Stresses in Multi Layered Systems

Jin YOSHIMURA*
Shunsuke USHIO*
Teruo SUGAWARA*

Abstract

The analysis of elastic layered systems is required for the design and analysis of flexible pavements or rock mechanics. This paper presents a general analysis of three layered systems due to a variety of surface loading conditions and shows some numerical examples including a principal stress analysis.

1. Introduction

The analysis of stresses and displacements in a semi infinite elastic solid with a vertical surface load was originally established by Boussinesq.

Burmister[1], Fox[2], Jelinek[3] and others gave some solutions for a two layered systems and Acum[4] and Fox analized three layered systems under a uniformly distributed normal load. These solutions, however, are only for a limited case, that is, stresses and displacements in a vertical axis through a center of the normally distributed circular surface load ($r=0$) and in an interface between layers.

Snedden[5] has presented a method of analysis by integral transforms (Hankel transforms) for three dimensional problems in the theory of elasticity and successfully used to determine the distribution of stresses in a semi infinite solid and other interesting problems under various loading conditions. This method is, however, restricted to the axisymmetric case.

Muki[5] has extended and generalized Sneddon's method for the axisymmetric problems of the theory of elasticity to the asymmetric problems. This method is applicable to the analysis of layered systems. Schiffman[7] has applied Muki's method to several interesting problems.

This paper presents a general solution for the three layered systems. Stresses and displacements in any point of layered systems under a vertical and tangential load on its surface were obtained.

2. General Solutions

In an asymmetric cylindrical coordinate systems the stress tensor is

$$\sigma^i_{pq}(r, \theta, z) = \begin{bmatrix} \sigma^i_{rr} & \sigma^i_{r\theta} & \sigma^i_{rz} \\ \sigma^i_{r\theta} & \sigma^i_{\theta\theta} & \sigma^i_{\theta z} \\ \sigma^i_{rz} & \sigma^i_{\theta z} & \sigma^i_{zz} \end{bmatrix} \qquad (1)$$

and components of stresses are shown in Fig. 1.

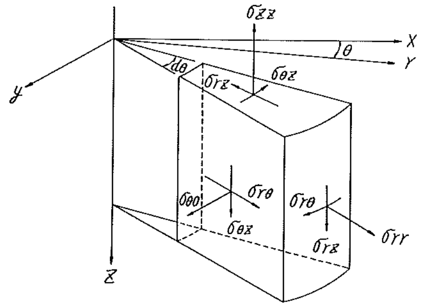

Fig. 1. Components of stresses in cylindrical coordinate

The displacement vector is

* Department of Civil Engineering, Faculty of Engineering, Hokkaido University, Sapporo, Japan.

$$u_p^i = \begin{bmatrix} u_r^i \\ u_\theta^i \\ u_z^i \end{bmatrix} . \tag{2}$$

The index (i) referes to the layer for which the stresses and displacements are being determined.

The equations of equilibrium in terms of the displacements, u_r, u_θ, u_z in the r, θ and z directions are

$$\left.\begin{aligned} &\nabla^2 u_r + \frac{1}{1-2\nu}\frac{\partial \Delta}{\partial r} - \frac{1}{r}\left(2\frac{\partial u_\theta}{r\cdot\partial\theta} + \frac{u_r}{r}\right) = 0 \\ &\nabla^2 u_\theta + \frac{1}{1-2\nu}\frac{\partial \Delta}{r\partial\theta} - \frac{1}{r}\left(\frac{u_\theta}{r} - 2\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) = 0 \\ &\nabla^2 u_z + \frac{1}{1-2\nu}\frac{\partial \Delta}{\partial z} = 0 , \end{aligned}\right\} \tag{3}$$

where ∇^2 and ν denote the Laplacian operater und Poisson's ratio respectively, and Δ is

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} . \tag{4}$$

Equations of equilibrium (3) are satisfied if we take in ith layer by defining stress functions Φ_i and Ψ_i for each layer,

$$\left.\begin{aligned} u_r^i &= \frac{1+\nu_i}{E_i}\left\{-\frac{\partial^2\Phi_i}{\partial r\partial z} + \frac{2}{r}\frac{\partial\Psi_i}{\partial\theta}\right\} \\ u_\theta^i &= \frac{1+\nu_i}{E_i}\left\{-\frac{1}{r}\frac{\partial^2\Phi_i}{\partial\theta\partial z} - 2\frac{\partial\Psi_i}{\partial r}\right\} \\ u_z^i &= \frac{1+\nu_i}{E_i}\left\{2(1-\nu_i)\nabla^2\Phi_i - \frac{\partial^2\Psi_i}{\partial z^2}\right\} , \end{aligned}\right\} \tag{5}$$

where,

$$\nabla^4\Phi_i = 0 , \qquad \nabla^2\Psi_i = 0 . \tag{6}$$

The components of stresses in each layer are :

$$\left.\begin{aligned} \sigma_{rr}^i &= \frac{\partial}{\partial z}\left\{\nu_i\nabla^2\Phi_i - \frac{\partial^2\Phi_i}{\partial r^2}\right\} + \frac{2}{r}\frac{\partial}{\partial\theta}\left\{\frac{\partial\Psi_i}{\partial r} - \frac{1}{r}\Psi_i\right\} \\ \sigma_{\theta\theta}^i &= \frac{\partial}{\partial z}\left\{\nu_i\nabla^2\Phi_i - \frac{1}{r^2}\frac{\partial^2\Phi_i}{\partial\theta^2} - \frac{1}{r}\frac{\partial\Phi_i}{\partial r}\right\} - \frac{2}{r}\frac{\partial}{\partial\theta}\left\{\frac{\partial\Psi_i}{\partial r} - \frac{1}{r}\Psi_i\right\} \\ \sigma_{zz}^i &= \frac{\partial}{\partial z}\left\{(2-\nu_i)\nabla^2\Phi_i - \frac{\partial^2\Phi_i}{\partial z^2}\right\} \\ \sigma_{r\theta}^i &= \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}\left\{\frac{\Phi_i}{r} - \frac{\partial\Phi_i}{\partial r}\right\} - 2\frac{\partial^2\Psi_i}{\partial r^2} - \frac{\partial^2\Psi_i}{\partial z^2} \\ \sigma_{\theta z}^i &= \frac{1}{r}\frac{\partial}{\partial\theta}\left\{(1-\nu_i)\nabla^2\Phi_i - \frac{\partial^2\Phi_i}{\partial z^2}\right\} - \frac{\partial^2\Psi_i}{\partial r\cdot\partial z} \\ \sigma_{rz}^i &= \frac{\partial}{\partial r}\left\{(1-\nu_i)\nabla^2\Phi_i - \frac{\partial^2\Phi_i}{\partial z^2}\right\} + \frac{1}{r}\frac{\partial^2\Psi_i}{\partial\theta\cdot\partial z} . \end{aligned}\right\} \tag{7}$$

