Eisenstein series for G_2 and the symmetric cube Bloch–Kato conjecture

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Abstract

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The purpose of this thesis is to construct nontrivial elements in the Bloch–Kato Selmer group of the symmetric cube of the Galois representation attached to a cuspidal holomorphic eigenform F of level 1. The existence of such elements is predicted by the Bloch–Kato conjecture. This construction is carried out under certain standard conjectures related to Langlands functoriality. The broad method used to construct these elements is the one pioneered by Skinner and Urban in [SU06a] and [SU06b].

The construction has three steps, corresponding to the three chapters of this thesis. The first step is to use parabolic induction to construct a functorial lift of F to an automorphic representation Π of the exceptional group G_2 and then locate every instance of this functorial lift in the cohomology of G_2. In Eisenstein cohomology, this is done using the decomposition of Franke–Schwermer [FS98]. In cuspidal cohomology, this is done assuming Arthur's conjectures in order to classify certain CAP representations of G_2 which are nearly equivalent to Π, and also using the work of Adams–Johnson [AJ87] to describe the Archimedean components of these CAP representations. This step works for F of any level, even weight $k \geq 4$, and trivial nebentypus, as long as the symmetric cube L-function of F vanishes at its central value. This last hypothesis is necessary because only then will the Bloch–Kato conjecture predict the existence of nontrivial elements in the symmetric cube Bloch–Kato Selmer group. Here this hypothesis is used in the case of Eisenstein cohomology to show the holomorphicity of certain Eisenstein series via the Langlands–Shahidi method, and in the case of cuspidal cohomology it is used to ensure that relevant discrete representations classified by Arthur's conjecture are cuspidal and not residual.

The second step is to use the knowledge obtained in the first step to p-adically deform a certain critical p-stabilization $\sigma(\Pi)$ of Π in a generically cuspidal family of automorphic representations of G_2. This is done using the machinery of Urban's eigenvariety [Urb11]. This machinery operates on the multiplicities of automorphic representations in certain cohomology groups; in particular, it can relate the location of Π in cohomology to the location of $\sigma(\Pi)$ in an overconvergent analogue of cohomology and, under favorable circumstances, use this information to p-adically deform $\sigma(\Pi)$ in a generically cuspidal family. We show that these circumstances are indeed favorable when the sign of the symmetric functional equation for F is -1, either under certain conditions on the slope of $\sigma(\Pi)$, or in general when F has level 1.

The third and final step is to, under the assumption of a global Langlands correspondence for cohomological automorphic representations of G_2, carry over to the Galois side the generically cuspidal family of automorphic representations obtained in the second step to obtain a family of Galois representations which factors through G_2 and which specializes to the Galois representation attached to Π. We then show this family is generically irreducible and make a Ribet-style construction of a particular lattice in this family. Specializing this lattice at the point corresponding to Π gives a three step reducible Galois representation into GL_7, which we show must factor through, not only G_2, but a certain parabolic subgroup of G_2. Using this, we are able to construct the desired element of the symmetric cube Bloch–Kato Selmer group as an extension appearing in this reducible representation. The fact that this representation factors through the aforementioned parabolic subgroup of G_2 puts restrictions on the extension we obtain and guarantees that it lands in the symmetric cube Selmer group and not the Selmer group of F itself. This step uses that F is level 1 to control ramification at places different from p, and to ensure that F is not CM so as to guarantee that the Galois representation attached to Π has three irreducible pieces instead of four.

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Introduction

Let p be a prime number, and write $G_{\mathbb{Q}}$ for the absolute Galois group of $\mathbb{Q}$. Let n be a positive integer, and let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a continuous Galois representation. We assume ρ is geometric in the sense of Fontaine–Mazur, and for simplicity we also assume that ρ is irreducible and that $n \geq 2$.

Attached to ρ are two objects, one of an analytic nature, and one of an arithmetic nature. On the analytic side one has the L-function of ρ, written $L(s, \rho)$. This is a holomorphic function which conjecturally has an analytic continuation to the entire complex plane. And on the arithmetic side one has the Bloch–Kato Selmer group $H^1_f(\mathbb{Q}, \rho^\vee(1))$, where $\rho^\vee$ denotes the dual of ρ and $\rho^\vee(1)$ denotes the twist of that by the cyclotomic character. This Selmer group is the group of all cohomology classes in $H^1(\mathbb{Q}, \rho^\vee(1))$ which are unramified at all primes $\ell \neq p$ and crystalline at p.

Then the Bloch–Kato conjecture predicts the following relationship between these two objects:

$$\mathrm{ord}_{s=0} L(s, \rho) = \dim_{\overline{\mathbb{Q}}_p} H^1_f(\mathbb{Q}, \rho^\vee(1)).$$

This conjecture is extremely far-reaching, and at the same time, extremely difficult. It is a sweeping generalization of the conjecture of Birch and Swinnerton-Dyer, and progress has only been made towards it in little pieces thus far.

So, what are the methods available to tackle a conjecture like this? How can one bridge the gap between two objects both from such distant mathematical worlds? Well, in some sense, there are two main methods to do this which work in opposite directions: There is the Euler system method, which can be used in special cases to establish inequalities like

$$\mathrm{ord}_{s=0} L(s, \rho) \geq \dim_{\overline{\mathbb{Q}}_p} H^1_f(\mathbb{Q}, \rho^\vee(1)),$$

and there is what one might call the modular method, which can be used in special cases to establish inequalities in the other direction,

$$\operatorname{ord}_{s=0} L(s,\rho) \le \dim_{\overline{\mathbb{Q}}_p} H^1_f(\mathbb{Q}, \rho^\vee(1)).$$

Much progress has been made recently in using the Euler system method pioneered by Thaine, Kolyvagin, Rubin and others. For ρ coming from a modular form, the work of Kato [Kat04] constructs an Euler system which is able to be used to establish implications of the form

$$L(0,\rho) \neq 0 \quad \Longrightarrow \quad H^1_f(\mathbb{Q}, \rho^\vee(1)) = 0.$$

Here, ρ has been normalized so that the central critical point of its L-function is located at $s = 0$.

