Integral Transform and Generalized Sampling Theorem

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Abstract

A generalization of the sampling theorem is presented under the consideration of the reciprocity relation of integral transforms.

New sampling formulae, being based on the generalized sampling theorem presented here, are also given, one of which includes Shannon's sampling theorem as a special case.

Zusammenfassung

Unter Berücksichtigung der Reziprozitätsrelation der Integraltransformierten ist eine Verallgemeinerung des Abtast-theorems (sampling-theorems) angegeben.

Auf Grund des hieran angegebenen verallgemeinerten Abtast-theorems sind neuen Abtast-formeln auch präsentiert, eine von denen das Shannon'sche Abtast-theorem als einen besonderen Fall enthält.

§ 1. Preliminaries

The generalization of the sampling theorem and the reconstruction of a band-limited function from its sampled values and derivatives were made by Kohlenberg[1), Fogel[2), Jagerman and Fogel[3), Bond and Cahn[4), and Linden and Abramson[5). The sampling theorem was also generalized by Balakrishnan[6) to the case of a continuous-parameter stochastic process. On the other hand, it was pointed out that the sampling intervals need not be uniformly distributed[7).

In the previous paper[8), the authors presented a generalized sampling theorem along the line of consideration of the reciprocity relation of integral transforms and gave some of its examples, which include Someya-Shannon's sampling theorem[9)~11) as a special case. There the sampling intervals were not uniformly distributed. Recently the authors were informed that the sampling theorem was also generalized by Isomiti[12) in connection with the generalized frequency domain. Some of his results agreed with those given by the authors[8). Here in the present paper, the authors wish to discuss the series-expansion of functions in an orthogonal set of functions, and also to give some of the new sampling formulae.

§ 2. Generalization of Sampling Theorem

Let $f(t)$ belong to L_p, and let the following reciprocity relations hold:

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$$F(s) = \mathscr{L}_s \cdot f \equiv \int_A K(s, t) f(t)\, dt\,, \qquad \text{for} \quad s \in B \tag{1}$$

and

$$f(t) = \mathscr{L}_t^{-1} \cdot F \equiv \int_B \bar{K}(t, s)\, F(s)\, ds\,. \qquad \text{for} \quad t \in A \tag{2}$$

From (1) and (2), we obtain at once:

$$\int_A K(s, t)\, \bar{K}(t, \sigma)\, dt = \delta(s - \sigma)\,, \tag{3}$$

and

$$\int_B \bar{K}(t, s)\, K(s, \tau)\, ds = \delta(t - \tau)\,, \tag{4}$$

with delta function $\delta(t)$.
Further we assume that

$$F(s) = 0\,. \qquad \text{for} \quad s \notin D \subseteq B \tag{5}$$

Accordingly, from (2) and (5) we obtain the following expression:

$$f(t) = \mathscr{L}_t^{-1} \cdot F = \int_D \bar{K}(t, s)\, F(s)\, ds\,. \tag{6}$$

Let $F(s)$ be expanded in a complete orthogonal set of functions:

$$\left\{ \phi_n(s)\,;\ \int_D \phi_m(s)\, \phi_n(s)\, ds = \gamma_m \cdot \delta_{m,n} \quad (m, n = \text{integers}) \right\}, \tag{7}$$

in the domain $D \subseteq B$, i. e.

$$F(s) = \sum_n a_n \phi_n(s)\,. \qquad \text{for} \quad s \in D \subseteq B \tag{8}$$

The expression (8), being multiplied by $\phi_m(s)$ and integrated over D, gives:

$$\int_D F(s) \phi_m(s)\, ds = \sum_n a_n \cdot \int_D \phi_m(s) \phi_n(s)\, ds = \sum_n a_n \cdot \gamma_m \cdot \delta_{m,n} = \gamma_m \cdot a_m\,. \tag{9}$$

From (6), (7), (8), and (9), we obtain:

$$\begin{aligned} f(t) &= \mathscr{L}_t^{-1} \cdot F = \sum_n a_n \cdot \mathscr{L}_t^{-1} \cdot \phi_n \\ &= \sum_n \left(\frac{1}{\gamma_n} \int_D F(s)\, \phi_n(s)\, ds \right) \cdot \int_D \bar{K}(t, s)\, \phi_n(s)\, ds\,. \end{aligned} \tag{10}$$

If the functions $f(t)$, $\{\phi_n(s)\}$, and the integral kernels $K(s, t)$ and $\bar{K}(t, s)$, are given, then we can construct a series-expansion of $f(t)$ by means of (10).
If we can take:

$$\phi_n(s) = \bar{K}(\lambda_n, s)\,, \qquad \text{for} \quad s \in D \tag{11}$$

with constants λ_n, i. e. if the kernel $\bar{K}(\lambda_n, s)$ can be put equal to $\phi_n(s)$, then the expression (10) is simplified into:

$$f(t) = \sum_n \left(\frac{1}{\gamma_n} \int_D F(s)\, \bar{K}(\lambda_n, s)\, ds \right) \cdot \int_D \bar{K}(t, s) \bar{K}(\lambda_n, s)\, ds$$

$$= \sum_n \frac{1}{\gamma_n} f(\lambda_n) \cdot \int_D \widetilde{K}(t, s)\,\widetilde{K}(\lambda_n, s)\,ds\,, \tag{12}$$