The steress functions Φ_i and Ψ_i can be written in the following forms,

$$\left.\begin{aligned} \Phi_i(r, \theta, z) &= \sum_{n=0}^{\infty} \Phi_n^i(r, z) \cos n\theta \\ \Phi_i(r, \theta, z) &= \sum_{n=0}^{\infty} \Psi_n^i(r, z) \sin n\theta \,. \end{aligned}\right\} \tag{8}$$

Φ_n^i and Ψ_n^i are the solutions of the following partial differential equations

$$\nabla_n^4 \Phi_n^i = \left(\frac{\partial^2}{\partial r^2} + \frac{1\partial}{r\partial r} - \frac{n^2}{r^2} + \frac{\partial^2}{\partial z^2}\right)^2 \Phi_n^i = 0 \tag{9}$$

$$\nabla_n^2 \Psi_n^i = \left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2} + \frac{\partial^2}{\partial z^2}\right) \Psi_n^i = 0 \,. \tag{10}$$

Applying Hankel transform to Eq. (9) and (10), we have

$$\int_0^\infty r\nabla_n^4 \Phi_n^i J_n(mr)\, dr = \left(\frac{d^2}{dz^2} - m^2\right)^2 \int_0^\infty r\Phi_n^i J_n(mr)\, dr = 0 \tag{11}$$

$$\int_0^\infty r\nabla_n^2 \Psi_n^i J_n(mr)\, dr = \left(\frac{d^2}{dz^2} - m^2\right) \int_0^\infty r\Psi_n^i J_n(mr)\, dr = 0 \,. \tag{12}$$

If we write,

$$G_n^i(m, z) = \int_0^\infty r\Phi_n^i(r, z)\, J_n(mr)\, dr \tag{13}$$

$$H_n^i(m, z) = \int_0^\infty r\Psi_n^i(r, z)\, J_n(mr)\, dr \tag{14}$$

where, $J_n(mr)$ is n^{th} Bessel function,
Hankel transforms G_n^i and H_n^i must be the solutions of the ordinary differential equations

$$\left(\frac{d^2}{dz^2} - m^2\right)^2 G_n^i(m, z) = 0 \,, \tag{15}$$

$$\left(\frac{d^2}{dz^2} - m^2\right) H_n^i(m, z) = 0 \,. \tag{16}$$

The general solutions of Eq. (15) and (16) are:

$$G_n^i(m, z) = \left\{A_n^i(m) + B_n^i(m)\cdot z\right\} e^{-mz} + \left\{C_n^i(m) + D_n^i(m)\cdot z\right\} e^{mz} \tag{17}$$

$$H_n^i(m, z) = E_n^i(m)\, e^{-mz} + F_n^i(m)\, e^{mz} \tag{18}$$

where the arbitrary constants $A_n^i(m) \cdots F_n^i(m)$ are to be determnied from the given boundary conditions on the surface and interface between layers. Once these constants have been determined, G_n^i and H_n^i are known functions of z and of the parameter m, and the expressions for Φ_n^i and Ψ_n^i may then be obtained by means of the Hankel inversion theorem

$$\Phi_n^i(r, z) = \int_0^\infty mG_n^i(m, z)\, J_n(mr)\, dm \tag{19}$$

$$\Psi_n^i(r, z) = \int_0^\infty mH_n^i(m, z)\, J_n(mr)\, dm. \tag{20}$$

Next, transformations of expressions for the displacements and stress components into relations including G^i_n, H^i_n and their derivatives are required. These results are as follows,

$$
\left.
\begin{aligned}
u^i_r &= \frac{1+\nu_i}{2E_i}\sum_{n=1}^{\infty}\left\{U^i_{n+1}-V^i_{n-1}\right\}\cos n\theta \\
u^i_\theta &= \frac{1+\nu_i}{2E_i}\sum_{n=0}^{\infty}\left\{U^i_{n+1}+V^i_{n-1}\right\}\sin n\theta \\
u^i_z &= \frac{1+\nu_i}{2E_i}\sum_{n=0}^{\infty}\left\{\int_0^\infty m\left[(1-2\nu_i)\frac{d^2G^i_n}{dz^2}-2(1-\nu_i)m^2G^i_n\right]J_n(mr)\,dm\right\}\cos n\theta \\
\sigma^i_{rr} &= \sum_{n=0}^{\infty}\left\{\int_0^\infty m\frac{dW^i_n}{dz}J_n(mr)\,dm-\frac{n+1}{2r}U^i_{n+1}-\frac{n-1}{2r}V^i_{n-1}\right\}\cos n\theta \\
\sigma^i_{\theta\theta} &= \sum_{n=0}^{\infty}\left\{\nu_i\int_0^\infty m\left[\frac{d^3G^i_n}{dz^3}-m^2\frac{dG^i_n}{dz}\right]J_n(mr)\,dm+\frac{n+1}{2r}U^i_{n+1}+V^i_{n-1}\right\}\cos n\theta \\
\sigma^i_{zz} &= \sum_{n=0}^{\infty}\left\{\int_0^\infty m\left[(1-\nu_i)\frac{d^3G^i_n}{dz^3}-(2-\nu_i)m^2\frac{dG^i_n}{dz}\right]J_n(mr)\,dm\right\}\cos n\theta \\
\sigma^i_{r\theta} &= \sum_{n=0}^{\infty}\left\{\int_0^\infty m^3H^i_nJ_n(mr)\,dm-\frac{n+1}{2r}U^i_{n+1}+\frac{n-1}{2r}V^i_{n-1}\right\}\sin n\theta \\
\sigma^i_{\theta z} &= \frac{1}{2}\sum_{n=0}^{\infty}\left\{\int_0^\infty m^2\left[W^i_n+\frac{dH^i_n}{dz}\right]J_{n+1}(mr)\,dr+\right. \\
&\qquad\left.+\int_0^\infty m^2\left[W^i_n-\frac{dH^i_n}{dz}\right]J_{n-1}(mr)\,dm\right\}\sin n\theta \\
\sigma^i_{rz} &= \frac{1}{2}\sum_{n=0}^{\infty}\left\{\int_0^\infty m^2\left[W^i_n+\frac{dH^i_n}{dz}\right]J_{n+1}(mr)\,dm-\right. \\
&\qquad\left.-\int_0^\infty m^2\left[W^i_n-\frac{dH^i_n}{dz}\right]J_{n-1}(mr)\,dm\right\}\cos n\theta
\end{aligned}
\right\} \tag{21}
$$

where

$$
\left.
\begin{aligned}
U^i_{n+1} &= \int_0^\infty m^2\left[\frac{dG^i_n}{dz}+2H^i_n\right]J_{n+1}(mr)\,dm \\
V^i_{n-1} &= \int_0^\infty m^2\left[\frac{dG^i_n}{dz}-2H^i_n\right]J_{n-1}(mr)\cdot dm \\
W_n &= \nu_i\frac{d^2G^i_n}{dz^2}+(1-\nu_i)\,m^2G^i_n\,.
\end{aligned}
\right\} \tag{22}
$$

3. Stresses and displacements in three layered systems under normally distributed circular loading on its surface

The solution of a layered system problem consists of determining the constants A^i_n through F^i_n for each layer. This will determine the functions G^i_n and H^i_n for each layer. Then, stresses are determined by Eq. (21).

