Minimal submanifolds immersed in a complex projective space

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Contents

1 Introduction

In 1968, Simons [34] gave the formula for the square of the length of the second fundamental form A of a compact n-dimensional minimal submanifold M in a real space form $M^{n+p}(k)$ of constant curvature k. The specific expression of the formula is the following:

$$\frac{1}{2}\Delta|A|^2 = nk|A|^2 - \sum_{a,b}(\mathrm{tr}A_{v_a}A_{v_b})^2 + \sum_{a,b}\mathrm{tr}[A_{v_a}, A_{v_b}]^2 + |\nabla A|^2,$$

where $\{v_1, \cdots, v_p\}$ is an orthonormal basis of the normal vector space. Here we denote by $|\cdot|$ the length of a tensor with respect to the induced metric g on M and by $[\ ,\ ]$ the commutator.

As an application, Simons proved that if the second fundamental form A of a compact n-dimensional minimal submanifold M in S^{n+p} satisfies $|A|^2 < n/(2-1/p)$, then M is totally geodesic. Moreover, Chern, do Carmo and Kobayashi [7] proved that if the second fundamental form A satisfies $|A|^2 = n/(2-1/p)$, then M is the Clifford hypersurface or the Veronese surface in S^4. For minimal submanifolds in the sphere, the Simons type formula was studied by many authors, and many interesting results are given (e.g. [31], [42], [40]).

For minimal submanifolds of complex space forms, there are some pinching theorems with respect to the sectional curvature, Ricci curvature, scalar curvature and so on. For example, for the study of complex submanifolds in a complex space form, Ogiue [27] and Tanno [37] showed the Simons type formula for the square of the length of the second fundamental form. The Simons type formulas for minimal totally real submanifolds and minimal generic submanifolds are given by Chen-Ogiue [5], Yano-Kon [46], respectively.

For general submanifolds of a complex space form, a direct extension of Simons' methods for the sphere to the complex projective space CP^m as an ambient space has some difficulties (see Lawson [24]). So many authors push known theorems on the sphere down to CP^m by using the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{i'} & S^{2m+1} \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{i} & CP^m, \end{array}$$

where $\pi : S^{2m+1} \longrightarrow CP^m$ is the standard fibration, N and M are submanifolds of S^{2m+1} and CP^m, respectively, such that the immersion i' is a diffeomorphism on the fibres (e.g. [24], [29], [46]).

In this paper, we give pinching theorems for general real submanifolds in a complex space form without this method.

In section 2, we prepare some definitions and basic formulas for the submanifolds in a complex space form. In section 3 and section 4, we compute the Simons type formula and its useful modification for general submanifolds in a complex space form $M^m(c)$. Using the formula in the previous section, in section 5 and section 6, we give pinching theorems in terms of the square of the length of the second fundamental form without the assumption that the existence of the above commutative diagram for the standard fibration. We prove the following

Theorem 5.7 ([23]). *Let M be an n-dimensional compact minimal submanifold of a complex space form $M^m(c), c > 0$, of codimension $p = 2m - n$. If the second fundamental form A satisfies*

$$|A|^2 \leq \frac{c}{4}\Big(\frac{n+1}{2-1/p} - 2p\Big),$$

then M is a totally geodesic complex submanifold $M^{n/2}(c)$ or a real hypersurface of $M^m(c)$ with $|A|^2 = (n-1)c/4$.

This theorem is an extension of the pinching theorem with respect to the square of the length of the second fundamental form of compact minimal submanifolds in CP^m given by Yano-Kon [45, Theorem 3.2, p.150].

In the next place, we study some pinching theorems for the sectional curvature of minimal submanifolds in a complex space form. For compact minimal submanifolds in S^{n+p}, complex submanifolds in CP^m and totally real submanifolds in CP^m, there are many results of the pinching problems for the sectional curvature (e.g. [6], [10], [28], [33], [39]). In 1980, Kon [17] proved that if the sectional curvature of a compact minimal real hypersurface of CP^m satisfies $K \geq 1/(2m-1)$, then M is the geodesic minimal hypersphere. In section 7, we improve this theorem. We prove the following

Theorem 7.2 ([21]). *Let M be an n-dimensional compact minimal submanifold in a complex projective space CP^m with flat normal connection. If the second fundamental form A satisfies $\sum_{a=1}^{2m-n} \mathrm{tr} A_{fv_a}^2 \geq 16|FP|^2$, and if the* *sectional curvature K of M satisfies $K \geq 1/n$, then M is the geodesic minimal hypersphere $\pi(S^1(\sqrt{1/2m}) \times S^{2m-1}(\sqrt{(2m-1)/2m}))$ in CP^m.*

The above tensor fields F, P and f are defined in Definition 2.2.

We also prove that if the sectional curvature K of an n-dimensional compact minimal submanifold M in CP^m satisfies $K \geq 3/n$, then M is the complex projective space $CP^{n/2}$ under the assumption that the normal connection of M is semi-flat (Theorem 7.5). The semi-flatness of the normal connection of a submanifold in a complex projective space is closely related to the flatness of the normal connection of the corresponding submanifold in the sphere (Definition 2.4, Lemma 2.13).

Pinching problems for the Ricci curvature of minimal submanifolds in S^{n+p} or CP^m are also studied ([8], [18]). In section 8 and section 9, we consider the pinching problems with respect to the Ricci tensor of minimal submanifolds in CP^m.

Section 8 is devoted to prove a reduction theorem of the codimension of a compact n-dimensional minimal proper CR submanifold M in CP^m. We prove that if the Ricci curvature of M is equal or greater than $n-1$, then M is a real hypersurface of some $CP^{(n+1)/2}$ in CP^m (Theorem 8.1). Using this result, in section 9, we improve the pinching theorem given by Kon [18]. We prove the following

Theorem 9.3 ([22]). *Let M be a compact n-dimensional minimal CR submanifold of a complex projective space CP^m which is not a complex submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$ for any vector X tangent to M, then M is congruent to one of the following*:

(a) *a totally geodesic real projective space RP^n of CP^m,*

(b) *a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m,*

(c) *a real hypersurface of some $CP^{(n+1)/2}$ in CP^m which lies on a tube of radius $\pi/4$ over certain Kähler submanifold N with principal curvatures $\cot\theta$, $0 < \theta \leq \pi/12$.*

Each submanifold is precisely described in Definition 2.1 and Definition 2.3. Using this theorem, we classify compact n-dimensional minimal CR submanifolds immersed in CP^m whose Ricci tensor S satisfies $S(X,X) \geq$

$(n-1)g(X,X)+g(PX,PX)$ for any vector field X (Theorem 9.6).

It is an interesting and important problem to determine real hypersurfaces of complex space forms with respect to some conditions for the holomorphic distribution on real hypersurfaces. For instance, Kimura [12] classified real hypersurfaces of a complex projective space CP^n, $n \geq 3$, on which the sectional curvature of the holomorphic 2-plane spanned by a unit tangent vector orthogonal to the structure vector field ξ is constant. In the last section, we give a characterization for totally η-umbilical real hypersurfaces and ruled real hypersurfaces of a complex projective space with respect to the condition of the second fundamental form on the holomorphic distribution (Theorem 10.5, [20]) and a characterization for pseudo-Einstein real hypersurfaces of a complex projective space with respect to that of the Ricci tensor (Theorem 10.3, [19]).

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2 Preliminaries

Let $M^m(c)$ denote the complex space form of complex dimension m (real dimension $2m$) with constant holomorphic sectional curvature c. We denote by J the almost complex structure of $M^m(c)$. The Hermitian metric of $M^m(c)$ is denoted by g.

Let M be a real n-dimensional manifold immersed in $M^m(c)$. We denote by the same g the Riemannian metric on M induced from that of $M^m(c)$, and by p the codimension of M, that is, $p = 2m - n$. We denote by $\tilde{\nabla}$ the Levi-Civita connection in $M^m(c)$ and by ∇ the connection induced on M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \qquad \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the normal connection. We call both A and B the *second fundamental form* of M which are related by $g(B(X, Y), V) = g(A_V X, Y)$. The second fundamental form B is symmetric. A normal vector field V on M is said to be parallel if $D_X V = 0$ for any vector field X tangent to M.

For the second fundamental form B, we define ∇B, the covariant derivative of B, by

$$(\nabla_X B)(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M. If $\nabla_X B = 0$ for all X, then the second fundamental form B of M is said to be parallel. This is equivalent to the condition $\nabla_X A = 0$ for all X, where $\nabla_X A$ is defined by

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V(\nabla_X Y).$$

We notice the relation

$$g((\nabla_X B)(Y, Z), V) = g((\nabla_X A)_V Y, Z).$$

Definition 2.1. The *mean curvature vector* μ of M is defined to be $\mu = (1/n)\mathrm{tr}B$, where $\mathrm{tr}B$ is the trace of B, that is, $\mathrm{tr}B = \sum_i B(e_i, e_i)$, $\{e_i\}$ being an orthonormal basis for the tangent space $T_x(M)$ at x. If $\mu = 0$, then M is said to be *minimal*. A submanifold M is said to be totally geodesic if the second fundamental form vanishes identically.

For $x \in M$, the *first normal space* $N_1(x)$ is the orthogonal complement in $T_x(M)^\perp$ of the set $N_0(x) = \{V \in T_x(M)^\perp : A_V = 0\}$. If $D_X V \in N_1(x)$ for any vector field V with $V_x \in N_1(x)$ and any vector field X of M at x, then the first normal space $N_1(x)$ is said to be parallel with respect to the normal connection.

We next give some fundamental formulas on M induced from the action of the almost complex structure J of $M^m(c)$ to the tangent space and the normal space of M.

Definition 2.2. For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX. For any vector field V normal to M, we put

$$JV = tV + fV,$$

where tV is the tangential part of JV and fV the normal part of JV.

Then P is a $(1,1)$-tensor field on M and F is a normal bundle valued 1-form on M. P and f are skew-symmetric with respect to g and $g(FX, V) = -g(X, tV)$. We also have

$$\begin{aligned} &P^2 = -I - tF, \quad FP + fF = 0, \\ &Pt + tf = 0, \quad f^2 = -I - Ft. \end{aligned}$$

We notice that $|FP| = |fF| = |Pt| = |tf|$, where $|\cdot|$ denotes the length of a tensor with respect to g.

We define the covariant derivatives of P, F, t and f by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$, $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$, respectively. We then have

$$\begin{aligned} &(\nabla_X P)Y = A_{FY}X + tB(X,Y), \quad (\nabla_X F)Y = -B(X, PY) + fB(X,Y), \\ &(\nabla_X t)V = -PA_V X + A_{fV}X, \quad (\nabla_X f)V = -FA_V X - B(X, tV). \end{aligned}$$

The Riemannian curvature tensor $\tilde{R}$ of a complex space form $M^m(c)$ is defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z,$$

are given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= \frac{c}{4}\Big(g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &\quad - g(JX,Z)JY + 2g(X,JY)JZ\Big)\end{aligned}$$

for any vector fields X, Y and Z of $M^m(c)$. Let R be the Riemannian curvature tensor of M which is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for any vector fields X, Y and Z tangent to M. The *equation of Gauss* and the *equation of Codazzi* are given respectively by

$$\begin{aligned}R(X,Y)Z &= \frac{c}{4}\Big(g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY \\ &\quad -2g(PX,Y)PZ\Big) + A_{B(Y,Z)}X - A_{B(X,Z)}Y\end{aligned}$$

and

$$\begin{aligned}&(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad = \frac{c}{4}\Big(g(PY,Z)FX - g(PX,Z)FY + 2g(X,PY)FZ\Big).\end{aligned}$$

The Ricci tensor field S of M is the covariant tensor field of degree 2 defined as $S(X,Y) = \sum_i g(R(e_i,X)Y,e_i)$. Then we have

$$\begin{aligned}S(X,Y) &= \frac{c}{4}\Big((n-1)g(X,Y) + 3g(PX,PY)\Big) \\ &\quad + \sum_a \mathrm{tr}A_a g(A_aX,Y) - \sum_a g(A_a^2X,Y),\end{aligned}$$

where A_a is the second fundamental form in the direction of v_a, $\{v_1,\cdots,v_p\}$ being an orthonormal basis for the normal space $T_x(M)^\perp$ at x.

Definition 2.3. If the Ricci tensor S is of the form $S = ag$, where a is a function, then M is said to be *Einstein.* Moreover, a real hypersurface M of CP^m is called a *pseudo-Einstein* if the Ricci tensor S is of the form $S(X,Y) = ag(X,Y) + bg(X,\xi)g(Y,\xi)$, where $\xi = -JN$ for the unit normal vector field N and b is a function.

It is known that any real hypersurface of CP^m is not Einstein. Accordingly the notion of pseudo-Einstein is necessary.

The scalar curvature $r = \sum_i S(e_i, e_i)$ of M is given by

$$r = \frac{c}{4}\Big((n-1)n - 3\mathrm{tr}P^2\Big) + \sum_a (\mathrm{tr}A_a)^2 - |A|^2,$$

where $|A|^2 = \sum_a \mathrm{tr}A_a^2$.

We define the sectional curvature of a 2-dimensional subspace σ of T_pM by $K(u,v) = g(R(u,v)v,u)$, where $\{u,v\}$ denotes an orthonormal basis for σ.

The curvature tensor $R^\perp$ of the normal bundle $T(M)^\perp$ of M is defined by

$$R^\perp(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}Z,$$

where X and Y are vector fields tangent to M and V is a vector field normal to M. Then we have the *equation of Ricci*:

$$\begin{aligned} g(R^\perp(X,Y)U,V) &+ g([A_V, A_U]X,Y) \\ &= \frac{c}{4}\Big(g(FY,U)g(FX,V) - g(FX,U)g(FY,V) \\ &\quad +2g(X,PY)g(fU,V)\Big), \end{aligned}$$

where [,] denotes the commutator and $[A_V, A_U] = A_V A_U - A_U A_V$.

Definition 2.4. Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. If the normal curvature tensor $R^\perp$ of M satisfies

$$R^\perp(X,Y)U = \frac{1}{2}cg(X,PY)fU$$

for any vector fields X and Y tangent to M and any vector field U normal to M, then the normal connection of M is said to be *semi-flat*. If the normal curvature tensor $R^\perp$ of M vanishes identically, then the normal connection of M is said to be *flat* [46, p.224].

From the equation of Ricci, we have

Lemma 2.5. *Let M be an n-dimensional submanifold in $M^m(c)$. If the normal connection of M is flat, then*

$$\sum_{a,b}|[A_a,A_b]|^2 = \frac{c^2}{16}\Big(2\sum_{a,b}(g(tv_a,tv_a)g(tv_b,tv_b)-g(tv_a,tv_b)^2)$$
$$-8\sum_a g(tfv_a,tfv_a)+4\sum_{i,a}g(Pe_i,Pe_i)g(fv_a,fv_a)\Big),$$
$$\sum_{i,a}g([A_{fa},A_a]e_i,Pe_i) = 2\sum_a \mathrm{tr}A_aA_{fa}P$$
$$= \frac{c}{2}\Big(\sum_a g(tfv_a,tfv_a)-\sum_{i,a}g(Pe_i,Pe_i)g(fv_a,fv_a)\Big),$$
$$\sum_{a,b}g([A_a,A_b]tv_a,tv_b) = \sum_{a,b}(g(A_atv_b,A_btv_a)-g(A_atv_a,A_btv_b))$$
$$= \frac{c}{4}\Big(\sum_{a,b}(g(tv_a,tv_a)g(tv_b,tv_b)-g(tv_a,tv_b)^2)-2\sum_a g(tfv_a,tfv_a)\Big),$$

where we have put $A_{fa}=A_{fv_a}$.

Proof. By the equation of Ricci, we have

$$[A_a,A_b]e_i = \frac{c}{4}\Big(g(e_i,tv_a)tv_b-g(e_i,tv_b)tv_a-2g(fv_b,v_a)Pe_i\Big).$$

Hence we obtain

$$\begin{aligned}\sum_{a,b}|[A_a,A_b]|^2 &= \sum_{a,b,i}g([A_a,A_b]e_i,[A_a,A_b]e_i)\\ &= \frac{16}{c^2}\Big(2\sum_{a,b}g(tv_a,tv_a)g(tv_b,tv_b)-2\sum_{a,b}g(tv_a,tv_b)^2\\ &\quad+\sum_a 8g(Ptv_a,tfv_a)+4\sum_{a,i}g(Pe_i,Pe_i)g(fv_a,fv_a)\Big).\end{aligned}$$

Since $Pt=-tf$, we have the first equation. By the similar computation, we obtain the other equations. *q.e.d.*

By the similar method, we have

Lemma 2.6. *Let M be an n-dimensional submanifold in $M^m(c)$. If the normal connection of M is semi-flat, then*

$$\sum_{a,b}|[A_a,A_b]|^2 = \frac{c^2}{8}\sum_{a,b}(g(tv_a,tv_a)g(tv_b,tv_b)-g(tv_a,tv_b)^2),$$

$$\sum_{i,a} g([A_{fa}, A_a]e_i, Pe_i) = \frac{c}{2}\sum_a g(tfv_a, tfv_a),$$
$$\sum_{a,b} g([A_a, A_b]tv_a, tv_b) = \frac{c}{4}\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2).$$

Here we recall some classes of submanifolds of a Kähler manifold $\tilde{M}$ with almost complex structure J.

Definition 2.7.

(a) If $JT_x(M)^\perp \subset T_x(M)$ for any point x of M, then we call M a *generic submanifold* of $\tilde{M}$.

(b) If $JT_x(M) \subset T_x(M)$ for any point x of M, then we call M a *complex submanifold* of $\tilde{M}$.

(c) If $JT_x(M) \subset T_x(M)^\perp$ for any point x of M, then we call M a *totally real submanifold* of $\tilde{M}$.

Remark 2.8. M is a complex submanifold if and only if F and t vanishes identically. M is totally real if and only if P vanishes identically.

Definition 2.9 (Bejancu [2]). A submanifold M of a Kähler manifold $\tilde{M}$ with almost complex structure J is called a CR *submanifold* of $\tilde{M}$ if there exists a differentiable distribution $\mathcal{D} : x \longrightarrow \mathcal{D}_x \subset T_x(M)$ on M satisfying the following conditions:

(a) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = \mathcal{D}_x$ for each $x \in M$, and

(b) the complementary orthogonal distribution $\mathcal{D}^\perp : x \longrightarrow \mathcal{D}_x^\perp \subset T_x(M)$ is anti-invariant, i.e., $J\mathcal{D}_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

Remark 2.10. By the definitions, if a submanifold M of $M^m(c)$ is generic, complex or totally real, then M is also a CR submanifold. Any real hypersurface of $M^m(c)$ is obviously a generic submanifold.

Lemma 2.11 ([46]). *Let M be a CR submanifold of a Kähler manifold $\tilde{M}$. Then*

$$FP = 0, \quad fF = 0, \quad tf = 0, \quad Pt = 0,$$
$$P^3 + P = 0, \quad f^3 + f = 0.$$

Theorem 2.12 ([46]). *In order for a submanifold M of a Kähler manifold*

$\tilde{M}$ to be a CR submanifold, it is necessary and sufficient that $FP = 0$.

For the study of submanifolds of a complex projective space CP^m with constant holomorphic sectional curvature 4, many authors use the method of the standard fibration to push known theorems on the sphere down to CP^m by considering the commutative diagram below (e.g. [24], [29], [46]).

Let S^{m+1} be a $(2m+1)$-dimensional unit sphere, i.e., $S^{2m+1} = \{z \in C^{m+1} : |z| = 1\}$. For any point $z \in S^{2m+1}$ we put $\xi = JZ$, where J denotes the almost complex structure of C^{m+1}. We consider the orthogonal projection $\pi' : T_z(C^{m+1}) \longrightarrow T_z(S^{2m+1})$. Putting $\phi = \pi' \cdot J$, we have a contact metric structure (ϕ, ξ, η, G) on S^{2m+1}, where η is a 1-form dual to ξ and G the standard metric tensor field on S^{2m+1} which satisfies $G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y)$. The contact metric structure satisfies $\eta(\xi) = 1$ and $\phi^2(X) = -X + \eta(X)\xi$. We see that S^{2m+1} is of constant curvature 1.

There exists a fibration $\pi : S^{2m+1} \longrightarrow CP^m$ that satisfies the following:

(a) The fibers are totally geodesic in M.

(b) At each point p of S^{2m+1} the differential π_* carries the normal space to the fiber at p isometrically onto the tangent space of CP^m at $\pi(p)$.

We call π the standard fibration. Let M be an n-dimensional submanifold in CP^m. Let N be an $(n+1)$-dimensional submanifold immersed in a $(2m+1)$-dimensional unit sphere S^{2m+1} such that the following diagram is commutative:

$$\begin{array}{ccc} N & \xrightarrow{i'} & S^{2m+1} \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{i} & CP^m, \end{array}$$

where the immersion i' is a diffeomorphism on the fibres.