by means of (6). The expression (12) gives a *generalized sampling theorem*[8]. The points at the variable t:

$$t = \lambda_n\,, \qquad (n = \text{integers}) \tag{13}$$

are called *sampling points*, and the function $g_n(t)$, defined by:

$$g_n(t) \equiv \mathscr{L}_t^{-1} \cdot \phi_n = \int_D \widetilde{K}(t, s)\,\phi_n(s)\,ds = \int_D \widetilde{K}(t, s)\,\widetilde{K}(\lambda_n, s)\,ds\,, \ (n = \text{integers}) \tag{14}$$

is the *sampling function.*

If the set of functions $\{\widetilde{K}(\lambda_n, s)\,;\ n=\text{integers}\}$ is not necessarily orthogonal, but complete and linearly independent, we can find normalized biorthogonal set $\{\psi_m(s)\,;\ m=\text{integers}\}$, in such a way that

$$\int_D \widetilde{K}(\lambda_m, s)\,\psi_n(s)\,ds = \delta_{m,n}\,, \tag{15}$$

is to be satisfied. From the completeness of the set $\{\widetilde{K}(\lambda_m, s)\,;\ m=\text{integers}\}$, we have:

$$\sum_n \widetilde{K}(\lambda_n, s)\,\psi_n(\sigma) = \delta(s-\sigma)\,. \tag{16}$$

Isomiti[12] expanded $F(s)$ in $\{\psi_n(s)\,;\ n=\text{integers}\}$, and obtained:

$$F(s) = \sum_n b_n \psi_n(s)\,, \qquad \text{for} \quad s \in D \subseteq B \tag{17}$$

with

$$b_n = \int_D \widetilde{K}(\lambda_n, s)\,F(s)\,ds\,. \tag{18}$$

From (6) and (18), we obtain at once:

$$b_n = \int_D \widetilde{K}(\lambda_n, s)\,F(s)\,ds = f(\lambda_n)\,. \tag{19}$$

Operating $\mathscr{L}_t^{-1}$ on (17), we obtain, by means of (6):

$$\begin{aligned} f(t) &= \mathscr{L}_t^{-1} \cdot F = \sum_n b_n \cdot \int_D \widetilde{K}(t, s)\,\psi_n(s)\,ds \\ &= \sum_n f(\lambda_n) \cdot \int_D \widetilde{K}(t, s)\,\psi_n(s)\,ds\,. \end{aligned} \tag{20}$$

The expression (18) or (19) becomes to:

$$\begin{aligned} f(\lambda_n) &= \int_D ds \cdot \widetilde{K}(\lambda_n, s) \cdot \int_A dt \cdot K(s, t) f(t) \\ &= \int_A dt \cdot f(t) \cdot \int_D ds \cdot \widetilde{K}(\lambda_n, s)\,K(s, t)\,, \end{aligned} \tag{21}$$

by means of (1). The expression (20) is the generalized sampling theorem obtained by Isomiti[12], and the expression

$$g_n(t)=\int_D \widetilde{K}(t,s)\,\psi_n(s)\,ds\,, \qquad (n=\text{integers}) \tag{22}$$

is the sampling function, with sampling points $t=\lambda_n$ (n=integers) and with

$$g_n(\lambda_m)=\int_D \widetilde{K}(\lambda_m,s)\,\psi_n(s)\,ds=\delta_{m,n}\,.$$

The idea of the generalized sampling theorem was also given by one of the present authors, Takizawa, in his book[13] of information theory.

If we take $\{\widetilde{K}(\lambda_n,s)\}$ to be orthogonal, then we have at once

$$\left\{\psi_n(s)\right\}=\left\{\phi_n(s)\right\}=\left\{\widetilde{K}(\lambda_n,s)\right\}, \tag{23}$$

as was given in (11), the expression (20) reduces to the sampling theorem (12), the dual vector $\{\psi_n(s)\}$ for $\{\widetilde{K}(\lambda_n,s)\}$ reducing to $\{\widetilde{K}(\lambda_n,s)\}$ itself.

The corresponding expressions of $F(t)$ for (20) and (21) are as follows:

$$F(t)=\sum_n F(\mu_n)\cdot\int_E K(t,s)\,\psi_n(s)\,ds\,, \tag{24}$$

and

$$F(\mu_n)=\int_B dt\cdot F(t)\cdot\int_E ds\cdot K(\mu_n,s)\,\widetilde{K}(s,t)\,, \tag{25}$$

with the sampling function:

$$p_n(t)=\int_E K(t,s)\,\tilde{\varphi}_n(s)\,ds\,, \tag{26}$$

and the sampling points $t=\mu_n$ (n=integers), under the condition that for the complete set $\{K(\mu_n,t)\,;\ n=\text{integers}\}$ we have a set of biorthogonal functions $\{\tilde{\varphi}_n(s)\,;\ n=\text{integers}\}$ in the domain $E\subseteq A$, i.e.

$$\int_E K(\mu_m,t)\,\tilde{\varphi}_n(t)\,dt=\delta_{m,n}\,. \qquad \text{for} \quad t\in E\subseteq A \tag{27}$$

From the completeness of the set $\{K(\mu_m,s)\,;\ m=\text{integers}\}$, it follows that the expression:

$$\sum_n K(\mu_n,s)\,\tilde{\varphi}_n(\sigma)=\delta(s-\sigma)\,, \tag{28}$$

holds.