For the boundary conditions and continuity conditions, transformed stresses and transformed surface load are used.

In the case of axisymmetric normal load as shown in Fig. 2, we can put $n=0$, and H_n^i vanishes.
The surface loading conditions are

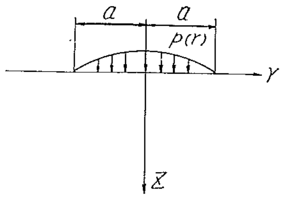

Fig. 2.

$$\left.\begin{array}{ll} \sigma_{zz}^1(r,\theta)=-p(r) & 0\leqq r\leqq a \\ \sigma_{zz}^1(r,\theta)=0 & r>a \\ \sigma_{rz}^1(r,\theta)=0 & 0\leqq r<\infty \end{array}\right\} \tag{23}$$

In the case of $p(r)=p_0=\text{const.}$, transformed surface load is

$$M(m)=-\int_0^\infty rp(r)\,J_0(mr)\,dr$$
$$=-p_0\int_0^\infty rJ_0(mr)\,dr=-\frac{p_0a}{m}\,J_1(ma). \tag{24}$$

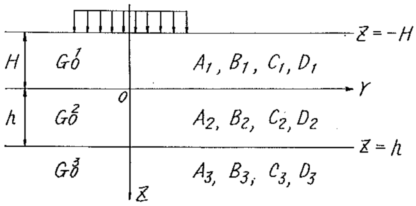

Fig. 3.

A coordinate system for the three layered system are shown in Fig. 3. In the case of flexible loading, boundary conditions at an interface are as follows,

$$\left.\begin{array}{llll} \text{for} & z=-H: & \sigma_{zz}^1=M(m), & \sigma_{rz}^1=0 \\ \text{for} & z=0: & \sigma_{zz}^1=\sigma_{zz}^2, & \sigma_{rz}^1=\sigma_{rz}^2 \\ & & u_z^1=u_z^2, & u_r^1=u_r^2 \\ \text{for} & z=h: & \sigma_{zz}^2=\sigma_{zz}^3, & \sigma_{rz}^2=\sigma_{rz}^3 \\ & & u_z^2=u_z^3, & u_r^2=u_r^3. \end{array}\right\} \tag{25}$$

As the third layer is semi-infinite, the stresses and displacements must vanish at infinity. This condition is accomplished if

$$C_n^3=D_n^3=0. \tag{26}$$

These conditions (25) are represented by the following equations.

$$\left.\begin{array}{l} m^3A_1e^{mH}+(1-2\nu_1-mH)\,m^2B_1e^{mH}-m^3C_1e^{-mH}+ \\ \quad +(1-2\nu_1+mH)\,m^2De^{-mH}=M(m) \\ m^3A_1e^{mH}-(2\nu_1+mH)\,m^2B_1e^{mH}+m^3C_1e^{-mH}+(2\nu_1-mH)\,m^2D_1e^{-mH}=0 \\ m^3A_1+(1-2\nu_1)\,m^2B_1-m^3C_1+(1-2\nu)\,m^2D_1= \\ \quad =m^3A_2+(1-2\nu_2)\,m^2B_2-m^3C_2+(1-2\nu_2)\,m^2D_2 \\ m^3A_2e^{-mh}+(1-2\nu_2+mh)\,m^2B_2e^{-mh}-m^3C_2e^{mh}+(1-2\nu_2-mh)\,m^2D_2e^{mh}= \\ \quad =m^3A_3e^{-mh}+(1-2\nu_3+mh)\,B_3e^{-mh}-m^3C_3e^{mh}+(1-2\nu_3-mh)\,m_2D_3e^{mh} \\ m^3A_1-(2\nu_1m^2B_1)+m^3C_1+2\nu_1m^2D_1=m^3A_2-2\nu_2m^2B_2+m^3C_2+2\nu_2m^2D_2 \\ m_3A_2e^{-mh}-(2\nu_2-mh)\,m^2B_2e^{-mh}+m^3C_2e^{mh}+(2\nu_2+mh)\,m^2D_2e^{mh}= \\ \quad =m^3A_3e^{-mh}-(2\nu_3-mh)\,m^2B_3e^{-mh}+m^3C_3e^{mh}+(2\nu_3+mh)\,m^2D_3e^{mh} \\ \frac{1+\nu_1}{E_1}\left\{-m^2A_1-(2m-4m\nu_1)\,B_1-m^2C_1+(2m-4m\nu_1)\,D_1\right\}= \end{array}\right\} \tag{27}$$

$$= \frac{1+\nu_2}{E_2}\left\{-m^2A_2-(2m-4m\nu_2)B_2-m^2C_2+(2m-4m\nu_2)D_2\right\}$$

$$\frac{1+\nu_2}{E_2}\left\{-m^2A_2e^{-mh}-(2m+m^2h-4m\nu_2)B_2e^{-mh}-m^2C_2e^{mh}+(2m-m^2h-\right.$$

$$\left.-4m\nu_2)D_2e^{mh}\right\}=\frac{1+\nu_3}{E_3}\left\{-m^2A_3e^{-mh}-(2m+m^2h--4m\nu_3)B_3e^{-mh}-\right.$$

$$\left.-m_2C_3e^{mh}+(2m-m^2h-4m\nu_3)D_3e^{mh}\right\}$$

$$\frac{m(1+\nu_1)}{E_1}\left\{-mA_1+B_1+mC_1+D_1\right\}=\frac{m(1+\nu_2)}{E_2}\left\{-mA_2+B_2+mC+D_2\right\}$$

$$\frac{m(1+\nu_2)}{E_2}\left\{-mA_2e^{-mh}+(1-mh)B_2e^{-mh}+mC_2e^{mh}+(1+mh)D_2e^{mh}\right\}=$$

$$=\frac{m(1+\nu_3)}{E_3}\left\{-mA_3e^{-mh}+(1-mh)B_3e^{-mh}+mC_3e^{mh}+(1+mh)D_3e^{mh}\right\}.$$