The horizontal lift with respect to the connection η is denoted by $*$. Then $(JX)^* = \phi X^*$ and $G(X^*, Y^*) = g(X, Y)^*$ for any vectors X and Y tangent to CP^m. A submanifold N in S^{2m+1} is tangent to the totally geodesic fibre of π and the structure vector field ξ is tangent to N.

Let α be the second fundamental form of N in S^{2m+1}. Then we have the relations of the second fundamental form α of N and B of M:

$$\alpha(X^*, Y^*) = B(X, Y)^*, \quad \alpha(\xi, \xi) = 0.$$

Moreover, we have

$$(\nabla_{X^*}\alpha)(Y^*, Z^*) = [(\nabla_X B)(Y, Z) + g(PX, Y)FZ + g(PX, Z)FY]^*,$$

$$(\nabla_{X^*}\alpha)(Y^*, \xi) = [fB(X,Y) - B(X,PY) - B(Y,PX)]^*,$$
$$(\nabla_{X^*}\alpha)(\xi, \xi) = -2(FPX)^*$$

for any vectors X, Y and Z tangent to M. From the third equation, we see that if the second fundamental form α of N is parallel, then $FP = 0$ and M is a CR submanifold of CP^m by Theorem 2.12.

We denote by $\mu' = (1/(n+1))\mathrm{tr}\alpha$ the mean curvature vector field of N, and by $\mu = (1/n)\mathrm{tr}B$ the mean curvature vector field of M. Then we have

$$\mu' = \frac{n}{n+1}\mu^*, \quad D'_{X^*}\mu' = \frac{n}{n+1}(D_X\mu)^*, \quad D'_\xi\mu' = (f\mu)^*$$

where D' is the normal connection of N. Thus the mean curvature vector field μ' of N is parallel if and only if the mean curvature vector field μ of M is parallel and $f\mu = 0$.

Let $K^\perp$ be the curvature tensor of the normal bundle of N. Then we have

$$G(K^\perp(X^*, Y^*)V^*, U^*) = [g(R^\perp(X,Y)V,U) - 2g(X,PY)g(fV,U)]^*,$$
$$G(K^\perp(X^*, \xi)V^*, U^*) = g((\nabla_X f)V, U)^*$$

for any vectors X and Y tangent to M and any vectors V and U normal to M. Therefore, we have the following lemma (see [29], [30], [46]).

Lemma 2.13. *The normal connection of N in S^{2m+1} is flat if and only if the normal connection of M in CP^m is semi-flat and $\nabla f = 0$.*

Example 2.14. In this setting, we put

$$N = S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k), \quad n+1 = \sum_{i=1}^k m_i, \quad 1 = \sum_{i=1}^k r_i^2,$$

where $m_1, \cdots, m_k$ are odd numbers. Then $n + k$ is also odd. The second fundamental form α of N is parallel in S^{2m+1}. We can see that $M = \pi(N)$ is a generic submanifold in $CP^{(n+k-1)/2}$ with flat normal connection. Especially, $\pi(S^1(r_1) \times S^n(r_2))$ is a geodesic hypersphere in $CP^{(n+1)/2}$. Moreover, M is a CR submanifold in CP^m ($m > (n+k-1)/2$) with semi-flat normal connection and $\nabla f = 0$.

If $r_i = (m_i/(n+1))^{1/2}$ ($i = 1, \cdots, k$), then M is a generic minimal submanifold in $CP^{(n+k-1)/2}$. Then we have $|A|^2 = \sum_a \mathrm{tr}A_a^2 = (n-1)q$,

$q = k - 1$.

If M is a complex submanifold in CP^m, the normal connection of M is semi-flat if and only if M is totally geodesic (see [9]).

Example 2.15. The natural imbedding of CP^n into CP^m is induced from the inclusion of C^{n+1} into C^{m+1}, i.e. $(z^0, \cdots, z^n) \longrightarrow (z^0, \cdots, z^n, 0, \cdots, 0)$. It gives rise to a complex submanifold. The natural imbedding of RP^n into CP^m is induced from the inclusion of R^{n+1} into C^{m+1}, i.e. $(x^0, \cdots, x^n) \longrightarrow (x^0, \cdots, x^n, 0, \cdots, 0)$. It gives rise to a totally real submanifold. We remark that both are totally geodesic.

Conversely, an n-dimensional complete totally geodesic submanifold M of CP^m is either a complex projective space $CP^{n/2}$ or a real projective space RP^n of constant curvature 1 (see [1]).

Example 2.16. Let $z^0, z^1, \cdots, z^m$ be a homogeneous coordinates of CP^m. The *complex quadric* Q^{m-1} is a complex hypersurface of CP^m defined by the equation

$$(z^0)^2 + (z^1)^2 + \cdots + (z^m)^2 = 0.$$

Then Q^{m-1} is a Kähler manifold. Moreover, Q^{m-1} is an Einstein manifold with Ricci curvature $2(m-1)$. Smith [35] proved that CP^n and the complex quadric Q^n are the only complete complex Einstein hypersurfaces in CP^{n+1}.

Example 2.17. For an integer k and for $0 < r < \pi/2$, we define $M(k, r)$ in $S^{2m+1} \subset C^{m+1}$ by

$$\sum_{j=0}^{k} |z_j|^2 = \cos^2 r, \qquad \sum_{j=k+1}^{m} |z_j|^2 = \sin^2 r.$$

$M(k, r)$ is the standard product $S^{2k+1}(\cos r) \times S^{2l+1}(\sin r)$, $l = m-k-1$. We consider the standard fibration $\pi : S^{2m+1} \longrightarrow CP^m$, where S^{2m+1} denotes the unit sphere. Then $M^c(k, r) = \pi(M(k, r))$ is a real hypersurface in CP^m. For an integer $1 \le k \le m-2$, we see that $M^c(k, r)$ is the tube of radius r over CP^k (see [3]).

When r satisfies $\cos r = \sqrt{(2k+1)/(2m)}$ and $\sin r = \sqrt{(2l+1)/(2m)}$, $M^c(k, r)$ is a minimal real hypersurface of CP^m.

Moreover, we see that $M^c(k,r)$ is a pseudo-Einstein real minimal hypersurface of CP^m if and only if $k = l = (m-1)/2$ and $r = \pi/4$. Then the Ricci tensor S satisfies $S(X,Y) = (2m-2)g(X,Y) + 2g(PX,PY)$ [46, p.376].

There are many pinching results with respect to the length of second fundamental form, Ricci curvature, sectional curvature of compact minimal submanifolds in the sphere. In the last of this section, we recall some of them. With respect to the pinching theorem for the length of the second fundamental form, Peng and Terng [31] proved the following: Let M be a compact minimal hypersurface of S^{n+1} with constant scalar curvature. There exists a constant $\epsilon(n) > 1/(12n)$ such that if $n \leq |A|^2 \leq n + \epsilon(n)$ then $|A|^2 = n$, so that M is a generalized Clifford torus.

Yang and Cheng [42] proved that, for a compact minimal hypersurface M with constant scalar curvature in S^{n+1}, if $|A|^2 > n > 3$, then $|A|^2 > n + n/3$. In particular, if the shape operator A of M in S^{n+1} with respect to a unit normal vector satisfying $\mathrm{tr}A^3$ =constant, then $|A|^2 \geq n + 2n/3$.

For an n-dimensional compact minimal manifold M in S^{n+p} with $p \geq 2$, Xia [40] proved the following:

(1) If n is even and $|A|^2 \leq n(3n-2)/(5n-4)$, then M is either totally geodesic or the Veronese surface in S^4;

(2) If n is odd and $|A|^2 \leq n(3n-5)/(5n-9)$, then

(2-a) when $n > 5$, M is totally geodesic in S^{n+p};

(2-b) when $n = 5$, M is either totally geodesic or homeomorphic to S^5 and $|A|^2 = 25/8$ on M; and

(2-c) when $n = 3$, $|A|^2$ is identically equal to 0 or 2; in the latter case M is diffeomorphic to S^3 or RP^3.

Itoh [10] proved that if $f : M \longrightarrow S^{n+p}$ is a minimal full isometric immersion of a compact orientable Riemannian n-manifold into S^{n+p} and if the sectional curvature K of M satisfies $K \geq n(n+1)/2$, then either M is totally geodesic or M is of constant sectional curvature $n(n+1)/2$ and f is given by the second standard immersion of an n -sphere of sectional curvature $n(n+1)/2$. Chen and Zou [6] showed that if the sectional curvature satisfies $K \geq 1/2 - 1/(3p)$, then either M is totally geodesic or the Veronese surface in S^4.

Ejiri [8] showed that if the Ricci tensor of an n-dimensional compact minimal submanifold of S^{n+p} $(n \geq 4)$ satisfies $S \geq (n-2)g$, then M is totally geodesic, or $n = 2m$ and M is

$$S^m(\sqrt{1/2}) \times S^m(\sqrt{1/2}) \subset S^{n+1} \subset S^{n+p}$$

embedded in a standard way, or M is a 2-dimensional complex projective CP^2 of constant holomorphic sectional curvature $4/3$ which is isometrically immersed in a totally geodesic S^7 via Hermitian harmonic functions of degree one.

3 Laplacian

We compute the Laplacian for the square of the length of the second fundamental form A of an n-dimensional submanifold M immersed in a complex space form $M^m(c)$. In the following, we put $\nabla_i = \nabla_{e_i}$ and $D_i = D_{e_i}$, where $\{e_i\}$ being an orthonormal basis of $T_x(M)$. We use the following (see Simons [34])

Lemma 3.1. *Let M be a submanifold of a locally symmetric Riemannian manifold $\bar{M}$. If the mean curvature vector field of M is parallel, then*

$$\begin{aligned}
&g((\nabla^2 B)(X,Y),V) = \sum_i g((\nabla_i\nabla_i B)(X,Y),V)\\
&= \sum_i \Big(2g(\bar{R}(e_i,Y)B(X,e_i),V) + 2g(\bar{R}(e_i,X)B(Y,e_i),V)\\
&\quad - g(A_V X, \bar{R}(e_i,Y)e_i) - g(A_V Y, \bar{R}(e_i,X)e_i) + g(\bar{R}(e_i,B(X,Y))e_i,V)\\
&\quad + g(\bar{R}(B(e_i,e_i),X)Y,V) - 2g(A_V e_i, \bar{R}(e_i,X)Y)\Big)\\
&\quad + \sum_a \Big(\mathrm{tr}A_a g(A_V A_a X,Y) - \mathrm{tr}A_a A_V g(A_a X,Y) + 2g(A_a A_V A_a X,Y)\\
&\quad - g(A_a^2 A_V X,Y) - g(A_V A_a^2 X,Y)\Big).
\end{aligned}$$

We compute the equation of Lemma 3.1 for an n-dimensional minimal submanifold M in $M^m(c)$. We notice that $M^m(c)$ is locally symmetric. Using the expression of the curvature tensor $\tilde{R}$ of $M^m(c)$, we have the equation of Lemma 3.1 in the following:

$$\begin{aligned}
&g((\nabla^2 B)(X,Y),V)\\
&\quad = \sum_i g((\nabla_i\nabla_i B)(X,Y),V)\\
&\quad = \frac{c}{2}\Big(-g(A_{FY}X,tV) - g(A_{FX}Y,tV)\\
&\qquad + \sum_i g(Y,tV)g(A_{Fe_i}e_i,X) + \sum_i g(X,tV)g(A_{Fe_i}e_i,Y)\\
&\qquad -2g(A_{fV}X,PY) - 2g(A_{fV}Y,PX)\Big) \qquad\qquad (3.1)\\
&\qquad +\frac{c}{4}\Big(ng(A_V X,Y) - 3g(A_V X,P^2Y) - 3g(A_V Y,P^2X)\\
&\qquad +3g(A_{FtV}X,Y)\Big) - \frac{3c}{2}g(A_V PX,PY)
\end{aligned}$$

$$+\sum_a\Big(-\operatorname{tr}A_aA_Vg(A_aX,Y)+2g(A_aA_VA_aX,Y)$$
$$-g(A_a^2A_VX,Y)-g(A_VA_a^2X,Y)\Big).$$

We generally have $g((\nabla^2B)(X,Y),V)=g((\nabla^2A)_VX,Y)$. Hence we obtain

$$\begin{aligned}
&g(\nabla^2A,A)\\
&=\frac{nc}{4}|A|^2-\frac{3c}{4}\sum_{a,b}\mathrm{tr}A_aA_bg(tv_a,tv_b)-\frac{c}{4}\sum_a(\mathrm{tr}A_a)^2\\
&\quad-\frac{3c}{2}\sum_a\mathrm{tr}P^2A_a^2+\frac{3c}{2}\sum_a(\mathrm{tr}A_aP)^2+\frac{3c}{4}\sum_{a,b}\mathrm{tr}A_bg(A_atv_a,tv_b)\\
&\quad+c\sum_{a,b}\Big(g(A_atv_b,A_btv_a)-g(A_atv_a,A_btv_b)\Big)-2c\sum_a\mathrm{tr}A_aA_{fa}P\\
&\quad+\sum_{a,b}\Big(\mathrm{tr}A_b\mathrm{tr}A_a^2A_b-(\mathrm{tr}A_aA_b)^2+2\mathrm{tr}(A_aA_b)^2-2\mathrm{tr}A_a^2A_b^2\Big),
\end{aligned}$$

where we put $A_{fa}=A_{fv_a}$. Substituting equations:

$$\sum_{a,b}\mathrm{tr}A_aA_bg(tv_a,tv_b)=-\sum_a\mathrm{tr}A_{Ftv_a}A_a=|A|^2-\sum_a\mathrm{tr}A_{fa}^2, \quad (3.2)$$

$$2\sum_{a,b}(\mathrm{tr}A_a^2A_b^2-\mathrm{tr}(A_aA_b)^2)=-\sum_{a,b}\mathrm{tr}[A_a,A_b]^2, \quad (3.3)$$

$$2\sum_a(\mathrm{tr}(A_aP)^2-\mathrm{tr}A_a^2P^2)=\sum_a|[P,A_a]|^2 \quad (3.4)$$

into the equation above, we have the following theorems.

Theorem 3.2. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$ with parallel mean curvature vector field. Then we have*

$$\begin{aligned}
&g(\nabla^2A,A)\\
&=\frac{(n-3)c}{4}|A|^2+\frac{3c}{4}\sum_a\mathrm{tr}A_{fa}^2-\frac{c}{4}\sum_a(\mathrm{tr}A_a)^2+\frac{3c}{4}\sum_{a,b}\mathrm{tr}A_bg(A_atv_a,tv_b)\\
&\quad+c\sum_{a,b}(g(A_atv_b,A_btv_a)-g(A_atv_a,A_btv_b))-2c\sum_a\mathrm{tr}A_aA_{fa}P\\
&\quad+\frac{3c}{4}\sum_a|[P,A_a]|^2+\sum_{a,b}\mathrm{tr}[A_a,A_b]^2+\sum_{a,b}(\mathrm{tr}A_b\mathrm{tr}A_a^2A_b-(\mathrm{tr}A_aA_b)^2).
\end{aligned}$$

Theorem 3.3. *Let M be an n-dimensional minimal submanifold of a complex space form $M^m(c)$. Then we have*

$$\begin{aligned}
&g(\nabla^2 A, A) \\
&= \frac{(n-3)c}{4}|A|^2 + \frac{3c}{4}\sum_a \mathrm{tr} A_{fa}^2 \\
&\quad + c\sum_{a,b}(g(A_a tv_b, A_b tv_a) - g(A_a tv_a, A_b tv_b)) - 2c\sum_a \mathrm{tr} A_a A_{fa} P \\
&\quad + \frac{3c}{4}\sum_a |[P, A_a]|^2 + \sum_{a,b} \mathrm{tr}[A_a, A_b]^2 - \sum_{a,b}(\mathrm{tr} A_a A_b)^2.
\end{aligned}$$

Next, we give the Simons type integral formula for a compact minimal submanifold in CP^m with flat normal connection. We use the following lemma ([4, p.81]).

Lemma 3.4. *Let M be a minimal submanifold in a Riemannian manifold $\bar{M}$. Then*

$$\begin{aligned}
&(\nabla^2 B)(X, Y) = \sum_i (\nabla_i \nabla_i B)(X, Y) \\
&= \sum_i \Big((R(e_i, X)B)(e_i, Y) + (\bar{\nabla}_X(\bar{R}(e_i, Y)e_i)^\perp)^\perp + (\bar{\nabla}_i(\bar{R}(e_i, X)Y)^\perp)^\perp \Big),
\end{aligned}$$

where $\{e_1, \cdots, e_n\}$ denotes an orthonormal basis of $T_x(M)$, and $\bar{\nabla}$ is the Levi-Civita connection in $\bar{M}$.

We compute the equation in Lemma 3.4 for an n-dimensional minimal submanifold M in a complex projective space CP^m of constant holomorphic sectional curvature 4. Since CP^m is locally symmetric, using the expression of the curvature tensor $\bar{R}$ of CP^m, we have

$$\begin{aligned}
&\sum_i (\bar{\nabla}_X(\bar{R}(e_i, Y)e_i)^\perp)^\perp \\
&= \sum_i \Big(\bar{R}(B(X, e_i), Y)e_i + \bar{R}(e_i, B(X, Y))e_i + \bar{R}(e_i, Y)B(X, e_i) \Big)^\perp \\
&\quad - \sum_i B(X, (\bar{R}(e_i, Y)e_i)^T), \\
&= 3\Big(fB(X, PY) + FtB(X, Y) - B(X, P^2Y) + FA_{FY}X \Big),
\end{aligned}$$

$$
\begin{aligned}
&\sum_i (\bar\nabla_{e_i}(\bar R(e_i,X)Y)^\perp)^\perp \\
&\quad = \sum_i \Big(\bar R(B(e_i,e_i),X)Y + \bar R(e_i,B(e_i,X))Y + \bar R(e_i,X)B(e_i,Y)\Big)^\perp \\
&\qquad - \sum_i B(e_i,(\bar R(e_i,X)Y)^T) \\
&\quad = FA_{FX}Y - FA_{FY}X + fB(X,PY) + 2fB(PX,Y) \\
&\qquad -3B(PX,PY) - 2\sum_i g(A_{Fe_i}e_i,X)FY - \sum_i g(A_{Fe_i}e_i,Y)FX,
\end{aligned}
$$

where $(\bar R(e_i,X)Y)^T)$ denotes a tangential part of $\bar R(e_i,X)Y$. Thus we obtain

$$
\begin{aligned}
&g(\nabla^2 B, B) = \sum_{i,j,k} g((\nabla_{e_i}\nabla_{e_i}B)(e_j,e_k), B(e_j,e_k)) \\
&= \sum_{i,j,a} g((R(e_i,e_j)A)_a e_i, A_a e_j) + 3\Big(\sum_a \mathrm{tr} A_{Ftv_a}A_a \\
&-2\sum_a \mathrm{tr} A_a A_{fa} P - \sum_a \mathrm{tr} P^2 A_a^2 + \sum_a \mathrm{tr}(A_a P)^2 \\
&+\sum_{a,b} g(A_a tv_a, tv_b)\mathrm{tr} A_b + \sum_{a,b}(g(A_a tv_b, A_b tv_a) - g(A_a tv_a, A_b tv_b))\Big).
\end{aligned}
$$

Using (3.2) and (3.4), we have

Lemma 3.5. *Let M be an n-dimensional minimal submanifold in CP^m. Then*

$$
\begin{aligned}
&g(\nabla^2 B, B) = g(\nabla^2 A, A) \\
&\quad = \sum_{i,j,a} g((R(e_i,e_j)A)_a e_i, A_a e_j) \\
&\qquad +3\Big(-\sum_a \mathrm{tr} A_a^2 + \sum_a \mathrm{tr} A_{fa}^2 - 2\sum_a \mathrm{tr} A_a A_{fa} P + \frac{1}{2}\sum_a |[P,A_a]|^2 \\
&\qquad +\sum_{a,b}(g(A_a tv_b, A_b tv_a) - g(A_a tv_a, A_b tv_b))\Big).
\end{aligned}
$$

We prepare the following lemma.

Lemma 3.6. *Let M be an n-dimensional minimal submanifold in CP^m. If U is a parallel section in the normal bundle of M, then*

$$
\mathrm{div}(\nabla_{tU} tU) = (n-1)g(tU,tU) + 3g(PtU,PtU) - \sum_a g(A_a tU, A_a tU)
$$

$$+\mathrm{tr}A_{fU}^2 - \mathrm{tr}A_U^2 - 2\mathrm{tr}A_U A_{fU} P + \sum_a g(A_U tv_a, A_U tv_a) + \frac{1}{2}|[P, A_U]|^2.$$

Proof. For any vector field X on a Riemannian manifold, we generally have the equation ([43])

$$\begin{aligned} &\mathrm{div}(\nabla_X X) - \mathrm{div}((\mathrm{div}X)X) \\ &= S(X,X) + \frac{1}{2}|L_X g|^2 - |\nabla X|^2 - (\mathrm{div}X)^2, \end{aligned} \tag{3.5}$$

where S denotes the Ricci tensor and $(L_X g)(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y)$.