§ 3. Properties of Sampling Functions and Related Integrals

Now, let us take sampling functions $\{g_n(t)\,;\ n=\text{integers}\}$, $\{h_n(t)\,;\ n=\text{integers}\}$, $\{p_n(t)\,;\ n=\text{integers}\}$, and $\{q_n(t)\,;\ n=\text{integers}\}$, defined as follows:

$$g_n(t)\equiv\int_D \widetilde{K}(t,s)\,\psi_n(s)\,ds\,, \tag{29}$$

$$h_n(t)\equiv\int_D K(s,t)\,\phi_n(s)\,ds=\int_D K(s,t)\,\widetilde{K}(\lambda_n,s)\,ds\,, \tag{30}$$

$$p_n(t) \equiv \int_E K(t, s)\, \tilde{\varphi}_n(s)\, ds\,, \tag{31}$$

and

$$q_n(t) \equiv \int_E \widetilde{K}(s, t)\, K(\mu_n, s)\, ds\,. \tag{32}$$

From (3) and (4), we have:

$$\int_A K(\sigma, t)\, g_n(t)\, dt = \int_D \phi_n(s)\cdot\delta(\sigma - s)\, ds = \begin{cases} \phi_n(\sigma)\,, & \text{for} \quad \sigma \in D \\ 0\,, & \text{for} \quad \sigma \notin D \end{cases} \tag{33}$$

$$\int_A \widetilde{K}(t, \sigma)\, h_n(t)\, dt = \int_D \phi_n(s)\cdot\delta(\sigma - s)\, ds = \begin{cases} \widetilde{K}(\lambda_n, \sigma)\,, & \text{for} \quad \sigma \in D \\ 0\,, & \text{for} \quad \sigma \notin D \end{cases} \tag{34}$$

$$\int_B \widetilde{K}(\sigma, t)\, p_n(t)\, dt = \int_E \tilde{\varphi}_n(s)\cdot\delta(\sigma - s)\, ds = \begin{cases} \tilde{\varphi}_n(\sigma)\,, & \text{for} \quad \sigma \in E \\ 0\,, & \text{for} \quad \sigma \notin E \end{cases} \tag{35}$$

and

$$\int_B K(t, \sigma)\, q_n(t)\, dt = \int_E K(\mu_n, s)\cdot\delta(\sigma - s)\, ds = \begin{cases} K(\mu_n, \sigma)\,, & \text{for} \quad \sigma \in E \\ 0\,, & \text{for} \quad \sigma \notin E \end{cases} \tag{36}$$

By means of (3) and (15), we obtain:

$$\int_A g_m(t)\, h_n(t)\, dt = \int_D ds \cdot \int_D d\sigma \cdot \phi_m(s)\, \phi_n(\sigma)\cdot\delta(\sigma - s)$$

$$= \int_D \phi_m(s)\, \phi_n(s)\, ds = \int_D \widetilde{K}(\lambda_n, s)\, \phi_m(s)\, ds = \delta_{m,n}\,, \tag{37}$$

for any integers m and n. This expression (37) shows that any function of $\{g_n(t)\,;\ n = \text{integers}\}$ and any function of $\{h_n(t)\,;\ n = \text{integers}\}$ are *mutually orthogonal* in the domain D. Similarly, we obtain, from (4) and (27):

$$\int_B p_m(t)\, q_n(t)\, dt = \int_E \tilde{\varphi}_m(s)\, K(\mu_n, s)\, ds = \delta_{m,n}\,. \tag{38}$$

We shall return to the expressions (20), (21), (24), and (25). They are written as follows:

$$f(t) = \sum_n f(\lambda_n)\, g_n(t)\,, \tag{39}$$

$$f(\lambda_n) = \int_A f(t)\, h_n(t)\, dt\,, \tag{40}$$

$$F(t) = \sum_n F(\mu_n)\, p_n(t)\,, \tag{41}$$

and

$$F(\mu_n) = \int_B F(t)\, q_n(t)\, dt\,, \tag{42}$$

in terms of (29)～(32).

Let $\xi(t)$ be any function which can be expanded in $\{g_n(t)\,;\ n = \text{integers}\}$, i.e.

$$\xi(t) = \sum_n c_n g_n(t)\,, \tag{43}$$

with constants c_n, then the integral operator:

$$\mathscr{T} \equiv \int_A d\tau \cdot \sum_n g_n(t)\, h_n(\tau) = \int_A d\tau \cdot \delta(t-\tau)\,, \tag{44}$$

applied to (43), is the *identical operator*, in the sense that

$$\mathscr{T} \cdot \xi(\tau) = \xi(t)\,. \tag{45}$$

The proof of (45) is quite obvious, if we put (43) into (45) and refer (37).

