frac{1}{3}f(z_n)\cdot\frac{g_n^{(3)}}{g_n^{(2)}}\right\}\right]\cdot\frac{2!\,g(z)}{(z-z_n)^2\cdot g_n^{(2)}}+K\cdot g(z),\tag{2-1}$$

with $g(z)$ an entire function, having zeros of second order at points $z=z_n$ ($n=$ = integers). Constant K is given by (III′) in (T-2-2) or (III″) in (T-2-4), and the summation over n covers all the sampling points $z=z_n$ (n = integers).

If function $g(z)$ can be expressed in terms of $\phi(z)$ as in (TI-2-27) or (T-2-5) with $m=1$, *i.e.*

$$g(z)=\phi^2(z),\qquad \text{(with simple zeros of } \phi(z) \text{ at } z=z_n)$$

formula (2-1) implies:

$$f(z)=\sum_n\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-f(z_n)\cdot\frac{\phi_n^{(2)}}{\phi_n'}\right\}\right]\cdot\frac{\phi^2(z)}{(z-z_n)^2\cdot(\phi_n')^2}+K\cdot\phi^2(z),\tag{2-1′}$$

where $\phi_n'=\phi'(z_n)$ and $\phi_n^{(2)}=\phi^{(2)}(z_n)$, with $z=z_n$ (n = integers) simple zeros of $\phi(z)$.

1) Let us take a polynomial of degree s:

$$\phi(z)=\prod_{k=1}^{s}(z-\eta_k),\qquad \text{(all the } \eta_k\text{'s are distinct)}\tag{2-2}$$

in (T-2-5), and we shall put

$$g(z)=\phi^2(z)=\prod_{k=1}^{s}(z-\eta_k)^2,\qquad \text{(all the } \eta_k\text{'s are distinct)}\tag{2-3}$$

in (T-2-3). Then, we have formula (T-2-6), (T-2-10), (2-1), or (2-1′). Function $g(z)$ has s zeros of second order at $z=\eta_m$ ($m=1, 2, 3, \cdots, s$). We calculate:

$$g^{(2)}(\eta_m)=2\left(\phi'(\eta_m)\right)^2,\qquad \phi'(\eta_m)=\prod_{\substack{k=1\\k\neq m}}^{s}(\eta_m-\eta_k),$$

$$g^{(3)}(\eta_m)=6\ \phi'(\eta_m)\cdot\phi^{(2)}(\eta_m),\tag{2-4}$$

$$\phi^{(2)}(\eta_m)=2\sum_{\substack{p=1\\p\neq m}}^{s}(\eta_m-\eta_1)(\eta_m-\eta_2)\cdots(\widehat{\eta_m-\eta_p})\cdots(\eta_m-\eta_{m-1})(\eta_m-\eta_{m+1})\cdots(\eta_m-\eta_s)\,, \tag{2-5}$$

where the symbol $\hat{\eta}_p$ indicates the omission of the factor η_p in the product. Accordingly we have

$$\frac{\phi^{(2)}(\eta_m)}{\phi'(\eta_m)}=2\sum_{\substack{p=1\\p\neq m}}^{s}\frac{1}{\eta_m-\eta_p}\,. \tag{2-5'}$$

From (2-1) or (2-1′), we obtain:

$$f(z)=\sum_{m=1}^{s}\left[f(\eta_m)+(z-\eta_m)\cdot\left\{f'(\eta_m)-f(\eta_m)\cdot 2\sum_{\substack{p=1\\p\neq m}}^{s}\frac{1}{\eta_m-\eta_p}\right\}\right]\cdot\prod_{\substack{k=1\\k\neq m}}^{s}\frac{(z-\eta_k)^2}{(\eta_m-\eta_k)^2}+K\cdot\prod_{k=1}^{s}(z-\eta_k)^2\,, \tag{2-6}$$

provided that

$$\lim_{z\to\infty}\frac{f(z)}{\prod_{k=1}^{s}(z-\eta_k)^2}=K\,. \tag{2-7}$$

2) If we take:

$$g(z)=\phi^2(z)=\cos^2(\alpha\cdot\cos^{-1}\beta z)\,,\qquad(\alpha\cdot\beta\neq0) \tag{2-8}$$

with

$$\phi(z_n)=\cos(\alpha\cdot\cos^{-1}\beta z_n)=0\,,\quad i.e.,\quad z_n=\frac{1}{\beta}\cdot\cos\left(\frac{2n+1}{2\alpha}\pi\right),\qquad(n=\text{integers}) \tag{2-9}$$

we have

$$\phi'_n=\phi'(z_n)=(-1)^n\cdot\frac{\alpha\beta}{\sin\{(2n+1)\pi/(2\alpha)\}}\,, \tag{2-10}$$

$$\phi^{(2)}_n=\phi^{(2)}(z_n)=(-1)^n\cdot\frac{\alpha\beta^2\cdot\cos\{(2n+1)\pi/(2\alpha)\}}{\sin^3\{(2n+1)\pi/(2\alpha)\}}\,. \tag{2-11}$$

We put (2-9)~(2-11) into (2-1′), and obtain

$$f(z)=\sum_n\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-f(z_n)\cdot\frac{\beta\cdot\cos\{(2n+1)\pi/(2\alpha)\}}{\sin^2\{(2n+1)\pi/(2\alpha)\}}\right\}\right]\cdot\frac{\sin^2\{(2n+1)\pi/(2\alpha)\}\cdot\cos^2(\alpha\cdot\cos^{-1}\beta z)}{\alpha^2\cdot(\beta z-\cos\{(2n+1)\pi/(2\alpha)\})^2}+K\cdot\cos^2(\alpha\cdot\cos^{-1}\beta z)\,, \tag{2-12}$$

provided that

$$\lim_{z\to\infty}\frac{f(z)}{\cos^2(\alpha\cdot\cos^{-1}\beta z)}=K\,. \tag{2-13}$$

For a positive integer s, function $\phi(z)=\cos(s\cdot\cos^{-1}\beta z)$ is a Chebyscheff polynomial of degree s, and is expressed by $\phi(z)=\cos(s\theta)$, with $\beta z=\cos\theta$. Then the summation over n in (2–12) covers from $n=0$ to $n=s-1$, with $\beta z_n=\cos\theta_n=$ $=\cos\{(2n+1)\pi/(2s)\}$. Expression (2–12) reads:

$$f\left(\frac{\cos\theta}{\beta}\right)=\sum_{n=0}^{s-1}\left[f\left(\frac{\cos\theta_n}{\beta}\right)+\frac{\cos\theta-\cos\theta_n}{\beta}\cdot\left\{f'\left(\frac{\cos\theta_n}{\beta}\right)-f\left(\frac{\cos\theta_n}{\beta}\right)\cdot\frac{\beta\cdot\cos\theta_n}{\sin^2\theta_n}\right\}\right]\cdot$$
$$\cdot\frac{\sin^2\theta_n\cdot\cos^2(s\theta)}{s^2\cdot(\cos\theta-\cos\theta_n)^2}+K\cdot\cos^2(s\theta)\,. \qquad (2\text{–}14)$$

3) We shall take

$$g(z)=\psi^2(z)=\sin^2(\beta z+\gamma)\,, \qquad (\beta\neq 0) \qquad (2\text{–}15)$$

with

$$\psi(z_n)=\sin(\beta z_n+\gamma)=0\,, \qquad i.e.,\quad z_n=(n\pi-\gamma)/\beta\,, \qquad (n=\text{integers}) \qquad (2\text{–}16)$$

and obtain:

$$\psi'_n=\psi'(z_n)=(-1)^n\cdot\beta\,, \qquad (2\text{–}17)$$

$$\psi_n^{(2)}=\psi^{(2)}(z_n)=0\,. \qquad (2\text{–}18)$$

From (2–1′) with (2–16)∼(2–18), the sampling formula reads:

$$f(z)=\sum_{n=-\infty}^{+\infty}\left[f\left(\frac{n\pi-\gamma}{\beta}\right)+\frac{\beta z+\gamma-n\pi}{\beta}\cdot f'\left(\frac{n\pi-\gamma}{\beta}\right)\right]\cdot\frac{\sin^2(\beta z+\gamma-n\pi)}{(\beta z+\gamma-n\pi)^2}\,, \qquad (2\text{–}19)$$

under condition (T–4–17), *i.e.*

$$\frac{1}{|\beta|}<\frac{2}{W}\,, \qquad (2\text{–}20)$$

if $f(z)$ is band-limited with maximum frequency W in the sense of Fourier spectrum. Expression (2–19) for $\gamma=0$ was also obtained by Jagerman-Fogel[4] and Linden-Abramson[5].

As an example of (2–19), we put $f(z)=$const, and refer to condition (III′) in (T–2–4) with (T–4–10′), and obtain:

$$1=\sum_{n=-\infty}^{+\infty}\frac{\sin^2(\beta z+\gamma-n\pi)}{(\beta z+\gamma-n\pi)^2}\,. \qquad (\beta\neq 0) \qquad (2\text{–}21)$$

Referring to expression (TH–1–28), *i.e.*

$$1=\sum_{n=-\infty}^{+\infty}\frac{\sin(\beta z+\gamma-n\pi)}{\beta z+\gamma-n\pi}\,, \qquad (\beta\neq 0)$$

and comparing this expression with (2–21), we obtain:

$$0=\sum_{\substack{m,n=-\infty\\ m\neq n}}^{+\infty}\frac{\sin(\beta z+\gamma-m\pi)}{\beta z+\gamma-m\pi}\cdot\frac{\sin(\beta z+\gamma-n\pi)}{\beta z+\gamma-n\pi}\,. \qquad (\beta\neq 0) \qquad (2\text{–}21')$$

4) Let us take

$$g(z)=\psi^2(z)=\xi^2(z)\cdot\phi^2(z)\,, \qquad (2\text{–}22)$$

with two entire functions $\xi(z)$ and $\phi(z)$, having simple zeros z_n (n=integers) and z_m (m=integers) respectively, *i. e.*

$$\xi(z_n)=0\,, \quad \text{and} \quad \phi(z_m)=0\,. \qquad (z_n \neq z_m) \tag{2-23}$$

For (2-22) we obtain :

$$\psi'(z_n)=\xi'_n\cdot\phi_n\,, \qquad \psi'(z_m)=\xi_m\cdot\phi'_m\,, \tag{2-24}$$

$$\psi^{(2)}(z_n)=\xi^{(2)}_n\cdot\phi_n+2\xi'_n\cdot\phi'_n\,, \tag{2-25}$$

$$\psi^{(2)}(z_m)=2\xi'_m\cdot\phi'_m+\xi_m\cdot\phi^{(2)}_m\,, \tag{2-26}$$

with $\xi'_n=\xi'(z_n)$, $\xi'_m=\xi'(z_m)$, $\phi'_n=\phi'(z_n)$, $\phi'_m=\phi'(z_m)$, etc.