Much more recently, for Galois representations attached to certain automorphic representations of GSp_4, there is also the work of Loeffler, Zerbes, and their collaborators, which ultimately culminated in [LZ20]. There, they establish an implication as above, but for ρ coming from certain automorphic representations of GSp_4. Both these works also succeed in establishing the corresponding inclusions in the Iwasawa main conjecture for these Galois representations.

On the other hand, the modular direction, which was pioneered in the paper of Ribet [Rib76] where the converse to Herbrand's theorem is proved, has also seen progress recently. Besides the work of Mazur–Wiles [MW84] and Wiles [Wil90], which establish Iwasawa's main conjecture for $\mathbb{Q}$ and for totally real fields respectively, there is the work of Skinner–Urban [SU14]. There they prove the Iwasawa main conjecture for modular forms under certain hypotheses. But before this, Skinner and Urban [SU06a] proved implications converse to the one above for modular forms; so if again ρ is the Galois representation attached to a modular form, normalized so that the central critical point of its L-function is located at $s = 0$, then Skinner and Urban prove under certain hypotheses that

$$L(0,\rho) = 0 \quad \Longrightarrow \quad H^1_f(\mathbb{Q}, \rho^\vee(1)) \neq 0.$$

In work yet unpublished (though see [SU06b]) Skinner and Urban also prove the same implication for Galois representations attached to certain automorphic representations of unitary groups of mixed signature.

The broad method used in the works [SU06a] and [SU06b] is roughly the same, and we will call it the Skinner–Urban method. This thesis should be viewed as another instance of this method, and we will establish an implication like the one just above, for a particular family of representations ρ, under certain assumptions. But before we come to this, let us describe now how the Skinner–Urban method works in general.

The method

Let M now be a reductive group over $\mathbb{Q}$. For simplicity, we assume M is split, although the method can work for nonsplit groups as well. Let π be an automorphic representation of the adelic points $M(\mathbb{A})$ of M, and assume π is "nice" enough to have attached to it a p-adic Galois representation ρ_π. We will not be too precise about what this means here, but, this Galois representation should be a continuous representation

$$\rho_\pi : G_{\mathbb{Q}} \to M^\vee(\overline{\mathbb{Q}}_p),$$

where $M^\vee$ is the dual reductive group of M. The behavior of ρ_π, when restricted to a decomposition group $G_{\mathbb{Q}_\ell}$ at a prime ℓ, should be determined by the local nature of the automorphic representation π at ℓ.

Now let R be a representation

$$R : M^\vee \to \mathrm{GL}_n$$

of the dual group $M^\vee$. Then $R \circ \rho_\pi$ should, in particular, be geometric, and therefore we may expect that a suitable version of the modular method mentioned above might yield a proof of the following implication towards the Bloch–Kato conjecture for $R \circ \rho_\pi$:

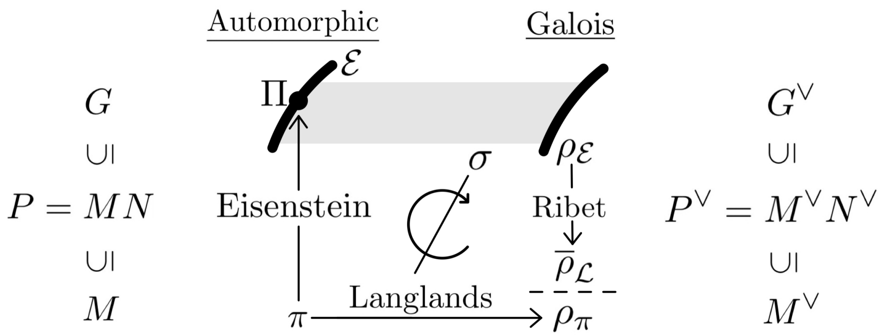
$$L(s_0, \pi, R) = 0 \quad \Longrightarrow \quad H^1_f(\mathbb{Q}, (R \circ \rho_\pi)^\vee(-n_0 + 1)) \neq 0.$$

Here, $s = s_0$ is the central point for the L-function $L(s, \pi, R)$, and n_0 is an integer such that

$$L(s + s_0, \pi, R) = L(s + n_0, R \circ \rho_\pi),$$

which we assume does exist.

The Skinner–Urban method is an incarnation of the modular method which, under certain favorable circumstances, can prove implications like the one just above. It may be described by the following diagram, whose pieces we explain just below.



The Skinner–Urban method

We first explain the groups on the outer edges of this diagram. Although the target implication makes no use of any groups other than M and $M^\vee$ (and GL_n, though this is just part of the data of the representation R) the Skinner–Urban method, and indeed any version of the modular method, must pass though a larger reductive group G. So we fix another reductive group G and assume that we can embed M as the Levi of some maximal parabolic subgroup $P \subset G$. Then $M^\vee$ occurs as a Levi subgroup of a maximal parabolic subgroup, which we denote $P^\vee$, in the dual group $G^\vee$ of G; if P is given by omitting one node of the Dynkin diagram for G, say corresponding to a simple root γ, then $P^\vee$ is the parabolic subgroup of $G^\vee$ which is obtained by omitting the node corresponding to the coroot $\gamma^\vee$ from the Dynkin diagram for $G^\vee$. We write $N^\vee$ for the unipotent radical of $P^\vee$.

Then bottom row of the diagram is the process described above; we assume we can use some version of the global Langlands correspondence to attach to π the Galois representation ρ_π. Now we must describe how to traverse the diagram by taking the arch above the bottom row.

The first step is to construct a functorial lift Π of π from M to G. This lift Π will be an automorphic representation of $G(\mathbb{A})$ obtained from π via some process of parabolic induction. A natural way to obtain such a Π is through Langlands's theory of Eisenstein series, and this is why the corresponding arrow is labelled "Eisenstein."