Considering (4) and (44), we shall take

$$F^*(s) = \int_A \widetilde{K}(t, s) f(t)\, dt\,, \tag{46}$$

with

$$f(t) = \int_B K(s, t)\, F^*(s)\, ds\,, \tag{47}$$

and

$$f^*(\lambda_n) = \int_A g_n(t) f(t)\, dt\,, \tag{48}$$

with

$$f(t) = \sum_n f^*(\lambda_n)\, h_n(t)\,. \tag{49}$$

Then we obtain, from (2), (3), (47), (39), (49), and (37), the following expressions:

$$\begin{aligned} \|f\|^2 &= \int_A dt \cdot \int_B ds \cdot K(s, t)\, F^*(s) \cdot \int_B d\sigma \cdot \widetilde{K}(t, \sigma)\, F(\sigma) \\ &= \int_B ds \cdot F^*(s) \cdot \int_B d\sigma \cdot F(\sigma) \cdot \delta(s-\sigma) = \int_B F^*(s) F(s)\, ds\,, \end{aligned} \tag{50}$$

and

$$\begin{aligned} \|f\|^2 &= \int_A dt \cdot \sum_m \sum_n f^*(\lambda_m)\, h_m(t) \cdot f(\lambda_n)\, g_n(t) \\ &= \sum_m \sum_n f^*(\lambda_m) f(\lambda_n) \cdot \delta_{m,n} = \sum_n f^*(\lambda_n) f(\lambda_n)\,, \end{aligned} \tag{51}$$

with the norm $\|f\|$ of function $f(t)$:

$$\|f\|^2 \equiv \int_A [f(t)]^2\, dt\,. \tag{52}$$

The expressions (37), (45), and the generalized Parseval equalities (50) and (51), were given by Isomiti[12].

Now we shall differentiate (39) with regard to t, then we have

$$f'(t) = \sum_n f(\lambda_n)\, g_n'(t)\,. \tag{53}$$

On the other hand, we apply the generalized sampling theorem (39) to $f'(t)$, then we obtain:

$$f'(t) = \sum_n f'(\lambda_n)\, g_n(t)\,, \tag{54}$$

with

$$f'(\lambda_n)=\left[\frac{df(t)}{dt}\right]_{t=\lambda_n}.$$

From (53) and (54), we have

$$f'(t)=\sum_n f'(\lambda_n)\,g_n(t)=\sum_n f(\lambda_n)\,g_n'(t)\,. \tag{55}$$

Accordingly, from the m-th derivative of $f(t)$, we obtain:

$$f^{(m)}(t)=\sum_n f^{(m-r)}(\lambda_n)\,g_n^{(r)}(t)=\sum_n f^{(m-k)}(\lambda_n)\,g_n^{(k)}(t)\,, \tag{56}$$

for any integers r and k $(0\leqslant r\leqslant m,\ 0\leqslant k\leqslant m)$, and with

$$f^{(m)}(z)=\frac{d^m}{dz^m}f(z)\,, \qquad (m=\text{integers})$$

and

$$g_n^{(k)}(z)=\frac{d^k}{dz^k}\,g_n(z)\,. \qquad (k=\text{integers})$$

Here, we shall mention an expansion-formula, which contains both Taylor's and sampling formulae. If a function $f(t)$ is expressed in (39), i.e.

$$f(t)=\sum_n f(\lambda_n)\,g_n(t)\,, \tag{57}$$

and if we can expand $f(t+\xi)$ in a Taylor series, as

$$f(t+\xi)=\sum_{m=0}^{+\infty}\frac{1}{m\,!}\,\xi^m\cdot f^{(m)}(t)\,, \tag{58}$$

then we have an expansion-formula:

$$f(t+\xi)=\sum_{m=0}^{+\infty}\sum_n\frac{1}{m\,!}\,\xi^m\cdot f^{(m-r)}(\lambda_n)\,g_n^{(r)}(t) \tag{59}$$

$$=\sum_{m=0}^{+\infty}\sum_n\frac{1}{m\,!}\,\xi^m\cdot f^{(m-k)}(\lambda_n)\,g_n^{(k)}(t)\,, \tag{60}$$

for any integers r and k $(0\leqslant r\leqslant m,\ 0\leqslant k\leqslant m)$.

For the special case of $\xi=0$, the expressions (59) and (60) reduce to (57). While, for the special case of $t=\lambda_s$ (with s fixed) in (59) and (60), we have

$$f(\lambda_s+\xi)=\sum_{m=0}^{+\infty}\sum_n\frac{1}{m\,!}\,\xi^m\cdot f^{(m-r)}(\lambda_n)\,g_n^{(r)}(\lambda_s)$$

$$=\sum_m\sum_n\frac{1}{m\,!}\,\xi^m\cdot f^{(m)}(\lambda_n)\,g_n(\lambda_s)=\sum_m\sum_n\frac{1}{m\,!}\,\xi^m\cdot f^{(m)}(\lambda_n)\cdot\delta_{n,s}$$

$$=\sum_{m=0}^{+\infty}\frac{1}{m\,!}\,\xi^m\cdot f^{(m)}(\lambda_s)\,, \tag{61}$$

and

$$f(t)=\sum_n\sum_{m=0}^{+\infty}\frac{1}{m\,!}\,(t-\lambda_s)^m\cdot f^{(m-r)}(\lambda_n)\,g_n^{(r)}(\lambda_s)\,, \tag{62}$$

for any integers r $(0\leqslant r\leqslant m)$.