Suppose that U is a parallel section of the normal bundle of M. From the equation of Gauss, we have

$$S(tU, tU) = (n-1)g(tU, tU) + 3g(PtU, PtU) - \sum_a g(A_a tU, A_a tU).$$

On the other hand, since $(\nabla_X t)V = -PA_V X + A_{fV} X$ for any V normal to M, we have $\nabla_X(tU) = -PA_U X + A_{fU} X$. This implies $\mathrm{div}(tU) = \mathrm{tr}A_{fU} = 0$. We also have

$$\begin{aligned} |\nabla tU|^2 &= \mathrm{tr}A_{fU}^2 + \mathrm{tr}A_U^2 - 2\mathrm{tr}A_U A_{fU} P - \sum_a g(A_U tv_a, A_U tv_a), \\ |L_{tU} g|^2 &= |[P, A_U]|^2 + 4\mathrm{tr}A_{fU}^2 - 8\mathrm{tr}A_U A_{fU} P. \end{aligned}$$

Substituting these equations into (3.5), we have our lemma. *q.e.d.*

Lemma 3.7. *Let M be an n-dimensional minimal submanifold in CP^m with flat normal connection. Then*

$$\begin{aligned} &-g(\nabla^2 A, A) - 2\sum_{i,a} g(Pe_i, Pe_i) g(tv_a, tv_a) - 2\sum_i g(FPe_i, FPe_i) \\ &+\frac{1}{2}\Big(\sum_a \mathrm{tr}A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr}A_a A_{fa} P\Big) \\ &+\sum_{a,b}\big(g(tv_a, tv_a) g(tv_b, tv_b) - g(tv_a, tv_b)^2\big) \\ &= \sum_a \mathrm{tr}A_a^2 - \sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) \\ &\quad +8\sum_i g(FPe_i, FPe_i) - \frac{1}{2}\sum_a \mathrm{tr}A_{fa}^2 - 2\sum_a \mathrm{div}(\nabla_{tv_a} tv_a). \end{aligned}$$

Proof. By a straightforward computation, we obtain

$$\sum_a g(tfv_a, tfv_a) = \sum_a g(Ptv_a, Ptv_a) = \sum_i g(FPe_i, FPe_i), \quad (3.6)$$

$$\begin{aligned} &\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2) \quad (3.7) \\ &= (n-1)\sum_a g(tv_a, tv_a) - \sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) \\ &\quad + \sum_a g(Ptv_a, Ptv_a). \end{aligned}$$

Thus, using Lemma 2.5 and Lemma 3.5, we have

$$\begin{aligned} &-g(\nabla^2 A, A) \\ &= -\sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) + 3\sum_a \mathrm{tr} A_a^2 - 3\sum_a \mathrm{tr} A_{fa}^2 \\ &\quad +6\sum_a \mathrm{tr} A_a A_{fa} P - \frac{3}{2}\sum_a |[P, A_a]|^2 - 2(n-1)\sum_a g(tv_a, tv_a) \\ &\quad +2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) + 4\sum_a g(Ptv_a, Ptv_a) \\ &\quad -\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2). \end{aligned}$$

Since the normal connection of M is flat, we can choose an orthonormal basis $\{v_a\}$ of $T(M)^\perp$ such that $Dv_a = 0$ for all a. Thus, from Lemma 3.6, we have

$$\begin{aligned} \mathrm{div}(\nabla_{tv_a} tv_a) &= (n-1)g(tv_a, tv_a) + 3g(Ptv_a, Ptv_a) \\ &\quad +\mathrm{tr} A_{fa}^2 - \mathrm{tr} A_a^2 - 2\mathrm{tr} A_a A_{fa} P + \frac{1}{2}|[P, A_a]|^2. \end{aligned}$$

From these equations, we have our assertion. *q.e.d.*

Next, for the later use, we compute the Laplacian for the square of the length of F of an n-dimensional submanifold M immersed in $M^m(c)$.

Lemma 3.8. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$ with parallel mean curvature vector field. Then we have*

$$\begin{aligned} \Delta|F|^2 &= 3c|Pt|^2 + 4\sum_a \mathrm{tr} A_{fa}^2 - 4\sum_a \mathrm{tr} A_a A_{fa} P \\ &\quad - 4\sum_{a,b} g(A_a tv_b, A_a tv_b) + 4\sum_{a,b} g(A_a tv_b, A_b tv_a). \end{aligned}$$

Proof. First we compute

$$
\begin{aligned}
\frac{1}{2}\Delta|F|^2 &= \frac{1}{2}\sum_{i,j}\nabla_j\nabla_j g(Fe_i, Fe_i) \\
&= \sum_{i,j}\nabla_j g((\nabla_j F)e_i, Fe_i) \\
&= \sum_{j,a}(\nabla_j g(A_a e_j, Ptv_a) + \nabla_j g(A_{fa}e_j, tv_a)) \\
&= \sum_{j,a}(g((\nabla_j A)_a e_j, Ptv_a) + g(A_{D_j v_a}e_j, Ptv_a) + g(A_a e_j, (\nabla_j P)tv_a) \\
&\quad + g(A_a e_j, P(\nabla_j t)v_a) + g(A_a e_j, PtD_j v_a) + g((\nabla_j A)_{fa}e_j, tv_a) \\
&\quad + g(A_{D_j f v_a}e_j, tv_a) + g(A_{fa}e_j, (\nabla_j t)v_a) + g(A_{fa}e_j, tD_j v_a)).
\end{aligned}
$$

Since the mean curvature vector field of M is parallel, using the equation of Codazzi, we have

$$
\begin{aligned}
\sum_j g((\nabla_j A)_a e_j, X) &= \sum_j g((\nabla_j A)_a X, e_j) \\
&= \sum_j g((\nabla_X A)_a e_j, e_j) - \frac{3c}{4}g(PX, tv_a) \\
&= -\frac{3c}{4}g(PX, tv_a).
\end{aligned}
$$

Moreover, using formulas for ∇P and ∇t, we obtain our equation.

4 Integral formulas

In this section we give integral formulas for a compact submanifold in a complex space form $M^m(c)$, $c > 0$, with respect to the square of the length of the second fundamental form A ([23]).

We notice that second fundamental form A_V can be considered as a symmetric (n, n)-matrix for any vector V normal to M. For an orthonormal basis $\{e_i\}$ of the tangent space $T_x(M)$ and an orthonormal basis $\{v_a\}$ of the normal space $T_x(M)^\perp$, we put $A_a e_i = \sum_k h^a_{ik} e_k$. Let H_a, $a = 1, \cdots, p$, be a symmetric $(n+1, n+1)$-matrix defined as

$$H_a = \left(\begin{array}{ccc|c} & & & \mu^a_1 \\ & A_a & & \vdots \\ & & & \mu^a_n \\ \hline \mu^a_1 & \ldots & \mu^a_n & 0 \end{array}\right) = \left(\begin{array}{ccc|c} h^a_{11} & \cdots & h^a_{1n} & \mu^a_1 \\ \vdots & \ddots & \vdots & \vdots \\ h^a_{n1} & \cdots & h^a_{nn} & \mu^a_n \\ \hline \mu^a_1 & \ldots & \mu^a_n & 0 \end{array}\right),$$

where $\mu^a_i = -(\sqrt{c}/2) g(tv_a, e_i)$. In the following, we put $|H|^2 = \sum_a \mathrm{tr} H_a^2$.

The main purpose of this section is to prove the following

Theorem 4.1. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$ with parallel mean curvature vector field. Then*

$$\begin{aligned} & -g(\nabla^2 A, A) - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) \\ & + \frac{3c}{4}\sum_a (\mathrm{tr} A^2_{fa} + |[P, A_a]|^2 - 4\mathrm{tr} A_a A_{fa} P) + \frac{3c^2}{4}|FP|^2 \\ & = -\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 + \sum_{a.b} (\mathrm{tr} H_a H_b)^2 - \frac{(n+1)c}{4}|H|^2 + \frac{c}{4}\Delta|F|^2 \\ & \quad + \frac{c}{4}\sum_a (\mathrm{tr} H_a)^2 - \sum_{a,b} \mathrm{tr} H_b \mathrm{tr} H_a^2 H_b + \sum_{a,b} \mathrm{tr} H_b \mathrm{tr}((H_a H_b - H_b H_a) H_a E), \end{aligned}$$

where

$$E = \left(\begin{array}{ccc|c} & & & 0 \\ & 0 & & \vdots \\ & & & 0 \\ \hline 0 & \ldots & 0 & 1 \end{array}\right).$$

Remark. In Theorem 4.1, if the mean curvature vector field μ of M satisfies $f\mu = 0$, then $\sum_{a,b} \mathrm{tr} H_b \mathrm{tr}((H_a H_b - H_b H_a) H_a E) = 0$. For the condition $f\mu = 0$, see section 2.

Before we prove Theorem 4.1, by the consequence of this theorem, we state the following theorems.

Theorem 4.2. *Let M be an n-dimensional minimal submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\begin{aligned}
&-g(\nabla^2 A, A) - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) \\
&+ \frac{3c}{4}\sum_a(\mathrm{tr}A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr}A_a A_{fa} P) + \frac{3c^2}{4}|FP|^2 \\
&\quad = -\sum_{a,b}\mathrm{tr}[H_a, H_b]^2 + \sum_{a.b}(\mathrm{tr}H_a H_b)^2 - \frac{(n+1)c}{4}|H|^2 + \frac{c}{4}\Delta|F|^2.
\end{aligned}$$

If M is compact, then $\int_M |\nabla A|^2 = -\int_M g(\nabla^2 A, A)$ (see [34]). Thus we have

Theorem 4.3. *Let M be an n-dimensional compact submanifold of a complex space form $M^m(c)$, $c > 0$, with parallel mean curvature vector field. Then*

$$\begin{aligned}
&\int_M \Big(|\nabla A|^2 - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) \\
&+ \frac{3c}{4}\sum_a(\mathrm{tr}A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr}A_a A_{fa} P) + \frac{3c^2}{4}|FP|^2\Big) \\
&= \int_M \Big(-\sum_{a,b}\mathrm{tr}[H_a, H_b]^2 + \sum_{a,b}(\mathrm{tr}H_a H_b)^2 - \frac{(n+1)c}{4}|H|^2 \\
&+ \frac{c}{4}\sum_a(\mathrm{tr}H_a)^2 - \sum_{a,b}\mathrm{tr}H_b\mathrm{tr}H_a^2 H_b + \sum_{a,b}\mathrm{tr}H_b\mathrm{tr}((H_a H_b - H_b H_a)H_a E)\Big).
\end{aligned}$$

Theorem 4.4. *Let M be an n-dimensional compact minimal submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\int_M \Big(|\nabla A|^2 - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2)$$

$$+\frac{3c}{4}\sum_a(\mathrm{tr}A_{fa}^2+|[P,A_a]|^2-4\mathrm{tr}A_aA_{fa}P)+\frac{3c^2}{4}|FP|^2\Big)$$
$$=\int_M\Big(-\sum_{a,b}\mathrm{tr}[H_a,H_b]^2+\sum_{a,b}(\mathrm{tr}H_aH_b)^2-\frac{(n+1)c}{4}|H|^2\Big).$$

To prove Theorem 4.1, we prepare some lemmas.

Lemma 4.5 *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c>0$. Then*

$$\begin{aligned}-\sum_{a,b}\mathrm{tr}[H_a,H_b]^2 \;=\; &\sum_{a,b}\Big(-\mathrm{tr}[A_a,A_b]^2\\ &+c(g(A_atv_b,A_atv_b)-g(A_atv_b,A_btv_a))\\ &+c(g(A_atv_a,A_btv_b)-g(A_atv_b,A_btv_a))\\ &+\frac{c^2}{8}(g(tv_a,tv_a)g(tv_b,tv_b)-g(tv_a,tv_b)^2)\Big).\end{aligned}$$

Proof. By the straightforward computation, we have

$$\begin{aligned}&-\sum_{a,b}\mathrm{tr}[H_a,H_b]^2\\ &= 2\sum_{a,b}\mathrm{tr}H_a^2H_b^2-2\sum_{a,b}\mathrm{tr}(H_aH_b)^2\\ &= 2\sum_{a,b}\Big(\sum_{i,j,k,l}h_{ik}^ah_{kj}^ah_{jl}^bh_{li}^b+2\sum_{i,j,l}h_{jl}^bh_{li}^b\mu_i^a\mu_j^a+\sum_{i,j,k}h_{ik}^ah_{kj}^a\mu_j^b\mu_i^b\\ &\quad+\sum_{i,j}\mu_i^a\mu_j^a\mu_j^b\mu_i^b+2\sum_{j,k,l}h_{jk}^ah_{lj}^b\mu_k^a\mu_l^b+(\sum_k(\mu_k^a)^2)(\sum_l(\mu_l^b)^2)\\ &\quad-\sum_{i,j,k,l}h_{ik}^ah_{kj}^bh_{jl}^ah_{li}^b-\sum_{i,j,l}h_{jl}^ah_{li}^b\mu_i^a\mu_j^b-\sum_{i,j,k}h_{ik}^ah_{kj}^b\mu_j^a\mu_i^b\\ &\quad-\sum_{i,j}\mu_i^a\mu_j^b\mu_j^a\mu_i^b-2\sum_{j,k,l}h_{jk}^ah_{lj}^b\mu_k^b\mu_l^a-(\sum_k\mu_k^a\mu_k^b)^2\Big).\end{aligned}$$

Since $A_ae_i=\sum_k h_{ik}^ae_k$ and $\mu_i^a=-(\sqrt{c}/2)g(tv_a,e_i)$, we have

$$-\sum_{a,b}\mathrm{tr}[A_a,A_b]^2=2\sum_{a,b}(\sum_{i,j,k,l}h_{ik}^ah_{kj}^ah_{jl}^bh_{li}^b-\sum_{i,j,k,l}h_{ik}^ah_{kj}^bh_{jl}^ah_{li}^b),$$
$$\sum_{a,b}g(A_atv_b,A_atv_b)=\sum_{a,b}g(A_atv_b,e_i)g(A_atv_b,e_i)$$

$$= \frac{4}{c}\sum_{a,b}\sum_{i,k,l} h^a_{ik}h^a_{il}\mu^b_k\mu^b_l,$$

$$\sum_{a,b} g(A_a tv_b, A_b tv_a) = \frac{4}{c}\sum_{a,b}\sum_{i,k,l} h^a_{ik}h^b_{il}\mu^b_k\mu^a_l,$$

$$\sum_{a,b} g(A_a tv_a, A_b tv_b) = \frac{4}{c}\sum_{a,b}\sum_{i,k,l} h^a_{ik}h^b_{il}\mu^a_k\mu^b_l,$$

$$\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2)$$

$$= \frac{16}{c^2}\sum_{a,b}\Big((\sum_k \mu^a_k)^2(\sum_l \mu^b_l)^2 - (\sum_k \mu^a_k\mu^b_k)^2\Big).$$

From these equations we have our equation. *q.e.d.*

We also have

Lemma 4.6. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\sum_{a,b}(\mathrm{tr}H_aH_b)^2 = \sum_{a,b}(\mathrm{tr}\, A_aA_b)^2 + c|A|^2 - c\sum_a \mathrm{tr}A^2_{fa} + \frac{c^2}{4}|Ft|^2.$$

Lemma 4.7. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$|H|^2 = |A|^2 + \frac{c}{2}|t|^2.$$

Lemma 4.8. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$, $c > 0$. Then*

$$\frac{c}{4}\sum_a(\mathrm{tr}A_a)^2 - \frac{3c}{4}\sum_{a,b}\mathrm{tr}A_b g(A_a tv_a, tv_b) - \sum_{a,b}\mathrm{tr}A_b\mathrm{tr}A^2_aA_b$$

$$= \frac{c}{4}\sum_a(\mathrm{tr}H_a)^2 - \sum_{a,b}\mathrm{tr}H_b\mathrm{tr}H^2_aH_b + \sum_{a,b}\mathrm{tr}H_b\mathrm{tr}((H_aH_b - H_bH_a)H_aE).$$

Proof. From the definition of H_a, we have $\mathrm{tr}H_a = \mathrm{tr}A_a$. Next, by the straightforward computation, we have the following equations

$$\sum_a \mathrm{tr}H^2_aH_b = \sum_a(\sum_{i,j,k} h^a_{ik}h^a_{kj}h^b_{ji} + \sum_{i,j} h^b_{ji}\mu^a_i\mu^a_j + 2\sum_{i,j} h^a_{ji}\mu^a_i\mu^b_j),$$

$$\sum_a \mathrm{tr}((H_aH_b - H_bH_a)H_aE) = \sum_a(\sum_{i,j} h^b_{ji}\mu^a_i\mu^a_j - \sum_{i,j} h^a_{ji}\mu^a_i\mu^b_j).$$

Thus we have

$$\sum_{a,b} \mathrm{tr} H_b \mathrm{tr} H_a^2 H_b + \sum_{a,b} \mathrm{tr} H_b \mathrm{tr}((H_a H_b - H_b H_a) H_a E)$$
$$= \sum_{a,b} (\mathrm{tr} H_b)(\sum_{i,j,k} h_{ik}^a h_{kj}^a h_{ji}^b + 3 \sum h_{ji}^a \mu_i^a \mu_j^b).$$

On the other hand, we have

$$\frac{3c}{4} \sum_{a,b} (\mathrm{tr} A_b) g(A_a t v_a, t v_b) + \sum_{a,b} \mathrm{tr} A_b \mathrm{tr} A_a^2 A_b$$
$$= \sum_{a,b} (\mathrm{tr} H_b)(3 \sum_{i,j} h_{ji}^a \mu_i^a \mu_j^b + \sum_{i,j,k} h_{ik}^a h_{kj}^a h_{ji}^b).$$

From these equations, we have our equation. *q.e.d.*

From Theorem 3.2 and Lemmas 4.5-4.8, we have Theorem 4.1.

5 Pinching theorems of the square of the length of the second fundamental form

We give some pinching theorems with respect to the square of the length of the second fundamental form A, the square of the length of H and the scalar curvature r ([23]). We prepare some inequalities.

Lemma 5.1. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. Then*

$$|\nabla A|^2 \geq \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2).$$

Proof. We put

$$T_1(X, Y, Z) = (\nabla_X B)(Y, Z) + \frac{c}{4}(g(PX, Y)FZ + g(PX, Z)FY).$$

Then

$$\begin{aligned} |T_1^2| &= |\nabla B|^2 + \frac{c^2}{8}\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) + \frac{c^2}{8}\sum_i g(FPe_i, FPe_i) \\ &\quad + c\sum_{i,j} g((\nabla_i B)(Pe_i, e_j), Fe_j). \end{aligned}$$

From the equation of Codazzi, we obtain

$$\begin{aligned} &\sum_{i,j} g((\nabla_i B)(Pe_i, e_j), Fe_j) \\ &\quad = \sum_{i,j} g((\nabla_j B)(e_i, Pe_i), Fe_j) - \frac{c}{4}\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) \\ &\quad\quad - \frac{c}{4}\sum_i g(FPe_i, FPe_i). \end{aligned}$$

Since B is symmetric and P is skew-symmetric, the first term in the right hand side of the equation vanishes. So we have our assertion. *q.e.d.*

Lemma 5.2. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$ with parallel mean curvature vector field. If the equality*

$$|\nabla A|^2 = \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2)$$

holds, then M is a CR submanifold or $c = 0$.

Proof. By the proof of Lemma 5.1, the equation holds if and only if $T_1 = 0$. Suppose that $T_1 = 0$. Then we have

$$D_X(\mathrm{tr}B) = \sum_i (\nabla_X B)(e_i, e_i) = -\frac{c}{2} FPX.$$

Since the mean curvature vector field of M is parallel, we see that $D_X(\mathrm{tr}B) = 0$. When $c \neq 0$, we have $FP = 0$. Then, from Theorem 2.12, M is a CR submanifold. *q.e.d.*

Lemma 5.3. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. Then*

$$\sum_a \mathrm{tr}A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr}A_a A_{fa} P \geq 0.$$

Proof. We put

$$T_2(X, Y) = fB(X, Y) - B(X, PY) - B(PX, Y).$$

Then we have

$$\begin{aligned} |T_2|^2 &= \sum_{i,j} |fB(e_i, e_j) - B(e_i, Pe_j) - B(Pe_i, e_j)|^2 \\ &= \sum_a \mathrm{tr}A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr}A_a A_{fa} P. \end{aligned}$$

Thus we have our inequality. *q.e.d.*

Remark. From the consideration in section 2 and Lemma 5.2, we see that the conditions $T_1 = 0$, $T_2 = 0$ and $FP = 0$ for a submanifold M of CP^m correspond to the notion of the second fundamental form α of a submanifold of S^{2m+1} is parallel. Moreover, if $T_2 = 0$, we see that $f\mu = 0$. When M is a generic submanifold, the condition $T_1 = 0$ was studied by Yano-Kon [44].