In this case, expression (2-1′) implies :

$$\begin{aligned} f(z)=&\sum_n\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-f(z_n)\cdot\left(\frac{\xi''(z_n)}{\xi'(z_n)}+2\frac{\phi'(z_n)}{\phi(z_n)}\right)\right\}\right]\cdot \\ &\cdot\frac{\xi^2(z)\cdot\phi^2(z)}{(z-z_n)^2\cdot\{\xi'(z_n)\cdot\phi(z_n)\}^2}+ \\ &+\sum_m\left[f(z_m)+(z-z_m)\cdot\left\{f'(z_m)-f(z_m)\cdot\left(2\frac{\xi'(z_m)}{\xi(z_m)}+\frac{\phi''(z_m)}{\phi'(z_m)}\right)\right\}\right]\cdot \\ &\cdot\frac{\xi^2(z)\cdot\phi^2(z)}{(z-z_m)^2\cdot\{\xi(z_m)\cdot\phi'(z_m)\}^2}+K\cdot\xi^2(z)\cdot\phi^2(z)\,, \end{aligned} \tag{2-27}$$

provided that

$$\lim_{z\to\infty}\frac{f(z)}{\xi^2(z)\cdot\phi^2(z)}=K\,. \tag{2-28}$$

As an example of (2-27), we shall take

$$g(z)=\psi^2(z)=\xi^2(z)\cdot\phi^2(z)=\sin^2(\beta z+\gamma)\cdot\prod_{k=1}^{s}(z-\eta_k)^2\,,$$
$$(\beta\neq 0,\ \text{all the } \eta_k\text{'s are distinct}) \tag{2-29}$$

with

$$\xi(z_n)=\sin(\beta z_n+\gamma)=0\,, \qquad i.\,e.,\quad z_n=(n\pi-\gamma)/\beta\,, \qquad (n=\text{integers}) \tag{2-30}$$

and we have

$$\xi'(z_n)=(-1)^n\cdot\beta\,, \quad \xi''(z_n)=0\,, \quad \phi'(z_n)=\sum_{p=1}^{s}(z_n-\eta_1)(z_n-\eta_2)\cdots(\widehat{z_n-\eta_p})\cdots(z_n-\eta_s)\,,$$

$$\xi'(\eta_m)=\beta\cdot\cos(\beta\eta_m+\gamma)\,, \quad \phi'(\eta_m)=\prod_{\substack{k=1\\k\neq m}}^{s}(\eta_m-\eta_k)\,, \tag{2-31}$$

$$\phi''(\eta_m)=2\sum_{\substack{p=1\\p\neq m}}^{s}(\eta_m-\eta_1)(\eta_m-\eta_2)\cdots(\widehat{\eta_m-\eta_p})\cdots(\eta_m-\eta_{m-1})(\eta_m-\eta_{m+1})\cdots(\eta_m-\eta_s)\,, \tag{2-32}$$

where the symbol $\hat{\eta}_p$ indicates the omission of the factor η_p in the product. Expression (2-27) with (2-30)～(2-32) reads :

$$f(z)=\sum_{n=-\infty}^{+\infty}\left[f\left(\frac{n\pi-\gamma}{\beta}\right)+\frac{\beta z+\gamma-n\pi}{\beta}\cdot\left\{f'\left(\frac{n\pi-\gamma}{\beta}\right)-f\left(\frac{n\pi-\gamma}{\beta}\right)\cdot 2\sum_{p=1}^{s}\frac{1}{z_n-\eta_p}\right\}\right]\cdot$$
$$\cdot\frac{\sin^2(\beta z+\gamma-n\pi)}{(\beta z+\gamma-n\pi)^2}\cdot\prod_{k=1}^{s}\frac{(z-\eta_k)^2}{(z_n-\eta_k)^2}+$$
$$+\sum_{m=1}^{s}\left[f(\eta_m)+(z-\eta_m)\cdot\left\{f'(\eta_m)-2f(\eta_m)\cdot\left(\beta\cot(\beta\eta_m+\gamma)+\sum_{\substack{p=1\\p\neq m}}^{s}\frac{1}{\eta_m-\eta_p}\right)\right\}\right]\cdot$$
$$\cdot\frac{\sin^2(\beta z+\gamma)}{\sin^2(\beta\eta_m+\gamma)}\cdot\prod_{\substack{k=1\\k\neq m}}^{s}\frac{(z-\eta_k)^2}{(\eta_m-\eta_k)^2}, \tag{2-33}$$

provided that $\lim_{z\to\infty} f(z)\Big/\left\{\sin^2(\beta z+\gamma)\cdot\prod_{k=1}^{s}(z-\eta_k)^2\right\}=0$, and with $z_n=(n\pi-\gamma)/\beta$ ($n=$ =integers), all the η_m's and z_n's being distinct.

5) If we take

$$g(z)=\phi^2(z)=\Big[z\cdot\sin(\beta z)-A\cdot\cos(\beta z)\Big]^2, \qquad (\beta\cdot A\neq 0) \tag{2-34}$$

or

$$g(z)=\phi^2(z)=\Big[z\cdot\cos(\beta z)-B\cdot\sin(\beta z)\Big]^2, \qquad (\beta\cdot B\neq 0) \tag{2-35}$$

with constants A, B, and β, we obtain, from (2–1′), respectively

$$f(z)=\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty}\left[f(\lambda_n)+(z-\lambda_n)\cdot\left\{f'(\lambda_n)-f(\lambda_n)\cdot\frac{4\beta\cdot\cos^2(\beta\lambda_n)}{2\beta\lambda_n+\sin(2\beta\lambda_n)}\right\}\right]\cdot$$
$$\cdot\frac{4\cos^2(\beta\lambda_n)\cdot[z\cdot\sin(\beta z)-A\cdot\cos(\beta z)]^2}{(z-\lambda_n)^2\cdot[2\beta\lambda_n+\sin(2\beta\lambda_n)]^2}, \tag{2-36}$$

provided that $\lim_{z\to\infty} f(z)/[z\cdot\sin(\beta z)-A\cdot\cos(\beta z)]^2=0$; or

$$f(z)=\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty}\left[f(\mu_n)+(z-\mu_n)\cdot\left\{f'(\mu_n)-f(\mu_n)\cdot\frac{4\beta\cdot\sin^2(\beta\mu_n)}{2\beta\mu_n-\sin(2\beta\mu_n)}\right\}\right]\cdot$$
$$\cdot\frac{4\sin^2(\beta\mu_n)\cdot[z\cdot\cos(\beta z)-B\cdot\sin(\beta z)]^2}{(z-\mu_n)^2\cdot[2\beta\mu_n-\sin(2\beta\mu_n)]^2}+$$
$$+\Big[f(0)+z\cdot f'(0)\Big]\cdot\frac{[z\cdot\cos(\beta z)-B\cdot\sin(\beta z)]^2}{(1-\beta\cdot B)^2\cdot z^2}, \qquad (\beta\cdot B\neq 1) \tag{2-37}$$

provided that $\lim_{z\to\infty} f(z)/[z\cdot\cos(\beta z)-B\cdot\sin(\beta z)]^2=0$, where λ_n and μ_n ($n=$integers) are simple roots of the following equations, being arranged in ascending order of magnitude with increasing n:

$$\phi(\lambda_n)=\lambda_n\cdot\sin(\beta\lambda_n)-A\cdot\cos(\beta\lambda_n)=0, \qquad (\beta\cdot A\neq 0) \tag{2-38}$$

and

$$\phi(\mu_n)=\mu_n\cdot\cos(\beta\mu_n)-B\cdot\sin(\beta\mu_n)=0. \qquad (\beta\cdot B\neq 0) \tag{2-39}$$

The roots λ_n and μ_n are taken to be positive for $n>0$, and negative for $n<0$, and we put $\lambda_0=\mu_0=0$. λ_0 is not a zero of $g(z)$ in (2–34) for $A\neq 0$, while μ_0 is a zero of second order of $g(z)$ in (2–35) for $B\neq 0$.

For the limiting cases: $A \to 0$ in (2–34) and $B \to 0$ in (2–35), we obtain respectively the following formulae from (2–1):

$$f(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[f\left(\frac{n\pi}{\beta}\right)+\frac{\beta z-n\pi}{\beta}\cdot\left\{f'\left(\frac{n\pi}{\beta}\right)-f\left(\frac{n\pi}{\beta}\right)\cdot\frac{2\beta}{n\pi}\right\}\right]\cdot\frac{(\beta z)^2\cdot\sin^2(\beta z-n\pi)}{n^2\cdot\pi^2\cdot(\beta z-n\pi)^2}+$$
$$+\left[f(0)+z\cdot f'(0)+\frac{1}{2}z^2\cdot\left\{f''(0)+\frac{2}{3}\beta^2\cdot f(0)\right\}+\frac{1}{6}z^3\cdot\left\{f'''(0)+2\beta^2\cdot f'(0)\right\}\right]\cdot$$
$$\cdot\frac{\sin^2(\beta z)}{(\beta z)^2}, \tag{2–40}$$

provided that $\lim_{z\to\infty} f(z)/\{z^2\cdot\sin^2(\beta z)\}=0$; and

$$f(z)=\sum_{n=-\infty}^{+\infty}\left[f\left(\frac{2n+1}{2\beta}\pi\right)+\left(z-\frac{2n+1}{2\beta}\pi\right)\cdot\left\{f'\left(\frac{2n+1}{2\beta}\pi\right)-f\left(\frac{2n+1}{2\beta}\pi\right)\cdot\frac{4\beta}{(2n+1)\pi}\right\}\right]\cdot$$
$$\cdot\frac{4(\beta z)^2\cdot\cos^2(\beta z)}{(2n+1)^2\cdot\pi^2\cdot(\beta z-(2n+1)\pi/2)^2}+\left[f(0)+z\cdot f'(0)\right]\cdot\cos^2(\beta z), \tag{2–41}$$

provided that $\lim_{z\to\infty} f(z)/\{z^2\cdot\cos^2(\beta z)\}=0$.

We shall note that $z^2\cdot\sin^2(\beta z)$ has a zero of fourth order, and $z^2\cdot\cos^2(\beta z)$ has a zero of second order, respectively at the origin.

Examples of (2–40) and (2–41) read as follows:

$$\sin(\alpha z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[\sin\left(\frac{n\pi\alpha}{\beta}\right)+\frac{\beta z-n\pi}{\beta}\cdot\left\{\alpha\cdot\cos\left(\frac{n\pi\alpha}{\beta}\right)-\frac{2\beta}{n\pi}\cdot\sin\left(\frac{n\pi\alpha}{\beta}\right)\right\}\right]\cdot$$
$$\cdot\frac{(\beta z)^2\cdot\sin^2(\beta z-n\pi)}{n^2\cdot\pi^2\cdot(\beta z-n\pi)^2}+\left[\alpha z-\frac{\alpha}{6}z^3\cdot(\alpha^2-2\beta^2)\right]\cdot\frac{\sin^2(\beta z)}{(\beta z)^2},\qquad(|\alpha|<|\beta|) \tag{2–42}$$

$$\cos(\alpha z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[\cos\left(\frac{n\pi\alpha}{\beta}\right)-\frac{\beta z-n\pi}{\beta}\cdot\left\{\alpha\cdot\sin\left(\frac{n\pi\alpha}{\beta}\right)+\frac{2\beta}{n\pi}\cdot\cos\left(\frac{n\pi\alpha}{\beta}\right)\right\}\right]\cdot$$
$$\cdot\frac{(\beta z)^2\cdot\sin^2(\beta z-n\pi)}{n^2\cdot\pi^2\cdot(\beta z-n\pi)^2}+\left[1-\frac{1}{2}z^2\cdot\left(\alpha^2-\frac{2}{3}\beta^2\right)\right]\cdot\frac{\sin^2(\beta z)}{(\beta z)^2},\qquad(|\alpha|<|\beta|) \tag{2–43}$$

