Numerical Analysis of FitzHugh-Nagumo Neurons on Random Networks

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We investigate a model of randomly copuled neurons. The elements are FitzHgh-Nagumo excitable neurons. The interactions between them are the mixture of excitatory and inhibitory. When all interactions are excitatory, a rest state is globally stable due to the excitability of neurons. Increasing the number of inhibitory connections, we observe the phase transition from the rest state to an oscillatory state. An analytical description for the critical point of the transition is obtained by means of random matrix theories for an infinite number of neurons, and the result is in good agreement with numerical simulation.

§1. Introduction

In recent years, many studies of complex networks have been reported.[1] However, the dynamics of excitable elements on complex networks has not yet studied. In this paper, we study the system of N coupled excitable neurons on random networks in order to investigate the effect of network structure on neural firing. The connections between the neurons are the mixture of excitatory and inhibitory ones.

When all the interactions between neurons are excitatory, the rest state is globally stable. Increasing the number of inhibitory connections, we observe the phase transition from the rest state to an oscillatory state. The asymptotic description of the critical point of the transition is obtained analytically in the limit of infinite number of neurons. The transition depends on the ratio of inhibitory connections and a coupling strength for larger networks.

§2. Model

The model we consider is the following randomly coupled FitzHugh-Nagumo elements.[2),3)]

$$\dot{u_i} = u_i(u_i - \alpha)(1 - u_i) - v_i + \frac{K}{N}\sum_{i \neq j} \kappa_{ij}(u_j - u_i) \ ,$$
$$\dot{v_i} = \tau(u_i - \gamma v_i) \ ,$$

where α, τ, γ are parameters, K is a coupling strength and N is the number of elements. An adjacency matrix $\{\kappa_{ij}\}$ defines the network structure, i.e., the connections between the elements. The components of the adjacency matrix are randomly determined by

$$\kappa_{ij} = \begin{cases} 1 & \text{with probability} \quad 1 - p \ , \\ -1 & \text{with probability} \quad p \ , \end{cases}$$

where p is the probability that a connection between two elements is inhibitory. Thus, the interactions between the elements are the mixture of excitatory and inhibitory. Employing $\alpha = 0.01$, $\tau = 0.001$ and $\gamma = 1.0$, each element is excitable. When $p = 0$, i.e., all connections are excitatory, the rest state $(u_i, v_i) = (0, 0)$ is globally stable. Increasing p, that is the ratio of the number of inhibitory connections to the number of all connections, we observe the phase transition from the rest state to the oscillatory state.

§3. Numerical simulations

In this section, we investigate numerically the transition from the rest state to the oscillatory state by changing the parameters K and p. For smaller values of K and p, all neurons soon fall into the rest state. On the other hand, beyond certain critical parameters, most of the firing patterns for given adjacency matrixes converge to periodical excitations. In addition, many periodic solutions coexist in the same parameters, K and p and a fixed network $\{\kappa_{i,j}\}$. Depending on the initial conditions, one of the firing patterns are selected. Typical periodic firing patterns with a fixed $\{\kappa_{i,j}\}$ are depicted in Fig. 1.

In order to characterize the transition from the rest state to the oscillatory state by changing the parameters p and K, we introduce the following normalized norm as an order parameter:

$$l_2 = \lim_{T\to\infty} \frac{1}{T}\sqrt{\int_0^T \frac{1}{N}\sum_{i=1}^{N}(u_i^2 + v_i^2)dt}\ .$$

Increasing coupling strength K with fixed p, we observe that the rest state becomes unstable and a spontaneous firing appears. The transition point, of course, depends on the matrix $\{\kappa_{i,j}\}$ and the system size N. However, the order parameter averaged over the different adjacent matrix with fixed p, $L_2 = \langle l_2 \rangle_\kappa$, converges to a finite value.

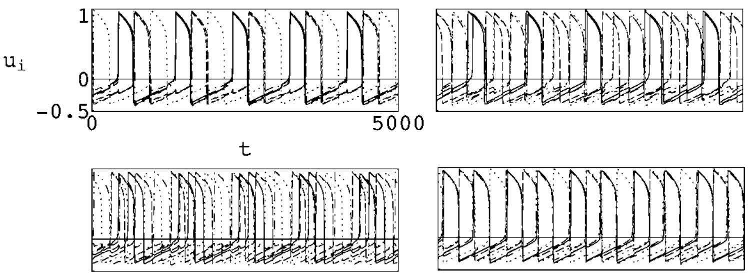

Fig. 1. Typical time evolutions of u_i. Many firing patterns coexist in the same parameters with fixed network structure $\{\kappa_{i,j}\}$. The parameters are $\alpha = 0.01$, $\tau = 0.001$, $\gamma = 1.0$, $p = 0.75$, $K = 0.1$ and $N = 5$. Depending on the initial conditions, dynamics falls into a periodic state.