Lemma 5.4. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$ with parallel mean curvature vector field. If $T_1 = 0$ and $T_2 = 0$, then $|A|^2$ and $|H|^2$ are constant.*

Proof. Since $T_1 = 0$, Lemma 5.1 implies

$$(\nabla_X B)(Y,Z) = -\frac{c}{4}(g(PX,Y)FZ + g(PX,Z)FY).$$

Moreover, by Lemma 5.2, M is a CR submanifold, and hence $|t|$ is constant. We notice that $|A|^2 = \sum_{a,i} g(A_a e_i, A_a e_i) = \sum_{i,j} g(B(e_i,e_j), B(e_i,e_j)) = |B|^2$. Then we have

$$\nabla_X |A|^2 = 2\sum_{i,j} g((\nabla_X B)(e_i,e_j), B(e_i,e_j)) = c\sum_a g(A_a PX, tv_a).$$

Since $T_2 = 0$, we also have $fB(X,Y) = B(PX,Y) + B(X,PY)$. Hence we obtain $\sum_a g(A_a X, tfv_a) = \sum_a g(A_a PX, tv_a) + \sum_a g(A_a X, Ptv_a)$. From Lemma 2.11 and Lemma 4.7, we see that $|A|^2$ and $|H|^2$ are constant. *q.e.d.*

We need the following lemma (see Chern-do Carmo-Kobayashi [7]).

Lemma 5.5. *Let A and B be symmetric (n,n)-matrices. Then*

$$-\mathrm{tr}(AB - BA)^2 \leq 2\mathrm{tr}A^2\mathrm{tr}B^2,$$

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\bar{A}$ and $\bar{B}$ respectively, while

$$\bar{A} = \left(\begin{array}{cc|c} 0 & 1 & \\ & & 0 \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \qquad \bar{B} = \left(\begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ 0 & -1 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

Moreover, if A_1, A_2 and A_3 are (n,n)-symmetric matrices and if

$$-\mathrm{tr}(A_iA_j - A_jA_i)^2 = 2\mathrm{tr}A_i^2\mathrm{tr}A_j^2, \qquad 1 \leq i,j \leq 3,$$

then at least one of the matrices A_i must be zero.

Using these lemmas, we prove following

Theorem 5.6. *Let M be an n-dimensional compact minimal submanifold of a complex space form $M^m(c)$, $c > 0$. If H satisfies*

$$|H|^2 \leq \frac{(n+1)c}{8 - 4/p},$$

then M is a totally geodesic complex submanifold $M^{n/2}(c)$ or a real hypersurface of $M^m(c)$ with $|A|^2 = (n-1)c/4$.

Proof. Using Lemma 5.5, for a suitable choice of an orthonormal basis $\{v_a\}$, we have

$$\begin{aligned}
&\sum_{a,b}(\mathrm{tr}H_aH_b)^2 - \sum_{a,b}\mathrm{tr}[H_a,H_b]^2 \\
&\quad\le \sum_a(\mathrm{tr}H_a^2)^2 + 2\sum_{a\neq b}\mathrm{tr}H_a^2\mathrm{tr}H_b^2 \\
&\quad= 2(\sum_a \mathrm{tr}H_a^2)^2 - \sum_a(\mathrm{tr}H_a^2)^2 \\
&\quad= (2-\frac{1}{p})(\sum_a \mathrm{tr}H_a^2)^2 - \frac{1}{p}\sum_{a>b}(\mathrm{tr}H_a^2 - \mathrm{tr}H_b^2)^2 \\
&\quad\le (2-\frac{1}{p})|H|^4.
\end{aligned}$$

From Theorem 4.4, Lemma 5.1 and Lemma 5.3, we obtain

$$\begin{aligned}
0 \;\le\; & \int_M \Big(|\nabla A|^2 - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) \\
& + \frac{3c}{4}\sum_a(\mathrm{tr}A_{fa}^2 + |[P,A_a]|^2 - 4\mathrm{tr}A_aA_{fa}P) + \frac{3c^2}{4}|FP|^2\Big) \\
\le\; & \int_M \Big((2-\frac{1}{p})|H|^2 - \frac{(n+1)c}{4}\Big)|H|^2.
\end{aligned}$$

Thus we see that if $|H|^2 \le (n+1)c/(8-4/p)$, then $FP = 0$ and M is a CR submanifold by Theorem 2.12. Moreover, we have $|\nabla A|^2 = (c^2/8)(n-q)q$, where $q = |t|^2 = \sum_a g(tv_a, tv_a)$. Then Lemma 5.1 and Lemma 5.3 imply that $T_1 = 0$ and $T_2 = 0$. Therefore, by Lemma 5.4, $|A|^2$ and $|H|^2$ are constant. Consequently we see that $|H|^2 = (n+1)c/(8-4/p)$ or $|H|^2 = 0$.

Suppose that $|H|^2 = 0$. From Lemma 4.7, we have $A_a = 0$ and $tv_a = 0$ for all v_a. Thus M is a totally geodesic complex submanifold, that is, M is a complex space form $M^{n/2}(c)$ of $M^m(c)$.

Next we suppose that $|H|^2 = (n+1)c/(8-4/p)$. Since $\sum_{a>b}(\mathrm{tr}H_a^2 - \mathrm{tr}H_b^2)^2 = 0$, we have $\mathrm{tr}H_a^2 = \mathrm{tr}H_b^2$ for any $a \neq b$. Thus, from Lemma 5.5, we have $p = 1$ or $p = 2$.

Suppose that $p = 2$. If $\dim\mathcal{D}^\perp = 0$, then M is a complex submanifold of $M^m(c)$. Hence we have $PA_a + A_aP = 0$ and $A_{fa} = PA_a$ (c.f. [44]). On the other hand, we obtain $\mathrm{tr}A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr}A_a A_{fa} P = 0$. Thus we see that $A_a = 0$ for all a and that M is a totally geodesic complex submanifold $M^{n/2}(c)$ of $M^m(c)$.

If there exist vector fields $X \in \mathcal{D}^\perp$ and $V \in N$, where N is the orthogonal complement of $J\mathcal{D}_x^\perp$ in $T_x(M)^\perp$, then $JX \in J\mathcal{D}^\perp$ and $JV \in N$. So we have $\dim T_x(M)^\perp \geq 3$. This is a contradiction. Thus we see that if $\dim \mathcal{D}^\perp \neq 0$, then $\dim N = 0$, that is, M is a generic submanifold of $M^m(c)$.

Suppose that $\dim \mathcal{D}^\perp \neq 0$. Since M is generic, we have $fv = 0$ for any $v \in T_x(M)^\perp$. Then, we obtain

$$\sum_a (\mathrm{tr}A_{fa}^2 + |[P, A_a]|^2 - 4\mathrm{tr}A_a A_{fa} P) = \sum_a |[P, A_a]|^2 = 0,$$

that is, $A_a P = P A_a$ for all v_a. Changing the order of the orthonormal basis $\{e_i\}$ of $T_x(M)$, we suppose $e_1, e_2 \in \mathcal{D}_x^\perp$, $e_3, \cdots, e_n \in \mathcal{D}_x$ and $v_a = Je_a$ $(a = 1, 2)$. Since $A_a P = P A_a$ for all a, we have

$$g(A_a tV, PX) = -g(A_a PtV, X) = 0$$

for any tangent vector field X and normal vector field V. So we have $g(A_a e_i, e_j) = h_{i,j}^a = h_{j,i}^a = 0$ for $i = 1, 2$ and $j \geq 3$. Since $\mathrm{rank} H_a = 2$ and $\mathrm{tr}A_a = 0$ for $a = 1, 2$, the matrices H_a $(a = 1, 2)$ are represented as

$$H_1 = \left(\begin{array}{cc|ccc|c} 0 & h_{12}^1 & & & & \sqrt{c}/2 \\ h_{12}^1 & 0 & & 0 & & 0 \\ \hline & & & & & 0 \\ 0 & & & 0 & & \vdots \\ & & & & & 0 \\ \hline \sqrt{c}/2 & 0 & 0 & \ldots & 0 & 0 \end{array}\right)$$

and

$$H_2 = \left(\begin{array}{cc|ccc|c} 0 & h_{12}^2 & & & & 0 \\ h_{12}^2 & 0 & & 0 & & \sqrt{c}/2 \\ \hline & & & & & 0 \\ 0 & & & 0 & & \vdots \\ & & & & & 0 \\ \hline 0 & \sqrt{c}/2 & 0 & \ldots & 0 & 0 \end{array}\right).$$

By Lemma 5.5, there exist an orthogonal matrix $T = (t_{ij})$ and scalars α and β such that $TH_1T^{-1} = \alpha\bar{A}$ and $TH_2T^{-1} = \beta\bar{B}$. By the straightforward computation, we have $t_{11} = 0, t_{12} = 0, t_{21} = 0$ and $t_{22} = 0$. Hence we obtain $A_a = 0$ $(a = 1, 2)$.

On the other hand, from Lemma 4.7 and $\sum_a g(tv_a, tv_a) = p = 2$, we have

$$|A|^2 = |H|^2 - c = \frac{(n-5)c}{6}.$$

Consequently, we have $n = 5$ and hence $2m = 7$. Thus is a contradiction. Hence we see that if $|H|^2 = (n+1)c/(8-4/p)$, then M is a real hypersurface with $|A|^2 = (n-1)c/4$. Thus we have our theorem. *q.e.d.*

From Theorem 5.6, we have

Theorem 5.7. *Let M be an n-dimensional compact minimal submanifold of a complex space form $M^m(c), c > 0$. If the second fundamental form A satisfies*

$$|A|^2 \leq \frac{c}{4}\Big(\frac{n+1}{2-1/p} - 2p\Big),$$

then M is a totally geodesic complex submanifold $M^{n/2}(c)$ or a real hypersurface of $M^m(c)$ with $|A|^2 = (n-1)c/4$.

Proof. Since $p \geq |t|^2$, we have

$$|A|^2 \leq \frac{c}{4}\Big(\frac{n+1}{2-1/p} - 2|t|^2\Big),$$

from Lemma 4.7, we obtain $|H|^2 \leq (n+1)c/(8-4/p)$. Thus, from Theorem 5.6, we have our conclusion. *q.e.d.*

Remark. This theorem is an extension of the pinching theorem with respect to the square of the length of the second fundamental form of compact minimal submanifolds in CP^m given by Yano-Kon [45, Theorem 3.2, p.150]. If M is a real hypersurface of $M^m(c)$ with $|A|^2 = (n-1)c/4$, we see that $PA_a = A_aP$. Then M has at most three constant principal curvatures. When the ambient manifold $M^m(c)$ is $CP^{(n+1)/2}$ of constant holomorphic sectional curvature 4, a compact minimal real hypersurface of M with the second fundamental form A which satisfies $|A|^2 = n-1$ is equivalent to $\pi(S^{2p+1}(((2p+1)/(n+1))^{1/2}) \times S^{2q+1}(((2q+1)/(n+1))^{1/2}))$, $2(p+q) = n$.

Corollary 5.8. *Let M be an n-dimensional compact minimal submanifold of $M^m(c), c > 0$. If the scalar curvature r of M satisfies*

$$r \geq \frac{c}{4}\Big(n(n+2) - \frac{n+1}{2-1/p}\Big),$$

then M is a totally geodesic complex submanifold $M^{n/2}(c)$.

Proof. Since the scalar curvature r of M is given by

$$r = \frac{c}{4}\Big((n-1)n + 3|P|^2\Big) - |A|^2,$$

Lemma 4.7 implies

$$\begin{aligned} r &= \frac{c}{4}\Big(n(n-1) + 3|P|^2\Big) + \frac{c}{2}|t|^2 - |H|^2 \\ &= \frac{c}{4}\Big(n(n+2) - |t|^2\Big) - |H|^2 \\ &\leq \frac{n(n+2)c}{4} - |H|^2. \end{aligned}$$

Hence we see that if r satisfies the inequality in the statement, then $|H|^2 \leq (n+1)c/(8-4/p)$. By the proof of Theorem 5.6, M is a totally geodesic complex submanifold $M^{n/2}(c)$ or a real hypersurface with $|H|^2 = (n+1)c/4$ of $M^m(c)$. When M is a real hypersurface with $|H|^2 = (n+1)c/4$, we have $r = (n^2+n-2)c/4$. This is a contradiction. Thus we have our conclusion. *q.e.d.*

6 Semi-flat normal connection

Let M be a n-dimensional submanifold of a complex space form $M^m(c)$. We consider the condition that the normal connection of M is *semi-flat*, that is, the normal curvature tensor $R^\perp$ of M satisfies $R^\perp(X,Y)U = (c/2)g(X,PY)fU$ for any vector fields X and Y tangent to M and any vector field U normal to M. We put

$$S_1(X,Y) = g([A_V, A_U]X, Y) - \frac{1}{4}c(g(FY,U)g(FX,V) - g(FX,U)g(FY,V)).$$

By the straightforward computation using the equation of Ricci, the normal connection of M is semi-flat if and only if $S_1 = 0$. Thus we have the following two lemmas.

Lemma 6.1. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. The normal connection of M is semi-flat if and only if the following equation holds*

$$\begin{aligned} &-\sum_{a,b} \mathrm{tr}[A_a, A_b]^2 - c\sum_{a,b} g([A_a, A_b]tv_a, tv_b) \\ &\quad + \frac{1}{8}c^2 \sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2) = 0. \end{aligned}$$

Proposition 6.2. *Let M be an n-dimensional submanifold of a complex space form $M^m(c)$. Then we have*

$$|S_1|^2 = -\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 - \frac{c}{2}|\nabla f|^2.$$

Proof. From Lemma 2.6 and Lemma 4.5, we have

$$\begin{aligned} |S_1|^2 &= -\sum_{a,b} \mathrm{tr}[A_a, A_b]^2 - c\sum_{a,b} g([A_a, A_b]tv_a, tv_b) \\ &\quad + \frac{c^2}{8}\sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2) \\ &= -\sum_{a,b} \mathrm{tr}[H_a, H_b]^2 - c\sum_{a}(g(A_a tv_b, A_a tv_b) - g(A_a tv_b, A_b tv_a)). \end{aligned}$$

Since $(\nabla_X f)V = -FA_V X - B(X, tV)$, we obtain

$$|\nabla f|^2 = 2\sum_{a}(g(A_a tv_b, A_a tv_b) - g(A_a tv_b, A_b tv_a)).$$

From these equations we have our result. *q.e.d.*

Theorem 6.3. *Let M be an n-dimensional compact minimal submanifold with semi-flat normal connection of a complex space form $M^m(c)$, $c > 0$. If $|H|^2 \leq (n-2)c/4$, then M is a totally geodesic complex submanifold $M^{n/2}(c)$ of $M^m(c)$.*

Proof. From Lemma 2.6 and Lemma 3.8, we have

$$\Delta|F|^2 = 2c|Pt|^2 + 4\sum_a \mathrm{tr} A_{fa}^2 - 2|\nabla f|^2.$$

Hence, from Theorem 4.2 and Lemma 2.6, we have

$$\begin{aligned}
&\sum_{a,b}(\mathrm{tr} H_a H_b)^2 - \frac{(n-2)c}{4}|H|^2 \\
&= -g(\nabla^2 A, A) - \frac{c^2}{8}(|P|^2|t|^2 + |FP|^2) + \frac{3c}{4}(|H|^2 - \sum_a \mathrm{tr} A_{fa}^2) + \frac{c}{4}|[P, A_a]|^2 \\
&\quad + \frac{c}{2}(\sum_a \mathrm{tr} A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr} A_a A_{fa} P).
\end{aligned}$$

Thus we have, by Lemma 5.1 and Lemma 5.3,

$$\int_M \Big(\sum_{a,b}(\mathrm{tr} H_a H_b)^2 - \frac{(n-2)c}{4}|H|^2\Big) \geq 0.$$

We now choose an orthonormal basis $\{v_a\}$ such that $\mathrm{tr} H_a H_b = 0$ for $a \neq b$. Then $\sum_{a,b}(\mathrm{tr} H_a H_b)^2 = \sum_a (\mathrm{tr} H_a^2)^2 \leq (\sum_a \mathrm{tr} H_a^2)^2$. Hence we have

$$\int_M \Big(|H|^2 - \frac{(n-2)c}{4}\Big)|H|^2 \geq 0.$$

From Lemma 5.2, M is a CR submanifold of $M^m(c)$. By a similar method of the proof in Theorem 5.6, we see that if $|H|^2 \leq (n-2)c/4$, then $|H|^2 = (n-2)c/4$ or $|H|^2 = 0$. When $|H|^2 = 0$, M is a totally geodesic complex space form $M^{n/2}(c)$ of $M^m(c)$. We suppose that $|H|^2 = (n-2)c/4$. Then, we have

$$\sum_a |[P, A_a]|^2 = 0, \qquad |H|^2 = |A|^2 + \frac{c}{2}|t|^2 = \sum_a \mathrm{tr} A_{fa}^2.$$

Since $|A|^2 = \sum_a \mathrm{tr} A_a^2 \geq \sum_a \mathrm{tr} A_{fa}^2$, we have $t = 0$. Thus M is a complex submanifold of $M^m(c)$. Then we generally see that $PA_a + A_aP = 0$ for all a. Combining this to $PA_a = A_aP$, we have $PA_a = 0$, and hence $A_a = 0$, $n = 2$. Consequently, M is a totally geodesic complex space form $M^{n/2}(c)$ of $M^n(c)$. *q.e.d.*

From Theorem 6.3, we have the following results.

Theorem 6.4. *Let M be an n-dimensional compact minimal submanifold with semi-flat normal connection of $M^m(c)$. If $|A|^2 \leq (n-2p-2)c/4$, then M is a totally geodesic complex submanifold $M^{n/2}(c)$ of $M^m(c)$.*

Corollary 6.5. *Let M be an n-dimensional compact minimal submanifold with semi-flat normal connection of $M^m(c)$. If the scalar curvature r of M satisfies $r \geq (n^2+n+2)c/4$, then M is a complex space form $M^{n/2}(c)$ of $M^m(c)$.*

We next prove a reduction theorem of the codimension of a submanifold of a complex space form.

Theorem 6.6. *Let M be an n-dimensional submanifold with semi-flat normal connection of a complex space form $M^m(c), c > 0$. If $\nabla f = 0$, then M is a totally geodesic complex submanifold of $M^m(c)$ or a generic submanifold of some $M^{n+q}(c)$ in $M^m(c)$.*

Proof. From the assumptions, Lemma 3.8 implies

$$\Delta|F|^2 = 2c|Pt|^2 + 4\sum_a \mathrm{tr} A_{fa}^2.$$

Moreover, we see that $|f|^2$ is constant by $\nabla f = 0$. Then $|t|^2$ and $|F|^2$ are also constant. Hence we have $A_{fa} = 0$ and $Pt = 0$. This means that M is a CR submanifold. If $t = 0$, M is a totally geodesic complex submanifold, that is, complex space form $M^{n/2}(c)$. If $t \neq 0$, then we have $g(D_XV, fU) = -g(V, (\nabla_X f)U) = 0$ for any vector field V in $FT(M)$. Thus D_XV is in $FT(M)$. Therefore, $FT(M)$ is the parallel subbundle in the normal bundle $T(M)^\perp$. From this and $A_{fa} = 0$, we have our assertion (see [4, Lemma 5.9]). *q.e.d.*

Remark. In [46, Theorem 3.14, p.236], it was proved that if an n-dimensional compact minimal CR submanifold M of CP^m with semi-flat normal connection and $\nabla f = 0$ satisfies $|A|^2 \leq (n-1)q$, then M is $CP^{n/2}$, or M is a generic minimal submanifold of some $CP^{(n+q)/2}$ in CP^m and is $\pi(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k))$, $\quad n+1 = \sum_{i=1}^k m_i$, $\quad 1 = \sum_{i=1}^k r_i^2$, $\quad q = k-1$, where $m_1, \cdots, m_k$ are odd numbers. Then $n+k$ is also odd.

From Proposition 6.2, we see that $H_a H_b = H_b H_a$ for all a and b if and only if the normal connection of M is semi-flat and $\nabla f = 0$.