Solving these 10 simultameous equations for $\nu=0.5$, the ten constants can be determined and represented in the following,

$$A_1m^2=-\frac{M(m)}{m\Delta}\left\{(1+mH)e^{mH}-K(1-mH)e^{-mH}+\left[-N(1+K^2)(1+\right.\right.$$

$$\left.+2m^2h^2-mH)+2mhN(1-K^2)(1-mH)+4mHm^2h^2K^2N\right]e^{-mH-2mh}+$$

$$+2KN(1+2m^2h^2)(1+mH)e^{mH-2mh}-KN^2(1-mH)e^{-mH-4mh}+$$

$$\left.+K^2N^2(1+mH)e^{mH-4mh}\right\} \tag{28. 1}$$

$$B_1m=-\frac{M(m)}{m\Delta}\left\{e^{mH}-(1-2mH)Ke^{-mH}-KN^2(1-2mH)e^{-mH-4mH}+\right.$$

$$+2KN(1+2m^2h^2)e^{mH-2mh}-\left[N(1-2mH)+K^2N(1-2mH)-2Nmh+\right.$$

$$\left.\left.+K^2N(1-2mh)4m^2h^2+2K^2Nmh\right]e^{-mH-2mh}+K^2N^2e^{mH-4mh}\right\} \tag{28. 2}$$

$$C_1m^2=\frac{M(m)}{m\Delta}\left\{-K(1+mH)e^{mH}+K^2(1-mH)e^{-mH}+N\left[(1+K^2)(1+\right.\right.$$

$$\left.+2m^2h^2+mH)+2(1-K^2)mh(1+mH)+4K^2m^3Hh^2\right]e^{mH-2mh}+$$

$$+N^2(1-mH)e^{-mH-4mh}+2KN(1+2m^2h^2)(1-mH)e^{-mH-2mh}-$$

$$\left.-KN^2(1+mH)e^{mH-4mh}\right\} \tag{28. 3}$$

$$D_1m=\frac{M(m)}{m\Delta}\left\{K(1+2mH)e^{mH}-K^2e^{-mH}+KN^2(1+2mH)e^{mH-4mh}+\right.$$

$$+\left[K^2N(1+2mH)+K^2N(1+2mH)4m^2h^2+2Nmh++N(1+2mH)-\right.$$

$$\left.-2K^2Nmh\right]e^{mH-2mh}-2KN(1+2m^2h^2)e^{-mH-2mh}-N^2e^{-mH-4mh} \tag{28. 4}$$

$$A_2m^2=-\frac{M(m)}{m\Delta}(1-K)\left\{(1+mH)e^{mH}-K(1-mH)e^{-mH}+KN\times\right.$$

$$\times\Big[1-2mh+2m^2h^2+mH(1-2mh+4m^2h^2)\Big]\,e^{mH-2mh}-$$

$$-N\Big[1-2mh+2m^2h^2-mH\,(1-2mh)\Big]\,e^{-mH-2mh}\Big\} \qquad (28.5)$$

$$B_2m=-\frac{M(m)}{m\varDelta}(1-K)\Big\{e^{mH}-K(1-2mH)\,e^{-mH}+KN\Big[1-2mH(1+2mH)\Big]\times$$

$$\times\,e^{mH-2mh}-N(1-2mh-2mH)\,e^{-mH-2mh}\Big\} \qquad (28.6)$$

$$C_2m^2=\frac{M(m)}{m\varDelta}\,(1-K)\Big\{-N\Big[(1+2mh)\,(1+mH)+2m^2h^2\Big]\,e^{mH-2mh}+$$

$$+N^2(1-mH)e^{-mH-4mh}+KN\Big[1+2mh+2m^2h^2-mH(1+2mh+4m^2h_2)\Big]\times$$

$$\times\,e^{-mH-2mh}-KN^2(1+mH)\,e^{mH-4mh}\Big\} \qquad (28.7)$$

$$D_2m=\frac{M(m)}{m\varDelta}\,(1-K)\Big\{N(1+2mh+2mH)\,e^{mH-2mh}-N^2e^{-mH-4mh}-$$

$$-KN\Big[1+2mh(1-2mH)\Big]\,e^{-mH-2mh}+KN^2(1+2mH)\,e^{mH-4mh}\Big\} \qquad (28.8)$$

$$A_3m^2=-\frac{M(m)}{m\varDelta}\,(1-K)\,(1-N)\Big\{(1+mH)\,e^{mH}-K(1-mH)\,e^{-mH}-$$

$$-N\Big[1+2m^2h^2-2mh-mH\,(1-2mh)\Big]e^{-mH-2mh}+KN\Big[1-2mh+$$

$$+2m^2h^2+mH(1-2mh+4m^2h^2)\Big]\,e^{mH-2mh}\Big\} \qquad (28.9)$$

$$B_3m=-\frac{M(m)}{m\varDelta}\,(1-K)\,(1-N)\Big\{e^{mH}-K(1-2mH)\,e^{-mH}+KN\Big[1-$$

$$-2mh(1+2mH)\Big]\,e^{mH-2mh}-N(1-2mh-2mH)\,e^{-mH-2mh}\Big\} \qquad (28.10)$$

$$C_3m^2=0$$

$$D_3m^2=0$$

where,

$$\left.\begin{aligned}
&M(m)=-\frac{p_0a}{m}\,J_1(ma)\\
&K=\frac{1-k}{1+k}\,,\qquad k=\frac{E_2}{E_1}\\
&N=\frac{1-n}{1+n}\,,\qquad n=\frac{E_3}{E_2}\\
&\varDelta=-\Big\{e^{2mH}-2K(1+2m^2H^2)+K^2e^{-2mH}-2N(1+2m^2h^2)\,e^{-2mh}+\\
&\quad+N^2e^{-2mH-4mh}-\Big[K^2N(1+4m^2H^2)+4K^2N(1+4m^2H^2)\,m^2h^2+\\
&\quad+K^2N+8N(1-K^2)\,m^2Hh+4Nm^2H^2\Big]e^{-2mh}+2KN(1+2m^2h^2)\times\\
&\quad(e^{2mH}e^{-2mh}+e^{-2mH-2mh})-2KN^2(1+2m^2H^2)\,e^{-4mh}+K^2N^2e^{2mH-4mh}\Big\}.
\end{aligned}\right\} \qquad (29)$$

Substituting these twelve constants from Eq. (26) and (28) into Eq. (21), desired expressions for stresses and displacements in any point in layered systems are obtained. Take σ_{zz}^1 and u_z^1 as an instance, they are written as follows,