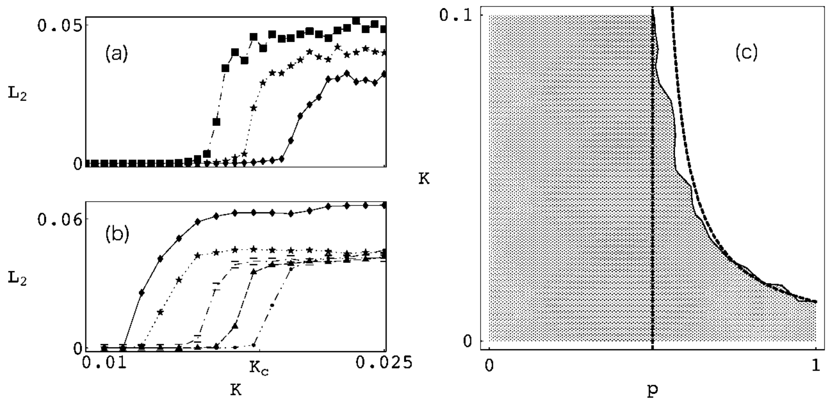

Fig. 2. The phase transition from the rest state to the oscillatory state. (a) The order parameter L_2 as a function of coupling strength K. The transitions are clearly observed. square: $p = 0.75$, star: $p = 0.8$, rhombic: $p = 0.85$. (b) The system size dependency of the order parameter for $p = 0.75$. The critical points converge to $K_c \sim 0.022$, which agrees well with theoretical prediction for $N \to \infty$. Rhombic: $N = 10$, star: $N = 20$, square: $N = 50$, triangle: $N = 100$, and circle: $N = 200$. (c) The order parameter L_2 in (p, K)-parameters space is shown. The gray region shows $L_2 < 0.01$, in which the rest state is globally stable. Dotted lines show the critical line obtained by linear stability analysis and $p = 1/2$, i.e., the asymptotic critical value for $K, N \to \infty$. $N = 200$ and the other parameters are the same as in Fig. 1

The K dependency of the order parameter L_2 is shown in Fig. 2(a) for different values of p. It is clearly shown that the transition from the rest state to the oscillatory state occurs, and that the critical point depends on p. The dependency of the system size N for the transition is also shown in Fig. 2(b). The critical value of K converges the value $K_c \sim 0.022$ with the increase of the system size. The phase diagram in (p, K)-parameters space is shown in Fig. 2(c), where the gray region represents that the rest state is globally stable. The critical line agrees well with the theoretical prediction $K = K_c(p)$ represented by dotted line, which is discussed in the next section.

Near the critical parameters, the number of firing neurons is very small and most of the elements remain the rest state, i.e., the firing is localized. Indeed $\rho = \max_i u_i / \sum_i u_i$, which measures the localization of the firing, takes larger value along the critical line. As the coupling strength K increases with fixed $\{\kappa_{i,j}\}$, ρ is decreased, indicating that many neurons start to fire.

§4. Linear stability analysis

In this section, we briefly describe the result of the linear stability analysis. The eigenequation of our model is given by

$$\left\| (\lambda + \gamma\tau) \left[(\lambda + \alpha) I - \frac{K}{N} \left\{ \kappa_{i,j}^{'} \right\} \right] + \tau I \right\| = 0 \; , \tag{4·1}$$

where I represents the unit matrix and

$$\{\kappa'_{i,j}\} = \begin{cases} \kappa_{i,j} & \text{if } i \neq j \ , \\ -\sum_j \kappa_{i,j} & \text{if } i = j \ . \end{cases}$$

Therefore, the eigenvalues of the above equation are expressed as

$$\lambda_{i\pm} = -\frac{1}{2}\left\{-(\alpha+\gamma\tau-K\xi_i) \pm \sqrt{(\alpha+\gamma\tau-K\xi_i)^2 - 4(\alpha\gamma\tau+\tau-K\xi_i\gamma\tau)}\right\}, \quad (4{\cdot}2)$$

where ξ_i represents the eigenvalues of $\{\kappa'_{i,j}\}$. Using the results obtained by Bray and Rodgers,[4] we find that the distribution of eigenvalues of $\{\kappa'_{i,j}\}$ approaches to $\delta(2p-1)$ in the limit $N \to \infty$. Therefore the Hopf bifurcation occurs at

$$K(2p-1) = \alpha + \gamma\tau \ . \quad (4{\cdot}3)$$

The critical line $K_c(p) = (\alpha+\gamma\tau)/(2p-1)$ is shown in Fig. 2(c), which is in good agreement with numerical simulation for $N = 200$. This analysis also shows that oscillation does not appear at $p \leq p_c = 1/2$. It means that the ratio of inhibitory coupling must be larger than 1/2 so that the spontaneous firing appears.

§5. Conclusion

We have investigated the transition from the rest state $(u_i, u_j) = (0, 0)$ to the oscillatory state in randomly coupled neurons. Because each element is excitable, the rest state is a globally stable solution when all connections between elements are excitatory. Increasing p, the ratio of the number of inhibitory connections, we have found numerically that the firing patterns appear beyond a critical value. Numerics showed that most of the firing patterns are periodic. In addition, these solutions are multistable, that is, many stable periodic solutions coexist in the same parameters with fixed adjacent matrix. Using the linear stability analysis, we showed that there exists the critical value $p_c = 1/2$, below which the rest state is stable for any coupling strength K. An asymptotic estimation for the critical line in (p, K)-parameters space is obtained for $N \to \infty$ by means of random matrix theories, which is in good agreement with numerical simulation.

References

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