The expression (61) is nothing but the usual Taylor expansion of $f(\lambda_s+\xi)$, while the expression (62) gives $f(t)$ in terms of the derivatives of sampling functions $g_n(\lambda_s)$ at a fixed point λ_s.

§ 4. Examples of the Sampling Formulae derived from the Generalized Sampling Theorem

Example 1

We shall take the Fourier transforms in (1) and (2):

$$K(s,t)=\frac{1}{2\pi}\exp[ist], \qquad A=(-\infty,+\infty),$$
$$\widetilde{K}(t,s)=\exp[-its], \qquad B=(-\infty,+\infty),$$
$$\phi_n(s)=\widetilde{K}(\lambda_n,s)=\exp[-i\lambda_n s], \qquad D=(-\beta,\beta),$$
$$\lambda_n=\frac{n-k}{\beta}\pi, \qquad n=\text{integers}, \qquad k=\text{real number given arbitrarily},$$
$$\int_{-\beta}^{\beta}\exp[-i\lambda_m s]\cdot\exp[+i\lambda_n s]\,ds=\gamma_m\cdot\delta_{m,n},$$
$$\gamma_n=2\beta,$$

then the sampling theorem (12) reads as follows:

$$f(t)=\frac{1}{2\beta}\sum_{n=-\infty}^{+\infty}f\left(-\frac{n-k}{\beta}\pi\right)\cdot\frac{2\sin(\beta t+(n-k)\pi)}{t+\dfrac{n-k}{\beta}\pi}$$
$$=\sum_{n=-\infty}^{+\infty}f\left(\frac{n+k}{\beta}\pi\right)\cdot\frac{\sin(\beta t-(n+k)\pi)}{\beta t-(n+k)\pi}. \tag{63}$$

The expression is nothing but Someya's sampling theorem[9] given in 1949. The expression (63) gives $f(t)$ in terms of the sampling function $\sin(\beta t-(n+k)\pi)/(\beta t-(n+k)\pi)$, with values of $f((n\pi+k\pi)/\beta)$ at sampling points $t=(n+k)\pi/\beta$ (n = integers). If we put $k=0$ in (63), we have Shannon's sampling theorem[10),11)].

Example 2

We shall take the Fourier cosine-transforms for (1) and (2):

$$K(s,t)=\frac{2}{\pi}\cos(st), \qquad A=(0,+\infty),$$
$$\widetilde{K}(t,s)=\cos(ts), \qquad B=(0,+\infty),$$
$$\phi_n(s)=\widetilde{K}(\lambda_n,s)=\cos(\lambda_n s), \qquad D=(0,\beta),$$
$$\gamma_n=\frac{\beta}{2}\left\{1+\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}\right\},$$

where λ_n's are the roots (arranged in ascending order of magnitude) of the equation:

$$\lambda_n\tan(\lambda_n\beta)=M, \tag{64}$$

with a constant M. Then the expression (12) for an even function $f(t)$, takes form:

$$f(t)=\frac{2}{\beta}\sum_{n}\frac{1}{1+\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\left(\int_0^\beta F(s)\cos(\lambda_n s)\,ds\right)\cdot\int_0^\beta \cos(ts)\cos(\lambda_n s)\,ds$$

$$=\frac{2}{\beta}\sum_{n} f(\lambda_n)\cdot\frac{\cos(\lambda_n\beta)}{1+\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\tan(\beta t)-M}{t^2-\lambda_n^2}\cos(\beta t), \tag{65}$$

with sampling points $t=\lambda_n$ (n=integers), and the sampling function:

$$g_n(t)=\frac{2}{\beta}\cdot\frac{\cos(\lambda_n\beta)}{1+\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\tan(\beta t)-M}{t^2-\lambda_n^2}\cos(\beta t). \tag{66}$$

Example 3

We shall take the Fourier sine-transforms for (1) and (2):

$$K(s,t)=\frac{2}{\pi}\sin(st), \qquad A=(0,+\infty),$$

$$\widetilde{K}(t,s)=\sin(ts), \qquad B=(0,+\infty),$$

$$\phi_n(s)=\widetilde{K}(\lambda_n,s)=\sin(\lambda_n s), \qquad D=(0,\beta),$$

$$\gamma_n=\frac{\beta}{2}\left\{1-\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}\right\},$$

where λ_n's are the roots (arranged in ascending order of magnitude) of the equation:

$$\lambda_n\cot(\lambda_n\beta)=N, \tag{67}$$

with a constant N. Then the expression (12) for an odd function $f(t)$, becomes to:

$$f(t)=\frac{2}{\beta}\sum_{n}\frac{1}{1-\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\left(\int_0^\beta F(s)\sin(\lambda_n s)\,ds\right)\cdot\int_0^\beta \sin(ts)\sin(\lambda_n s)\,ds$$

$$=-\frac{2}{\beta}\sum_{n} f(\lambda_n)\cdot\frac{\sin(\lambda_n\beta)}{1-\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\cot(\beta t)-N}{t^2-\lambda_n^2}\sin(\beta t), \tag{68}$$

with sampling points $t=\lambda_n$ (n=integers), and the sampling function:

$$g_n(t)=-\frac{2}{\beta}\cdot\frac{\sin(\lambda_n\beta)}{1-\dfrac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\cot(\beta t)-N}{t^2-\lambda_n^2}\sin(\beta t). \tag{69}$$

The formulae (65) and (68) were obtained by Kroll[14] in connection with the solution of an integral equation.