$$
\begin{aligned}
\sigma_{zz}^1 = & -\int_0^\infty \frac{M(m)\cdot m}{\Delta}\cdot\Big\{e^{m(H-z)}(1+mH+mz)-Ke^{-m(H+z)}\Big[1-mH+ \\
& +mz(1-2mH)\Big]-N\cdot\Big[(1+K^2)(1-mH)+2(1+K^2)m^2h^2- \\
& -2(1-K^2)(1-mH)mh-4K^2m^3Hh^2+mz\Big[K^2(1-2mH)(1+4m^2h^2+) \\
& +2K^2mh+1-2mH-2mh\Big]\Big]e^{-m(H+2h+z)}+2KN(1+2m^2h^2)(1+mH+ \\
& +mz)e^{m(H-2h-z)}-KN^2e^{-m(H+4h+z)}\Big[1-mH+mz(1-2mH)\Big]+ \\
& +K^2N^2e^{m(H-4h-z)}(1+mH+mz)-K\Big[1+mH-mz(1+2mh)\Big]e^{m(H+z)}+ \\
& +K^2(1-mH-mz)e^{-m(H-z)}-KN^2\Big[1+mH-mz(1+2mH)\Big]e^{m(H-4h+z)}+ \\
& +2KN(1+2m^2h^2)(1-mH-mz)e^{-m(H+2h-z)}+N^2e^{-m(H+4h-z)}(1-mH- \\
& -mz)+N\Big[1-(1+K^2)(1+2m^2h^2+mH)-2(1-K^2)(1+mH)mh- \\
& -4m^3Hh^2K^2+mz\Big[K^2(1+2mH)(1+4m^2h^2)-2K^2mh\Big]+1+2mH+ \\
& +2mh\Big]e^{m(H-2h+z)}\Big\}J_0(mr)\,dm
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
u_z^1 = & -\frac{1.5}{E_1}\int_0^\infty \frac{M(m)}{m\cdot\Delta}\Big\{-(1+mH+mz)e^{m(H-z)}+K\Big[1+mz(1-2mH)- \\
& -mH\Big]e^{-m(H+z)}+Ne^{-m(H+2h+z)}\Big[(1+K^2)(1+2m^2h^2-mH)-2(1-K^2)(1- \\
& -mH)mh-4K^2m^3Hh^2+mz\Big[(1-2mH)+K^2(1-2mH)-2(1-K^2)mh+ \\
& +4K^2(1-2mH)m^2h^2\Big]\Big]-2KNe^{m(H-2h-z)}(1+2m^2h^2)(1+mH+mz)+ \\
& +KN^2e^{-m(H+4h+z)}\Big[1-mH+mz(1-2mH)\Big]-K^2N^2e^{-m(H-4h-z)}(1+mH+ \\
& +mz)-Ke^{m(H+z)}\Big[1+mH-mz(1+2mH)\Big]+K^2e^{-m(H-z)}(1-mH-mz)+ \\
& +Ne^{m(H-h+z)}\Big[-(1+K^2)(1+2m^2h^2+mH)-2(1-K^2)(1+mH)mh- \\
& -4K^2m^3Hh^2+mz\Big[K^2(1+2mH)(1+4m^2h^2)+(1-K^2)2mh+)1+2mH)\Big]\Big]+ \\
& +N^2e^{-m(H+4h-z)}(1-mH-mz)+2KNe^{-m(H+2h-z)}(1+2m^2h^2)(1-mH-mz)- \\
& -KN^2e^{m(H-4h+z)}\Big[1+mH-mz(1+2mH)\Big]\Big\}mJ_0(mr)\,dm.
\end{aligned}
\tag{31}
$$

If $E_2=E_3$, that is, $N=0$ in Eq. (29), then expressions for the stress components σ_{zz}^1 and displacement u_z^1 shown in Eq. (30) and Eq. (31) are reduced to the expressions for two layered systems. They are

$$
\begin{aligned}
\sigma_{zz}^1 = & \int_0^\infty \frac{mM(m)}{\Delta}\Big\{K^2e^{m(z-h)}(1-mh-mz)+Ke^{m(z+h)}\Big[mz(1+2mh)- \\
& -(1+mh)\Big]-Ke^{-m(z+h)}\Big[mz(1-2mh)+(1-mh)\Big]+ \\
& +e^{-m(z-h)}(1+mh+mz)\Big\}J_0(mr)\,dm
\end{aligned}
$$

$$\left.\begin{aligned} u_z^1 &= \frac{-1.5}{E_1}\int_0^\infty \frac{M(m)}{\Delta}\Big\{(1+mh+mz)\,e^{m(h-z)}+K\Big[-1+mh-mz(1- \\ &\quad -2mh)\Big]e^{-m(h+z)}+K\Big[1+mh-mz(1+2mh)\Big]e^{m(h+z)}+K^2(mh+mz- \\ &\quad -1)\,e^{m(z-h)}\Big\}J_0(mr)\,dm \\ \text{where, } \Delta &= e^{2mh}-2K(1+2m^2h^2)+K^2e^{-mh} \\ M(m) &= -\frac{p_0a}{m}J_1(ma). \end{aligned}\right\} \quad (32)$$

Further, if we take $E_2/E_1=1$, $h=0$, that is, $K=0$, $\Delta=1$, we have the solutions for semi-infinite solid as follows,

$$\begin{aligned} \sigma_{zz}^1 &= -p_0\int_0^\infty aJ_1(ma)\,e^{-mz}(1+mz)\,dm \\ &= -p_0\left[a\frac{\sqrt{a^2+z^2}-z}{a\sqrt{a^2+z^2}}+az\frac{2a\Gamma(3/2)}{\sqrt{\pi}\,(a^2+z^2)^{\frac{3}{2}}}\right] \\ &= -p_0\left[1-\frac{z^3}{(a^2+z^2)^{\frac{3}{2}}}\right]. \end{aligned}$$

This result coincides with Boussineq's solution of the problem. In the case of the displacement u_z (Eq. 31), letting $K=0$, $h=0$, $r=0$, $z=-h$, then Eq. (31) is reduced to the expression of the displacement of the surface at its loading center. From Eq. (31) we have

$$\begin{aligned} u_z^1 &= \frac{1.5}{E}p_0a\int_0^\infty \frac{1}{m}J_1(ma)\,dm \\ &= \frac{1.5}{E}p_0a \end{aligned}$$

This result coincides with Boussinesq's solution of the semi-infinite solid problem.

4. Stresses and displacements in three layered systems under tangential load on the surface

We consider the stress distribution in three layered systems under the action of a tangential load applied to the suface. If the tangential load is uniformlly distributed over a circular area (Fig. 4), the boundary conditions on the surface become

$$\left.\begin{aligned} &\sigma_{rz}(r,\theta,0) = -q(r)\cos\theta && r<a \\ &\sigma_{\theta z}(r,\theta,0) = q(r)\sin\theta && r<a \\ &\sigma_{rz}(r,\theta,0) = \sigma_{\theta z}(r,\theta)=0 && r>a \\ &\sigma_{zz}(r,\theta,0)=0 && 0\leq r\leq\infty \end{aligned}\right\} \quad (33)$$

Transformed tangential load $q(r)$ is

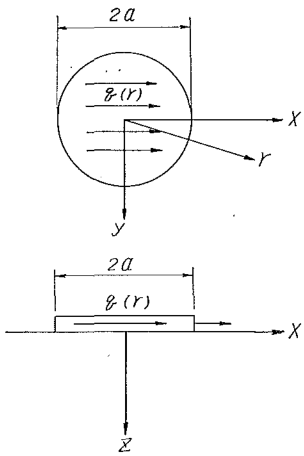

Fig. 4.

$$N(m)=\int_0^\infty rq(r)\,J_0(mr)\,dr$$

and in the case of $q(r)=q_0=\text{const.}$

$$N(m)=q_0\int_0^a rJ_0(mr)\,dr$$

$$=\frac{q_0a}{m}\,J_1(ma)\,.$$

Putting $n=1$ for the series expansion for the stresses and displacements, we can obtain the solutions for this problem.