7 Pinching problem of the sectional curvature

In this section we give some pinching theorems with respect to the sectional curvature of the compact minimal submanifold in a complex projective space ([21]). If M is compact, we have $\int_M |\nabla A|^2 = -\int_M g(\nabla^2 A, A)$ (see [34]). Therefore Lemma 3.7 implies

Theorem 7.1. *Let M be an n-dimensional compact minimal submanifold in a complex projective space CP^m with flat normal connection. Then*

$$\begin{aligned}
&\int_M \Big(|\nabla A|^2 - 2\sum_{i,a} g(Pe_i, Pe_i)g(tv_a, tv_a) - 2\sum_i g(FPe_i, FPe_i)\\
&\quad + \frac{1}{2}(\sum_a \mathrm{tr}A_{fa}^2 + \sum_a |[P, A_a]|^2 - 4\sum_a \mathrm{tr}A_a A_{fa} P)\\
&\quad + \sum_{a,b}(g(tv_a, tv_a)g(tv_b, tv_b) - g(tv_a, tv_b)^2)\Big)\\
&= \int_M \Big(\sum_a \mathrm{tr}A_a^2 - \sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j)\\
&\qquad + 8\sum_i g(FPe_i, FPe_i) - \frac{1}{2}\sum_a \mathrm{tr}A_{fa}^2\Big).
\end{aligned}$$

Theorem 7.2. *Let M be an n-dimensional compact minimal submanifold in a complex projective space CP^m with flat normal connection. If the second fundamental form A satisfies $\sum_a \mathrm{tr}A_{fa}^2 \geq 16|FP|^2$, and if the sectional curvature K of M satisfies $K \geq 1/n$, then M is the geodesic minimal hypersphere $\pi(S^1(\sqrt{1/2m}) \times S^{2m-1}(\sqrt{(2m-1)/2m}))$ in CP^m.*

Proof. From Lemma 5.1 and Lemma 5.3, we see that the left-hand side of the equation in Theorem 7.1 is non-negative. Next we prove that the right-hand side of this is non-positive.

Choosing an orthonormal basis $\{e_i\}$ of $T_x(M)$ such that $A_a e_i = h_i^a e_i$, $i = 1, \cdots, n$, we have

$$\begin{aligned}
&\sum_{i,j} g((R(e_i, e_j)A)_a e_i, A_a e_j)\\
&\qquad = \sum_{i,j} g(R(e_i, e_j)A_a e_i, A_a e_j) - \sum_{i,j} g(A_a R(e_i, e_j)e_i, A_a e_j)
\end{aligned}$$

$$= \frac{1}{2} \sum_{i,j} (h_i^a - h_j^a)^2 K_{ij},$$

where K_{ij} denotes the sectional curvature of M with respect to the section spanned by e_i and e_j. Since $K_{ij} \geq 1/n$, we obtain

$$\sum_{i,j} g((R(e_i, e_j)A)_a e_i, A_a e_j) \geq \frac{1}{2n} \sum_{i,j} (h_i^a - h_j^a)^2 \geq \mathrm{tr} A_a^2.$$

The left-hand side of this inequality is independent of the choice of an orthonormal basis $\{e_i\}$. Hence we have

$$\sum_a \mathrm{tr} A_a^2 - \sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) \leq 0.$$

Consequently, Theorem 7.1, Lemma 5.1 and Lemma 5.3 imply

$$|\nabla A|^2 - 2 \sum_{i,a} g(Pe_i, Pe_i) g(tv_a, tv_a) - 2 \sum_a g(Ptv_a, Ptv_a) = 0, \quad (7.1)$$

$$\sum_{a,b} (g(tv_a, tv_a) g(tv_b, tv_b) - g(tv_a, tv_b)^2) = 0, \quad (7.2)$$

$$8 \sum_a g(FPe_i, FPe_i) - \frac{1}{2} \sum_a \mathrm{tr} A_{fa}^2 = 0. \quad (7.3)$$

By (7.1) and Lemma 5.2, M is a CR submanifold. Thus, from (7.3), we have $A_{fa} = 0$ for all v_a. On the other hand, (7.2) implies $q = 1$ or $q = 0$.

Suppose that $q = 1$. Using Lemma 2.5, we obtain

$$\sum_{i,a} g([A_{fa}, A_a] e_i, Pe_i) = -2h(p-1) = 0.$$

When $p = 1$, from the theorem in [17], M is a geodesic minimal hypersphere. When $h = 0$, we have $n = q = 1$ and $K = 0$. This is a contradiction.

We next suppose that $q = 0$. Then M is a complex submanifold and $n = h$. On the other hand, again using Lemma 2.5, we have $hp = 0$, and hence $h = 0$. This is a contradiction. *q.e.d.*

When M is a CR minimal submanifold, by Theorem 2.12, we have $FP = 0$. Hence the condition $\sum_a \mathrm{tr} A_{fa}^2 \geq 16|FP|^2$ in Theorem 7.2 is automatically satisfied. So we have

Theorem 7.3. *Let M be an n-dimensional compact minimal CR submanifold in a complex projective space CP^m with flat normal connection. If the sectional curvature K of M satisfies $K \geq 1/n$, then M is the geodesic minimal hypersphere $\pi(S^1(\sqrt{1/2m}) \times S^{2m-1}(\sqrt{(2m-1)/2m}))$ in CP^m.*

Next we give pinching theorems for minimal submanifolds in CP^m with semi-flat normal connection. Using (3.6), (3.7), Lemma 2.6 and Lemma 3.5, we have

Lemma 7.4. *Let M be an n-dimensional compact minimal submanifold in CP^m with semi-flat normal connection. Then*

$$\begin{aligned}
&\int_M \Big(|\nabla A|^2 - 2\sum_{i,a} g(Pe_i, Pe_i) g(tv_a, tv_a) - 2\sum_i g(FPe_i, FPe_i) \\
&+\frac{3}{2}(\sum_a \mathrm{tr} A_{fa}^2 + \sum_a |[P, A_a]|^2 - \sum_a 4\mathrm{tr} A_a A_{fa} P) \\
&+4\sum_i g(FPe_i, FPe_i)\Big) \\
&\quad = \int_M \Big(-\sum_{i,j,a} g((R(e_i, e_j)A)_a e_i, A_a e_j) + 3\sum_a \mathrm{tr} A_a^2 \\
&\quad -\frac{3}{2}\sum_a \mathrm{tr} A_{fa}^2 - 2(n-1)\sum_a g(tv_a, tv_a) \\
&\quad -\sum_{a,b}(g(tv_a, tv_a) g(tv_b, tv_b) - g(tv_a, tv_b)^2)\Big).
\end{aligned}$$

From this, we have

Theorem 7.5. *Let M be an n-dimensional compact minimal submanifold in a complex projective space CP^m with semi-flat normal connection. If the sectional curvature K of M satisfies $K \geq 3/n$, then M is the complex projective space $CP^{\frac{n}{2}}$ in CP^m.*

Proof. From Lemma 5.1 and Lemma 5.3, we see that the left-hand side of the equation in Lemma 7.4 is non-negative. Next we prove that the right-hand side of this is non-positive.

Since $K_{ij} \geq 3/n$, by a similar method in the proof of Theorem 7.2, we obtain

$$-\sum_{i,j,a} g((R(e_i,e_j)A)_a e_i, A_a e_j) + 3\sum_a \mathrm{tr}A_a^2 \le 0.$$

Consequently, we have

$$\frac{3}{2}\sum_a \mathrm{tr}A_{fa}^2 + 2(n-1)\sum_a g(tv_a, tv_a) = 0.$$

Thus, we obtain $A_{fa} = 0$ for all v_a and $t = 0$. Therefore M is a complex submanifold in CP^m and $A_a = 0$ for all v_a. Thus M is a real n-dimensional totally geodesic complex submanifold in CP^m, that is, $CP^{\frac{n}{2}}$. *q.e.d.*

Next we give a pinching theorem for a compact minimal CR submanifold in CP^m with semi-flat normal connection.

Theorem 7.6. *Let M be a compact n-dimensional minimal CR submanifold in a complex projective space CP^m with semi-flat normal connection. If the sectional curvature K of M satisfies $K \ge 1/n$, then M is a totally geodesic complex projective space $CP^{n/2}$ or a geodesic minimal hypersphere $\pi(S^1(\sqrt{1/(n+1)}) \times S^n(\sqrt{n/(n+1)}))$ of some $CP^{(n+1)/2}$ in CP^m.*

Proof. Since M is a CR submanifold in CP^m, we can take an orthonormal basis $\{v_a\}$ of $T_x(M)^\perp$ such that $\{v_1, \cdots, v_q\}$ form an orthonormal basis of $FT_x(M)$ and $\{v_{q+1}, \cdots, v_p\}$ form an orthonormal basis of $fT_x(M)^\perp$.

If $q = 0$, M is a complex submanifold in CP^m. Then the normal connection of M is semi-flat if and only if M is a totally geodesic complex projective space $CP^{n/2}$ by a theorem of Ishihara [9].

We next suppose that $q \ge 1$. Since the normal connection of M is semi-flat, we have $A_{fV}PX = 0$ and $A_{fV}tU = \beta tU$ for any vector X tangent to M and any vectors U, V normal to M (see Chen [4, Lemma 5.3, Lemma 5.6]). Thus, by the minimality of M, we see that $\beta = 0$ and $A_{fV} = 0$.

Let V be in $FT(M)$. Then we have

$$\begin{aligned} g(fD_XV, fU) &= -g((\nabla f)V, fU) \\ &= g(FA_VX, fU) + g(B(X,tV), fU) \\ &= g(A_{fU}X, tV) = 0. \end{aligned}$$

This means that $FT(M)$ is parallel, that is, D_XV is in $FT(M)$. Moreover, we have $R^\perp(X,Y)V = 0$ for any $V \in FT(M)$. So we can choose an orthonormal basis $\{v_\lambda\}$ in such a way that $D_X v_\lambda = 0$, $\lambda = 1, \cdots, q$. We notice that $\nabla_X t v_\lambda = -PA_\lambda X$. Since P is skew-symmetric and A_λ is symmetric, we have $\mathrm{div}(tv_\lambda) = -\mathrm{tr} PA_\lambda = 0$.

From Lemma 2.6 and Lemma 3.5, we obtain

$$\begin{aligned} g(\nabla^2 A, A) &= \sum_{i,j,\lambda} g((R(e_i, e_j)A)_\lambda e_i, A_\lambda e_j) \\ &\quad +3(-\sum_a \mathrm{tr} A_\lambda^2 + \frac{1}{2}\sum_a |[P, A_\lambda]|^2) + 3q(q-1). \end{aligned}$$

On the other hand, Lemma 3.6 implies

$$\sum_\lambda \mathrm{div}(\nabla_{tv_\lambda} tv_\lambda) = (n-1)q - \sum_\lambda \mathrm{tr} A_\lambda^2 + \frac{1}{2}\sum_\lambda |[P, A_\lambda]|^2.$$

Using these equations, we have

$$\begin{aligned} &-g(\nabla^2 A, A) - 2hq + \frac{1}{2}\sum_\lambda |[P, A_\lambda]|^2 + q(q-1) \\ &\quad = \sum_\lambda \mathrm{tr} A_\lambda^2 - \sum_{i,j,\lambda} g((R(e_i, e_j)A)_\lambda e_i, A_\lambda e_j) - 2\sum_\lambda \mathrm{div}(\nabla_{tv_\lambda} tv_\lambda). \end{aligned}$$

Thus we have

$$\begin{aligned} &\int_M \Big(|\nabla A|^2 - 2hq + \frac{1}{2}\sum_\lambda |[P, A_\lambda]|^2 + q(q-1)\Big) \\ &\quad = \int_M \Big(\sum_\lambda \mathrm{tr} A_\lambda^2 - \sum_{i,j,\lambda} g((R(e_i, e_j)A)_\lambda e_i, A_\lambda e_j)\Big). \end{aligned}$$

By Lemma 5.1, we see that the left-hand side of this equation is non-negative. Next we prove that the right-hand side of the equation above is non-positive. By a similar method in the proof of Theorem 7.2, we have

$$\sum_\lambda \mathrm{tr} A_\lambda^2 - \sum_{i,j,\lambda} g((R(e_i, e_j)A)_\lambda e_i, A_\lambda e_j) \leq 0.$$

Consequently, we obtain

$$|\nabla A|^2 = 2hq, \quad PA_\lambda = A_\lambda P, \quad q(q-1) = 0.$$

Hence we have $q = 1$ and M is a real hypersurface in some $CP^{(n+1)/2}$ in CP^m (cf. [46, p.227]). Therefore, using Theorem 7.3, we have our result (see also

[17]). *q.e.d.*

If $n > p + 2$, we see that $\nabla f = 0$ and M is a CR submanifold in CP^m with the second fundamental form A which satisfies $A_{fV} = 0$ for any vector V normal to M (see Okumura [29], [30]). Therefore, Theorem 7.6 implies

Theorem 7.7. *Let M be a compact n-dimensional minimal submanifold in CP^m with semi-flat normal connection. If the sectional curvature K of M satisfies $K \geq 1/n$, and if $n > p+2$, then M is a totally geodesic complex projective space $CP^{n/2}$ or a geodesic minimal hypersphere $\pi(S^1(\sqrt{1/(n+1)}) \times S^n(\sqrt{n/(n+1)}))$ of some $CP^{(n+1)/2}$ in CP^m.*

8 Reduction of the codimension

In this section we prove the following reduction theorem of a codimension ([22]). If a CR submanifold satisfies $\dim \mathcal{D} > 0$ and $\dim \mathcal{D}^\perp > 0$, then it is said to be proper.

Theorem 8.1. *Let M be a compact n-dimensional minimal proper CR submanifold of a complex projective space CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$ for any vector X tangent to M, then M is a real hypersurface of some $CP^{(n+1)/2}$ in CP^m.*

First of all, we prove

Lemma 8.2. *Let M be a compact n-dimensional minimal CR submanifold of CP^m which is not a complex submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then M is a real projective space RP^n or $q = 1$, that is,* $\dim \mathcal{D}_x^\perp = 1$.

Proof. Since M is minimal, by the assumption, we have

$$\begin{aligned} S(X,X) &- (n-1)g(X,X) \\ &= 3g(PX,PX) - \sum_a g(A_a^2 X, X) \geq 0. \end{aligned} \tag{8.1}$$

If $P = 0$, then M is a totally real submanifold of CP^m. Moreover the above inequality implies that $A_a = 0$ for all a. So M is totally geodesic in CP^m, and hence M is a real projective space RP^n by a theorem of Abe [1].

We next suppose $P \neq 0$. For any normal vector fields U and V, we have $A_U tV = 0$. Thus we obtain

$$0 = (\nabla_X A)_U tV - A_U P A_V X + A_U A_{fV} X,$$

from which

$$\begin{aligned} g((\nabla_X A)_U Y, tV) &= g((\nabla_X A)_U tV, Y) \\ &= g(A_U P A_V X, Y) - g(A_U A_{fV} X, Y). \end{aligned}$$

By the equation of Codazzi, we have

$$\begin{aligned} -2g(X,PY)g(tU,tV) &= g(A_U P A_V X, Y) + g(A_V P A_U X, Y) \\ &\quad - g(A_U A_{fV} X, Y) + g(A_{fV} A_U X, Y). \end{aligned} \tag{8.2}$$

Since $\sum_a g(tv_a, tv_a) = q$, we obtain

$$\begin{aligned}2\sum_a g(A_aPA_aX, PX) - \sum_a g((A_aA_{fa} - A_{fa}A_a)X, PX)\\ = 2qg(PX, PX).\end{aligned}$$

On the other hand, we have

$$S(PX, PX) = (n+2)g(PX, PX) - \sum_a g(A_aPX, A_aPX).$$

From these equations, we obtain

$$\begin{aligned}&\sum_a g(A_aPX, A_aPX)\\ &= \sum_a g(A_aPA_aX, PX) - \frac{1}{2}\sum_a((A_aA_{fa} - A_{fa}A_a)X, PX)\\ &\quad +(n+2-q)g(PX, PX) - S(PX, PX).\end{aligned}$$

Thus we have, for any orthonormal basis $\{e_i\}$ of $T_x(M)$,

$$\begin{aligned}&\frac{1}{2}\sum_a |[P, A_a]|^2 \qquad (8.3)\\ &= (n+2-q)h - \sum_i S(Pe_i, Pe_i) + \frac{1}{2}\sum_a \mathrm{tr}P(A_aA_{fa} - A_{fa}A_a)\\ &= -hq + \sum_a \mathrm{tr}A_a^2 + \sum_a \mathrm{tr}PA_aA_{fa}.\end{aligned}$$

Since $S(Pe_i, Pe_i) \geq n-1$, we have $\sum_a \mathrm{tr}A_a^2 \leq 3h$. From these equations, we see that

$$\frac{1}{2}\sum_a |[P, A_a]|^2 \leq h(3-q) + \sum_a \mathrm{tr}PA_aA_{fa}.$$

We take a basis $\{v_1, \cdots, v_p\}$ of $T_x(M)^\perp$ such that $\{v_1, \cdots, v_q\}$ is an orthonormal basis of $FT_x(M)$ and $\{v_{q+1}, \cdots, v_p\}$ is that of N_x. By (8.2), we have $\sum_{\lambda=q+1}^p \mathrm{tr}PA_\lambda A_{f\lambda} = \sum_{\lambda=q+1}^p \mathrm{tr}A_\lambda PA_\lambda P$. From these and

$$\frac{1}{2}\sum_{a=1}^p |[P, A_a]|^2 = \frac{1}{2}\sum_{x=1}^q |[P, A_x]|^2 + \sum_{\lambda=q+1}^p \mathrm{tr}A_\lambda PA_\lambda P - \sum_{\lambda=q+1}^p \mathrm{tr}P^2A_\lambda^2,$$

we obtain

$$\begin{aligned}0 &\leq \frac{1}{2}\sum_{x=1}^q |[P, A_x]|^2 + \sum_{i=1}^n \sum_{\lambda=q+1}^p g(A_\lambda Pe_i, A_\lambda Pe_i)\\ &\leq h(3-q).\end{aligned}$$

Thus we see that $q \leq 3$. Suppose $q = 3$. Then we have $PA_x = A_xP$ for $x = 1, 2, 3$ and $A_\lambda P = 0$ for $\lambda = 4, \cdots, p$. Hence we have $A_{fV}PX$=0 for any normal vector field V and tangent vector field X. From (8.2), we have

$$2g(PX, PY)g(tV, tU) = g(A_UA_VPX, PY) + g(A_VA_UPX, PY)$$

for any tangent vector fields X, Y and normal vector fields $U, V \in FT_x(M)$. So we obtain $A_x^2X = X$ and $g(A_xX, A_yX) = g(X, X)g(tv_x, tv_y)$ for any $X \in H$ and $x, y = 1, 2, 3$. From this, for a fixed x, taking a tangent vector $Y \neq 0$ which satisfies $A_xY = kY$, $k = \pm 1$, we obtain

$$g(A_xY, A_yY) = kg(Y, A_yY) = 0, \quad x \neq y.$$

Thus we have $g(Y, A_yY) = 0$. This is a contradiction.

Suppose $q = 2$. We have $A_{fx} = 0$ for $x = 1, 2$. Then we obtain

$$\begin{aligned}
&\sum_{x,i,j} g(\nabla_j tv_a, e_i)g(e_j, \nabla_i tv_a) \\
&\quad = \sum_{x,i,j} g(-PA_xe_j + tD_jv_x, e_i)g(-PA_xe_i + tD_iv_x, e_j) \\
&\quad = -\sum_{x,j} g(PA_xe_j, A_xPe_j) + \sum_{x,i,j} g(tD_jv_x, e_i)g(tD_iv_x, e_j) \\
&\quad = \sum_x \mathrm{tr}(PA_x)^2 + \sum_{x,y,z} g(D_{tz}v_x, v_y)(D_{ty}v_x, v_z) \\
&\quad = \sum_x \mathrm{tr}(PA_x)^2 + \sum_{x,y} g(D_{ty}v_x, v_y)^2,
\end{aligned}$$

where $x, y, z = 1, 2$ and $D_{tx} = D_{tv_x}$. On the other hand, we have

$$\begin{aligned}
\sum_x (\mathrm{div} tv_x)^2 &= \sum_{x,i,j} g(\nabla_i tv_x, e_i)g(\nabla_j tv_x, e_j) \\
&= \sum_{x,i,j} g(-PA_xe_i + tD_iv_a, e_i)g(-PA_xe_j + tD_jv_x, e_j) \\
&= \sum_{x,i,j} g(tD_iv_x, e_i)g(tD_jv_x, e_j) \\
&= \sum_{x,y} g(D_{ty}v_x, v_y)^2.
\end{aligned}$$

Since S satisfies

$$\begin{aligned}
&\mathrm{div}(\nabla_X X) - \mathrm{div}((\mathrm{div}X)X) \\
&\quad = S(X, X) + \sum_{i,j} g(\nabla_j X, e_i)g(e_j, \nabla_i X) - (\mathrm{divX})^2
\end{aligned}$$

for any tangent vector field X (cf. [46; p.44]), we have

$$\begin{aligned}
&\sum_x \Big(\mathrm{div}(\nabla_{tx} tv_x) - \mathrm{div}((\mathrm{div} tv_x) tv_x)\Big) \\
&\quad = \sum_x S(tv_x, tv_x) + \sum_x \mathrm{tr}(PA_x)^2 \\
&\quad = 2(n-1) + \frac{1}{2}\sum_x |[P, A_x]|^2 + \sum_x \mathrm{tr}(P^2 A_x^2) \\
&\quad = 2(n-1) - 2h + \sum_x \mathrm{tr} A_x^2 + \sum_x \mathrm{tr} P A_x A_{fx} + \sum_x \mathrm{tr}(P^2 A_x^2) \\
&\quad \geq 2.
\end{aligned}$$

Here we used (8.3) and $fv_x = 0$. Since M is compact, this is a contradiction. So we have $q = 1$. *q.e.d.*

If M is proper, then $h > 0$ and $q > 0$. Thus we have

Lemma 8.3. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then $q = 1$, that is,* $\mathrm{dim}\mathcal{D}_x^\perp = 1$.