Example 4

Let us take the Hankel transforms of order $\nu\geqslant -1/2$ for (1) and (2), and take the Fourier-Bessel series[15] for $F(s)$:

$$\sqrt{t}\,f(t)\in L_1(0,+\infty),$$

$$K(s,t)=tJ_\nu(st), \qquad A=(0,+\infty),$$

$$\widetilde{K}(t, s) = sJ_\nu(ts)\,, \qquad B = (0, +\infty)\,,$$
$$\phi_n(s) = J_\nu(j_n s)\,, \qquad D = (0, \beta)\,,$$

with the orthogonality relation:

$$\int_0^\beta sJ_\nu(j_m s)\, J_\nu(j_n s)\, ds = \gamma_m \cdot \delta_{m,n}\,,$$

and

$$\gamma_n = \frac{\beta^2}{2} J^2_{\nu+1}(j_n\beta)\,,$$

where $j_n\beta$ ($n=1, 2, 3, \cdots$) are the positive zeros of $J_\nu(z)$, being arranged in ascending order of magnitude, i.e.

$$J_\nu(j_n\beta) = 0\,. \tag{70}$$

The expression (12) takes form:

$$\begin{aligned} f(t) &= \frac{2}{\beta^2}\sum_{n=1}^{+\infty} \frac{1}{J^2_{\nu+1}(j_n\beta)} \left(\int_0^\beta sF(s)\, J_\nu(j_n s)\, ds\right)\cdot \int_0^\beta sJ_\nu(ts)\, J_\nu(j_n s)\, ds \\ &= -\frac{2}{\beta}\sum_{n=1}^{+\infty} f(j_n)\cdot \frac{j_n}{J_{\nu+1}(j_n\beta)}\cdot \frac{J_\nu(\beta t)}{t^2 - j_n^2}\,, \end{aligned} \tag{71}$$

with sampling points $t = j_n$ (n = positive integers), and the sampling function:

$$g_n(t) = -\frac{2}{\beta}\cdot \frac{j_n}{J_{\nu+1}(j_n\beta)}\cdot \frac{J_\nu(\beta t)}{t^2 - j_n^2}\,. \tag{72}$$

The formula (71) was already given by the present authors[8].

Example 5

We shall take the Hankel trasforms for (1) and (2) just as in Example 4. The function $F(s)$ is assumed to be expanded in the Deni expansion[16] in the domain $D=(0, \beta)$. We shall take

$$K(s, t) = tJ_\nu(st)\,, \qquad A = (0, +\infty)\,,$$
$$\widetilde{K}(t, s) = sJ_\nu(ts)\,, \qquad B = (0, +\infty)\,,$$
$$\phi_n(s) = J_\nu(\lambda_n s)\,, \qquad D = (0, \beta)\,,$$

with the orthogonality relation:

$$\int_0^\beta sJ_\nu(\lambda_m s)\, J_\nu(\lambda_n s)\, ds = \gamma_m \cdot \delta_{m,n}\,,$$

and

$$\gamma_n = \frac{1}{2\lambda_n^2}\left\{(\lambda_n^2 + h^2)\,\beta^2 - \nu^2\right\} J_\nu^2(\lambda_n\beta)\,,$$

where λ_n's ($n=1, 2, 3, \cdots$) are the positive roots (arranged in ascending order of magnitude) of the following equation:

$$\lambda_n J'_\nu(\lambda_n\beta) + hJ_\nu(\lambda_n\beta) = 0\,, \tag{73}$$

with a constant h.

Then the expression (12) becomes:

$$f(t)=2\sum_{n=1}^{+\infty}\frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\,\beta^2-\nu^2\}\,J_\nu^2(\lambda_n\beta)}\left(\int_0^\beta sF(s)\,J_\nu(\lambda_n s)\,ds\right)\cdot\int_0^\beta sJ_\nu(ts)\,J_\nu(\lambda_n s)\,ds$$

$$=-2\beta\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\,\beta^2-\nu^2\}\,J_\nu(\lambda_n\beta)}\cdot\frac{tJ_\nu'(\beta t)+hJ_\nu(\beta t)}{t^2-\lambda_n^2}\,, \tag{74}$$

with sampling points $t=\lambda_n$ (n=positive integers), and the sampling function:

$$g_n(t)=-2\beta\frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\,\beta^2-\nu^2\}\,J_\nu(\lambda_n\beta)}\cdot\frac{tJ_\nu'(\beta t)+hJ_\nu(\beta t)}{t^2-\lambda_n^2}\,. \tag{75}$$

The formula (74) was already given by Kroll[14].

Let us tend h to infinity in the expressions (73) and (74), then we see that they reduce to the following expressions, respectively:

$$J_\nu(\lambda_n\beta)=0\,, \tag{76}$$

and

$$f(t)=-\frac{2}{\beta}\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n}{J_{\nu+1}(\lambda_n\beta)}\cdot\frac{J_\nu(\beta t)}{t^2-\lambda_n^2}\,. \tag{77}$$

The expressions (76) and (77) are nothing but the expressions (70) and (71), respectively. Accordingly, the expression (71) is a limiting case of (74).