From here we must make a p-adic deformation of the functorial lift Π in a family $\mathcal{E}$ of automorphic representations which is generically cuspidal. This is often accomplished using tools from the p-adic theory of automorphic forms, such as the eigenvariety; this is why the family is labelled $\mathcal{E}$. This is also usually where the hypothesis that $L(s_0, \pi, R) = 0$ is used; one usually needs this to know that Π has nice enough properties at the archimedean place to show that Π can be p-adically deformed.

Each cuspidal member of $\mathcal{E}$ should have attached to it a Galois representation, again by some version of the global Langlands correspondence, and these Galois representations should fit into a p-adic family of Galois representations, denoted $\rho_{\mathcal{E}}$ in the diagram. In practice, this usually means there is an affinoid algebra $\mathcal{A}$ over some p-adic field which parametrizes the family $\mathcal{E}$, and $\rho_{\mathcal{E}}$ is a Galois representation

$$\rho_{\mathcal{E}} : G_{\mathbb{Q}} \to G^{\vee}(\mathrm{Frac}(\mathcal{A}))$$

into the points of $G^{\vee}$ over the fraction field of $\mathcal{A}$. The representation $\rho_{\mathcal{E}}$ should be continuous in a certain sense, and should specialize to the Galois representation of Π at the point of $\mathcal{E}$ corresponding to Π. We note that the Galois representation attached to Π is just the composition ρ_{π} with the inclusion of $M^{\vee}$ into $G^{\vee}$, due to the functorial nature of the Langlands correspondence.

The final step is to traverse the arrow labelled "Ribet" in the diagram. This is done by choosing a specific lattice $\mathcal{L}$ in $\rho_{\mathcal{E}}$ and specializing that lattice at the point of $\mathcal{E}$ corresponding to Π. If done correctly, this specialization $\overline{\rho}_{\mathcal{L}}$ will only give back the Galois representation of Π up to semisimplification, and will factor in a nontrivial way through the parabolic subgroup $P^{\vee}$, but not through the Levi subgroup $M^{\vee}$. The failure for $\overline{\rho}_{\mathcal{L}}$ to factor through $M^{\vee}$ should be measured by a cocycle σ, which should provide a nontrivial element in the Bloch–Kato Selmer group

$$H^1_f(\mathbb{Q}, (R \circ \rho_{\pi})^{\vee}(-n_0 + 1)),$$

as desired.

Now we have not actually explained yet how the representation R fits into this picture. We need certain pieces of "numerology" to be satisfied by the objects at play here, and one of them is the following.

The Levi $M^{\vee}$ acts on the unipotent radical $N^{\vee}$ of $P^{\vee}$ by the adjoint action. Under this action, the Jordan–Hölder filtration breaks $N^{\vee}$ into graded pieces $N^{\vee}_1, \ldots, N^{\vee}_r$, each a representation of

$M^\vee$. As part of the aforementioned numerology, we must require that the center $N_1^\vee$ of $N^\vee$ is one-dimensional, and that $N_r^\vee \cong R$ as representations of $M^\vee$. This requirement is what allows the Skinner–Urban method to see the representation R.

Perhaps the most basic instance of the Skinner–Urban method is when $M = \mathrm{GL}_1 \times \mathrm{GL}_1$ and $G = \mathrm{GL}_2$. Then one can prove the now classical implication

$$\zeta(-m) = 0 \quad \implies \quad H^1_f(\mathbb{Q}, \mathbb{Q}_p(m+1)) \neq 0,$$

for even integers $m > 0$. This is done by constructing an Eisenstein series E for GL_2, and the hypothesis that $\zeta(-m) = 0$ implies that E is holomorphic, as the nonholomorphic part of its constant term will be a multiple of $\zeta(-m)$ and will therefore vanish. Therefore one can put a critical p-stabilization of E in a Coleman family $\mathcal{E}$, and the criticality implies this family is generically cuspidal. Correspondingly, on the Galois side, one has a family of Galois representations $\rho_{\mathcal{E}}$. Constructing a particular lattice $\mathcal{L}$ in $\rho_{\mathcal{E}}$ and specializing at the point corresponding to the Eisenstein series E gives a representation $\overline{\rho}_{\mathcal{L}}$ given in matrix form by

$$\overline{\rho}_{\mathcal{L}} \sim \begin{pmatrix} \chi_{\mathrm{cyc}} & * \\ 0 & \chi_{\mathrm{cyc}}^{-m} \end{pmatrix}.$$

with $*$ nonzero. One then shows that this $*$ is the desired cocycle in $H^1_f(\mathbb{Q}, \mathbb{Q}_p(m+1))$. The (unpublished) notes of Skinner [Ski09] for the 2009 CMI summer school contains this argument in detail.

Notice that, if we let P be the upper triangular Borel of GL_2, then we can identify $M = M^\vee = \mathrm{GL}_1 \times \mathrm{GL}_1$, as well as $\mathrm{GL}_2 = \mathrm{GL}_2^\vee$ and $P = P^\vee$. Then the representation $\overline{\rho}_{\mathcal{L}}$ above factors through $P^\vee$, and the nontriviality of $*$ says exactly that it does not factor further through $M^\vee$.

Another instance of this method was carried out by Skinner–Urban in [SU06a], where $M = \mathrm{GL}_2 \times \mathrm{GL}_1$ is embedded in $G = \mathrm{GSp}_4$ as the Levi of the Siegel parabolic subgroup. There they use Saito–Kurokawa lifts instead of Eisenstein series, and they obtain results towards the Bloch–Kato conjecture for a modular form.

As mentioned above, yet another instance of this method is carried out by Skinner–Urban in unpublished work, though see [SU06b]. There, M is (up to center) a unitary group of mixed signature (a, b) and G is a unitary group of signature $(a+1, b+1)$. The group M is not split over $\mathbb{Q}$, and correspondingly, the Galois representations and Selmer groups thereof are defined over an imaginary quadratic field K.

In all three of these cases, the representation of the Levi $M^\vee$ is the standard one.