$$J_\nu(\alpha z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[J_\nu\left(\frac{n\pi\alpha}{\beta}\right)+\frac{\beta z-n\pi}{\beta}\cdot\left\{\alpha\cdot J'_\nu\left(\frac{n\pi\alpha}{\beta}\right)-\frac{2\beta}{n\pi}\cdot J_\nu\left(\frac{n\pi\alpha}{\beta}\right)\right\}\right]\cdot$$
$$\cdot\frac{(\beta z)^2\cdot\sin^2(\beta z-n\pi)}{n^2\cdot\pi^2\cdot(\beta z-n\pi)^2}+\left[J_\nu(0)+\alpha z\cdot J'_\nu(0)+\frac{1}{2}z^2\cdot\left\{\alpha^2\cdot J''_\nu(0)+\frac{2}{3}\beta^2\cdot J_\nu(0)\right\}+\right.$$
$$\left.+\frac{1}{6}z^3\cdot\left\{\alpha^3\cdot J'''_\nu(0)+2\alpha\beta^2\cdot J'_\nu(0)\right\}\right]\cdot\frac{\sin^2(\beta z)}{(\beta z)^2},\qquad(|\alpha|<|\beta|) \tag{2–44}$$

$$\sin(\alpha z)=\sum_{n=-\infty}^{+\infty}\left[\sin\frac{(2n+1)\pi\alpha}{2\beta}+\left(z-\frac{2n+1}{2\beta}\pi\right)\cdot\left\{\alpha\cdot\cos\frac{(2n+1)\pi\alpha}{2\beta}-\right.\right.$$
$$\left.\left.-\frac{4\beta}{(2n+1)\pi}\cdot\sin\frac{(2n+1)\pi\alpha}{2\beta}\right\}\right]\cdot\frac{4(\beta z)^2\cdot\cos^2(\beta z)}{(2n+1)^2\cdot\pi^2\cdot(\beta z-(2n+1)\pi/2)^2}+$$
$$+\alpha z\cdot\cos^2(\beta z),\qquad(|\alpha|<|\beta|) \tag{2–45}$$

$$\cos(\alpha z)=\sum_{n=-\infty}^{+\infty}\left[\cos\frac{(2n+1)\pi\alpha}{2\beta}-\left(z-\frac{2n+1}{2\beta}\pi\right)\cdot\left\{\alpha\cdot\sin\frac{(2n+1)\pi\alpha}{2\beta}+\right.\right.$$
$$\left.\left.+\frac{4\beta}{(2n+1)\pi}\cdot\cos\frac{(2n+1)\pi\alpha}{2\beta}\right\}\right]\cdot\frac{4(\beta z)^2\cdot\cos^2(\beta z)}{(2n+1)^2\cdot\pi^2\cdot(\beta z-(2n+1)\pi/2)^2}+$$
$$+\cos^2(\beta z),\qquad(|\alpha|<|\beta|)\tag{2-46}$$

$$J_\nu(\alpha z)=\sum_{n=-\infty}^{+\infty}\left[J_\nu\left(\frac{2n+1}{2\beta}\pi\alpha\right)+\left(z-\frac{2n+1}{2\beta}\pi\right)\cdot\left\{\alpha\cdot J_\nu'\left(\frac{2n+1}{2\beta}\pi\alpha\right)-\right.\right.$$
$$\left.\left.-\frac{4\beta}{(2n+1)\pi}\cdot J_\nu\left(\frac{2n+1}{2\beta}\pi\alpha\right)\right\}\right]\cdot\frac{4(\beta z)^2\cdot\cos^2(\beta z)}{(2n+1)^2\cdot\pi^2\cdot(\beta z-(2n+1)\pi/2)^2}+$$
$$+\left[J_\nu(0)+\alpha z\cdot J_\nu'(0)\right]\cdot\cos^2(\beta z).\qquad(|\alpha|<|\beta|)\tag{2-47}$$

6) If we take Bessel function of integral order ν for the function $\phi(z)$ in (2–1′), *i.e.*

$$g(z)=\phi^2(z)=J_\nu^2(\beta z),\qquad(\beta\neq 0)\tag{2-48}$$

with a constant β, we obtain (*cf.* (T–1–1′)):

$$f(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-f(z_n)\cdot\frac{\beta\cdot J_\nu''(\beta z_n)}{J_\nu'(\beta z_n)}\right\}\right]\cdot$$
$$\cdot\frac{J_\nu^2(\beta z)}{(\beta z-\beta z_n)^2\cdot\{J_\nu'(\beta z_n)\}^2}+E(z),\tag{2-49}$$

$$E(z)=\sum_{s=0}^{2|\nu|-1}\sum_{j=0}^{s}\frac{f_0^{(j)}}{j!}\cdot\frac{H_0^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{J_\nu^2(\beta z)}{z^{2|\nu|}},\tag{2-50}$$

provided that $\lim_{z\to\infty} f(z)/J_\nu^2(\beta z)=0$. In (2–49) and (2–50), $f_0^{(j)}=f^{(j)}(z_0)=f^{(j)}(0)$, $H_0^{(s-j)}=H^{(s-j)}(z_0)=H^{(s-j)}(0)$, and βz_n (n=integers) are zeros of Bessel function $J_\nu(z)$, *i.e.*, $J_\nu(\beta z_n)=0$, where $z_0=0$ is a zero of $|\nu|$-th order at the origin. In case $\nu=0$, $E(z)$ reduces to a null function.

When we put $f(z)=J_\mu(\alpha z)$ with μ=integer in (2–49), we obtain:

$$J_\mu(\alpha z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[J_\mu(\alpha z_n)+(z-z_n)\cdot\left\{\alpha\cdot J_\mu'(\alpha z_n)-J_\mu(\alpha z_n)\cdot\frac{\beta\cdot J_\nu''(\beta z_n)}{J_\nu'(\beta z_n)}\right\}\right]\cdot$$
$$\cdot\frac{J_\nu^2(\beta z)}{(\beta z-\beta z_n)^2\cdot\{J_\nu'(\beta z_n)\}^2}+E(z).\qquad(0\neq|\alpha|<|\beta|)\tag{2-49′}$$

In case $\mu=\nu=0$, expression (2–49′) reduces to the following one, if we put $z=0$:

$$1=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[J_0(\alpha z_n)\cdot\left\{1+z_n\cdot\frac{d}{dz_n}\log J_1(\beta z_n)\right\}+\alpha z_n\cdot J_1(\alpha z_n)\right]\cdot\frac{1}{\{\beta z_n\cdot J_1(\beta z_n)\}^2}$$
$$=2\sum_{n=1}^{+\infty}\left[J_0(\alpha z_n)\cdot\left\{1+z_n\cdot\frac{d}{dz_n}\log J_1(\beta z_n)\right\}+\alpha z_n\cdot J_1(\alpha z_n)\right]\cdot\frac{1}{\{\beta z_n\cdot J_1(\beta z_n)\}^2},$$
$$(0\neq|\alpha|<|\beta|)\tag{2-49″}$$

where βz_n (n=integers) are zeros of $J_0(z)$, and z_m ($m=1, 2, 3, \cdots$) are positive zeros, with $z_0=0$ being not a zero of $J_0(\beta z)$.

7) We shall take

$$g(z)=\phi^2(z)=\left[z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)\right]^2, \qquad (\beta\cdot h\neq 0) \tag{2-51}$$

in (2–1), with an integer ν and constants β and h. Expression (2–1′) reads, with reference to (T–1–1′),

$$f(z)=\sum_n\left[f(\lambda_n)+(z-\lambda_n)\cdot\left\{f'(\lambda_n)+\frac{1}{\lambda_n}\cdot f(\lambda_n)\cdot\frac{(h^2-\lambda_n^2)\,\beta^2-\nu^2}{(h^2+\lambda_n^2)\,\beta^2-\nu^2}\right\}\right]\cdot$$
$$\cdot\frac{(\beta\lambda_n)^2\cdot[z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)]^2}{(z-\lambda_n)^2\cdot[(h^2+\lambda_n^2)\,\beta^2-\nu^2]^2\cdot J_\nu^2(\beta\lambda_n)}+G(z), \tag{2-52}$$

$$G(z)=\sum_{s=0}^{2|\nu|-1}\sum_{j=0}^{s}\frac{f_0^{(j)}}{j!}\cdot\frac{H_0^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{[z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)]^2}{z^{2|\nu|}}, \tag{2-53}$$

provided that $\lim_{z\to\infty} f(z)/[z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)]^2=0$. In (2–52), the values λ_n ($n=$ =integers) are roots of the following equation:

$$\phi(\lambda_n)=\lambda_n\cdot J'_\nu(\beta\lambda_n)+h\cdot J_\nu(\beta\lambda_n)=0, \tag{2-54}$$

where λ_0 is a zero of $|\nu|$-th order at the origin. In case $\nu=0$, $G(z)$ reduces to a null function.

If we tend h to infinity in (2–51)∼(2–54), we obtain the expressions which are essentially the same as (2–48)∼(2–50).

8) Further, let us take a linear combination $T_\mu(x, z)$ of Bessel function $J_\mu(z)$ and Neumann function $Y_\mu(z)$ of order μ:

$$T_\mu(x,z)=Y_\nu(x)\cdot J_\mu(z)-J_\nu(x)\cdot Y_\mu(z), \tag{2-55}$$

and put

$$g(z)=\phi^2(z)=T_\nu^2(\alpha z,\beta z), \qquad (\alpha\cdot\beta\neq 0,\ \alpha\neq\beta,\ \nu=\text{integer}) \tag{2-56}$$

in (2–1), with constants α and β. Expression (2–1′) reads

$$f(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[f(\lambda_n)+(z-\lambda_n)\cdot\left\{f'(\lambda_n)+f(\lambda_n)\cdot\left[\frac{1}{\lambda_n}-\frac{d}{d\lambda_n}\log\left\{J_\nu^2(\alpha\lambda_n)-J_\nu^2(\beta\lambda_n)\right\}\right]\right\}\right]\cdot$$
$$\cdot\frac{\pi^2\cdot\lambda_n^2\cdot J_\nu^2(\alpha\lambda_n)\cdot J_\nu^2(\beta\lambda_n)}{4\{J_\nu^2(\alpha\lambda_n)-J_\nu^2(\beta\lambda_n)\}^2}\cdot\frac{T_\nu^2(\alpha z,\beta z)}{(z-\lambda_n)^2}, \tag{2-57}$$

with

$$\phi'_n=\phi'(\lambda_n)=\frac{-2}{\pi\lambda_n\cdot J_\nu(\alpha\lambda_n)\cdot J_\nu(\beta\lambda_n)}\cdot\left\{J_\nu^2(\alpha\lambda_n)-J_\nu^2(\beta\lambda_n)\right\},$$

$$\phi''_n=\phi''(\lambda_n)=\frac{2}{\pi\lambda_n\cdot J_\nu(\alpha\lambda_n)\cdot J_\nu(\beta\lambda_n)}\cdot\left[\frac{1}{\lambda_n}\cdot\left\{J_\nu^2(\alpha\lambda_n)-J_\nu^2(\beta\lambda_n)\right\}-\right.$$
$$\left.-\frac{d}{d\lambda_n}\left\{J_\nu^2(\alpha\lambda_n)-J_\nu^2(\beta\lambda_n)\right\}\right],$$

provided that $\lim_{z\to\infty} f(z)/T_\nu^2(\alpha z,\beta z)=0$. The values λ_n (n=integers) are roots of the equation:

$$\phi(\lambda_n) = T_\nu(\alpha\lambda_n, \beta\lambda_n) = 0\,, \tag{2-58}$$

i. e.

$$Y_\nu(\alpha\lambda_n)\cdot J_\nu(\beta\lambda_n) = J_\nu(\alpha\lambda_n)\cdot Y_\nu(\beta\lambda_n)\,, \tag{2-59}$$

being arranged in ascending order of magnitude with increasing n, where $\lambda_0=0$ is not a zero of $T_\nu(\alpha z, \beta z)$. We referred to expression (2–59) and the Wronskian of cylinder functions $J_\nu(z)$ and $Y_\nu(z)$, in order to obtain the sampling formula (2–57).