If we put $\nu=1/2$ in the expression (74), then we obtain

$$f(t)=-2\beta\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n^2\sqrt{\lambda_n}}{\left\{(\lambda_n^2+h^2)\,\beta^2-\frac{1}{4}\right\}\sin(\lambda_n\beta)}\cdot\frac{t\cos(\beta t)+\left(h-\frac{1}{2\beta}\right)\sin(\beta t)}{\sqrt{t}\,(t^2-\lambda_n^2)}\,, \tag{78}$$

where λ_n's are positive roots of the following equations, being arranged in ascending order of magnitude:

$$\lambda_n\cos(\lambda_n\beta)+\left(h-\frac{1}{2\beta}\right)\sin(\lambda_n\beta)=0\,. \tag{79}$$

In the expression (78), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/\sqrt{t}$ and $f(\lambda_n)/\sqrt{\lambda_n}$ respectively, then we obtain:

$$f(t)=-2\beta\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n^2}{\left\{(\lambda_n^2+h^2)\,\beta^2-\frac{1}{4}\right\}\sin(\lambda_n\beta)}\cdot\frac{t\cot(\beta t)+\left(h-\frac{1}{2\beta}\right)}{t^2-\lambda_n^2}\sin(\beta t)\,. \tag{80}$$

Further, if we put

$$h-\frac{1}{2\beta}=-N\,, \tag{81}$$

in the expressions (79) and (80), then the expressions (79) and (80) reduce to (67) and (68), respectively.

If we put

$$h=\frac{1}{2\beta}, \qquad N=0, \tag{82}$$

in the expressions (80) and (79), then we obtain:

$$f(t)=-\frac{2}{\beta}\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{1}{\sin(\lambda_n\beta)}\cdot\frac{t\cos(\beta t)}{t^2-\lambda_n^2}, \tag{83}$$

with

$$\cos(\lambda_n\beta)=0, \tag{84}$$

i. e.

$$\lambda_n=\frac{2n-1}{2\beta}\pi, \qquad (n=1, 2, 3, \cdots) \tag{85}$$

and

$$\sin(\lambda_n\beta)=(-1)^{n+1}. \qquad (n=1, 2, 3, \cdots) \tag{86}$$

Then the expression (83) reduces to:

$$f(t)=2\sum_{n=1}^{+\infty} f\left(\frac{2n-1}{2\beta}\pi\right)\cdot(-1)^n\cdot\frac{\beta t\cos(\beta t)}{\beta^2t^2-\dfrac{(2n-1)^2\pi^2}{4}}. \tag{87}$$

The expression (87) corresponds to Someya's sampling theorem (63) for an odd function $f(t)$, if we take $k=\pm 1/2$.

If we put $\nu=-1/2$ in (74), we obtain:

$$f(t)=2\beta\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{\lambda_n^2\sqrt{\lambda_n}}{\left\{(\lambda_n^2+h^2)\beta^2-\dfrac{1}{4}\right\}\cos(\lambda_n\beta)}\cdot\frac{t\sin(\beta t)-\left(h-\dfrac{1}{2\beta}\right)\cos(\beta t)}{\sqrt{t}\,(t^2-\lambda_n^2)}, \tag{88}$$

with

$$\lambda_n\sin(\lambda_n\beta)-\left(h-\frac{1}{2\beta}\right)\cos(\lambda_n\beta)=0. \tag{89}$$

In the expression (88), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/\sqrt{t}$ and $f(\lambda_n)/\sqrt{\lambda_n}$, respectively, then we obtain:

$$f(t)=2\beta\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{\lambda_n^2}{\left\{(\lambda_n^2+h^2)\beta^2-\dfrac{1}{4}\right\}\cos(\lambda_n\beta)}\cdot\frac{t\tan(\beta t)-\left(h-\dfrac{1}{2\beta}\right)}{t^2-\lambda_n^2}\cos(\beta t). \tag{90}$$

Further, if we put

$$h-\frac{1}{2\beta}=M, \tag{91}$$

in the expressions (89) and (90), then the expressions (89) and (90) reduce to (64) and (65), respectively.

If we put

$$h=\frac{1}{2\beta}, \qquad M=0, \tag{92}$$

in the expressions (90) and (89), then we obtain the following expressions:

$$f(t)=\frac{2}{\beta}\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{1}{\cos(\lambda_n\beta)}\cdot\frac{t\sin(\beta t)}{t^2-\lambda_n^2}, \tag{93}$$

with

$$\sin(\lambda_n\beta)=0, \tag{94}$$

i. e.

$$\lambda_n=\frac{n\pi}{\beta}, \qquad (n=1, 2, 3, \cdots) \tag{95}$$

and

$$\cos(\lambda_n\beta)=(-1)^n. \qquad (n=1, 2, 3, \cdots) \tag{96}$$

Then the expression (93) reduces to:

$$f(t)=2\sum_{n=1}^{+\infty} f\left(\frac{n\pi}{\beta}\right)\cdot(-1)^n\cdot\frac{\beta t\sin(\beta t)}{\beta^2t^2-n^2\pi^2}. \tag{97}$$

The expression (97) corresponds to Shannon's sampling theorem for an even function $f(t)$.