In the following, we shall prove that the first normal space of M is just $FH^\perp$ and is of dimension 1 under the condition of Lemma 8.3. To prove this, we prepare some lemmas.

Lemma 8.4. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then the following hold*:

(*a*) $\nabla f = 0$.

(*b*) *For any X tangent to M and any $V \in FH^\perp$, we have $D_X V \in FH^\perp$.*

(*c*) *For any X tangent to M and any $U \in N$, we have $D_X U \in N$.*

Proof. By the proof of Lemma 8.2, if the Ricci tensor S of a minimal CR submanifold M satisfies $S(X,X) \geq (n-1)g(X,X)$ for any tangent vector field X, then $A_U tV = 0$ for any U and V normal to M. Thus we have

$$\begin{aligned}
g((\nabla_X f)V, U) &= -g(FA_V X, U) - g(B(X, tV), U) \\
&= -g(X, A_V tU) - g(A_U tV, X) \\
&= 0
\end{aligned}$$

for any X tangent to M and any U and V normal to M. Thus f is parallel.

Since M is proper, by Lemma 8.3, we have $\dim\mathcal{D}_x^\perp = 1$. Let V be a vector field in $FH^\perp$. Then we have $g(D_X V, fU) = -g(V, (\nabla_X f)U) = 0$ for any vector field $U \in N$. Hence we have (b).

Next we prove (c). For any vector field U in N, there exists U' in N that satisfy $U = fU'$. Hence we have

$$D_X U = D_X(fU') = fD_X U'.$$

Consequently, we have $D_X U \in N$. *q.e.d.*

Lemma 8.5. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then the second fundamental form A satisfies the following*:

(*a*) $A_v P A_v = P$, *where v is a unit vector field in $FH^\perp$.*

(*b*) $|[P, A_v]|^2 = 2\mathrm{tr}A_v^2 - 2(n-1)$, *where v is a unit vector field in $FH^\perp$.*

(*c*) $A_V A_U = A_U A_V$ *for any $V \in FH^\perp$ and $U \in N$.*

(*d*) $PA_U = A_{fU}$ *and* $PA_U + A_U P = 0$ *for any $U \in N$.*

Proof. By Lemma 8.3, we have $\dim\mathcal{D}_x^\perp = 1$. Let $\{v_1, \cdots, v_p\}$ be an orthonormal basis of $T_x(M)^\perp$ such that $v_1 = v \in F\mathcal{D}_x^\perp$ and $v_2, \cdots, v_p \in N_x$.

By (8.2) and $fv = 0$, we have

$$2g(A_v P A_v X, Y) = -2g(X, PY)g(tv, tv)$$

for any X and Y tangent to M. Thus we have (a). Using this, we have (b) by a straightforward computation.

Next we prove (c). From the equation of Ricci and Lemma 8.4 (b), we have

$$\begin{aligned} &g([A_U, A_V]X, Y) \\ &\quad = g(Y, tV)g(X, tU) - g(X, tV)g(Y, tU) - 2g(X, PY)g(V, fU) \\ &\quad = 0 \end{aligned}$$

for any X and Y tangent to M and $V \in FH^\perp$, $U \in N$. Thus we have $A_V A_U = A_U A_V$.

From the Weingarten formula and Lemma 8.4 (a), we have

$$\tilde{\nabla}_X JU = \tilde{\nabla}_X fU = -A_{fU}X + D_X fU = -A_{fU}X + fD_X U.$$

On the other hand, since $\tilde{\nabla} J = 0$, we obtain

$$\tilde{\nabla}_X JU = J\tilde{\nabla}_X U = -PA_U X - FA_U X + fD_X U,$$

thus we have $PA_U = A_{fU}$. Since A_{fU} is symmetric and P is skew-symmetric, we obtain $PA_U + A_U P{=}0$. Hence we have (d). *q.e.d.*

Using Theorem 3.3 and Lemma 8.5, we next compute the Laplacian for the square of the length of the second fundamental form of the minimal submanifold in CP^m whose Ricci tensor satisfies $S(X,X) \geq (n-1)g(X,X)$ for any tangent vector field X.

Lemma 8.6. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then*

$$\begin{aligned} g(\nabla^2 A, A) &= (n+3)\mathrm{tr}A_v^2 + (n+4)\sum_a \mathrm{tr}A_{fa}^2 - 6(n-1) \\ &\quad - \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b} (\mathrm{tr}A_a A_b)^2. \end{aligned}$$

Proof. From Lemma 8.5, we have $\sum_a \mathrm{tr}A_a A_{fa} P = \sum_a \mathrm{tr}A_{fa}^2$. Next we compute $\sum_a |[P, A_a]|^2$. Using Lemma 8.5, we obtain

$$\begin{aligned} \sum_a |[P, A_a]|^2 &= |[P, A_v]|^2 + \sum_{a \geq 2} |[P, A_a]|^2 \\ &= -2(n-1) + 2\mathrm{tr}A_v^2 + 4\sum_a \mathrm{tr}A_{fa}^2. \end{aligned}$$

From these equations and Theorem 3.3, we have our result. *q.e.d.*

Lemma 8.7. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then*

$$\begin{aligned} \sum_j g((\nabla^2 A)_v e_j, A_v e_j) &= (n+3)\mathrm{tr}A_v^2 - 6(n-1) - (\mathrm{tr}A_v^2)^2, \\ \sum_{a \geq 2, j} g((\nabla^2 A)_a e_j, A_a e_j) &= \sum_a \mathrm{tr}A_{fa}^2 - \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b \geq 2} (\mathrm{tr}A_a A_b)^2. \end{aligned}$$

Proof. From (3.1) and Lemma 8.5, we have

$$\begin{aligned}
&\sum_j g((\nabla^2 A)_v e_j, A_v e_j) \\
&= \sum_j g((\nabla^2 B)(e_j, A_v e_j), v) \\
&= ng \sum_j g(A_v e_j, A_v e_j) - 3\sum_j g(A_v e_j, P^2 A_v e_j) \\
&\quad -3\sum_j g(A_v^2 e_j, P^2 e_j) - 3\sum_j g(A_v e_j, A_v e_j) - 6\sum_j g(A_v P e_j, P A_v e_j) \\
&\quad +\sum_{a,j}(-\mathrm{tr} A_a A_v g(A_a e_j, A_v e_j) + 2g(A_a A_v A_a e_j, A_v e_j) \\
&\quad\quad -g(A_a^2 A_v e_j, A_v e_j) - g(A_v A_a^2 e_j, A_v e_j)) \\
&= (n-3)\mathrm{tr} A_v^2 + 3|[P, A_v]|^2 - \sum_a (\mathrm{tr}\, A_a A_v)^2 + \sum_a |[A_a, A_v]|^2 \\
&= (n+3)\mathrm{tr} A_v^2 - 6(n-1) - (\mathrm{tr} A_v^2)^2.
\end{aligned}$$

Here we used the fact that $\sum_{a\geq 2}(\mathrm{tr} A_a A_v)^2 = 0$, which is proved by Lemma 8.5 (c), (d). From this equation and Lemma 8.6, we have

$$\begin{aligned}
&\sum_{a\geq 2, j} g((\nabla^2 A)_a e_j, A_a e_j) \\
&\quad = g(\nabla^2 A, A) - \sum_j g((\nabla^2 A)_v e_j, A_v e_j) \\
&\quad = (n+4)\sum_a \mathrm{tr} A_{fa}^2 - \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b}(\mathrm{tr} A_a A_b)^2 + (\mathrm{tr} A_v^2)^2 \\
&\quad = (n+4)\sum_a \mathrm{tr} A_{fa}^2 - \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b\geq 2}(\mathrm{tr} A_a A_b)^2.
\end{aligned}$$

Hence we have our equation. *q.e.d.*

Next we give inequalities for $\sum_{a,b} |[A_a, A_b]|^2$ and $\sum_{a,b\geq 2}(\mathrm{tr} A_a A_b)^2$ in the equation in Lemma 8.7.

Lemma 8.8. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n-1)g(X, X)$, then*

$$\sum_{a,b} |[A_a, A_b]|^2 \leq 4\sum_a \mathrm{tr} A_{fa}^2,$$

$$\sum_{a,b\geq 2}(\mathrm{tr}A_aA_b)^2 \leq \frac{1}{2}(\sum_a \mathrm{tr}A_{fa}^2)^2.$$

Proof. From (8.1), we have $3g(PX, PX) \geq \sum_a g(A_aX, A_aX)$ for any X tangent to M. On the other hand, by Lemma 8.5, we have

$$\begin{aligned}
&\sum_{i,a} g(A_v^2A_{fa}e_i, A_{fa}e_i)\\
&= \sum_{i,a} g(A_vA_{fa}A_ve_i, A_{fa}e_i) = \sum_{i,a\geq 2} g(A_vPA_aA_ve_i, A_{fa}e_i)\\
&= \sum_{i,a\geq 2} g(A_vPA_vA_ae_i, A_{fa}e_i) = \sum_{i,a\geq 2} g(PA_ae_i, PA_ae_i).
\end{aligned}$$

From these and Lemma 8.5, we obtain

$$\begin{aligned}
3\sum_a \mathrm{tr}A_{fa}^2 &= 3\sum_{i,a} g(PA_{fa}e_i, PA_{fa}e_i)\\
&\geq \sum_{i,a,b} g(A_bA_{fa}e_i, A_bA_{fa}e_i)\\
&= \sum_{i,a} g(A_vA_{fa}e_i, A_vA_{fa}e_i) + \sum_{i,a,b} g(A_{fa}^2A_{fb}^2e_i, e_i)\\
&= \sum_{i,a\geq 2} g(PA_ae_i, PA_ae_i) + \frac{1}{2}\sum_{a,b}|[A_a, A_b]|^2\\
&= \sum_a \mathrm{tr}A_{fa}^2 + \frac{1}{2}\sum_{a,b}|[A_a, A_b]|^2,
\end{aligned}$$

from which $4\sum_a \mathrm{tr}A_{fa}^2 \geq \sum_{a,b}|[A_a, A_b]|^2$. Hence we have our first inequality. In the next place, we take a basis $\{v, v_2, \cdots, v_{p'}, v_{p'+1} = fv_2, \cdots, v_p = fv_{p'}\}$ $(p = 2p'+1)$ of $T_x(M)^\perp$ such that $\sum_{a,b\geq 2}^p(\mathrm{tr}A_aA_b)^2 = \sum_{a=2}^p(\mathrm{tr}A_a^2)^2$. Since $\mathrm{tr}A_a^2 = \mathrm{tr}A_{fa}^2$ for $a \geq 2$, we have

$$\begin{aligned}
\sum_{a=2}^p(\mathrm{tr}A_a^2)^2 &= 2\sum_{a=2}^{p'}(\mathrm{tr}A_a^2)^2\\
&= 2\Big((\sum_{a=2}^{p'}\mathrm{tr}A_a^2)^2 - \sum_{a,b\geq 2, a\neq b}^{p'}\mathrm{tr}A_a^2\mathrm{tr}A_b^2\Big).
\end{aligned}$$

On the other hand, we have

$$(\sum_{a=2}^p \mathrm{tr}A_a^2)^2 = (2\sum_{a=2}^{p'}\mathrm{tr}A_a^2)^2 = 4(\sum_{a=2}^{p'}\mathrm{tr}A_a^2)^2.$$

Therefore we obtain

$$\sum_{a=2}^{p}(\mathrm{tr}A_a^2)^2 = \frac{1}{2}(\sum_{a=2}^{p}\mathrm{tr}A_a^2)^2 - 2\sum_{a,b\geq 2, a\neq b}^{p'}\mathrm{tr}A_a^2\mathrm{tr}A_b^2 \leq \frac{1}{2}(\sum_{a=2}^{p}\mathrm{tr}A_a^2)^2,$$

from which we have $\sum_{a,b\geq 2}^{p}(\mathrm{tr}A_aA_b)^2 \leq (1/2)(\sum_a \mathrm{tr}A_{fa}^2)^2$. Hence we have the second inequality. *q.e.d.*

Using Lemma 8.3-Lemma 8.8, we prove the following lemma.

Lemma 8.9. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then $A_{fa} = 0$ for all a.*

Proof. From Lemma 8.7 and Lemma 8.8, we have

$$\begin{aligned}
&\frac{1}{2}\Delta(\sum_a \mathrm{tr}A_{fa}^2)\\
&\quad= \sum_{a\geq 2,i} g((\nabla^2 A)_a e_i, A_a e_i) + \sum_{a\geq 2,i} g((\nabla A)_a e_i, (\nabla A)_a e_i)\\
&\quad\geq \sum_{a\geq 2,i} g((\nabla^2 A)_a e_i, A_a e_i)\\
&\quad= (n+4)\sum_a \mathrm{tr}A_{fa}^2 - \sum_{a,b\geq 2}|[A_a, A_b]|^2 - \sum_{a,b\geq 2}(\mathrm{tr}A_aA_b)^2\\
&\quad\geq (\sum_a \mathrm{tr}A_{fa}^2)\Big(n - \frac{1}{2}\sum_a \mathrm{tr}A_{fa}^2\Big).
\end{aligned}$$

On the other hand, since

$$\sum_i S(e_i, e_i) = (n+3)(n-1) - |A|^2 \geq (n-1)\sum_i g(e_i, e_i),$$

we have $|A|^2 = \mathrm{tr}A_v^2 + \sum_a \mathrm{tr}A_{fa}^2 \leq 3(n-1)$. From Lemma 8.5 (b), we have $\mathrm{tr}A_v^2 \geq n-1$. Hence we have $\sum_a \mathrm{tr}A_{fa}^2 \leq 2(n-1) < 2n$. Hence, by the theorem of E. Hopf, $\sum_a \mathrm{tr}A_{fa}^2$ is constant so that $\Delta(\sum_a \mathrm{tr}A_{fa}^2) = 0$. Thus we have $A_{fa} = 0$ for all a. *q.e.d.*

From Lemma 8.4 and Lemma 8.9, the first normal space of M is of dimension 1 and parallel. Hence we see that M is a real hypersurface of some totally geodesic complex projective space $CP^{(n+1)/2}$ in CP^m (cf. [46; p.227]). This theorem is an extension of the reduction theorem of the codimension of a generic minimal submanifold in CP^m given by Yamagata-Kon [41].

9 Pinching theorems of the Ricci curvature

We define the notion of the tube of a submanifold. For the local calculation, assume that N is an embedded real n-dimensional C^∞-submanifold of CP^m. For a normal vector field V of N, let $F(V)$ be a point in CP^m reached by traversing a distance $|V|$ along the geodesic in CP^m originating at the base point x of V with initial tangent vector V. A point $p \in CP^m$ is called a focal point of multiplicity $\nu > 0$ of (N, x) if $p = F(V)$ and the Jacobian of the map F from the normal bundle of N to CP^m has nullity ν at V. Let BN denote the bundle of unit normal vectors to N. The tube of radius r over N is defined by the map $\phi_r : BN \longrightarrow CP^m$ given by $\phi_r(V) = F(rV)$. For sufficiently small value of r at least, ϕ_r determines a real hypersurface of CP^m.

In the following, we take the unit normal vector field v of a real hypersurface M in CP^m, and we put $\xi = -Jv$. Then ξ is the unit tangent vector field of M and $P^2X = -X + g(X, \xi)\xi$, $P\xi = 0$. We also put $A_v = A$ to simplify the notation. Then $\nabla_X \xi = PAX$ for any vector field X tangent to M.

In 1982, Cecil and Ryan classified real hypersurfaces of a complex projective space CP^m with a princilpal curvature vector field ξ.

Proposition 9.1 ([3]). *Let M be a real hypersurface* (*with unit normal vector v*) *of a complex projective space CP^m on which ξ is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ and the focal map ϕ_r has constant rank on M. Then the following hold*:

(a) *M lies on a tube* (*in the direction $\eta = \gamma'(r)$, where $\gamma(r) = \exp_x(rv)$ and x is a base point of the normal vector v*) *of radius r over a certain Kähler submanifold N in CP^m.*

(b) *Let $\cot\theta$, $0 < \theta < \pi$, be a principal curvature of the second fundamental form A_η at $y = \gamma(r)$ of the Kähler submanifold N. Then the real hypersurface M has a principal curvature $\cot(r - \theta)$ at $x = \gamma(0)$.*

For the special case that the second fundamental form A satisfies $A\xi = 0$, Maeda proved the following

Proposition 9.2 ([26]). *Let M be a real hypersurface of a complex projective space CP^m. If $A\xi = 0$, except for the null set on which the focal map ϕ_r degenerates, M is locally congruent to one of the following*:

(a) *a homogeneous real hypersurface which lies on a tube of radius* $\pi/4$ *over a totally geodesic* CP^k $(1 \leq k \leq m-1)$,

(b) *a nonhomogeneous real hypersurface which lies on a tube of radius* $\pi/4$ *over a Kähler submanifold* N *with nonzero principal curvatures* $\neq \pm 1$.

Using these results, we prove the following

Theorem 9.3. *Let* M *be a compact* n*-dimensional minimal CR submanifold of a complex projective space* CP^m *which is not a complex submanifold of* CP^m. *If the Ricci tensor* S *of* M *satisfies* $S(X,X) \geq (n-1)g(X,X)$ *for any vector* X *tangent to* M, *then* M *is congruent to one of the following*:

(a) *a totally geodesic real projective space* RP^n *of* CP^m,

(b) *a pseudo-Einstein real hypersurface* $M^c((n-1)/4, \pi/4)$ *of some* $CP^{(n+1)/2}$ *in* CP^m,

(c) *a real hypersurface of some* $CP^{(n+1)/2}$ *in* CP^m *which lies on a tube of radius* $\pi/4$ *over certain Kähler submanifold* N *with principal curvatures* $\cot\theta$, $0 < \theta \leq \pi/12$.

Proof. We suppose that M is proper. Then Theorem 8.1 implies that M is a real hypersurface of some totally geodesic complex projective space $CP^{(n+1)/2}$ in CP^m. By the proof of Lemma 8.2, we have $A\xi = 0$. On the other hand, from Lemma 8.5, we obtain $APAX = PX$ for any X tangent to M. Thus we see that if $AX = \lambda X$, then $APX = (1/\lambda)PX$. Since $3g(PX,PX) \geq g(A^2X,X)$, we obtain $\lambda^2 \leq 3$. We also have $\mathrm{rank}A = n-1$ because $A\xi = 0$. A homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic CP^k is minimal if and only if $k = (n-1)/4$, that is, M is $M^c_{k,k}$. The principal curvatures of this real hypersurface are ± 1 (see [3; p.493]).

For a nonhomogeneous real hypersurface M which lies on a tube of radius $\pi/4$ over a Kähler submanifold N, by the condition $\lambda^2 \leq 3$ and (b) of Proposition 9.1, we have $\cot^2(\pi/4 - \theta) \leq 3$. Thus we have $0 < \theta \leq \pi/12$. Consequently, using Proposition 9.1 and Proposition 9.2, we have our theorem. *q.e.d.*

Remark. The author does not know examples of certain Kähler submanifold N having the properties required in case (c) in Theorem 9.3.

Corollary 9.4. *Let* M *be a compact* n*-dimensional minimal proper CR* *submanifold of a complex projective space CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then M is congruent to one of the following*:

(a) *a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m,*

(b) *a real hypersurface of some $CP^{(n+1)/2}$ in CP^m which lies on a tube of radius $\pi/4$ over certain Kähler submanifold N with principal curvatures* $\cot\theta$, $0 < \theta \leq \pi/12$.