If we replace $f(z)$ by $f(z)\cdot B_\nu(z)$ in (2–57), we obtain a sampling formula for $f(z)\cdot B_\nu(z)$, which reads:

$$f(z)\cdot B_\nu(z) = \sum_n \Bigg[f(\lambda_n)\cdot B_\nu(\lambda_n) + (z-\lambda_n)\cdot\Bigg\{f'(\lambda_n)\cdot B_\nu(\lambda_n) + f(\lambda_n)\cdot B'_\nu(\lambda_n) + f(\lambda_n)\cdot B_\nu(\lambda_n)\cdot \left[\frac{1}{\lambda_n} - \frac{d}{d\lambda_n}\log\left\{J_\nu^2(\alpha\lambda_n) - J_\nu^2(\beta\lambda_n)\right\}\right]\Bigg\}\Bigg]\cdot\frac{\pi^2\cdot\lambda_n^2\cdot J_\nu^2(\alpha\lambda_n)\cdot J_\nu^2(\beta\lambda_n)}{4\{J_\nu^2(\alpha\lambda_n) - J_\nu^2(\beta\lambda_n)\}^2}\cdot\frac{T_\nu^2(\alpha z, \beta z)}{(z-\lambda_n)^2}\,, \tag{2-60}$$

with a given function $B_\nu(z)$, provided that

$$\lim_{z\to\infty} f(z)\cdot B_\nu(z)/T_\nu^2(\alpha z, \beta z) = 0\,. \tag{2-61}$$

Taking

$$B_\nu(z) = J_\nu^2(z) + Y_\nu^2(z)\,, \tag{2-62}$$

in (2–60), we have a sampling formula slightly different from the one previously obtained by the authors[3], *i.e.*, (TH–1–71).

9) If we take

$$g(z) = \phi^2(z) = \sin^2(\beta z^2 + \gamma)\,, \qquad (\beta\cdot\gamma \neq 0) \tag{2-63}$$

in (2–1), we obtain

$$f(z) = \sum_n \left[f(z_n) + (z-z_n)\cdot\left\{f'(z_n) - \frac{1}{z_n}\cdot f(z_n)\right\}\right]\cdot\frac{\sin^2(\beta z^2+\gamma)}{4\beta^2\cdot z_n^2\cdot(z-z_n)^2}\,, \tag{2-64}$$

provided that $\lim_{z\to\infty} f(z)/\sin^2(\beta z^2+\gamma)=0$, where $z_n = \pm\sqrt{(n\pi-\gamma)/\beta}$ (n=integers), and the summation over n covers all the values of z_n, *i.e.*, positive and negative square roots.

In case $\gamma=0$, $\sin^2(\beta z^2+\gamma)$ has a zero of fourth order at the origin, and a modification of expression (2–64) should be required (*cf.* (T–1–1′)).

10) If we take $g(z)$ to be a product of two entire functions squared, such as $\sin^2(\beta z+\gamma)\cdot J_\nu^2(pz)$, $J_\nu^2(pz)\cdot\prod_{k=1}^{s}(z-\eta_k)^2$ etc., we obtain a sampling formula from (2–1) or (2–1′) by similar calculations as in (2–22)~(2–28).

Truncation error of the sampling expansion (2–1) can be easily estimated. The bound for truncation error[2] in case $m_n=1$ was already given by (T–6–21).

§ 3. Examples of Sampling Formulae making use of the Second Order Derivatives of Sampled Function

We shall refer to expressions (TI–2–1), (TI–2–1′), or (T–1–1)~(T–1–1″) mainly for the case $m_n=2$, which corresponds to the case for the sampling formula making use of the sampled second order derivatives of an entire function $f(z)$.

For the sake of convenience in printing, all the formulae and expressions in § 3 shall be set forth in the Appendix. This was done, in accordance with the suggestions by the editorial committee of this Memoirs.

For the case $m=2$ in (TI–2–27) or (T–2–3)~(T–2–3′), we have the sampling formula (TI–2–23) or (T–2–11), *i.e.* (3–1), with $g(z)$ an entire function, having zeros of third order at points $z=z_n$ (n=integers). Constant K is given by (III′) in (T–2–2) or (III″) in (T–2–4). The summation over n covers all the sampling points $z=z_n$ (n=integers).

If $g(z)$ can be expressed in terms of $\psi(z)$ as in (TI–2–27) or (T–2–5) with $m=2$, *i.e.* $g(z)=\psi^3(z)$, formula (3–1) implies (3–2), with $\psi'_n=\psi'(z_n)$, $\psi_n^{(2)}=\psi^{(2)}(z_n)$, and $\psi_n^{(3)}=\psi^{(3)}(z_n)$.

1) If we take $g(z)$ to be (3–3) in (T–2–3), we obtain expression (T–2–6), (T–2–11), (3–1) or (3–2). Function $g(z)$ has s zeros of third order at $z=\eta_m$ ($m=1, 2, 3, \cdots, s$). From (3–3) we have (3–4)~(3–6), where the symbol $\hat{\eta}$ indicates the omission of the factor η in the product. In the double summation in (3–6), two symbols $\hat{\eta}_p$ and $\hat{\eta}_r$ are used at the same time. They indicate the omission of both η_p and η_r in the product. Putting (3–4)~(3–6) into (3–2), we obtain (3–7), provided that $\lim_{z\to\infty} f(z)\Big/\prod_{k=1}^{s}(z-\eta_k)^3=K$.

2) We take (3–8), with (3–9)~(3–10). Putting (3–9)~(3–10) into (3–2), we have (3–11), provided that $\lim_{z\to\infty} f(z)/\cos^3(\alpha\cdot\cos^{-1}\beta z)=K$.

3) We shall take (3–12), with (3–13)~(3–14). Expression (3–2) with (3–13)~(3–14) implies (3–15), under condition (T–4–17), *i.e.* $1/|\beta|<3/W$, if function $f(z)$ is band-limited with maximum frequency W in the sense of Fourier spectrum.

If we put $f(z)$=const in (3–15) and refer to condition (III″) in (T–2–4) with (T–4–10′), we obtain (3–16).

4) If we take (3–17), with $\xi(z)$ and $\phi(z)$ entire functions, having simple zeros z_n (n = integers) and z_m (m = integers) respectively, *i.e.* $\xi(z_n)=0$ (n = integers) and $\phi(z_m)=0$ (m=integers), we have (3–18)~(3–20). We put (3–18)~(3–20) into (3–2), and obtain (3–21), provided that $\lim_{z\to\infty} f(z)/\{\xi^3(z)\cdot\phi^3(z)\}=0$.

As an example of (3–21), we shall take (3–22), with $\xi(z_n)=\sin(\beta z_n+\gamma)=0$, *i.e.* $z_n=(n\pi-\gamma)/\beta$ (n = integers), and we have (3–23)~(3–25). We put (3–23)~(3–25) into (3–21), and obtain (3–26), if $\lim_{z\to\infty} f(z)/g(z)=0$, where all the η_m's and z_n's are distinct.

5) If we take (3–27) or (3–28), with constants A, B, and β, then we obtain (3–29) or (3–30) from (3–2), provided that $\lim_{z\to\infty} f(z)/[z\cdot\sin(\beta z)-A\cdot\cos(\beta z)]^3=0$, or that $\lim_{z\to\infty} f(z)/[z\cdot\cos(\beta z)-B\cdot\sin(\beta z)]^3=0$, where λ_n and μ_n (n=integers) are the simple roots of equations (3–31) and (3–32). The roots λ_n and μ_n are taken to be positive for $n>0$, and negative for $n<0$, and we put $\lambda_0=\mu_0=0$. λ_0 is not a zero of $g(z)$ in (3–27) for $A\neq 0$, while μ_0 is a zero of third order of $g(z)$ in (3–28) for $B\neq 0$.

For the limiting cases: $A\to 0$ in (3–27) and $B\to 0$ in (3–28), we obtain formulae (3–33) and (3–34) from (T–1–1) or (3–2), respectively, provided that $\lim_{z\to\infty} f(z)/\{z^3\cdot\sin^3(\beta z)\}=0$, and that $\lim_{z\to\infty} f(z)/\{z^3\cdot\cos^3(\beta z)\}=0$. Here we shall note that $z^3\cdot\sin^3(\beta z)$ has a zero of six-th order, and $z^3\cdot\cos^3(\beta z)$ has a zero of third order, respectively at the origin.

Examples of (3–33) and (3–34) read (3–35)~(3–40).

6) We shall take Bessel function $J_\nu(\beta z)$ of integral order ν, and put (3–41) into (T–1–1), with a constant β. In this case expression (T–1–1) reads (3–42), provided that $\lim_{z\to\infty} f(z)/J_\nu^3(\beta z)=0$. In (3–42), $f_0^{(j)}=f^{(j)}(z_0)=f^{(j)}(0)$, $H_0^{(s-j)}=H^{(s-j)}(z_0)=H^{(s-j)}(0)$, and βz_n (n=integers) are zeros of Bessel function $J_\nu(z)$, *i.e.* $J_\nu(\beta z_n)=0$, where $z_0=0$ is a zero of $|\nu|$-th order at the origin. In case $\nu=0$, $E(z)$ is a null function.

When we put $f(z)=J_\mu(\alpha z)$ with μ=integer in (3–42), we obtain (3–43).

In case $\mu=\nu=0$, expression (3–43) with $z=0$, reduces to (3–44), where $J_0(\beta z_n)=0$ (n=integers), with $z_0=0$ being not a zero of $J_0(\beta z)$.

7) We take (3–45) in (T–1–1), with an integer ν and constants β and h. Expression (T–1–1) reads (3–46), provided that $\lim_{z\to\infty} f(z)/[z\cdot J_\nu'(\beta z)+h\cdot J_\nu(\beta z)]^3=0$. In (3–46), the values λ_n (n=integers) are roots of (3–47), where λ_0 is a zero of $|\nu|$-th order at the origin. In case $\nu=0$, $G(z)$ reduces to a null function.

If we tend h to infinity in (3–45)~(3–47), we have the expressions which are essentially the same as (3–41)~(3–44).

8) Further, we shall take (3–48), with $T_\nu(\alpha z, \beta z)$ defined by (2–55), α and β being constants. Expression (3–1) reads (3–49), provided that $\lim_{z\to\infty} f(z)/T_\nu^3(\alpha z, \beta z)=0$. The values λ_n (n=integers) are roots of equation (2–58) or (2–59), being arranged in ascending order of magnitude with n, where $\lambda_0=0$ is not a zero of $T_\nu(\alpha z, \beta z)$.

9) If we take (3–50), we have (3–51) from (3–2), provided that $\lim_{z\to\infty} f(z)/$ $/\sin^3(\beta z^2+\gamma)=0$, where $z_n=\pm\sqrt{(n\pi-\gamma)/\beta}$ (n=integers), and the summation over n covers all the values of z_n, *i.e.* positive and negative square roots. In case $\gamma=0$, expression (3–51) should be modified, because $\sin^3(\beta z^2)$ has a zero of six-th order at the origin.