This thesis

This thesis carries out the Skinner–Urban method for the first time in a particular case where the representation R is not the standard one. The setting is as follows.

Let F be a cuspidal holomorphic eigenform of level 1 and weight k. Then F gives rise to a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$. So our group M will be GL_2. Therefore $M^\vee$ is also GL_2. Now the representation R here will be a symmetric cube representation of GL_2; More precisely, we let R be the representation

$$\mathrm{Ad}^3 = \mathrm{Sym}^3(\mathrm{Std}) \otimes \det{}^{-1},$$

where Std is the standard representation of GL_2. We will then take G to be the exceptional group G_2, and we will apply the Skinner–Urban method when GL_2 is embedded as the Levi of the long root parabolic subgroup of G_2.

Now it follows from the hypothesis that F is level 1 that the L-function $L(1/2, \pi, \mathrm{Ad}^3) = 0$ always. Here $s = 1/2$ is the central point for this L-function. Let ρ_F be the Galois representation attached to F. The main theorem of this thesis is the following, and is proved via the Skinner–Urban method; see Theorem 3.5.3.3

Theorem. *Under Arthur's conjectures and a version of the global Langlands correspondence for* G_2*, the Bloch–Kato Selmer group*

$$H^1_f(\mathbb{Q}, (\mathrm{Ad}^3 \rho_F)^\vee(k/2))$$

is nontrivial.

The twist $k/2$ is the correct one to correspond to the central point of the L-function $L(s, \pi, \mathrm{Ad}^3)$.

We remark here that there has been a lot of interest lately in these symmetric cube Selmer groups. For example, there is the recent work of Haining Wang [Wan20] and the work of Loeffler–Zerbes [LZ20]. Both of these papers work in the Euler system direction, establishing upper bounds on the ranks of the symmetric cube Selmer groups that they study, as opposed to this thesis which works in the modular direction.

We will describe in more detail how the method of Skinner–Urban works in our case momentarily, including the difficulties encountered which lead to the assumptions made in the hypotheses of the theorem. But before we come to that, we should say how this thesis will be organized.

Organization of this thesis

The proof of the above theorem can be separated into three main steps. The first step creates a functorial lift Π of F to G_2 and locates every instance of Π in the cohomology of the locally symmetric spaces attached to G_2. The second step chooses a critical p-stabilization $\sigma(\Pi)$ of Π and p-adically deforms $\sigma(\Pi)$ in a generically cuspidal family. The third step carries this family to the Galois side and constructs a lattice in the Galois representation attached to this family whose specialization at the point corresponding to Π gives the desired cocycle in the correct Bloch–Kato Selmer group.

These three steps were originally written as three different papers, and were incorporated into this thesis each as one of the three chapters. As such, each chapter more or less stands alone, with a minimal amount of reference between them. Thus enough redundancy is built into the exposition that the reader can choose to read any one of these chapters without having read the others.

We now introduce the contents of each of these chapters.

Chapter 1, on cohomology

For the first chapter we can be less restrictive with our modular form F. So we let F be a cuspidal holomorphic eigenform with level N not divisible by the prime p, with even weight $k \geq 4$, and with trivial nebentypus. Then associated with F is a unitary cuspidal automorphic representation π_F of $\mathrm{GL}_2(\mathbb{A})$.

Now the group G_2, being a simple group of rank 2, has two simple roots. They are of different lengths, and we denote the long simple root by α and the short simple root by β. We let P_α be the long root parabolic subgroup of G_2; by definition, this is the parabolic subgroup whose Levi factor M_α contains the long root α. Then we have $M_\alpha \cong \mathrm{GL}_2$, and we may view π_F as a cuspidal automorphic representation of $M_\alpha(\mathbb{A})$.

We then consider the parabolic induction

$$\mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\pi_F \otimes \delta_{P_\alpha(\mathbb{A})}^{1/10+1/2}),$$

and we let $\Pi = \mathcal{L}_\alpha(\pi_F, 1/10)$ denote the Langlands quotient of this parabolic induction; it is the unique irreducible quotient of this induction. This representation Π will be the functorial lift which we will use in carrying out the Skinner–Urban method.

Let us explain briefly why we take this particular choice of Π as our functorial lift. The parabolic induction spaces of the form

$$\mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\pi_F \otimes \delta_{P_\alpha(\mathbb{A})}^{s+1/2}),$$

with s a complex variable, allow us to build Eisenstein series. And through the lens of the Langlands–Shahidi method, these Eisenstein series see the L-function $L(s, \pi_F, \mathrm{Ad}^3)$. A little more precisely, in the constant term of such an Eisenstein series one finds the expression

$$L(5s, \pi_F, \mathrm{Ad}^3).$$

So specializing to $s = 1/10$, one finds the L-value $L(1/2, \pi_F, \mathrm{Ad}^3)$ and hence an opportunity to apply the hypothesis that this L-value vanishes. This explains roughly why we must consider this induction space, and we must consider its Langlands quotient to have a maximally unramified irreducible automorphic representation to work with.

To motivate what we are going to do with Π in this chapter, we jump ahead a little and remark that the ultimate goal on the automorphic side is to p-adically deform a critical p-stabilization of Π. Doing this requires computing a certain *cuspidal overconvergent multiplicity*, and we will explain this in more detail when we discuss the second chapter of this thesis. But the upshot is that, in order to get a handle on the cuspidal overconvergent multiplicity of a critical p-stabilization of Π, we must first locate every instance of Π in the cohomology of the locally symmetric spaces for G_2. This is the purpose of this first chapter.

Now we fix an irreducible, finite dimensional representation E of the complex group $\mathrm{G}_2(\mathbb{C})$.

This gives rise to compatible local systems on the locally symmetric spaces attached to G_2. The cohomology groups of these local systems form a direct system whose direct limit is an admissible representation of the group of finite adelic points $\mathrm{G}_2(\mathbb{A}_f)$. By work of Franke [Fra98], this representation can be constructed in the following way.