In [25], Maeda proved that if the Ricci tensor S of a compact minimal real hypersurface M of CP^m satisfies $(2m-2)g(X,X) \leq S(X,X) \leq 2mg(X,X)$, then M is congruent to a pseudo-Einstein real hypersurface $M^c((m-1)/2, \pi/4)$ of CP^m. Combining this with Corollary 9.4, we have

Corollary 9.5. *Let M be a compact n-dimensional minimal proper CR submanifold of a complex projective space CP^m. If the Ricci tensor S satisfies $(n-1)g(X,X) \leq S(X,X) \leq (n+1)g(X,X)$, then M is congruent to a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m.*

Next we prove the following

Theorem 9.6. *Let M be a compact n-dimensional minimal CR submanifold of a complex projective space CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X) + g(PX,PX)$ for any vector X tangent to M, then M is congruent to one of the following:*

(*a*) *a totally geodesic real projective space RP^n of CP^m,*

(*b*) *a totally geodesic complex projective space $CP^{n/2}$ of CP^m,*

(*c*) *a complex $(n/2)$ dimensional complex quadric $Q^{(n/2)}$ of some $CP^{n/2+1}$ of CP^m,*

(*d*) *a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m,*

(*e*) *a real hypersurface of some $CP^{(n+1)/2}$ in CP^m which lies on a tube of radius $\pi/4$ over certain Kähler submanifold N with principal curvatures* $\cot\theta$, *where θ satisfies* $0 < \sin 2\theta \leq 1/3$.

For the proof of the theorem, we prepare some lemmas for complex submanifolds. We take an orthonormal basis $\{v_1, \cdots, v_p, v_{p+1} = fv_1, \cdots, v_{2p} =$

$fv_p\}$ of $T_x(M)^\perp$.

Lemma 9.7 ([14]). *Let M be a complex k-dimensional Kähler submanifold of a complex m-dimensional Kähler manifold $\bar{M}$. Then*

$$\frac{1}{k}|A|^4 \le \sum_{a,b=1}^{2p} |[A_a, A_b]|^2 \le |A|^4,$$

$$\frac{1}{2p}|A|^4 \le \sum_{a,b=1}^{2p} (\mathrm{tr} A_a A_b)^2 \le \frac{1}{2}|A|^4,$$

where $p = m - k$. If $\bar{M}$ is of constant holomorphic sectional curvature c, then M is Einstein if and only if $\sum_{a,b=1}^{2p} |[A_a, A_b]|^2 = |A|^4/k$.

From Lemma 3.1, we have,

Lemma 9.8. *Let M be a complex k-dimensional Kähler submanifold of CP^m. Then*

$$g(\nabla^2 A, A) = 2(k+2)|A|^2 - \sum_{a,b=1}^{2p} |[A_a, A_b]|^2 - \sum_{a,b=1}^{2p} (\mathrm{tr} A_a A_b)^2.$$

In the following we prove Theorem 9.6. From Theorem 8.1, if M is proper, then M is a real hypersurface of some $CP^{(n+1)/2}$ in CP^m.

Next we suppose that M is a complex $(n/2)$ dimensional complex submanifold of CP^m. Since M is complex minimal submanifold of CP^m, we have

$$S(X,Y) = (n+2)g(X,Y) - \sum_{a=1}^{2p} g(A_a^2 X, Y).$$

Thus we have $\sum_a^{2p} g(A_a^2 X, X) \le 2g(X,X)$, from which $|A|^2 \le 2n$. Moreover, we see that $2I - \sum_a A_a^2$ is a positive semi-definite operator. Since A_a is symmetric, $\sum_a A_a^2$ is positive semi-definite. The operators $\sum_a A_a^2$ and $2I - \sum_a A_a^2$ can be transformed simultaneously by an orthogonal matrix into diagonal forms at each point of M, thus we see that $(\sum_a A_a^2)(2I - \sum_a A_a^2)$ is positive semi-definite. Hence we have

$$\mathrm{tr}(\sum_{a=1}^{2p} A_a^2)^2 \le 2|A|^2 \le 4n. \tag{9.1}$$

On the other hand, we obtain

$$\sum_{a,b=1}^{2p} |[A_a, A_b]|^2 = 2\sum_{a,b=1}^{2p} \mathrm{tr} A_a^2 A_b^2 = 2\mathrm{tr}(\sum_{a=1}^{2p} A_a^2)^2.$$

Therefore we have $\sum_{a,b=1}^{2p} |[A_a, A_b]|^2 \leq 4|A|^2$. From Lemma 9.7, Lemma 9.8 and these equations, we have

$$\begin{aligned} \frac{1}{2}\Delta|A|^2 &= g(\nabla^2 A, A) + |\nabla A|^2 \\ &\geq g(\nabla^2 A, A) \geq |A|^2(n - \frac{1}{2}|A|^2) \geq 0. \end{aligned} \tag{9.2}$$

Hence, by the theorem of Hopf, $|A|^2$ is constant so that $\Delta|A|^2 = 0$. Thus we have $|A| = 0$ or $|A|^2 = 2n$. When $|A| = 0$, M is totally geodesic.

Next we suppose $|A|^2 = 2n$. By (9.1), we have $\mathrm{tr}(\sum_{a=1}^{2p} A_a^2)^2 = 4n$, which induces

$$\sum_{a,b=1}^{2p} |[A_a, A_b]|^2 = 8n = \frac{2|A|^4}{n}.$$

From Lemma 9.7, M is Einstein complex submanifold of CP^m.

For any $V \in N_0(x) = \{V \in T_x(M)^\perp : A_V = 0\}$, we have

$$\nabla_Y(A_V X) = (\nabla_Y A)_V X + A_{D_Y V} X + A_V(\nabla_Y X) = 0.$$

Hence we have $A_{D_Y V} X + (\nabla_Y A)_V X = 0$. Since the equality of (9.2) holds, we have $\nabla A = 0$, from which we see that N_0 is parallel with respect to the normal connection. Let $V \in N_0$ and $U \in N_1$. Then we have

$$Xg(U, V) = g(D_X U, V) + g(U, D_X V) = 0.$$

Hence we see that the first normal space is parallel with respect to the normal connection. On the other hand, since the equality of (9.2) holds, we have $\sum_{a,b=1}^{2p} (\mathrm{tr} A_a A_b)^2 = (1/2)|A|^4$. In the next place, we take a basis $\{v_1, \cdots, v_p, v_{p+1} = fv_1, \cdots, v_{2p} = fv_p\}$ of $T_x(M)^\perp$ such that $\sum_{a,b=1}^{2p} (\mathrm{tr} A_a A_b)^2 = \sum_{a=1}^{2p} (\mathrm{tr} A_a^2)^2$. Then we have

$$\sum_{a=1}^{2p} (\mathrm{tr} A_a^2)^2 = \frac{1}{2}|A|^4 - 2\sum_{a \neq b}^{p} (\mathrm{tr} A_a^2)(\mathrm{tr} A_b^2),$$

from which we have $\sum_{a\neq b}^{p}(\mathrm{tr}A_a^2)(\mathrm{tr}A_b^2) = 0$. Hence we have $\dim N_1 = 2$. Hence M is an Einstein complex hypersurface of some $CP^{n/2+1}$ in CP^m, that is, a complex quadric $Q^{n/2}$ of $CP^{n/2+1}$ (see [35]). From this and Theorem 9.3, we have our theorem. *q.e.d.*

We suppose that M is a compact n-dimensional minimal CR submanifold of a complex projective space CP^m. When the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X) + 2g(PX,PX)$ for any vector X tangent to M, the cases (c) and (e) in Theorem 9.6 do not occur. Thus we obtain

Theorem 9.9 ([18]). *Let M be a compact n-dimensional minimal CR submanifold of a complex projective space CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X) + 2g(PX,PX)$ for any vector X tangent to M, then M is equivalent to one of the following:*

(*a*) *a totally geodesic real projective space RP^n of CP^m,*

(*b*) *a totally geodesic complex projective space $CP^{n/2}$ of CP^m,*

(*c*) *a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m.*

10 Real hypersurfaces of a complex space form

In this section we first study the Ricci tensor on the holomorphic distribution on CR submanifolds in a complex space form and give a characterization of pseudo-Einstein real hypersurfaces ([19]).

Theorem 10.1. *Let M be an n-dimensional CR submanifold of a complex space form $M^m(c)$, $c \neq 0$, $h = \dim\mathcal{D}_x > 2$, with semi-flat normal connection. Suppose that the curvature tensor R and the Ricci tensor S satisfy $g((R(X,Y)S)Z,W) = 0$ for any tangent vectors $X, Y, Z, W \in \mathcal{D}_x$. Then we have*

$$g(SX,Y) = \frac{1}{h}(r - \sum_{a=1}^{q} g(Stv_a, tv_a))g(X,Y)$$

for any vectors $X, Y \in \mathcal{D}_x$, where r denotes the scalar curvature of M and $\{v_1, \cdots, v_q\}$ is an orthonormal basis of $J\mathcal{D}_x^\perp$.

Proof. Since $g((R(X,Y)S)Z,W) = 0$ for any tangent vectors $X, Y, Z, W \in \mathcal{D}_x$, the first Bianchi identity gives

$$g(R(X,Y)SZ + R(Y,Z)SX + R(Z,X)SY, W) = 0.$$

We take an orthonormal basis $\{e_1, \cdots, e_h, tv_1 := e_{h+1}, \cdots, tv_q := e_n\}$ of $T_x(M)$, where $\{e_1, \cdots, e_h\}$ is an orthonormal basis of $\mathcal{D}_x$ and $\{v_1, \cdots, v_q\}$ is an orthonormal basis of $J\mathcal{D}_x^\perp$. Then we have

$$g(\sum_{i=1}^{h} R(e_i, Pe_i)SX + \sum_{i=1}^{h} R(Pe_i, X)Se_i + \sum_{i=1}^{h} R(X, e_i)SPe_i, Y) = 0.$$

Since $Ptv_a = 0$ for $a = 1, \cdots, q$, we have

$$g(\sum_{i=1}^{n} R(e_i, Pe_i)SX + \sum_{i=1}^{n} R(Pe_i, X)Se_i + \sum_{i=1}^{n} R(X, e_i)SPe_i, Y) = 0.$$

Since we have

$$g(\sum_{i=1}^{n} R(Pe_i, X)Se_i, Y) = -g(\sum_{i=1}^{n} R(e_i, X)SPe_i, Y),$$

it follows that

$$\sum_{i=1}^{n} g(R(e_i, Pe_i)SX, Y) = 2\sum_{i=1}^{n} g(R(e_i, X)SPe_i, Y).$$

On the other hand, by the equation of Gauss, we obtain

$$\begin{aligned}
&\sum_i g(R(e_i, Pe_i)SX, Y) \\
&\quad = (-2h-4)cg(PSX, Y) + \sum_i g(A_{B(Pe_i,SX)}e_i, Y) \\
&\qquad - \sum_i g(A_{B(e_i,SX)}Pe_i, Y), \\
&2\sum_i g(R(e_i, X)SPe_i, Y) \\
&\quad = c\{-2g(PSX, Y) + 2g(PSPX, PY) + 4g(PX, PSPY) \\
&\qquad -2\sum_i g(SPe_i, Pe_i)g(PX, Y)\} + 2\sum_i g(A_{B(X,SPe_i)}e_i, Y) \\
&\qquad -2\sum_i g(A_{B(e_i,SPe_i)}X, Y).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&c\{(-2h-2)g(PSX, Y) - 2g(PSPX, PY) - 4g(PX, PSPY)\} \\
&\quad = -2c\sum_i g(SPe_i, Pe_i)g(PX, Y) + 2\sum_{i,a} g(A_a e_i, Y)g(A_a X, SPe_i) \\
&\qquad -2\sum_{i,a} g(A_a X, Y)g(A_a e_i, SPe_i) - 2\sum_{i,a} g(A_a e_i, Y)g(A_a Pe_i, SX).
\end{aligned}$$

Since the Ricci tensor S of M is given by

$$SX = (n-1)cX - 3cP^2X + \sum_a \mathrm{tr}A_a \cdot A_a X - \sum_a A_a^2 X,$$

we obtain, for $X, Y \in \mathcal{D}_x$,

$$\begin{aligned}
&\sum_{i,a} g(A_a e_i, Y)g(A_a X, SPe_i) - \sum_{i,a} g(A_a X, Y)g(A_a e_i, SPe_i) \\
&\quad - \sum_{i,a} g(A_a e_i, Y)g(A_a Pe_i, SX) \\
&= \sum_{i,a,b} \mathrm{tr}A_b g(A_a e_i, Y)g(A_a X, A_b Pe_i) - \sum_{i,a,b} g(A_a e_i, Y)g(A_a X, A_b^2 Pe_i) \\
&\quad - \sum_{i,a,b} \mathrm{tr}A_b g(A_a e_i, Y)g(A_a Pe_i, A_b X) + \sum_{i,a,b} g(A_a e_i, Y)g(A_a Pe_i, A_b^2 X) \\
&\quad - \sum_{i,a}(n-1)cg(A_a X, Y)g(A_a e_i, Pe_i) + 3\sum_{i,a} cg(A_a X, Y)g(A_a e_i, Pe_i)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i,a,b} \mathrm{tr} A_b g(A_a X, Y) g(A_a e_i, A_b P e_i) + \sum_{i,a,b} g(A_a X, Y) g(A_a e_i, A_b^2 P e_i) \\
= &- \sum_{a,b} \mathrm{tr} A_b g(A_a Y, P A_b A_a X) + \sum_{a,b} g(A_a Y, P A_b^2 A_a X) \\
&+ \sum_{a,b} \mathrm{tr} A_b g(A_a Y, P A_a A_b X) - \sum_{a,b} g(A_a Y, P A_a A_b^2 X) \\
&- \sum_{i,a,b} \mathrm{tr} A_b g(A_a X, Y) g(A_a e_i, A_b P e_i) + \sum_{i,a,b} g(A_a X, Y) g(A_a e_i, A_b^2 P e_i).
\end{aligned}
$$

Since the normal connection of M is semi-flat, the equation of Ricci gives

$$A_a A_b X = A_b A_a X$$

for any $X \in \mathcal{D}_x$. Therefore, the equation above vanishes identically. From these equations and the assumption $c \neq 0$, we have

$$
\begin{aligned}
&(h+1) g(PSX, Y) + g(PSPX, PY) + 2g(PX, PSPY) \\
&\quad = \sum_i g(SPe_i, Pe_i) g(PX, Y)
\end{aligned}
$$

for any $X, Y \in \mathcal{D}_x$. This implies

$$(h-1) g(PSX, Y) + g(SPX, Y) = \sum_i g(SPe_i, Pe_i) g(PX, Y).$$

Since $PX, PY \in \mathcal{D}_x$, we also have

$$(h-1) g(PSPX, PY) + g(SP^2X, PY) = \sum_i g(SPe_i, Pe_i) g(PX, Y),$$

and hence

$$(h-1) g(SPX, Y) + g(PSX, Y) = \sum_i g(SPe_i, Pe_i) g(PX, Y).$$

From these equations, we obtain

$$(h-2) g(SPX, PY) = (h-2) g(SX, Y).$$

Since $h > 2$, we have $g(SPX, PY) = g(SX, Y)$. Thus, by the definition of the scalar curvature r of M, we get

$$
\begin{aligned}
hg(SX, Y) &= \sum_i g(PSe_i, Pe_i) g(X, Y) \\
&= (r - \sum_{a=1}^{q} g(Stv_a, tv_a)) g(X, Y),
\end{aligned}
$$

which proves our assertion. *q.e.d.*

Let M be a real $(2m-1)$-dimensional hypersurface immersed in $M^m(c)$. We take the unit normal vector field N of M in $M^m(c)$ and define a tangent vector field ξ by $\xi = -JN$, which is called the structure vector field. We put $\eta(X) = g(X, \xi)$. As a corollary of Theorem 10.1, we have

Corollary 10.2. *Let M be a real hypersurface of a complex space form $M^m(c)$, $c \neq 0$, $m \geq 3$. Suppose that the curvature tensor R and the Ricci tensor S of M satisfy $g((R(X,Y)S)Z, W) = 0$ for any tangent vectors X, Y, Z and W orthogonal to ξ. Then we have*

$$g(SX, Y) = \frac{1}{2m-2}(r - g(S\xi, \xi))g(X, Y),$$

for any tangent vectors X and Y orthogonal to ξ, where r denotes the scalar curvature of M.

Theorem 10.3. *Let M be a real hypersurface of a complex space form $M^m(c)$, $c \neq 0$, $m \geq 3$. Then the curvature tensor R and the Ricci tensor S of M satisfy $g((R(X,Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ if and only if M is pseudo-Einstein.*

Proof. We suppose that M satisfies $g((R(X,Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ. We can choose an orthonormal basis $\{e_1, \cdots, e_{2m-2}, \xi\}$ of $T_x(M)$ such that the second fundamental form A is represented by a matrix form

$$A = \left(\begin{array}{ccc|c} \lambda_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_{2m-2} & h_{2m-2} \\ \hline h_1 & \cdots & h_{2m-2} & \alpha \end{array}\right).$$

Then, we have

$$\begin{aligned} Se_i &= (2n+1)ce_i - 3c\eta(e_i)\xi + hAe_i - A^2 e_i \\ &= ((2n+1)c + h\lambda_i - \lambda_i^2)e_i + h_i(h - \lambda_i - \alpha)\xi - \sum_{k=1}^{2m-2} h_i h_k e_k, \\ S\xi &= (2m+1)c\xi - 3c\eta(\xi)\xi + hA\xi - A^2\xi \end{aligned}$$

$$
\begin{aligned}
&= (2m-2)c\xi + h(\sum_{k=1}^{2m-2} h_k e_k + \alpha\xi) - A(\sum_{k=1}^{2m-2} h_k e_k + \alpha\xi) \\
&= \sum_{k=1}^{2m-2} h_k(h - \lambda_k - \alpha)e_k + ((2m-2)c + \alpha h - \sum_{k=1}^{2m-2} h_k^2 - \alpha^2)\xi.
\end{aligned}
$$

By Corollary 10.2, we have

$$
\begin{aligned}
g(Se_i, e_j) &= -h_i h_j = 0 \quad (i \neq j), &(10.1)\\
g(Se_i, e_i) &= \frac{1}{2n-2}(r - g(S\xi, \xi)) \quad (i = 1, \cdots 2m-2). &(10.2)
\end{aligned}
$$

Equation (10.1) shows that at most one h_i does not vanish. Thus we can assume that $h_i = 0$ for $i = 2, \cdots, 2m-2$. We set $a = g(Se_i, e_i)$. Then we have

$$
\begin{aligned}
&Se_1 = ae_1 + h_1(h - \lambda_1 - \alpha)\xi, \\
&Se_i = ae_i \quad (i = 2, \cdots, 2n-2), \qquad (10.3)\\
&S\xi = h_1(h - \lambda_1 - \alpha)e_1 + ((2m-2)c + \alpha h - h_1^2 - \alpha^2)\xi.
\end{aligned}
$$

Since $g((R(X,Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ, we have

$$
g(R(X,Y)SZ - SR(X,Y)Z, W) = 0.
$$

By the equation of Gauss, for any $j \geq 2$, we obtain

$$
\begin{aligned}
0 &= g(R(e_1, e_j)Se_1, e_j) - g(SR(e_1, e_j)e_1, e_j) \\
&= ag(R(e_1, e_j)e_1, e_j) + h_1(h - \lambda_1 - \alpha)g(R(e_1, e_j)\xi, e_j) \\
&\qquad - ag(R(e_1, e_j)e_1, e_j) \\
&= h_1(h - \lambda_1 - \alpha)g(R(e_1, e_j)\xi, e_j).
\end{aligned}
$$

By the equation of Gauss, we have

$$
\begin{aligned}
g(R(e_1, e_j)\xi, e_j) &= g(Ae_j, \xi)g(Ae_1, e_j) - g(Ae_1, \xi)g(Ae_j, e_j) \\
&= -h_1\lambda_j.
\end{aligned}
$$

Thus we see that $h_1^2\lambda_j(h - \lambda_1 - \alpha) = 0$ for $j \geq 2$. If $h_1(h - \lambda_1 - \alpha) \neq 0$, then we have $\lambda_j = 0$ for $j \geq 2$. Since $h = \mathrm{tr}A$, we have $h = \lambda_1 + \alpha$. This is a contradiction. So we have $h_1(h - \lambda_1 - \alpha) = 0$. By (10.3), we see that M is pseudo-Einstein and that $h_1 = 0$ (see [15]). Thus we see that, if $g((R(X,Y)S)Z, W) = 0$ for any tangent vector fields X, Y, Z and W orthogonal to ξ, then M is pseudo-Einstein.