10) If we take $g(z)$ to be a product of two entire functions, each of which has zeros of third order, such as $\sin^3(\beta z+\gamma)\cdot J_\nu^3(pz)$, $J_\nu^3(pz)\cdot\prod_{k=1}^{s}(z-\eta_k)^3$, etc., we obtain a sampling formula from (T–1–1), (3–1), or (3–2), by similar calculations as in (3–17)~(3–21).

Truncation error of the sampling expansion (3–1) can be easily obtained. The bound for truncation error[2] in case $m_n=2$ wasa lready given by (T–6–28).

Concluding Remarks

In this paper were given several new examples of sampling formulae, based on the generalized sampling theorem (TI–2–1), (T–1–1), or (T–2–3). Formulae which make use of the sampled zero-th order, first order, and second order derivatives, were mainly treated. It seems to the authors that almost all the formulae in this paper are quite new expansions. Other examples of the generalized sampling formulae which make use of the sampled higher order derivatives can be obtained similarly, based on the expression (TI-2-1), (TI-2-1′), (T-1-1)~(T-1-1″), (T-2-3), or (T-2-3′).

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Appendix

For $m=2$, we have (3-1):

$$f(z)=\sum_n\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-\frac{1}{4}\cdot f(z_n)\cdot\frac{g_n^{(4)}}{g_n^{(3)}}\right\}+\frac{1}{2}\cdot(z-z_n)^2\cdot\left\{f''(z_n)-\frac{1}{2}\cdot\right.\right.$$
$$\left.\left.\cdot f'(z_n)\cdot\frac{g_n^{(4)}}{g_n^{(3)}}+\frac{1}{2}\cdot f(z_n)\left[\frac{1}{4}\left(\frac{g_n^{(4)}}{g_n^{(3)}}\right)^2-\frac{1}{5}\,\frac{g_n^{(5)}}{g_n^{(3)}}\right]\right\}\right]\cdot\frac{3!\,g(z)}{(z-z_n)^3\cdot g_n^{(3)}}+K\cdot g(z)\,. \tag{3-1}$$

If $g(z)=\phi^3(z)$, with $\phi(z)$ an entire function having simple zeros at $z=z_n$ (n=integers),

$$f(z)=\sum_n\left[f(z_n)+(z-z_n)\cdot\left\{f'(z_n)-\frac{3}{2}f(z_n)\cdot\frac{\phi_n^{(2)}}{\phi_n'}\right\}+\frac{1}{2}(z-z_n)^2\cdot\left\{f''(z_n)-3f'(z_n)\cdot\right.\right.$$
$$\left.\left.\cdot\frac{\phi_n^{(2)}}{\phi_n'}+f(z_n)\cdot\left[3\left(\frac{\phi_n^{(2)}}{\phi_n'}\right)^2-\frac{\phi_n^{(3)}}{\phi_n'}\right]\right\}\right]\cdot\frac{\phi^3(z)}{(z-z_n)^3\cdot(\phi_n')^3}+K\cdot\phi^3(z)\,. \tag{3-2}$$

1) $g(z)=\phi^3(z)=\prod_{k=1}^{s}(z-\eta_k)^3$, (all the η_k's are distinct) (3-3)

$$\phi_m'=\prod_{k=1,k\neq m}^{s}(\eta_m-\eta_k)\,, \tag{3-4}$$

$$\phi_m^{(2)}=2\cdot\sum_{p=1,p\neq m}^{s}(\eta_m-\eta_1)(\eta_m-\eta_2)\cdots(\eta_m-\eta_{m-1})(\eta_m-\eta_{m+1})\cdots\widehat{(\eta_m-\eta_p)}\cdots(\eta_m-\eta_s)\,, \tag{3-5}$$

$$\phi_m^{(3)}=3\sum_{r=1,r\neq m}^{s}\sum_{p=1,p\neq m}^{s}(\eta_m-\eta_1)(\eta_m-\eta_2)\cdots(\eta_m-\eta_{m-1})(\eta_m-\eta_{m+1})\cdots$$
$$\cdots\widehat{(\eta_m-\eta_p)}\cdots\widehat{(\eta_m-\eta_r)}\cdots(\eta_m-\eta_s)\,, \tag{3-6}$$

$$f(z)=\sum_{m=1}^{s}\Big[f(\eta_m)+(z-\eta_m)\cdot\Big\{f'(\eta_m)-3f(\eta_m)\cdot\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_p)}\Big\}+$$
$$+\frac{1}{2}(z-\eta_m)^2\cdot\Big\{f''(\eta_m)-6f'(\eta_m)\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_p)}+f(\eta_m)\cdot\Big[3\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_p)^2}+$$
$$+9\sum_{r=1,r\neq m}^{s}\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_p)(\eta_m-\eta_r)}\Big]\Big\}\Big]\cdot\prod_{k=1,k\neq m}^{s}\frac{(z-\eta_k)^3}{(\eta_m-\eta_k)^3}+K\cdot\prod_{k=1}^{s}(z-\eta_k)^3. \quad (3\text{-}7)$$

2) $g(z)=\psi^3(z)=\cos^3(\alpha\cdot\cos^{-1}\beta z), \qquad (\alpha\cdot\beta\neq 0)$ (3-8)

$\psi(z_n)=\cos(\alpha\cdot\cos^{-1}\beta z_n)=0$, *i.e.* $z_n=(1/\beta)\cdot\cos\big((2n+1)\cdot\pi/(2\alpha)\big)$, $(n=\text{integers})$;

$\psi'(z_n)=(-1)^n\cdot\alpha\cdot\beta/(1-\beta^2\cdot z_n^2)^{1/2}$, (3-9)

$$\left.\begin{aligned}&\psi^{(2)}(z_n)=(-1)^n\cdot\alpha\cdot\beta^3\cdot z_n/(1-\beta^2\cdot z_n^2)^{3/2},\\&\psi^{(3)}(z_n)=(-1)^n\cdot(-\alpha^3\cdot\beta^3+\alpha^3\cdot\beta^5\cdot z_n^2+\alpha\cdot\beta^3+2\alpha\cdot\beta^5\cdot z_n^2)/(1-\beta^2\cdot z_n^2)^{5/2},\end{aligned}\right\}\quad(3\text{-}10)$$

$$f(z)=\sum_n(-1)^n\cdot\Big[f(z_n)+(z-z_n)\cdot\Big\{f'(z_n)-\frac{3}{2}f(z_n)\cdot\frac{\beta^2\cdot z_n}{(1-\beta^2\cdot z_n^2)}\Big\}+$$
$$+\frac{(z-z_n)^2}{2}\cdot\Big\{f''(z_n)-3f'(z_n)\cdot\frac{\beta^2\cdot z_n}{(1-\beta^2\cdot z_n^2)}+f(z_n)\cdot\frac{\beta^2\cdot(\alpha^2-1)}{(1-\beta^2\cdot z_n^2)}\Big\}\Big]\cdot$$
$$\cdot\frac{(1-\beta^2\cdot z_n^2)^{3/2}}{\alpha^3\cdot\beta^3}\cdot\frac{\cos^3(\alpha\cos^{-1}\beta z)}{(z-z_n)^3}+K\cdot\cos^3(\alpha\cos^{-1}\beta z). \quad (3\text{-}11)$$

3) $g(z)=\psi^3(z)=\sin^3(\beta z+\gamma), \qquad (\beta\neq 0)$ (3-12)

$\psi(z_n)=\sin(\beta z_n+\gamma)=0$, *i.e.* $\beta z_n=n\pi-\gamma$, $(n=\text{integers})$ (3-13)

$\psi'_n=(-1)^n\cdot\beta, \qquad \psi_n^{(2)}=0, \qquad \psi_n^{(3)}=(-1)^{n+1}\cdot\beta^3,$ (3-14)

$$f(z)=\sum_{n=-\infty}^{+\infty}\Big[f\Big(\frac{n\pi-\gamma}{\beta}\Big)+\frac{\beta z+\gamma-n\pi}{\beta}\cdot f'\Big(\frac{n\pi-\gamma}{\beta}\Big)+\frac{(\beta z+\gamma-n\pi)^2}{2\beta^2}\cdot$$
$$\cdot\Big\{f''\Big(\frac{n\pi-\gamma}{\beta}\Big)+\beta^2\cdot f\Big(\frac{n\pi-\gamma}{\beta}\Big)\Big\}\Big]\cdot\frac{\sin^3(\beta z+\gamma-n\pi)}{(\beta z+\gamma-n\pi)^3}. \quad (3\text{-}15)$$

$$1=\sum_{n=-\infty}^{+\infty}\Big[1+\frac{(\beta z+\gamma-n\pi)^2}{2}\Big]\cdot\frac{\sin^3(\beta z+\gamma-n\pi)}{(\beta z+\gamma-n\pi)^3}. \qquad (\beta\neq 0) \quad (3\text{-}16)$$

4) $g(z)=\psi^3(z)=\xi^3(z)\cdot\phi^3(z)$, (3-17)

$\psi'_n=\xi'(z_n)\cdot\phi(z_n), \qquad \psi'_m=\xi(z_m)\cdot\phi'(z_m),$ (3-18)

$$\left.\begin{aligned}&\psi_n^{(2)}=\xi^{(2)}(z_n)\cdot\phi(z_n)+2\xi'(z_n)\cdot\phi'(z_n),\\&\psi_m^{(2)}=2\xi'(z_m)\cdot\phi'(z_m)+\xi(z_m)\cdot\phi^{(2)}(z_m),\end{aligned}\right\}\quad(3\text{-}19)$$

$$\left.\begin{aligned}&\psi_n^{(3)}=\xi^{(3)}(z_n)\cdot\phi(z_n)+3\xi^{(2)}(z_n)\cdot\phi'(z_n)+3\xi'(z_n)\cdot\phi^{(2)}(z_n),\\&\psi_m^{(3)}=3\xi^{(2)}(z_m)\cdot\phi'(z_m)+3\xi'(z_m)\cdot\phi^{(2)}(z_m)+\xi(z_m)\cdot\phi^{(3)}(z_m),\end{aligned}\right\}\quad(3\text{-}20)$$

$$f(z)=\sum_n\Big[f(z_n)+(z-z_n)\cdot\Big\{f'(z_n)-\frac{3}{2}f(z_n)\cdot\Big(\frac{\xi''_n}{\xi'_n}+2\frac{\phi'_n}{\phi_n}\Big)\Big\}+\frac{(z-z_n)^2}{2}\cdot$$
$$\cdot\Big\{f''(z_n)-3\cdot f'(z_n)\cdot\Big(\frac{\xi''_n}{\xi'_n}+2\frac{\phi'_n}{\phi_n}\Big)+f(z_n)\cdot\Big[12\cdot\Big(\frac{\phi'_n}{\phi_n}\Big)^2+9\frac{\xi''_n}{\xi'_n}\cdot\frac{\phi'_n}{\phi_n}+3\Big(\frac{\xi''_n}{\xi'_n}\Big)^2-$$

$$-3\frac{\phi_n''}{\phi_n}-\frac{\xi_n^{(3)}}{\xi_n'}\Big]\Big\}\Big]\cdot\frac{\xi^3(z)\cdot\phi^3(z)}{(z-z_n)^3\cdot(\xi_n')^3\cdot\phi_n^3}+\sum_m\Big[f(z_m)+(z-z_m)\cdot\Big\{f'(z_m)-\frac{3}{2}f(z_m)\cdot$$