Let $\mathfrak{g}_2$ denote the complexified Lie algebra of G_2, and fix a maximal compact subgroup K_∞ in the group real points $\mathrm{G}_2(\mathbb{R})$. One can define a certain space $\mathcal{A}_E(\mathrm{G}_2)$ of automorphic forms for G_2 using E, in a way which we will not be precise about in this introduction. But it is a $\mathrm{G}_2(\mathbb{A}_f) \times (\mathfrak{g}_2, K_\infty)$-module, making its cohomology

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_E(\mathrm{G}_2) \otimes E)$$

a $\mathrm{G}_2(\mathbb{A}_f)$-module. By a conjecture of Borel, which was proved by Franke in his paper [Fra98], this module is exactly the direct limit discussed in the previous paragraph. Therefore, our goal will be to locate Π in the cohomology space displayed above.

Let $\mathcal{A}_E(\mathrm{G}_2)_{\mathrm{cusp}}$ be the space of cusp forms in $\mathcal{A}_E(\mathrm{G}_2)$. It has a natural complement $\mathcal{A}_E(\mathrm{G}_2)_{\mathrm{Eis}}$ which is built, in a way which can be made precise, from Eisenstein series, and the decomposition

$$\mathcal{A}_E(\mathrm{G}_2) = \mathcal{A}_E(\mathrm{G}_2)_{\mathrm{cusp}} \oplus \mathcal{A}_E(\mathrm{G}_2)_{\mathrm{Eis}}$$

is a decomposition of $\mathrm{G}_2(\mathbb{A}_f) \times (\mathfrak{g}_2, K_\infty)$-modules. We therefore get a decomposition

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_E(\mathrm{G}_2) \otimes E) = H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_E(\mathrm{G}_2)_{\mathrm{cusp}} \otimes E) \oplus H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_E(\mathrm{G}_2)_{\mathrm{Eis}} \otimes E)$$

as $\mathrm{G}_2(\mathbb{A}_f)$-modules. The first of these factors is called the *cuspidal cohomology* and the second is the *Eisenstein cohomology.* We then have the following result, which is a consequence of Theorem 1.5.3.3 of this thesis.

Theorem. *For the modular form F of level N, even weight $k \geq 4$, and trivial nebentypus, assume $L(1/2, \pi_F, \mathrm{Ad}^3) = 0$. Then there is one and only one representation E, which is of highest weight $\frac{k-4}{2}(2\alpha + 3\beta)$, for which the finite part $\Pi_f = \mathcal{L}_\alpha(\pi_F, 1/10)_f$ of our Langlands quotient appears as* *a subquotient of the Eisenstein cohomology*

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_E(\mathrm{G}_2)_{\mathrm{Eis}} \otimes E).$$

It appears exactly once in this cohomology space, in (middle) degree 4*, with multiplicity one.*

There are two steps to establishing this result. First, one must actually construct the representation Π_f as a subquotient of Eisenstein cohomology. This is made possible by a deeper analysis of the Eisenstein space $\mathcal{A}_E(\mathrm{G}_2)_{\mathrm{Eis}}$ as follows. For Q another parabolic subgroup of G_2, Franke and Schwermer [FS98] have defined an equivalence relation on the cuspidal automorphic representations of the Levi of Q. Let φ denote one of these equivalence classes. Then Franke and Schwermer construct a subspace

$$\mathcal{A}_{E,Q,\varphi}(\mathrm{G}_2) \subset \mathcal{A}_E(\mathrm{G}_2)_{\mathrm{Eis}}$$

out of Eisenstein series induced from the representations in φ, along with their residues and derivatives. (Actually Franke–Schwermer work much more generally on an arbitrary reductive group.) There is a decomposition

$$\mathcal{A}_E(\mathrm{G}_2)_{\mathrm{Eis}} = \bigoplus_Q \bigoplus_\varphi \mathcal{A}_{E,Q,\varphi}(\mathrm{G}_2),$$

where the first sum is over a fixed set of parabolic subgroups which represent the associate classes of proper parabolic subgroups of G_2.

If none the Eisenstein series arising in the construction of the space $\mathcal{A}_{E,Q,\varphi}(\mathrm{G}_2)$ have a pole, then the space $\mathcal{A}_{E,Q,\varphi}(\mathrm{G}_2)$ has a very nice and explicit $\mathrm{G}_2(\mathbb{A}_f) \times (\mathfrak{g}_2, K_\infty)$-module structure as a parabolically induced module. We can then compute the $(\mathfrak{g}_2, K_\infty)$-cohomology of this module explicitly in terms of a representation parabolically induced from the finite part of a representation in φ.

If $\varphi(\pi_F, 1/10)$ is the class for P which contains the representation $\pi_F \otimes \delta_{P_\alpha(\mathbb{A})}^{1/10}$, where π_F is the same as above, and E is the representation in the above theorem, then it turns out that none of the Eisenstein series appearing in the construction of $\mathcal{A}_{E,P,\varphi(\pi_F,1/10)}(\mathrm{G}_2)$ has a pole, and we can find Π_f as a quotient of the cohomology

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E,P,\varphi(\pi_F,1/10)}(\mathrm{G}_2) \otimes E).$$

This is where we use the hypothesis that $L(1/2, \pi_F, \mathrm{Ad}^3) = 0$; this vanishing allows us, via an examination of the constant term of our Eisenstein series, to conclude that these Eisenstein series do not have poles at $s = 1/10$.

This describes the first of the two steps we need to prove the above theorem. The second of these steps is to show that no other summand $\mathcal{A}_{E,Q,\varphi}(\mathrm{G}_2)$ of the decomposition above, besides the summand for $Q = P$ and $\varphi = \varphi(\pi_F, 1/10)$ just studied, contains any copy of Π_f in its cohomology. To do this, we need to study the cohomology of these summands in a way which is explicit enough to rule out an appearance of Π_f.