Taking the coordinate system as in Fig. 3, the boundary conditions for this problem become as follows

$$\left.\begin{array}{llll}
\text{for } z=-H: & \sigma^1_{zz}=0 & & \\
& \sigma^1_{rz}=-q(r)\cos\theta & r<a & \\
& \sigma^1_{\theta z}=q(r)\sin\theta & r<a & \\
& \sigma^1_{rz}=\sigma^1_{\theta z}=0 & r>a & \\
\text{for } \quad z=0: & \sigma^1_{zz}=\sigma^2_{zz}\,, & \sigma^1_{rz}=\sigma^2_{rz}\,, & \sigma^1_{\theta z}=\sigma^2_{\theta z} \\
& u^1_z=u^2_z\,, & u^1_r=u^2_r\,, & u^1_\theta=u^2_\theta \\
\text{for } \quad z=h: & \sigma^2_{zz}=\sigma^3_{zz}\,, & \sigma^2_{rz}=\sigma^3_{rz}\,, & \sigma^2_{\theta z}=\sigma^3_{\theta z} \\
& u^2_z=u^3_z\,, & u^2_r=u^3_r\,, & u^2_\theta=u^3_\theta\,.
\end{array}\right\} \tag{34}$$

After solving these 15 simultameous limear equations for $\nu=0.5$, we can obtain the following coefficients considering the semi-infinite solid of the third layer,

$$A_1m^2=\frac{N(m)}{\varDelta\cdot m}\Big\{mHe^{\lambda_1}\big[2KN(1+2m^2h^2)+e^{2mh}+K^2N^2e^{-2mh}\big]+$$
$$+e^{\lambda_2}\big[(1-K^2)2Nm^2h^2\big]-mHe^{\lambda_2}\big[Ke^{2mh}+N(1-2mh(+KN(1+2mh+$$
$$+4m^2h^2)+KN^2e^{-2mh}\big]\Big\} \tag{35. 1}$$

$$B_1m=\frac{N(m)}{\varDelta\cdot m}\Big\{e^{\lambda_1}\big[2KN(1+2m^2h^2)+e^{2mh}+K^2N^2e^{-2mh}\big]-e^{\lambda_2}\big[K^2N(1-2mh+$$
$$+4m^2h^2)+KN^2e^{-2mh}+Ke^{2mh}-mHe^{\lambda_2}\big[2K^2N(1+4m^2h^2)+2KN^2e^{-2mh}+$$
$$+2Ke^{2mh}+2N\big]\Big\} \tag{35. 2}$$

$$C_1m^2=\Big\{\frac{N(m)}{\varDelta\cdot m}\Big\{e^{\lambda_1}\big[(1-K^2)\,2Nm^2h^2\big]+mHe^{\lambda_1}\big[K^2N(1-2mh+4m^2h^2)+$$
$$+Ke^{2mh}+KN^2e^{-2mh}+N(1+2mh)\big]-mHe^{\lambda_2}\big[K^2e^{2mh}+KN+$$
$$+N^2e^{-2mh}+KN(1+4m^2h^2\big]\Big\} \tag{35. 3}$$

$$D_1m=\frac{N(m)}{\varDelta\cdot m}\Big\{e^{\lambda_1}\big[N(1-2mh)+Ke^{2mh}+K^2N(1+2mh+4m^2h^2)\big]-$$

$$-mHe^{\lambda_1}\Big[+2KN(1+4m^2h^2)+2KN^2e^{-2mh}+2Ke^{2mh}+2N\Big]-e^{\lambda_2}\Big[KN+$$
$$+K^2e^{2mh}+KN(1+4m^2h^2)+N^2e^{-2mh}\Big]\Big\} \qquad (35.4)$$

$$A_2m^2=\frac{N(m)}{\varDelta m}(1-K)\Big\{-e^{\lambda_1}\cdot 2KNm^2h^2+mHe^{\lambda_1}\Big[KN(1-2mh+4m^2h^2+$$
$$+e^{2mh}\Big]+e^{\lambda_2}\cdot 2Nm^2h^2-mHe^{\lambda_2}\Big[N(1-2mh)+Ke^{2mh}\Big]\Big\} \qquad (35.5)$$

$$B_2m=\frac{N(m)}{\varDelta\cdot m}(1-K)\Big\{e^{\lambda_1}\Big[e^{2mh}+KN(1+2mh)\Big]-mHe^{\lambda_1}\cdot 4KNmh-$$
$$-e^{\lambda_2}\Big[Ke^{2mh}+N(1+2mh)\Big]-mHe^{\lambda_2}(2Ke^{2mh}+2N)\Big\} \qquad (35.6)$$

$$C_2m^2=\frac{N(m)}{\varDelta\cdot m}(1-K)\Big\{e^{\lambda_1}\Big[2Nm^2h^2\Big]+mHe^{\lambda_1}\Big[KN^2e^{-2mh}+N(1+2mh)\Big]-$$
$$-e^{\lambda_2}\cdot 2KNm^2h^2-mHe^{\lambda_2}\Big[KN(1+2mh+4m^2h^2)+N^2e^{-2mh}\Big]\Big\} \qquad (35.7)$$

$$D_2m=\frac{N(m)}{\varDelta\cdot m}(1-K)\Big\{e^{\lambda_1}\Big[KN^2e^{-2mh}+N(1-2mh)\Big]-e^{\lambda_1}mH(2KN^2e^{-2mh}+$$
$$+2N)-e^{\lambda_2}\Big[N^2e^{-2mh}+KN(1-2mh)\Big]+mHe^{\lambda_2}\cdot 4KNmh\Big\} \qquad (35.8)$$

$$A_3m^2=\frac{N(m)}{\varDelta m}(1-K)(1-N)\Big\{-e^{\lambda_1}\cdot 2KNm^2h^2e^{-mh}+mHe^{\lambda_2}\cdot 2Nm^2h^2e^{-mh}+$$
$$+mHe^{\lambda_1}\Big[e^{mh}+KN(1-2mh+4m^2h^2)\,e^{-mh}\Big]-$$
$$-mHe^{\lambda_2}\Big[Ke^{mh}+N(1-2mh)\,e^{-mh}\Big]\Big\} \qquad (35.9)$$

$$B_3m=\frac{N(m)}{\varDelta\cdot m}(1-K)(1+N)\Big\{e^{\lambda_2}\Big[e^{mh}+KN(1+2mh)\,e^{-mh}\Big]-$$
$$-mHe^{\lambda_1}\cdot 4KNmhe^{-mh}-e^{\lambda_2}\Big[Ke^{mh}+N(1+2mh)\,e^{-mh}\Big]-$$
$$-mHe^{\lambda_2}\Big[2(Ne^{-mh}+Ke^{mh})\Big]\Big\} \qquad (35.10)$$