$$\cdot\Big(2\frac{\xi_m'}{\xi_m}+\frac{\phi_m''}{\phi_m'}\Big)\Big\}+\frac{1}{2}(z-z_m)^2\cdot\Big\{f''(z_m)-3f'(z_m)\cdot\Big(2\frac{\xi_m'}{\xi_m}+\frac{\phi_m''}{\phi_m'}\Big)+f(z_m)\cdot$$

$$\cdot\Big[12\Big(\frac{\xi_m'}{\xi_m}\Big)^2+9\frac{\xi_m'}{\xi_m}\cdot\frac{\phi_m''}{\phi_m'}+3\Big(\frac{\phi_m''}{\phi_m'}\Big)^2-3\frac{\xi_m''}{\xi_m}-\frac{\phi_m^{(3)}}{\phi_m'}\Big]\Big\}\Big]\cdot\frac{\xi^3(z)\cdot\phi^3(z)}{(z-z_m)^3\xi_m^3(\phi_m')^3}\cdot \tag{3-21}$$

$$g(z)=\psi^3(z)=\xi^3(z)\cdot\phi^3(z)=\sin^3(\beta z+\gamma)\cdot\prod_{k=1}^{s}(z-\eta_k)^3,$$

$$(\beta\neq 0,\ \text{all the } \eta_k\text{'s are distinct}) \tag{3-22}$$

$$\xi'(z_n)=(-1)^n\cdot\beta,\qquad \xi''(z_n)=0,\qquad \xi^{(3)}(z_n)=(-1)^{n+1}\cdot\beta^3, \tag{3-23}$$

$$\phi'(\eta_m)=\prod_{k=1,k\neq m}^{s}(\eta_m-\eta_k),$$

$$\phi''(\eta_m)=2\sum_{p=1,p\neq m}^{s}(\eta_m-\eta_1)(\eta_m-\eta_2)\cdots(\eta_m-\eta_{m-1})(\eta_m-\eta_{m+1})\cdots\widehat{(\eta_m-\eta_p)}\cdots(\eta_m-\eta_s), \tag{3-24}$$

$$\phi^{(3)}(\eta_m)=3\sum_{r=1,r\neq m}^{s}\sum_{p=1,p\neq m}^{s}(\eta_m-\eta_1)(\eta_m-\eta_2)\cdots(\eta_m-\eta_{m-1})(\eta_m-\eta_{m+1})\cdots$$

$$\cdots\widehat{(\eta_m-\eta_p)}\cdots\widehat{(\eta_m-\eta_r)}\cdots(\eta_m-\eta_s), \tag{3-25}$$

$$f(z)=\sum_{n=-\infty}^{+\infty}\Big[f\Big(\frac{n\pi-\gamma}{\beta}\Big)+\frac{\beta z+\gamma-n\pi}{\beta}\cdot\Big\{f'\Big(\frac{n\pi-\gamma}{\beta}\Big)-3f\Big(\frac{n\pi-\gamma}{\beta}\Big)\cdot\sum_{p=1}^{s}\frac{\beta}{(n\pi-\gamma-\beta\eta_p)}\Big\}+$$

$$+\frac{(\beta z+\gamma-n\pi)^2}{2\beta^2}\cdot\Big\{f''\Big(\frac{n\pi-\gamma}{\beta}\Big)-6f'\Big(\frac{n\pi-\gamma}{\beta}\Big)\cdot\sum_{p=1}^{s}\frac{\beta}{(n\pi-\gamma-\beta\eta_p)}+\beta^2f\Big(\frac{n\pi-\gamma}{\beta}\Big)\cdot$$

$$\cdot\Big[3\sum_{p=1}^{s}\frac{1}{(n\pi-\gamma-\beta\eta_p)^2}+9\sum_{r=1}^{s}\sum_{p=1}^{s}\frac{1}{(n\pi-\gamma-\beta\eta_p)(n\pi-\gamma-\beta\eta_r)}+1\Big]\Big\}\Big]\cdot$$

$$\cdot\frac{\sin^3(\beta z+\gamma-n\pi)}{(\beta z+\gamma-n\pi)^3}\cdot\prod_{k=1}^{s}\frac{(\beta z-\beta\eta_k)^3}{(n\pi-\gamma-\beta\eta_k)^3}+\sum_{m=1}^{s}\Big[f(\eta_m)+(z-\eta_m)\cdot\Big\{f'(\eta_m)-3f(\eta_m)\cdot$$

$$\cdot\Big[\beta\cdot\cot(\beta\eta_m+\gamma)+\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_p)}\Big]\Big\}+\frac{(z-\eta_m)^2}{2}\cdot\Big\{f''(\eta_m)-6f'(\eta_m)\Big[\beta\cdot\cot(\beta\eta_m+\gamma)+$$

$$+\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_k)}\Big]+f(\eta_m)\cdot\Big[3\beta^2+12\beta^2\cdot\cot^2(\beta\eta_m+\gamma)+18\beta\cdot\cot(\beta\eta_m+\gamma)\cdot$$

$$\cdot\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_p)}+3\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_p)^2}+9\sum_{r=1,r\neq m}^{s}\sum_{p=1,p\neq m}^{s}\frac{1}{(\eta_m-\eta_p)(\eta_m-\eta_r)}\Big]\Big\}\Big]\cdot$$

$$\cdot\frac{\sin^3(\beta z+\gamma)}{\sin^3(\beta\eta_m+\gamma)}\cdot\prod_{k=1,k\neq m}^{s}\frac{(z-\eta_k)^3}{(\eta_m-\eta_k)^3}. \tag{3-26}$$

5) $$g(z)=\psi^3(z)=\Big[z\cdot\sin(\beta z)-A\cdot\cos(\beta z)\Big]^3,\qquad(\beta\cdot A\neq 0) \tag{3-27}$$

$$g(z)=\psi^3(z)=\Big[z\cdot\cos(\beta z)-B\cdot\sin(\beta z)\Big]^3,\qquad(\beta\cdot B\neq 0) \tag{3-28}$$

$$f(z)=\sum_{n=-\infty,n\neq 0}^{+\infty}\Big[f(\lambda_n)+(z-\lambda_n)\cdot\Big\{f'(\lambda_n)-6f(\lambda_n)\cdot\frac{\beta\cdot\cos^2(\beta\lambda_n)}{2\beta\lambda_n+\sin(2\beta\lambda_n)}\Big\}+$$

$$+\frac{(z-\lambda_n)^2}{2}\cdot\left\{f''(\lambda_n)-12f'(\lambda_n)\cdot\frac{\beta\cdot\cos^2(\beta\lambda_n)}{2\beta\lambda_n+\sin(2\beta\lambda_n)}+2^2\cdot\beta^2\cdot f(\lambda_n)\cdot\right.$$

$$\left.\cdot\frac{(9\cos^4(\beta\lambda_n)+3\cos^2(\beta\lambda_n)+2\beta\lambda_n\cdot\sin(2\beta\lambda_n)+\beta^2\lambda_n^2)}{(2\beta\lambda_n+\sin(2\beta\lambda_n))^2}\right\}\Bigg]\cdot$$

$$\cdot\frac{2^3\cdot\cos^3(\beta\lambda_n)}{(2\beta\lambda_n+\sin(2\beta\lambda_n))^3}\cdot\frac{[z\sin(\beta z)-A\cdot\cos(\beta z)]^3}{(z-\lambda_n)^3}. \qquad (3\text{–}29)$$

$$f(z)=-\sum_{n=-\infty,n\neq 0}^{+\infty}\Bigg[f(\mu_n)+(z-\mu_n)\cdot\left\{f'(\mu_n)-6f(\mu_n)\cdot\frac{\beta\cdot\sin^2(\beta\mu_n)}{2\beta\mu_n-\sin(2\beta\mu_n)}\right\}+$$

$$+\frac{(z-\mu_n)^2}{2}\cdot\left\{f''(\mu_n)-12f'(\mu_n)\cdot\frac{\beta\cdot\sin^2(\beta\mu_n)}{2\beta\mu_n-\sin(2\beta\mu_n)}+\right.$$

$$\left.+2^2\cdot\beta^2\cdot f(\mu_n)\cdot\frac{(9\sin^4(\beta\mu_n)+3\sin^2(\beta\mu_n)-2\beta\mu_n\cdot\sin(2\beta\mu_n)+\beta^2\mu_n^2)}{(2\beta\mu_n-\sin(2\beta\mu_n))^2}\right\}\Bigg]\cdot$$

$$\cdot\frac{2^3\cdot\sin^3(\beta\mu_n)}{(2\beta\mu_n-\sin(2\beta\mu_n))^3}\cdot\frac{[z\cdot\cos(\beta z)-B\cdot\sin(\beta z)]^3}{(z-\mu_n)^3}+\Bigg[f(0)+z\cdot f'(0)+$$

$$+\frac{z^2}{2}\left\{f''(0)+\frac{3-\beta\cdot B}{1-\beta\cdot B}\beta^2\cdot f(0)\right\}\Bigg]\cdot\frac{[z\cdot\cos(\beta z)-B\cdot\sin(\beta z)]^3}{(1-\beta\cdot B)^3\cdot z^3}. \quad (\beta\cdot B\neq 1) \qquad (3\text{–}30)$$

$$\psi(\lambda_n)=\lambda_n\cdot\sin(\beta\lambda_n)-A\cdot\cos(\beta\lambda_n)=0, \qquad (\beta\cdot A\neq 0) \qquad (3\text{–}31)$$

$$\psi(\mu_n)=\mu_n\cdot\cos(\beta\mu_n)-B\cdot\sin(\beta\mu_n)=0, \qquad (\beta\cdot B\neq 0) \qquad (3\text{–}32)$$

$$f(z)=\frac{1}{\pi^3}\sum_{n=-\infty,n\neq 0}^{+\infty}\frac{1}{n^3}\Bigg[f\left(\frac{n\pi}{\beta}\right)+\frac{\beta z-n\pi}{\beta}\cdot\left\{f'\left(\frac{n\pi}{\beta}\right)-\frac{3\beta}{n\pi}\cdot f\left(\frac{n\pi}{\beta}\right)\right\}+\frac{(\beta z-n\pi)^2}{2\beta^2}\cdot$$

$$\cdot\left\{f''\left(\frac{n\pi}{\beta}\right)-\frac{6\beta}{n\pi}\cdot f'\left(\frac{n\pi}{\beta}\right)+\frac{\beta^2\cdot(12+n^2\pi^2)}{n^2\pi^2}\cdot f\left(\frac{n\pi}{\beta}\right)\right\}\Bigg]\cdot\frac{\beta^3\cdot z^3\cdot\sin^3(\beta z-n\pi)}{(\beta z-n\pi)^3}+$$

$$+\Bigg[f(0)+z\cdot f'(0)+\frac{z^2}{2}\cdot\left\{f''(0)+\beta^2\cdot f(0)\right\}+\frac{z^3}{3!}\cdot\left\{f^{(3)}(0)+3\beta^2\cdot f'(0)\right\}+\frac{z^4}{4!}\cdot$$

$$\cdot\left\{f^{(4)}(0)+6\beta^2\cdot f''(0)+\frac{17\beta^4}{5}\cdot f(0)\right\}+\frac{z^5}{5!}\left\{f^{(5)}(0)+10\beta^2\cdot f^{(3)}(0)+17\beta^4\cdot f'(0)\right\}\Bigg]\cdot$$