One runs into a problem here, as we only know the explicit structure of the space $\mathcal{A}_{E,Q,\varphi}(\mathrm{G}_2)$ as a parabolic induction when the Eisenstein series involved in its construction have no poles. But it may well be the case that certain Eisenstein series induced from φ do have poles. Luckily, following Franke [Fra98], Grobner [Gro13] has defined a filtration on these spaces whose graded pieces are parabolically induced modules whose cohomology can be explicitly studied.

So one just needs to show that Π_f doesn't appear in the cohomology of these graded pieces. To do this, we distinguish Π_f from the representations appearing in the cohomology of the graded pieces by assigning to them ℓ-adic Galois representations for a fixed prime ℓ. These Galois representations are only powerful enough to distinguish between near-equivalence classes of automorphic representations, that is, to tell them apart outside a set of finitely many primes. But actually this is enough for our purposes because we can appeal to strong multiplicity one theorems for the Levis of G_2.

The next thing to do would be to compute the multiplicity of Π_f in the cuspidal cohomology. This requires knowledge about the classification of CAP forms which are nearly equivalent to our Langlands quotient Π. However, not enough about such things is known unless we assume some standard conjectures related to those of Arthur. So this is what we do.

As explained by Gan and Gurevich [GG09], assuming such conjectures, under the hypothesis still that $L(1/2, \pi_F, \mathrm{Ad}^3) = 0$, precisely two kinds of CAP representations Π' with $\Pi'_f \cong \Pi_f$ should be able to appear in $\mathcal{A}_E(\mathrm{G}_2)_{\mathrm{cusp}}$, depending on the sign ϵ of the symmetric cube functional equation for π_F. They will appear with multiplicity one in either case. If $\epsilon = 1$, then $\Pi'_\infty \cong \Pi_\infty$, and hence this appears in cuspidal cohomology exactly once in each of degrees 3 and 5. But Gan and Gurevich do not describe Π'_∞ when $\epsilon = -1$. So we must do this ourselves.

One of the things which Gan and Gurevich explain, however, is that there is an Arthur parameter ψ for $\mathrm{G}_2(\mathbb{R})$ whose associated Arthur packet should consist of the two possible representations which can occur as Π'_∞. Upon examination of the parameter ψ, one sees that the one corresponding to $\epsilon = 1$ must indeed be $\Pi_\infty = \mathcal{L}_\alpha(\pi_F, 1/10)_\infty$, but Arthur's conjectures do not immediately tell us anything about the representation corresponding to $\epsilon = -1$.

However, for certain types of Arthur parameters ψ, Adams and Johnson have been able to construct packets AJ_ψ which satisfy the conclusion of Arthur's conjectures for real groups. We show that our parameter ψ is of this special type, and we explicitly compute AJ_ψ. We find the following, which is the content of Theorem 1.6.4.4 in this thesis.

Theorem. *The Adams–Johnson packet* AJ_ψ *contains the representation* $\mathcal{L}(\pi_F, 1/10)_\infty$ *and the quaternionic discrete series representation of* $\mathrm{G}_2(\mathbb{R})$ *of weight* $k/2$*, in the terminology of Gan–Gross–Savin [GGS02].*

Thus if $\epsilon = -1$, it follows that our CAP representation Π' should again be cohomological, appearing in the cuspidal cohomology of E exactly once in middle degree 4. Thus, in this case, Π_f appears in cohomology of E exactly twice, once in Eisenstein cohomology, and once in cuspidal cohomology, and both times in degree 4.

As a bonus, our methods also apply (even unconditionally) to GSp_4 in place of G_2. While not needed for the main results of this thesis, we carry this out in detail in this first chapter as well.

Chapter 2, on the p-adic deformation

We continue with our modular eigenform F of level N, even weight $k \geq 4$, and trivial nebentypus, as well as its associated automorphic representation π_F of $\mathrm{GL}_2(\mathbb{A})$, and the Langlands quotient Π on $\mathrm{G}_2(\mathbb{A})$. We assume p does not divide N, and we now fix a root α_p of the Hecke polynomial of F at p.

We would now like to p-adically deform a critical p-stabilization $\sigma(\Pi)$ of Π in a generically cuspidal family of cohomological automorphic representations of $\mathrm{G}_2(\mathbb{A})$. In the absence of a G_2-Shimura variety, our options for doing this are limited to the methods present in the paper [Urb11] of Urban on eigenvarieties for reductive groups with discrete series.

Making a p-adic deformation of a noncritical p-stabilization of an automorphic representation is not too difficult with Urban's methods, but when the p-stabilization is instead critical, this becomes significantly harder. And we do need to use a critical p-stabilization of Π in order to have in place certain pieces of numerology on the Galois side, as will be explained below when we discuss the third chapter.

Now the techniques in Urban's paper which allow us to make this p-adic deformation go through his theory of multiplicities. Urban defines certain local systems on the locally symmetric spaces of a reductive group with discrete series whose cohomology contains a subspace which can be considered a space whose constituents are overconvergent p-adic automorphic representations. If a p-stabilized automorphic representation appears in this space with a nonzero multiplicity, then it can be p-adically deformed.

There is furthermore a variant of this notion of multiplicity which allows us to detect when a p-stabilization of an automorphic representation deforms in a generically cuspidal family, as we would like to be the case for our Π. On the one hand, this cuspidal overconvergent multiplicity, as we call it, can be expressed as a difference between the overconvergent multiplicity just described and other overconvergent multiplicities coming from smaller Levi subgroups. On the other hand, Urban also relates the location of an automorphic representation in the classical cohomology of arithmetic groups to the (noncuspidal) overconvergent multiplicity of a p-stabilization of it and various "Weyl twists" of this p-stabilization.