$$C_3m^2=0 \qquad (35.11)$$

$$D_3m=0 \qquad (35.12)$$

$$E_1m=\frac{N(m)}{\delta\cdot m}(1+KNe^{-2mh}) \qquad (35.13)$$

$$F_1m=\frac{N(m)}{\delta\cdot m}(K+Ne^{-2mh}) \qquad (35.14)$$

$$E_2m=\frac{N(m)}{\delta m}(1-K) \qquad (35.15)$$

$$F_2m=\frac{N(m)}{\delta\cdot m}(1-K)\,Ne^{-2mh} \qquad (35.16)$$

$$E_3m=\frac{N(m)}{\delta m}(1-K)(1-N) \tag{35.17}$$

$$F_3m=0 \tag{35.18}$$

$$\left.\begin{aligned}
&\text{where,}\quad N(m)=\frac{q_0a}{m}J_1(ma)\\
&\qquad K=\frac{1-k}{1+k},\qquad k=\frac{E_2}{E_1}\\
&\qquad N=\frac{1-n}{1+n},\qquad n=\frac{E_3}{E_2}\\
&\qquad \lambda_1=mH+mh,\qquad \lambda_2=-mH-mh\\
&\Delta=-\Big\{e^{2mH}-2K(1+2m^2H^2)+K^2e^{-2mH}-2N(1+2m^2h^2)e^{-2mh}+N^2e^{-8m^2Hh}-\\
&\quad-\Big[(1+4m^2H^2)K^2N+4K^2N(1+4m^2H^2)m^2h^2+K^2N+8m^2Hh(1-\\
&\quad-K^2)N+4m^2H^2N\Big]e^{-2mh}+2KN(1+2m^2h^2)(e^{2m(H-h)}+e^{-2m(H+h)})-\\
&\quad-2KN^2(1+2m^2H^2)e^{-4mh}+K^2N^2e^{2m(H-2h)}\Big\}\\
&\delta=e^{mh}(1+KNe^{-2mh})-e^{-mh}(K+Ne^{-2mn})
\end{aligned}\right\} \tag{36}$$

Substituting Eq. (35) into Eq. (20), we obtain the solution for this problem. Take $\sigma^1_{\theta z}$ and σ^1_{rz} as an instance, they are written as follows,

$$\left.\begin{aligned}
&\left.\begin{matrix}\sigma^1_{\theta z}\\ \sigma^1_{rz}\end{matrix}\right\}=\frac{1}{2}\Bigg\{\int_0^\infty\frac{mN(m)}{\Delta}\cdot\Big[-e^{m(H-2h-z)}(1-mH-mz)\Big[2KN(1+2m^2h^2+e^{2mh}+\\
&\quad+K_1^2N^2e^{-2mn}\Big]+e^{-m(H+2h+z)}\Big[2N(1-K^2)m^2h^2+(1-mz)\Big[K^2N(1-\\
&\quad-2mh+4m^2h^2)+K_1^2N^2e^{-2mh}+Ke^{2mh}+N(1+2mh)\Big]\Big]-\\
&\quad-mHe^{-m(H+2h+z)}\Big[Ke^{2mh}+N(1-2mh(+K^2N(1+2mh+4m^2h^2)+\\
&\quad+KN^2e^{-2mh}-(1-mz)\Big[2K^2N(1+4m^2h^2)+2KN^2e^{-2mh}+2Ke^{2mh}+\\
&\quad+2N\Big]\Big]+e^{m(H-2h+z)}\Big[2N(1-K^2)m^2h^2+(1+mz)\Big[N(1-2mh)+Ke^{2mh}+\\
&\quad+K^2N(1+2mh+4m^2h^2)+KN^2e^{-2mh}\Big]\Big]+mHe^{m(H-2h+z)}\Big[K^2N(1-\\
&\quad-2mh+4m^2h^2)+Ke^{2mh}+KN^2e^{-2mk}+N(1+2mh)-(1+mz)\times\\
&\quad\times\Big[2K^2N(1+4m^2h^2)+2KN^2e^{-2mh}+2Ke^{2mh}+2N\Big]\Big]-e^{-m(H+2h-z)}(1+\\
&\quad+mH+mz)\Big[K^2e^{2mh}+N^2e^{-2mh}+2KN(1+2m^2h^2)\Big]\Big]\Big[J_2(mr)\pm\\
&\quad J_0(mr)\Big]dm+\int_0^\infty\frac{mN(m)}{\delta}\Big[e^{-mz}+KNe^{-2hm-mz}-Ke^{mz}-Ne^{-2mh+mz}\Big]\times\\
&\quad\Big[J_2(mr)\mp J_0(mr)\Big]dm\Bigg\}\begin{matrix}\sin\theta\\ \cos\theta\end{matrix}
\end{aligned}\right\} \tag{37}$$

Putting $N=0$, the results are reduced to the solutions of the two layered system. They are

$$\left.\begin{aligned}\left.\begin{matrix}\sigma^1_{\theta z}\\ \sigma^1_{rz}\end{matrix}\right\} = \frac{1}{2}\Big\{-\int_0^\infty \frac{mN(m)}{\Delta}\Big[e^{-m(z-h)}(mh-1+mh)+Ke^{-m(z+h)}\Big[1+mh-\\ -mz(1+2mh)\Big]+Ke^{m(z+h)}\Big[1-mh+mz(1-2mh)\Big]-K^2e^{m(z-h)}(mh+\\ +1+mz)\Big]\Big[J_2(mr)\pm J_0(mr)\Big]dm+\int_0^\infty \frac{mN(m)}{\delta}(e^{-mz}+Ke^{mz})\times\\ \times\Big[J_2(mr)\mp J_0(mr)\Big]dm\Big\}\begin{matrix}\sin\theta\\ \cos\theta\,.\end{matrix}\end{aligned}\right\} \quad (38)$$

Further, if we put $E_1/E_2=1$, ie, $K=0$, these expressions coincide with the solutions for a semi-infinite solid with a tangential surface load obtained by Muki.

5. Numerical Examples

Numerical calculations were carried out for various cases by a electronic computer. A calculation of a definite integration was done by the help of Simpson's formula. Many interesting results and important characteristics of the multi-layered systems were obtained. Here, only two examples of them are shown in Fig. 5 and Fig. 6. Fig. 5 shows the pressure bulbs and Fig. 6 shows the principal stresses in the two layered system subjected to a normal load and a tangential load simultaneously on its surface.