$$\cdot\frac{\sin^3(\beta z)}{(\beta z)^3}. \qquad (3\text{–}33)$$

$$f(z)=-\sum_{n=-\infty}^{+\infty}\Bigg[f\left(\frac{(2n+1)\pi}{2\beta}\right)+\frac{2\beta z-(2n+1)\pi}{2\beta}\cdot\left\{f'\left(\frac{(2n+1)\pi}{2\beta}\right)-\frac{6\beta}{(2n+1)\pi}\cdot\right.$$

$$\left.\cdot f\left(\frac{(2n+1)\pi}{2\beta}\right)\right\}+\frac{[2\beta z-(2n+1)\pi]^2}{8\beta^2}\cdot\left\{f''\left(\frac{(2n+1)\pi}{2\beta}\right)-\frac{12\beta}{(2n+1)\pi}\cdot\right.$$

$$\left.\cdot f'\left(\frac{(2n+1)\pi}{2\beta}\right)+\frac{\beta^2\cdot[48+(2n+1)^2\pi^2]}{(2n+1)^2\pi^2}\cdot f\left(\frac{(2n+1)\pi}{2\beta}\right)\right\}\Bigg]\cdot$$

$$\cdot\frac{64\beta^3\cdot z^3\cdot\cos^3(\beta z-n\pi)}{(2n+1)^3\cdot\pi^3\cdot[2\beta z-(2n+1)\pi]^3}+\Bigg[f(0)+z\cdot f'(0)+\frac{z^2}{2}\left\{f''(0)+\right.$$

$$\left.+3\beta^2\cdot f(0)\right\}\Bigg]\cdot\cos^3(\beta z). \qquad (3\text{–}34)$$

$$\sin(\alpha z)=\frac{1}{\pi^3}\sum_{n=-\infty,n\neq 0}^{+\infty}\frac{1}{n^3}\Bigg[\sin\left(\frac{\alpha n\pi}{\beta}\right)+\frac{\beta z-n\pi}{\beta}\cdot\left\{\alpha\cdot\cos\left(\frac{\alpha n\pi}{\beta}\right)-\frac{3\beta}{n\pi}\cdot\sin\left(\frac{\alpha n\pi}{\beta}\right)\right\}-$$

$$-\frac{(\beta z-n\pi)^2}{2\beta^2}\cdot\left\{\alpha^2\cdot\sin\left(\frac{\alpha n\pi}{\beta}\right)+\frac{6\alpha\cdot\beta}{n\pi}\cdot\cos\left(\frac{\alpha n\pi}{\beta}\right)-\frac{\beta^2\cdot(12+n^2\pi^2)}{n^2\pi^2}\cdot\sin\left(\frac{\alpha n\pi}{\beta}\right)\right\}\Bigg]\cdot$$
$$\cdot\frac{\beta^3\cdot z^3\cdot\sin^3(\beta z-n\pi)}{(\beta z-n\pi)^3}+\left[\alpha\cdot z-\frac{\alpha\cdot z^3}{3!}\cdot\left\{\alpha^2-3\beta^2\right\}+\frac{\alpha\cdot z^5}{5!}\cdot\left\{\alpha^4-10\cdot\alpha^2\cdot\beta^2+17\cdot\beta^4\right\}\right]\cdot$$
$$\cdot\frac{\sin^3(\beta z)}{(\beta z)^3},\qquad(|\alpha|<|\beta|),\tag{3-35}$$

$$\cos(\alpha z)=\frac{1}{\pi^3}\sum_{n=-\infty,n\neq 0}^{+\infty}\frac{1}{n^3}\Bigg[\cos\left(\frac{\alpha n\pi}{\beta}\right)-\frac{\beta z-n\pi}{\beta}\cdot\left\{\alpha\cdot\sin\left(\frac{\alpha n\pi}{\beta}\right)+\frac{3\beta}{n\pi}\cdot\cos\left(\frac{\alpha n\pi}{\beta}\right)\right\}-$$
$$-\frac{(\beta z-n\pi)^2}{2\beta^2}\cdot\left\{\alpha^2\cdot\cos\left(\frac{\alpha n\pi}{\beta}\right)-\frac{6\alpha\cdot\beta}{n\pi}\cdot\sin\left(\frac{\alpha n\pi}{\beta}\right)-\frac{\beta^2(12+n^2\pi^2)}{n^2\pi^2}\cdot\cos\left(\frac{\alpha n\pi}{\beta}\right)\right\}\Bigg]\cdot$$
$$\cdot\frac{\beta^3\cdot z^3\cdot\sin^3(\beta z-n\pi)}{(\beta z-n\pi)^3}+\left[1-\frac{z^2}{2}\left\{\alpha^2-\beta^2\right\}+\frac{z^4}{4!}\cdot\left\{\alpha^4-6\alpha^2\cdot\beta^2+\frac{17\beta^4}{5}\right\}\right]\cdot\frac{\sin^3(\beta z)}{(\beta z)^3},$$
$$(|\alpha|<|\beta|)\tag{3-36}$$

$$J_\nu(\alpha z)=\frac{1}{\pi^3}\sum_{n=-\infty,n\neq 0}^{+\infty}\frac{1}{n^3}\Bigg[J_\nu\left(\frac{\alpha n\pi}{\beta}\right)+\frac{\beta z-n\pi}{\beta}\cdot\left\{\alpha\cdot J_\nu'\left(\frac{\alpha n\pi}{\beta}\right)-\frac{3\beta}{n\pi}\cdot J_\nu\left(\frac{\alpha n\pi}{\beta}\right)\right\}+$$
$$+\frac{(\beta z-n\pi)^2}{2\beta^2}\cdot\left\{\alpha^2\cdot J_\nu''\left(\frac{\alpha n\pi}{\beta}\right)-\frac{6\alpha\beta}{n\pi}\cdot J_\nu'\left(\frac{\alpha n\pi}{\beta}\right)+\frac{\beta^2\cdot(12+n^2\pi^2)}{n^2\pi^2}\cdot J_\nu\left(\frac{\alpha n\pi}{\beta}\right)\right\}\Bigg]\cdot$$
$$\cdot\frac{\beta^3\cdot z^3\cdot\sin^3(\beta z-n\pi)}{(\beta z-n\pi)^3}+\Bigg[J_\nu(0)+\alpha z\cdot J_\nu'(0)+\frac{z^2}{2}\cdot\left\{\alpha^2\cdot J_\nu''(0)+\beta^2\cdot J_\nu(0)\right\}+$$
$$+\frac{z^3}{3!}\cdot\left\{\alpha^3\cdot J_\nu^{(3)}(0)+3\alpha\cdot\beta^2\cdot J_\nu'(0)\right\}+\frac{z^4}{4!}\cdot\left\{\alpha^4\cdot J_\nu^{(4)}(0)+6\alpha^2\cdot\beta^2\cdot J_\nu''(0)+\frac{17\beta^4}{5}\cdot J_\nu(0)\right\}+$$
$$+\frac{z^5}{5!}\cdot\left\{\alpha^5\cdot J_\nu^{(5)}(0)+10\alpha^3\cdot\beta^2\cdot J_\nu^{(3)}(0)+17\alpha\cdot\beta^4 J_\nu'(0)\right\}\Bigg]\cdot\frac{\sin^3(\beta z)}{(\beta z)^3},\qquad(|\alpha|<|\beta|)$$
$$\tag{3-37}$$

$$\sin(\alpha z)=-\sum_{n=-\infty}^{+\infty}\Bigg[\sin\frac{\alpha(2n+1)\pi}{2\beta}+\frac{2\beta z-(2n+1)\pi}{2\beta}\cdot\left\{\alpha\cdot\cos\frac{\alpha(2n+1)\pi}{2\beta}-\right.$$
$$\left.-\frac{6\beta}{(2n+1)\pi}\cdot\sin\frac{\alpha(2n+1)\pi}{2\beta}\right\}-\frac{[2\beta z-(2n+1)\pi]^2}{8\beta^2}\cdot\left\{\alpha^2\cdot\sin\frac{\alpha(2n+1)\pi}{2\beta}+\right.$$
$$\left.+\frac{12\alpha\cdot\beta}{(2n+1)\pi}\cdot\cos\frac{\alpha(2n+1)\pi}{2\beta}-\frac{\beta^2\cdot[48+(2n+1)^2\pi^2]}{(2n+1)^2\cdot\pi^2}\cdot\sin\frac{\alpha\cdot(2n+1)\pi}{2\beta}\right\}\Bigg]\cdot$$
$$\cdot\frac{64\beta^3}{(2n+1)^3\pi^3}\cdot\frac{z^3\cdot\cos^3(\beta z-n\pi)}{[2\beta z-(2n+1)\pi]^3}+\alpha\cdot z\cdot\cos^3(\beta z),\qquad(|\alpha|<|\beta|)\tag{3-38}$$

$$\cos(\alpha z)=-\sum_{n=-\infty}^{+\infty}\Bigg[\cos\frac{\alpha(2n+1)\pi}{2\beta}-\frac{2\beta z-(2n+1)\pi}{2\beta}\cdot\left\{\alpha\cdot\sin\frac{\alpha\cdot(2n+1)\pi}{2\beta}+\right.$$
$$\left.+\frac{6\beta}{(2n+1)\pi}\cdot\cos\frac{\alpha\cdot(2n+1)\pi}{2\beta}\right\}-\frac{[2\beta z-(2n+1)\pi]^2}{8\beta^2}\cdot\left\{\alpha^2\cdot\cos\frac{\alpha\cdot(2n+1)\pi}{2\beta}-\right.$$
$$\left.-\frac{12\alpha\cdot\beta}{(2n+1)\pi}\cdot\sin\frac{\alpha\cdot(2n+1)\pi}{2\beta}-\frac{\beta^2[48+(2n+1)^2\pi^2]}{(2n+1)^2\pi^2}\cdot\cos\frac{\alpha(2n+1)\pi}{2\beta}\right\}\Bigg]\cdot$$
$$\cdot\frac{64\beta^3\cdot z^3\cdot\cos^3(\beta z-n\pi)}{(2n+1)^3\pi^3\cdot\{2\beta z-(2n+1)\pi\}^3}+\left\{1-\frac{z^2}{2}\cdot(\alpha^2-3\beta^2)\right\}\cdot\cos^3(\beta z),\qquad(|\alpha|<|\beta|)$$
$$\tag{3-39}$$

$$J_\nu(\alpha z) = -\sum_{n=-\infty}^{+\infty}\Bigg[J_\nu\Big(\frac{\alpha(2n+1)\pi}{2\beta}\Big) + \frac{2\beta z-(2n+1)\pi}{2\beta}\cdot\Big\{\alpha\cdot J'_\nu\Big(\frac{\alpha\cdot(2n+1)\pi}{2\beta}\Big) - \frac{6\beta}{(2n+1)\pi}\cdot J_\nu\Big(\frac{\alpha(2n+1)\pi}{2\beta}\Big)\Big\} + \frac{[2\beta z-(2n+1)\pi]^2}{8\beta^2}\cdot\Big\{\alpha^2\cdot J''_\nu\Big(\frac{\alpha(2n+1)\pi}{2\beta}\Big) - \frac{12\alpha\cdot\beta}{(2n+1)\pi}\cdot J'_\nu\Big(\frac{\alpha(2n+1)\pi}{2\beta}\Big) + \frac{\beta^2\cdot[48+(2n+1)^2\pi^2]}{(2n+1)^2\pi^2}\cdot J_\nu\Big(\frac{\alpha(2n+1)\pi}{2\beta}\Big)\Big\}\Bigg]\cdot \frac{64\beta^3}{(2n+1)^3\pi^3}\,\frac{z^3\cdot\cos^3(\beta z-n\pi)}{[2\beta z-(2n+1)\pi]^3} + \Big[J_\nu(0)+\alpha\cdot z\cdot J'_\nu(0) + \frac{z^2}{2}\cdot\Big\{\alpha^2\cdot J''_\nu(0)+3\beta^2\cdot J_\nu(0)\Big\}\Big]\cos^3(\beta z). \qquad (|\alpha|<|\beta|) \tag{3-40}$$