Example 6

We shall take other Hankel transforms due to Weber[17] for (1) and (2), namely:

$$tf(t)\in L_1(p, +\infty), \qquad \alpha\geqslant p>0,$$

$$F(s)=\int_A K(s, t)f(t)\,dt, \qquad \text{for} \quad \nu\geqslant-\frac{1}{2} \tag{98}$$

and

$$G_\nu(\alpha t)f(t)=\int_B \widetilde{K}(t, s)\,F(s)\,ds, \qquad \text{for} \quad t\in A \tag{99}$$

with

$$K(s, t)=tT_\nu(t\alpha, st), \qquad A=(p, +\infty),$$

$$\widetilde{K}(t, s)=sT_\nu(t\alpha, ts), \qquad B=(\alpha, +\infty),$$

$$T_\mu(x, z)=Y_\nu(x)\,J_\mu(z)-J_\nu(x)\,Y_\mu(z), \tag{100}$$

and

$$G_\nu(z)=J_\nu^2(z)+Y_\nu^2(z), \tag{101}$$

under the condition that the integral:

$$\int_p^{+\infty} tf(t)\,dt<+\infty,$$

is absolutely convergent.

We shall take another Fourier-Bessel expansion[17] for $F(s)$ of the form:

$$F(s)=\sum_{n=1}^{+\infty} a_n\phi_n(s), \qquad \text{for} \quad s\in D\subseteq B$$

with

$$\phi_n(s)=T_\nu(\lambda_n\alpha,\lambda_n s), \qquad D=(\alpha,\beta),$$

$$\int_\alpha^\beta sT_\nu(\lambda_m\alpha,\lambda_m s)\,T_\nu(\lambda_n\alpha,\lambda_n s)\,ds=\gamma_m\cdot\delta_{m,n}, \tag{102}$$

and

$$\gamma_n=\frac{\beta^2}{2}\,T^2_{\nu+1}(\lambda_n\alpha,\lambda_n\beta)-\frac{\alpha^2}{2}\,T^2_{\nu+1}(\lambda_n\alpha,\lambda_n\alpha)=\frac{2}{\pi^2\lambda_n^2}\left[\frac{J_\nu^2(\lambda_n\alpha)}{J_\nu^2(\lambda_n\beta)}-1\right], \tag{103}$$

where λ_n's are the positive roots of the equation:

$$T_\nu(\lambda_n\alpha,\lambda_n\beta)=0, \tag{104}$$

i. e.

$$Y_\nu(\lambda_n\alpha)\,J_\nu(\lambda_n\beta)=J_\nu(\lambda_n\alpha)\,Y_\nu(\lambda_n\beta).$$

From (99), the expression (12) becomes to:

$$\begin{aligned} f(t)&=\frac{\pi^2}{2}\sum_{n=1}^{+\infty}\frac{\lambda_n^2J_\nu^2(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha)-J_\nu^2(\lambda_n\beta)}\cdot G_\nu(\lambda_n\alpha)\cdot f(\lambda_n)\cdot\beta\frac{d}{d\beta}\,T_\nu(\lambda_n\alpha,\lambda_n\beta)\cdot\frac{T_\nu(\alpha t,\beta t)}{(t^2-\lambda_n^2)\,G_\nu(\alpha t)}\\ &=-\pi\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n^2G_\nu(\lambda_n\alpha)\,J_\nu(\lambda_n\alpha)\,J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha)-J_\nu^2(\lambda_n\beta)}\cdot\frac{T_\nu(\alpha t,\beta t)}{(t^2-\lambda_n^2)\,G_\nu(\alpha t)}, \end{aligned} \tag{105}$$

with sampling points $t=\lambda_n$ $(n=1,2,3,\cdots)$, and the sampling function:

$$g_n(t)=-\pi\frac{\lambda_n^2G_\nu(\lambda_n\alpha)\,J_\nu(\lambda_n\alpha)\,J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha)-J_\nu^2(\lambda_n\beta)}\cdot\frac{T_\nu(\alpha t,\beta t)}{(t^2-\lambda_n^2)\,G_\nu(\alpha t)}. \tag{106}$$

In the expression (105), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/G_\nu(\alpha t)$ and $f(\lambda_n)/G_\nu(\alpha\lambda_n)$ respectively, then we obtain

$$f(t)=-\pi\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n^2J_\nu(\lambda_n\alpha)\,J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha)-J_\nu^2(\lambda_n\beta)}\cdot\frac{T_\nu(\alpha t,\beta t)}{t^2-\lambda_n^2}, \tag{107}$$

with sampling points $t=\lambda_n$ and the sampling function:

$$g_n(t)=-\pi\frac{\lambda_n^2J_\nu(\lambda_n\alpha)\,J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n\alpha)-J_\nu^2(\lambda_n\beta)}\cdot\frac{T_\nu(\alpha t,\beta t)}{t^2-\lambda_n^2}. \tag{108}$$

The expressions (63), (65), (68), (71), (74), (78), (80), (87), (88), (90), (97), (105), and (107), are examples of the sampling formulae derived from our generalized sampling theorem (12).

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