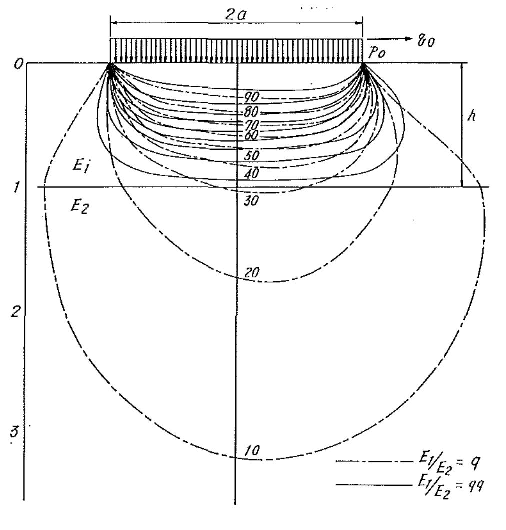

Fig. 5. (a)

Pressure bulb of σ_{zz}, (in the case of $a/h=1.0$) (Numerical values on each curve show the percentage to (p_0+q_0)) $(q_0=\frac{p_0}{2})$

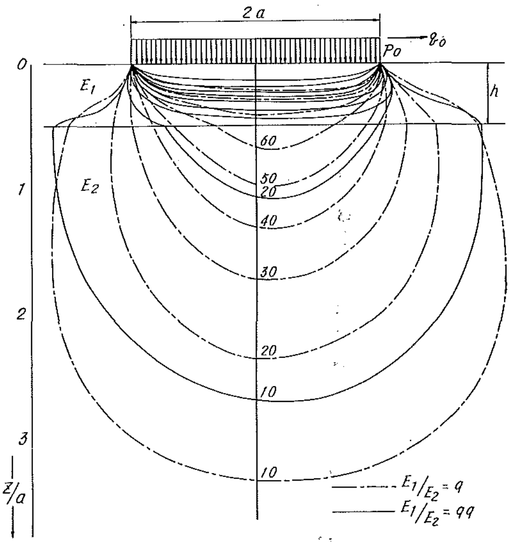

Fig. 5. (b)

Pressure bulb of σ_{zz} (in the case of $a/h=1.5$) $(q_0=\frac{p_0}{2})$

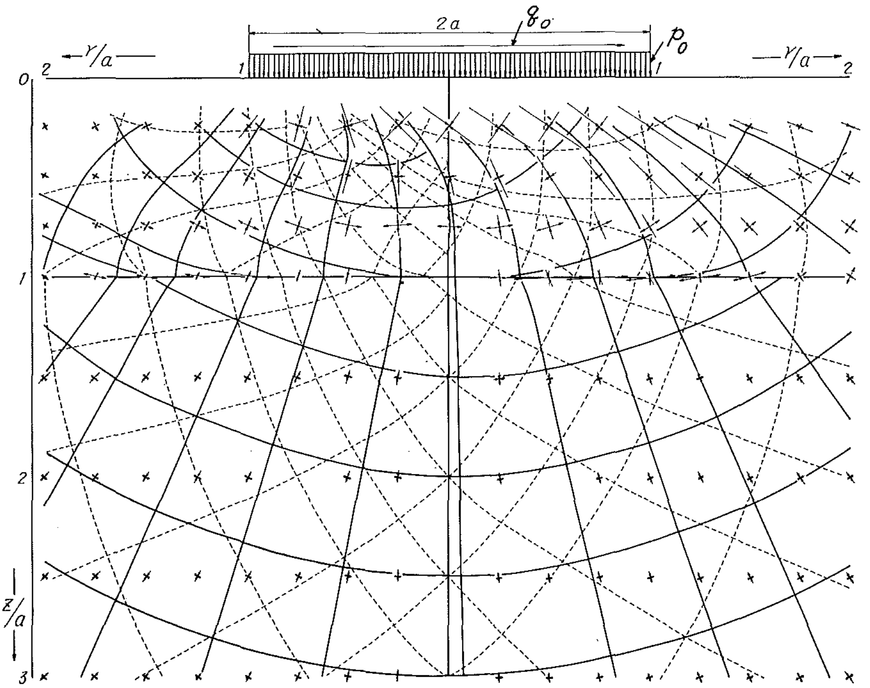

Fig. 6. (a) Principal stress line (solid curve) and Principal shearing stress line (dotted line). $(E_1/E_2=9)$ $(q_0 = \frac{p_0}{2})$

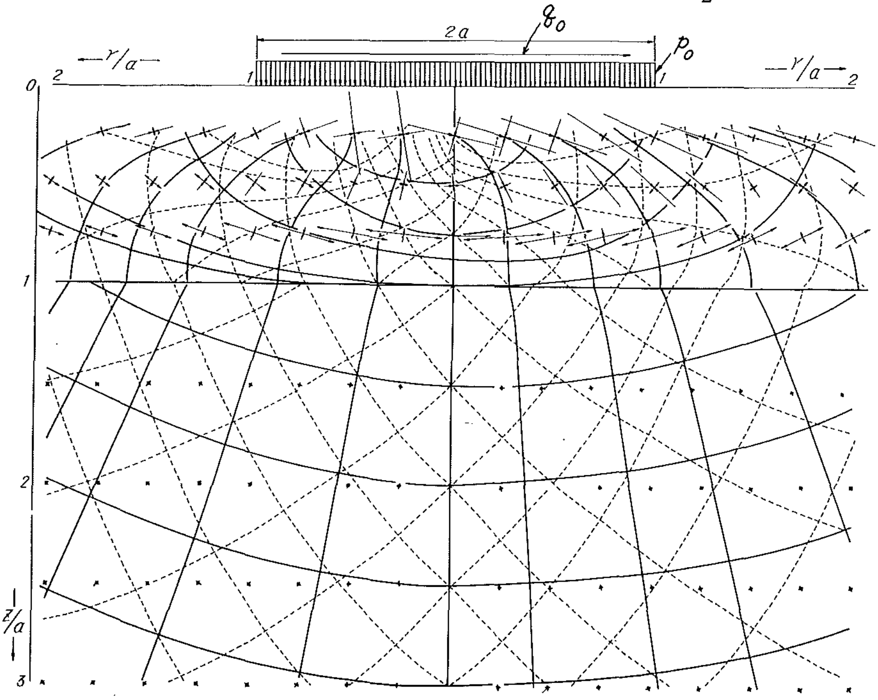

Fig. 6. (b) Principal stress line (solid curve) and Principal shearing stress line (dotted curve). $(E_1/E_2=99)$ $(q_0 = \frac{p_0}{2})$

References

1) Burmister, D. M.: "The theory of stresses and displacements in layered system and applications to the design of airport runways", Proc., 23rd-Annual Meating of H.R.B., Vol. 23 (1943).
"The general tohery of stresses and displacements in layered systems", J. App. phys. Vol. 16 (1945).

2) Fox, L.: "Computation of traffic stresses in a simple road structure", Road Research Tech. Paper No. 9 (1948).

3) Jelinek R., and Ranke, A.: "Berechnung der Spannungs verteilung in eimem Zweischichten system", Bautech. Heft. 2 (1970).

4) Acum, W. E. and Fox, L.: "Computation of Load Stresses in a Thee-Layer Elastic System," Geotechnique (1951),No. 4, pp. 293–300.

5) Sneddon, I. N.: Fourier Transforms (1951). McGraw-Hill.

6) Muki, R.: "Three dimensional Problem of elasticity theory for a semi-infinite solid and a thick plate", Progress in Solid Mechanics, pp. 400–439 (1956), North-Holl and Pub. Co.

7) Schiffmann, R. L.: "General analysis of stresses and displacements in layered systems", Int. Nat. Conf. on the structural design of Asphalt Pavements Proc. (1962).