6) $g(z) = J_\nu^3(\beta z), \qquad (\beta\neq 0)$ (3-41)

$$f(z) = \sum_{n=-\infty, n\neq 0}^{+\infty}\Bigg[f(z_n)+(z-z_n)\cdot\Big\{f'(z_n)+\frac{3}{2z_n}\cdot f(z_n)\Big\} + \frac{(z-z_n)^2}{2}\cdot\Big\{f''(z_n)+\frac{3}{z_n}\cdot f'(z_n)+\frac{(\beta^2 z_n^2+1-\nu^2)}{z_n^2}\cdot f(z_n)\Big\}\Bigg]\cdot\frac{(-1)}{J_{\nu+1}^3(\beta z_n)}\cdot\frac{J_\nu^3(\beta z)}{(\beta z-\beta z_n)^3} + E(z), \tag{3-42}$$

$$E(z) = \sum_{s=0}^{3|\nu|-1}\sum_{j=0}^{s}\frac{f_0^{(j)}}{j!}\cdot\frac{H_0^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{J_\nu^3(\beta z)}{z^{3|\nu|}}.$$

$$J_\mu(\alpha z) = \sum_{n=-\infty, n\neq 0}^{+\infty}\Bigg[J_\mu(\alpha z_n)+(z-z_n)\cdot\Big\{\alpha J'_\mu(\alpha z_n)+\frac{3}{2z_n}\cdot J_\mu(\alpha z_n)\Big\} + \frac{(z-z_n)^2}{2}\cdot\Big\{\alpha^2 J''_\mu(\alpha z_n)+\frac{3\alpha}{z_n}\cdot J'_\mu(\alpha z_n)+\frac{(\beta^2 z_n^2+1-\nu^2)}{z_n^2}\cdot J_\mu(\alpha z_n)\Big\}\Bigg]\cdot\frac{(-1)}{J_{\nu+1}^3(\beta z_n)}\cdot\frac{J_\nu^3(\beta z)}{(\beta z-\beta z_n)^3} + E(z), \qquad (0\neq|\alpha|<|\beta|). \tag{3-43}$$

$$1 = \sum_{n=-\infty, n\neq 0}^{+\infty}\Bigg[\frac{z_n^2}{2}\cdot\Big(\beta^2-\frac{\alpha^2}{2}\Big)\cdot J_0(\alpha z_n) - \frac{\alpha\cdot z_n}{2}\cdot J_1(\alpha z_n)+\frac{\alpha^2\cdot z_n^2}{4}\cdot J_2(\alpha z_n)\Bigg]\cdot\frac{1}{J_1^3(\alpha z_n)}\cdot\frac{1}{(\beta z_n)^3} = 2\sum_{n=1}^{+\infty}\Bigg[\frac{z_n^2}{2}\cdot\Big(\beta^2-\frac{\alpha^2}{2}\Big)\cdot J_0(\alpha z_n)-\frac{\alpha z_n}{2}\cdot J_1(\alpha z_n)+\frac{\alpha^2\cdot z_n^2}{4}\cdot J_2(\alpha z_n)\Bigg]\cdot\frac{1}{J_1^3(\alpha z_n)}\cdot\frac{1}{(\beta z_n)^3}, \qquad (0\neq|\alpha|<|\beta|). \tag{3-44}$$

7) $g(z) = \Big[z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)\Big]^3, \qquad (\beta\cdot h\neq 0)$ (3-45)

$$f(z) = \sum_{n=-\infty, n\neq 0}^{+\infty}\Bigg[f(\lambda_n)+(z-\lambda_n)\cdot\Big\{f'(\lambda_n)+\frac{3}{2}\cdot f(\lambda_n)\cdot\frac{[\beta^2(h^2-\lambda_n^2)-\nu^2]}{\lambda_n\cdot[\beta^2(h^2+\lambda_n^2)-\nu^2]}\Big\} + \frac{(z-\lambda_n)^2}{2}\cdot\Big\{f''(\lambda_n)+3f'(\lambda_n)\cdot\frac{[\beta^2(h^2-\lambda_n^2)-\nu^2]}{\lambda_n\cdot[\beta^2(h^2+\lambda_n^2)-\nu^2]} + f(\lambda_n)\cdot\frac{3[\beta^2(h^2-\lambda_n^2)-\nu^2]^2+[\beta^2(h^2+\lambda_n^2)-\nu^2]\cdot[\nu^4+2\nu^2+\beta^4\lambda_n^4+\beta^4h^2\lambda_n^2+2\beta^3h\lambda_n^2-\beta^2h^2\nu^2-2\beta^2h^2-2\beta^2\nu^2\lambda_n^2]}{\lambda_n^2[\beta^2\cdot(h^2+\lambda_n^2)-\nu^2]^2}\Big\}\Bigg]\cdot\frac{(-\beta\lambda_n)^3}{[\beta^2(h^2+\lambda_n^2)-\nu^2]^3\cdot J_\nu^3(\beta\lambda_n)}\cdot\frac{[zJ'_\nu(\beta z)+h\cdot J_\nu(\beta z)]^3}{(z-\lambda_n)^3} + G(z), \tag{3-46}$$

$$G(z)=\sum_{s=0}^{3|\nu|-1}\sum_{j=0}^{s}\frac{f_0^{(j)}}{j!}\cdot\frac{H_1^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{[z\cdot J_\nu'(\beta z)+h\cdot J_\nu(\beta z)]^3}{z^{3|\nu|}}.$$

$$\psi(\lambda_n)=\lambda_n\cdot J_\nu'(\beta\lambda_n)+h\cdot J_\nu(\beta\lambda_n)=0.\qquad(\beta\cdot h\neq 0)\tag{3-47}$$

8) $g(z)=\psi^3(z)=T_\nu^3(\alpha z,\beta z),\qquad(\nu=\text{integer},\ \alpha\cdot\beta\neq 0,\ \alpha\neq\beta)$ (3-48)

$$f(z)=\sum_{n=-\infty,n\neq 0}^{+\infty}\Bigg[f(\lambda_n)+(z-\lambda_n)\Bigg\{f'(\lambda_n)-\frac{3}{2}\cdot f(\lambda_n)\cdot\frac{(2\nu-1)(J_\nu^2(\beta\lambda_n)-J_\nu^2(\alpha\lambda_n))+\alpha\cdot\beta\cdot\pi\cdot\lambda_n^2J_\nu(\alpha\lambda_n)J_\nu(\beta\lambda_n)(Y_{\nu+1}(\alpha\lambda_n)\cdot J_{\nu+1}(\beta\lambda_n)-J_{\nu+1}(\alpha\lambda_n)\cdot Y_{\nu+1}(\beta\lambda_n))}{\lambda_n(J_\nu^2(\beta\lambda_n)-J_\nu^2(\alpha\lambda_n))}\Bigg\}+$$

$$+\frac{(z-\lambda_n)^2}{2}\cdot\Bigg\{f''(\lambda_n)-3f'(\lambda_n)\cdot\frac{(2\nu-1)(J_\nu^2(\beta\lambda_n)-J_\nu^2(\alpha\lambda_n))+\alpha\cdot\beta\pi\lambda_n^2J_\nu(\alpha\lambda_n)\cdot J_\nu(\beta\lambda_n)\cdot(Y_{\nu+1}(\alpha\lambda_n)\cdot J_{\nu+1}(\beta\lambda_n)-J_{\nu+1}(\alpha\lambda_n)\cdot Y_{\nu+1}(\beta\lambda_n))}{\lambda_n(J_\nu^2(\beta\lambda_n)-J_\nu^2(\alpha\lambda_n))}+$$

$$+f(\lambda_n)\cdot\Bigg[\frac{8\nu^2-6\nu+1}{\lambda_n^2}+\frac{12\alpha\cdot\beta\cdot\nu\cdot\pi\cdot J_\nu(\alpha\lambda_n)\cdot J_\nu(\beta\lambda_n)}{J_\nu^2(\beta\lambda_n)-J_\nu^2(\alpha\lambda_n)}\cdot$$

$$\cdot\Big(Y_{\nu+1}(\alpha\lambda_n)\cdot J_{\nu+1}(\beta\lambda_n)-J_{\nu+1}(\alpha\lambda_n)\cdot Y_{\nu+1}(\beta\lambda_n)\Big)+\frac{3\alpha^2\cdot\beta^2\cdot\pi^2\lambda_n^2J_\nu^2(\alpha\lambda_n)\cdot J_\nu^2(\beta\lambda_n)}{(J_\nu^2(\beta\lambda_n)-J_\nu^2(\alpha\lambda_n))^2}\cdot$$

$$\cdot\Big(Y_{\nu+1}(\alpha\lambda_n)\cdot J_{\nu+1}(\beta\lambda_n)-J_{\nu+1}(\alpha\lambda_n)\cdot Y_{\nu+1}(\beta\lambda_n)\Big)^2+$$

$$+\frac{(\alpha^2J_\nu^2(\beta\lambda_n)-\beta^2J_\nu^2(\alpha\lambda_n))+3(\beta^2J_\nu^2(\beta\lambda_n)-\alpha^2J_\nu^2(\alpha\lambda_n))-3\alpha\cdot\beta\cdot\pi\cdot J_\nu(\alpha\lambda_n)\cdot J_\nu(\beta\lambda_n)(Y_{\nu+1}(\alpha\lambda_n)\cdot J_{\nu+1}(\beta\lambda_n)-J_{\nu+1}(\alpha\lambda_n)\cdot Y_{\nu+1}(\beta\lambda_n))}{J_\nu^2(\beta\lambda_n)-J_\nu^2(\alpha\lambda_n)}\Bigg]\Bigg\}\Bigg]\cdot$$

$$\cdot\frac{\pi^3\lambda_n^3J_\nu^3(\alpha\lambda_n)\cdot J_\nu^3(\beta\lambda_n)}{8(J_\nu^2(\beta\lambda_n)-J_\nu^2(\alpha\lambda_n))^3}\cdot\frac{T_\nu^3(\alpha z,\beta z)}{(z-\lambda_n)^3}.\tag{3-49}$$

9) $g(z)=\psi^3(z)=\sin^3(\beta z^2+\gamma)\qquad(\beta\cdot\gamma\neq 0)$ (3-50)

$$f(z)=\sum_n\Bigg[f(z_n)+(z-z_n)\cdot\Big\{f'(z_n)-f(z_n)\cdot\frac{3}{2z_n}\Big\}+\frac{(z-z_n)^2}{2}\cdot\Big\{f''(z_n)-f'(z_n)\frac{3}{z_n}+f(z_n)\Big(4\beta^2\cdot z_n^2+\frac{3}{z_n^2}\Big)\Big\}\Bigg]\frac{\sin^3(\beta z^2+\gamma-n\pi)}{8\beta^3\cdot z_n^3(z-z_n)^3}.\tag{3-51}$$