Conversely, if M is pseudo-Einstein, we have $SZ = aZ + b\eta(Z)\xi = aZ$ and $SW = aW$ for any tangent vectors Z and W orthogonal to ξ. Then we have $g((R(X,Y)S)Z, W) = g(R(X,Y)SZ, W) - g(SR(X,Y)Z, W) = 0$. *q.e.d.*

As an application of Theorem 10.3, we prove the following theorem (see [11], [13]).

Theorem 10.4 *There are no real hypersurfaces with $R(X,Y)S = 0$, semi-symmetric Ricci tensor, of a complex space form $M^m(c)$, $c \neq 0$, $m \geq 3$.*

Proof. We suppose that the Ricci tensor S of the real hypersurface M is semi-symmetric, that is, the curvature tensor and the Ricci tensor satisfy $R(X,Y)S = 0$ for any tangent vector fields X and Y. Then by Theorem 10.3, the real hypersurface M is pseudo-Einstein. Consequently, the Ricci tensor S satisfies $Se_i = ae_i$ for $i = 1, \cdots, 2m-2$ and $S\xi = (c(2n-2) + \alpha h - \alpha^2)\xi := b\xi$. Then, for any $i = 1, \cdots, 2m-2$, we have

$$\begin{aligned} 0 &= R(\xi, e_i)S\xi - SR(\xi, e_i)\xi \\ &= bR(\xi, e_i)\xi - SR(\xi, e_i)\xi \\ &= b\{-cg(\xi,\xi)e_i - g(A\xi,\xi)Ae_i\} \\ &\quad\; -S\{-cg(\xi,\xi)e_i - g(A\xi,\xi)Ae_i\} \\ &= -bce_i - b\alpha\lambda_i e_i + ace_i + a\alpha\lambda_i e_i \\ &= (a-b)(c + \alpha\lambda_i)e_i. \end{aligned}$$

Since $b \neq a$, we have $\lambda_i = -c/\alpha$, $i = 1, \cdots, 2m-2$. We put $\lambda = -c/\alpha$. Suppose that X is a unit vector field orthogonal to ξ. Then we have

$$\begin{aligned} \nabla_X \nabla_\xi \xi &= \nabla_X PA\xi = 0, \\ \nabla_\xi \nabla_X \xi &= \nabla_\xi PAX = \lambda \nabla_\xi PX \\ &= \lambda(\nabla_\xi P)X + \lambda P \nabla_\xi X \\ &= \lambda(\eta(X)A\xi - g(A\xi, X)\xi) + \lambda P \nabla_\xi X \\ &= \lambda P \nabla_\xi X, \\ \nabla_{[X,\xi]}\xi &= PA[X,\xi] \end{aligned}$$

$$
\begin{aligned}
&= PA\nabla_X \xi - PA\nabla_\xi X \\
&= PAPAX - PA\nabla_\xi X \\
&= \lambda^2 P^2 X - PA\nabla_\xi X \\
&= -\lambda^2 X - PA\nabla_\xi X.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
R(X,\xi)\xi &= \nabla_X\nabla_\xi\xi - \nabla_\xi\nabla_X\xi - \nabla_{[X,\xi]}\xi \\
&= -\lambda P\nabla_\xi X + \lambda^2 X + PA\nabla_\xi X.
\end{aligned}
$$

So we have

$$
\begin{aligned}
g(R(X,\xi)\xi, X) &= -\lambda g(P\nabla_\xi X, X) + \lambda^2 g(X,X) + g(PA\nabla_\xi X, X) \\
&= \lambda g(\nabla_\xi X, PX) + \lambda^2 g(X,X) - \lambda g(\nabla_\xi X, PX) \\
&= \lambda^2 g(X,X) = \lambda^2.
\end{aligned}
$$

By the equation of Gauss, we have $g(R(X,\xi)\xi, X) = c + \alpha\lambda = 0$. These equations imply $\lambda = 0$ and $c = 0$. This is a contradiction. So we have our theorem. *q.e.d.*

Remark. We can see that the totally η-umbilical pseudo-Einstein real hypersurfaces of CP^m and CH^m satisfies $c + \alpha\lambda \neq 0$ by a straightforward computation using principal curvatures of examples (see [13]). Here, we proved Theorem 10.4 by a slight general method.

We next consider the condition for the holomorphic distribution on real hypersurfaces such that the second fundamental form A of a real hypersurface M satisfies $g(AX, Y) = ag(X, Y)$ for any $X, Y \in \mathcal{D}$, a being a function, which includes the notion of totally η-umbilical real hypersurfaces, that is, the second fundamental form A satisfies $AX = aX + bg(X,\xi)\xi$ for some functions a and b, and is independent of the condition with respect to the structure vector field ξ (see [38]).

Let M be a real hypersurface of a complex space form $M^m(c)$, $c \neq 0$. If the distribution $\mathcal{D}$ is integrable and its integral manifold is a totally geodesic submanifold $M^{m-1}(c)$, then M is said to be *ruled real hypersurface*.

We prove the following theorem.

Theorem 10.5. *Let M be a real hypersurface of a complex space form $M^m(c)$, $c \neq 0$, $m \geq 3$. If the second fundamental form A of M satisfies $g(AX, Y) = ag(X, Y)$ for any $X, Y \in \mathcal{D}_x$, a being a function, then M is either totally η-umbilical or it is locally a ruled real hypersurface.*

To prove the theorem above, we prepare some lemmas.

Let M be a real hypersurface of $M^m(c)$, $c \neq 0$, $m \geq 3$. Suppose that the second fundamental form A satisfies $g(AX, Y) = ag(X, Y)$ for any $X, Y \in \mathcal{D}_x$. We can choose a local field of orthonormal basiss $\{e_1, \cdots, e_{2m-2}, \xi\}$ of M such that the second fundamental form A is represented by a matrix form

$$A = \left(\begin{array}{ccc|c} \lambda_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_{2m-2} & h_{2m-2} \\ \hline h_1 & \cdots & h_{2m-2} & \alpha \end{array} \right).$$

where we have put $h_i = g(Ae_i, \xi)$, $i = 1, \cdots, 2m-2$ and $b = g(A\xi, \xi)$.

First of all, we consider the case $a \neq 0$.

Lemma 10.6. *Let M be a real hypersurface of $M^m(c)$, $c \neq 0$, $m \geq 3$. Suppose that the second fundamental form A of M satisfies $g(AX, Y) = ag(X, Y)$, $a \neq 0$, for any $X, Y \in \mathcal{D}_x$. Then $h_1, \cdots, h_{2m-2}$ satisfy*

$$h_i g(\phi e_j, e_k) = h_j g(\phi e_k, e_i) = h_k g(\phi e_i, e_j)$$

for any $i \neq j, j \neq k, k \neq i$.

Proof. In the following, let i, j, k and l satisfy $i, j, k, l \leq 2m-2$. By the equation of Codazzi, we have

$$(\nabla_{e_i} A)e_j - (\nabla_{e_j} A)e_i = 2cg(e_i, \phi e_j)\xi.$$

Since $Ae_i = ae_i + h_i\xi$ for $i = 1, \cdots, 2m-2$, we have

$$\begin{aligned} &(\nabla_{e_i} A)e_j - (\nabla_{e_j} A)e_i \\ &= \nabla_{e_i} Ae_j - A\nabla_{e_i} e_j - \nabla_{e_j} Ae_i + A\nabla_{e_j} e_i \\ &= \nabla_{e_i}(ae_j + h_j\xi) - A\nabla_{e_i} e_j - \nabla_{e_j}(ae_i + h_i\xi) + A\nabla_{e_j} e_i \\ &= (e_i a)e_j + a\nabla_{e_i} e_j + (e_i h_j)\xi + h_j\phi Ae_i - A\nabla_{e_i} e_j \\ &\quad -(e_j a)e_i - a\nabla_{e_j} e_i - (e_j h_i)\xi - h_i\phi Ae_j + A\nabla_{e_j} e_i \\ &= 2cg(e_i, \phi e_j)\xi \end{aligned}$$

for any $i \neq j$. Thus, for any k such that $k \neq i$ and $k \neq j$, we have

$$\begin{aligned} 0 &= ag(\nabla_{e_i} e_j - \nabla_{e_j} e_i, e_k) + ag(h_j \phi e_i - h_i \phi e_j, e_k) \\ &\quad -g(\nabla_{e_i} e_j - \nabla_{e_j} e_i, Ae_k) \qquad (10.4) \\ &= ah_j g(\phi e_i, e_k) - ah_i g(\phi e_j, e_k) + h_k g(e_j, \nabla_{e_i} \xi) - h_k g(e_i, \nabla_{e_j} \xi) \\ &= ah_j g(\phi e_i, e_k) - ah_i g(\phi e_j, e_k) + h_k g(e_j, \phi Ae_i) - h_k g(e_i, \phi Ae_j) \\ &= ah_j g(\phi e_i, e_k) - ah_i g(\phi e_j, e_k) + 2ah_k g(e_j, \phi e_i). \end{aligned}$$

By this equation, we obtain

$$ah_k g(\phi e_j, e_i) - ah_j g(\phi e_k, e_i) + 2ah_i g(e_k, \phi e_j) = 0, \qquad (10.5)$$

$$ah_i g(\phi e_k, e_j) - ah_k g(\phi e_i, e_j) + 2ah_j g(e_i, \phi e_k) = 0. \qquad (10.6)$$

Since $a \neq 0$, the equations (10.4) and (10.5) imply $h_i(\phi e_j, e_k) = h_k g(\phi e_i, e_j)$. Using (10.6), we have

$$h_i g(\phi e_j, e_k) = h_j g(\phi e_k, e_i) = h_k g(\phi e_i, e_j).$$

q.e.d.

Lemma 10.7. *Let M be a real hypersurface of $M^m(c)$, $c \neq 0$, $m \geq 3$. Suppose that the second fundamental form A of M satisfies $g(AX, Y) = ag(X, Y)$, $a \neq 0$, for any $X, Y \in \mathcal{D}_x$. If $h_i = 0$ for some i, then $h_1 = \cdots = h_{2m-2} = 0$.*

Proof. Suppose that there exists h_i which satisfies $h_i = 0$. Then we have

$$h_j g(\phi e_k, e_i) = h_k g(\phi e_i, e_j) = 0$$

for any j and k such that $j \neq k$, $k \neq i$ and $i \neq j$. If there is a $h_j \neq 0$, then $g(\phi e_k, e_i) = 0$ for any k such that $k \neq i$ and $k \neq j$. Thus we have $e_i = \phi e_j$ or $e_i = -\phi e_j$. Since $h_k g(\phi e_i, e_j) = 0$, we have $h_k = 0$ for any k such that $k \neq i$ and $k \neq j$.

Let l satisfy $l \neq i$, $l \neq j$ and $l \neq k$. Since $h_k = 0$ and $h_i = 0$, we have

$$h_j g(\phi e_k, e_l) = h_k g(\phi e_l, e_j) = 0,$$
$$h_j g(\phi e_i, e_l) = h_i g(\phi e_l, e_j) = 0.$$

Since $h_j \neq 0$, e_l satisfies $g(\phi e_k, e_l) = 0$ for any $k \neq j$, $k \neq i$ and $g(\phi e_i, e_l) = 0$. Thus we obtain $e_l = \phi e_j$ or $e_l = -\phi e_j$. Then we have $e_i = e_l$ or $e_i = -e_l$.

This is a contradiction. So we see that if there is an $h_i = 0$, then $h_1 = \cdots = h_{2m-2} = 0$. *q.e.d.*

Lemma 10.8. *Let M be a real hypersurface of $M^m(c)$, $c \neq 0$, $m \geq 3$. Suppose that the second fundamental form A of M satisfies $g(AX, Y) = ag(X, Y)$, $a \neq 0$, for any $X, Y \in \mathcal{D}_x$. Then there exists i such that $h_i = 0$.*

Proof. Suppose that $h_1 \neq 0, \cdots, h_{2m-2} \neq 0$, and i, j, k and l are different for each other. By Lemma 10.6, we have

$$h_i g(\phi e_j, e_k) = h_j g(\phi e_k, e_i) = h_k g(\phi e_i, e_j), \tag{10.7}$$

$$h_j g(\phi e_k, e_l) = h_k g(\phi e_l, e_j) = h_l g(\phi e_j, e_k), \tag{10.8}$$

$$h_k g(\phi e_l, e_i) = h_l g(\phi e_i, e_k) = h_i g(\phi e_k, e_l), \tag{10.9}$$

$$h_l g(\phi e_i, e_j) = h_i g(\phi e_j, e_l) = h_j g(\phi e_l, e_i). \tag{10.10}$$

By (10.8) and (10.10), we obtain

$$\begin{aligned} h_i g(\phi e_j, e_k) &= \frac{h_i h_k}{h_l} g(\phi e_l, e_j) \\ &= -\frac{h_i h_k}{h_l} \times \frac{h_l}{h_i} g(\phi e_i, e_j) \\ &= -h_k g(\phi e_i, e_j). \end{aligned}$$

Since $h_i g(\phi e_j, e_k) = h_k g(\phi e_i, e_j)$, we have $h_i g(\phi e_j, e_k) = 0$. Since $h_i \neq 0$, we have $g(\phi e_j, e_k) = 0$ for any j and k such that $i \neq j$, $j \neq k$ and $k \neq i$. Here, we fix the index i. Then we obtain $e_k = \phi e_i$ or $e_k = -\phi e_i$ for any $k \neq i$. This is a contradiction. Consequently, we see that there is a h_i such that $h_i = 0$. *q.e.d.*

Proof of Theorem 10.5.

From Lemmas 10.6, 10.7 and 10.8, if $a \neq 0$, we have $h_i = 0$ for all i, and hence $A = aI + b\eta \otimes \xi$. Thus M is a totally η-umbilical real hypersurface.

We next suppose that $a = 0$. Then $g(AX, Y) = 0$ for any $X, Y \in \mathcal{D}$. Using the basic formulas from the Preliminaries, we easily check that, for any $X, Y \in \mathcal{D}$, we have

$$g(\nabla_X Y, \xi) = -g(Y, \phi AX) = g(AX, \phi Y) = 0.$$

From here we see that always $\nabla_X Y \in \mathcal{D}$ and the distribution $\mathcal{D}$ is integrable. Moreover, $\tilde{\nabla}_X Y = \nabla_X Y$, and hence the integral manifold of $\mathcal{D}$ is a totally geodesic complex submanifold of $M^m(c)$. Consequently, M is locally a ruled real hypersurface. This completes the proof of our theorem. *q.e.d.*

References

[1] K. Abe, Applications of Riccati type differential equation to Riemannian manifolds with totally geodesic distribution, Tôhoku Math. J. **25** (1973) 425–444.

[2] A. Bejancu, Geometry of CR-submanifolds, D. Reidel Publishing Company, Dordrecht, 1986.

[3] T. E. Cecil and P. J. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[4] B. Y. Chen, CR-submanifolds of a Kaehler manifold, II. J. Differential Geom. **16** (1981), 493-509.

[5] B. Y. Chen and K. Ogiue, On totally real submanifolds, Trans. Amer. Math. Soc. **193** (1974), 257-266.

[6] G. Chen and X. Zou, Rigidity of compact submanifolds in a unit sphere, Kodai Math. J. **18** (1995), 75-85.

[7] S. S. Chern, M. Do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length. Functional analysis and related fields, Proc. Conf. in Honor of Marshall Stone, Springer, Berlin, 1970.

[8] N. Ejiri, Compact minimal submanifolds of a sphere with positive Ricci curvature, J. Math. Soc. Japan **31** (1979), 251-256.

[9] I. Ishihara, Kaehler submanifolds satisfying a certain condition on normal connection, Atti della Accademia Nazionale dei Lincei. LXII (1977), 30-35.

[10] T. Itoh, Addendum to: "On Veronese manifolds", J. Math. Soc. Japan **30** (1978), 73-74.

[11] U-Hang Ki, H. Nakagawa and Y. J. Suh, Real hypersurfaces with harmonic Weyl tensor of a complex space form, Hiroshima Math. J. **20** (1990), 93-102.

[12] M. Kimura, Sectional curvatures of holomorphic planes on a real hypersurface in $P^n(c)$. Math. Ann. **276** (1987).

[13] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space III, Hokkaido Math. J. **22** (1993), 63-78.

[14] Masahiro Kon, On some complex submanifolds in Kaehler manifolds, Canad. J. Math. **26** (1974), 1442-1449.

[15] Masahiro Kon, Pseudo-Einstein real hypersurfaces in complex space forms, J. Differential Geom. **14** (1979), 339-354.

[16] Masahiro Kon, Generic minimal submanifolds of a complex projective space, Bull. London Math. Soc. **12** (1980), 355-360.

[17] Masahiro Kon, Real minimal hypersurfaces in a complex projective space, Proc. Amer. Math. Soc. **79** (1980), 285-288.

[18] Masahiro Kon, Minimal CR submanifolds immersed in a complex projective space, Geom. Dedicata **31** (1989), 357-368.

[19] Mayuko Kon, Ricci recurrent CR submanifolds of a complex space form, Tsukuba J. Math. **31** (2007), 233-252.

[20] Mayuko Kon, A characterization of totally η-umbilical real hypersurfaces and ruled real hypersurfaces of a complex space form, to appear in Czech. Math. J.

[21] Mayuko Kon, Pinching theorems for a compact minimal submanifold in a complex projective space, to appear in Bull. Austral. Math. Soc.

[22] Mayuko Kon, Compact minimal CR submanifolds of a complex projective space with positive Ricci curvature, preprint.

[23] Mayuko Kon, Minimal submanifolds of a complex space form, preprint.

[24] H. B. Lawson Jr, Rigidity theorems in rank-1 symmetric spaces, J. Differential Geom. **4** (1970), 349-357.

[25] S. Maeda, Real hypersurfaces of a complex projective space II, Bull. Austral. Math. Soc. **29** (1984), 123–127.

[26] S. Maeda, Ricci tensors of real hypersurfaces in a complex projective space, Proc. Amer. Math. Soc. **122** (1994), 1229–1235.

[27] K. Ogiue, Differential geometry of Kaehler submanifolds, Adv. in Math. **13** (1974), 73-114.

[28] Y. Ohnita, Totally real submanifolds with nonnegative sectional curvature, Proc. Amer. Math. Soc. **97** (1986), 474-478.

[29] M. Okumura, Normal curvature and real submanifold of the complex projective space, Geom. Dedicata **7** (1978), 509-517.

[30] M. Okumura, Submanifolds with L-flat normal connection of the complex projective space, Pacific J. Math. **78**, No. 2 (1978), 447-454.

[31] C. K. Peng and C. L. Terng, The scalar curvature of minimal hypersurfaces in spheres, Math. Ann. **266** (1983), 105-113.

[32] A. Ros, A characterization of seven compact Kaehler submanifolds by holomorphic pinching, Ann. of Math. **121** (1985), 377-382.

[33] Yi-B. Shen, On compact Kaehler submanifolds in CP^{n+p} with nonnegaive sectional curvature, Proc. Amer. Math. Soc. **123** (1995), 3507-3512.

[34] J. Simons, Minimal varieties in riemannian manifolds, Ann. of Math. **88** (1968), 62-105.

[35] B. Smyth, Differential geometry of complex hypersurfaces, Ann. of Math. **85** (1967), 246-266.

[36] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, J. Math. Soc. Japan **27** (1975), 43-53.

[37] S. Tanno, Compact complex submanifolds immersed in complex projective spaces, J. Differential Geom. **8** (1973), 629-641.

[38] Y. Tashiro and S. Tachibana, On Fubinian and C-Fubinian manifolds, Kôdai Math. Sem. Rep. **15** (1963), 176-183.

[39] F. Urbano, Totally real minimal submanifolds of a complex projective space, Proc. Amer. Math. Soc. **93** (1985), 332-334.

[40] C. Xia, A pinching theorem for minimal submanifolds in a sphere, Arch. Math. **57** (1991), 307-312.

[41] M. Yamagata and Masahiro Kon, Reduction of the codimension of a generic minimal submanifold immersed in a complex projective space, Collq. Math. **74** (1997), 185-190.

[42] H. C. Yang and Q. M. Cheng, Chern's Conjecture on minimal hypersurfaces, Math. Z. **227** (1998), 377-390.

[43] K. Yano, On harmonic and Killing vector fields, Ann. of Math. **55** (1952), 38-45.

[44] K. Yano and Masahiro Kon, Generic submanifolds, Ann. Mat. Pura Appl. **123** (1980), 59-92.

[45] K. Yano and Masahiro Kon, CR submanifolds of Kaehlerian and Sasakian Manifolds, Birkhauser Verlag, Boston, 1983.

[46] K. Yano and Masahiro Kon, Structures on manifolds, World Scientific Publishing, Singapore, 1984.