In the first chapter of this thesis, we will have located Π in classical cohomology under the assumption that $L(1/2, \pi_F, \mathrm{Ad}^3) = 0$, and under Arthur's conjectures. This gives a "classical multiplicity" for the critical p-stabilization $\sigma(\Pi)$ of Π which we will relate to the overconvergent multiplicities just mentioned. Then we will relate these overconvergent multiplicities to certain cuspidal overconvergent multiplicities of $\sigma(\Pi)$ by computing explicitly the "Eisenstein" multiplicities which come from smaller Levi subgroups. Compiling these computations gives the following result, a precise version of which appears as Theorem 2.3.1.11.

Theorem. *Assume the weight k of F is sufficiently large with respect to the p-adic valuation $v_p(\alpha_p)$. Assume also that $\epsilon(1/2, \pi_F, \mathrm{Ad}^3) = -1$. Then under Arthur's conjectures, we have that the cuspidal overconvergent multiplicity of the critical p-stabilization $\sigma(\Pi)$ of Π is at least 3. In particular, $\sigma(\Pi)$ has a p-adic deformation in a generically cuspidal family.*

The dependency of this theorem on Arthur's conjectures simply comes from the same dependency of the results about cuspidal cohomology from the first chapter on these conjectures.

The hypothesis that $\epsilon(1/2, \pi_F, \mathrm{Ad}^3) = -1$ seems to be necessary. In fact, we expect that the cuspidal overconvergent multiplicity of $\sigma(\Pi)$ is always exactly 1 more than the classical multiplicity of Π, due to a certain overconvergent Eisenstein multiplicity. By the results of the first chapter of this thesis, this classical multiplicity is 2 when $\epsilon(1/2, \pi_F, \mathrm{Ad}^3) = -1$ because Π appears once in degree 4 in Eisenstein cohomology and once in degree 4 in cuspidal cohomology. But when $\epsilon(1/2, \pi_F, \mathrm{Ad}^3) = +1$ but $L(1/2, \pi_F, \mathrm{Ad}^3) = 0$, then instead Π appears in degrees 3 and 5 in cuspidal cohomology. The multiplicities we are dealing with are alternating sums over the cohomological degree. So if $\epsilon(1/2, \pi_F, \mathrm{Ad}^3) = +1$, we find that the classical multiplicity becomes -1, cancelling with the overconvergent Eisenstein multiplicities which contribute.

In any case, we will specialize now to the case when F has level 1. This assumption will be more important on the Galois side, but it also helps us improve the above result in this case. When F is level 1, we always have $\epsilon(1/2, \pi_F, \mathrm{Ad}^3) = -1$, and we get the following result, which is made more precise in Theorems 2.3.2.1 and 2.3.2.3.

Theorem. *Assume F has level 1. Then under Arthur's conjectures, $\sigma(\Pi)$ admits a p-adic deformation in a generically cuspidal family.*

Unlike the theorem stated before it, the theorem above has no assumption on the slope $v_p(\alpha_p)$. It is proved by putting F in a Coleman family $\mathcal{F}$ and constructing representations analogous to $\sigma(\Pi)$ for the classical members of $\mathcal{F}$. For members which have sufficiently high weight, these representations will lie on Urban's eigenvariety and admit $\sigma(\Pi)$ as a limit point. Therefore, $\sigma(\Pi)$ lies on the eigenvariety as well.

The assumption that F has level 1 is used to ensure the other members of the Coleman family $\mathcal{F}$ also have level 1, and therefore the signs of their Ad^3-functional equations are all -1 as well.

We remark here that we have attached an appendix to this thesis that explains the results of [Urb11] on eigenvarieties which we need to use. It also corrects an error in that paper, and the correction of this error leads to a slight modification of some of the main results. We explain this as well in the appendix.

Chapter 3, on Galois representations

We retain the assumption that our form F has level 1. This implies in particular that F is not CM, and this will ensure that the Galois representation attached to F has big enough image.

Now let us deform our Langlands quotient Π in a p-adic family in accordance with the results of the second chapter of this thesis. This p-adic deformation is parametrized by a rigid analytic space $\mathfrak{Y}$ over $\mathbb{Q}_p$, and this space $\mathfrak{Y}$ is finite over a space $\mathfrak{X}$ of p-adic weights. For sufficiently regular classical weights λ, the specialization of $\mathfrak{Y}$ at a point y in $\mathfrak{Y}$ above λ gives a cuspidal automorphic representation π_y of $\mathrm{G}_2(\mathbb{A})$ of hyperspecial level.

We now must assume some version of the global Langlands correspondence for cohomological automorphic representations of $\mathrm{G}_2(\mathbb{A})$. Then π_y has attached to it a p-adic Galois representation which is unramified at all primes $\ell \neq p$ and crystalline at p. As G_2 is self dual as a reductive group, these Galois representations should factor through $\mathrm{G}_2(\overline{\mathbb{Q}}_p)$, and we assume this.

Now our Langlands quotient Π has attached to it the following Galois representation. Let $\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the usual Galois representation attached to F. We view the target group of ρ_F as $M_\beta(\overline{\mathbb{Q}}_p)$, where M_β is the Levi subgroup of the short root parabolic P_β. Since passage to the dual switches the long and short roots of G_2, the Galois representation attached to Π should factor through $M_\beta(\overline{\mathbb{Q}}_p)$ since Π was induced from the long root parabolic $P_\alpha(\mathbb{A})$ of $\mathrm{G}_2(\mathbb{A})$. In fact, the Galois representation attached to Π is given by the composition of $\rho_F(-(k-2)/2) : G_{\mathbb{Q}} \to M_\beta(\overline{\mathbb{Q}}_p)$ with the inclusion of M_β into G_2. We call this representation ρ_Π.

In this third chapter, we use the theory of pseudocharacters of Lafforgue [Laf18] to interpolate the G_2 Galois representations ρ_y from above. We specialize to the rigid analytic curve $\mathfrak{Z}$ in $\mathfrak{Y}$ containing Π and cut out by an appropriate line $\mathfrak{L}$ in weight space. Then the theory of pseudocharacters gives us a Galois representation $\rho_{\mathcal{Z}} : G_{\mathbb{Q}} \to \mathrm{G}_2(\overline{\mathrm{Frac}(\mathcal{O}(\mathfrak{Z}))})$, where $\mathcal{O}(\mathfrak{Z})$ is the ring of analytic functions on $\mathfrak{Z}$.

At this point we would like to construct a lattice in $\rho_{\mathfrak{Z}}$, but this isn't so meaningful for a representation of the Galois group into anything other than GL_n. So we compose $\rho_{\mathfrak{Z}}$ with the smallest fundamental representation of G_2, which is 7-dimensional. We denote it by R_7. This gives us a Galois representation $R_7 \circ \rho_{\mathfrak{Z}}$ which, after a series of reductions, we assume takes values in $\mathrm{GL}_7(\mathcal{O}(\mathfrak{Z}))$. It specializes at Π to a representation with semisimplification given by the following

block-diagonal matrix:
$$\begin{pmatrix} \rho_F(-(k-2)/2) & 0 & 0 \\ 0 & \mathrm{Ad}^2\,\rho_F & 0 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}.$$

This is just the Galois representation attached to Π composed with R_7. Here $\mathrm{Ad}^2 = \mathrm{Sym}^2(\mathrm{Std}) \otimes \det^{-1}$ denotes the usual three dimensional adjoint representation of GL_2.

Now we can construct our lattice, which we call $\mathcal{L}$. It is constructed so that its specialization $\overline{\mathcal{L}}$ at Π will have unique irreducible quotient $\rho_F(-k/2)$ as long as $R_7 \circ \rho_3$ is irreducible; if it is reducible, we show it breaks into a 4-dimensional piece and a 3 dimensional piece. Thus $\overline{\mathcal{L}}$ has one of the following shapes:

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & *_3 & *_2 \\ 0 & \mathrm{Ad}^2\,\rho_F & *_1 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix},$$

with $*_1$ and $*_2$ nontrivial, or

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \mathrm{Ad}^2\,\rho_F & *_3 & *_1 \\ 0 & \rho_F(-(k-2)/2) & *_2 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix},$$

again with $*_1$ and $*_2$ nontrivial, or, if $R_7 \circ \rho_3$ is reducible,

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & 0 & *_2 \\ 0 & \mathrm{Ad}^2\,\rho_F & 0 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix},$$

with $*_2$ nontrivial. We must then rule out these latter two cases.

To do this, we use that $R_7 \circ \rho_3$ factors through G_2. It turns out that G_2 is the stabilizer in GL_7 of a certain kind of alternating trilinear form, and we use this to put a nontrivial alternating trilinear form of a certain shape on $\overline{\mathcal{L}}$. We can then make some serious computations involving matrix coefficients using this trilinear form, along with some p-adic Hodge theory, to rule out the second and third cases above.

In a little more detail, in the second and third cases, the representation $\overline{\mathcal{L}}$ would admit as a quotient a nontrivial extension E of the form

$$E \sim \begin{pmatrix} \rho_F(-(k-2)/2) & *_2 \\ 0 & \rho_F(-k/2) \end{pmatrix}.$$

We obtain enough information about the alternating trilinear form on $\overline{\mathcal{L}}$ in order to put strong restrictions on the possible entries $*_2$ that can occur. In particular, we show that the exterior square $\wedge^2 E$ admits as a subrepresentation an extension of the form

$$\begin{pmatrix} \chi_{\mathrm{cyc}} & *_2' \\ 0 & 1 \end{pmatrix},$$

with $*_2'$ nontrivial if and only if $*_2$ is nontrivial. So to rule out the extension the second and third cases above, it is enough to show $*_2'$ must be zero.

To do this, we show it is crystalline, for then it would give a nontrivial cohomology class in the Bloch–Kato Selmer group

$$H^1_f(\mathbb{Q}, \overline{\mathbb{Q}}_p(1)),$$

which is a trivial group. This kind of argument is ubiquitous in the Skinner–Urban method, and appears in both [SU06a] and [SU06b], as well as [Urb13b], when showing that their lattices have the correct shape.

Here there is a small problem. We would like to use a Lemma of Kisin (Lemma 3.3.1.2 in this thesis) to obtain a certain crystalline period in $D_{\mathrm{crys}}(\wedge^2 E)$ which is interpolated from those of the representations $\wedge^2 \rho_y$ for classical points y in $\mathfrak{Z}$ with sufficiently regular weight. However, it turns out that Kisin's Lemma provides the wrong period. We would like to have a period with crystalline Frobenius eigenvalue 1, but Kisin's lemma only gives one period and it is not this one. But there is a way to work around this. We first prove the following result, which is made precise in Proposition 3.3.2.5 and Corollary 3.3.2.3 in this thesis.

Proposition. *Any extension E_0 of $G_{\mathbb{Q}_p}$-representations fitting into an exact sequence*

$$0 \to \rho_F(1) \to E_0 \to \rho_F \to 0$$

is semistable. Furthermore, the possible filtered (ϕ, N)-modules that can occur as $D_{\mathrm{st}}(E_0)$ can be explicitly described.

Applying Kisin's lemma to the extension E above, and to its dual, actually puts enough restrictions on the monodromy operator N_E for $D_{\mathrm{st}}(E)$ to show that $D_{\mathrm{st}}(E)^{\phi=1}$ is in the kernel of the monodromy operator $N_{\wedge^2 E}$ for $\wedge^2 E$. Therefore $D_{\mathrm{st}}(E)^{\phi=1}$ consists of crystalline periods, forcing E' to be crystalline and forcing $*_2$ to be trivial, thus ruling out the second and third possible shapes of $\overline{\mathcal{L}}$ above.

Therefore,

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & *_3 & *_2 \\ 0 & \mathrm{Ad}^2 \rho_F & *_1 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix},$$

and we even have at this point that $*_3$ is nontrivial as a by-product of the matrix coefficient computations that were necessary to rule out the other two shapes of $\overline{\mathcal{L}}$. Being in this case, one can then obtain enough information about the trilinear form on $\overline{\mathcal{L}}$ to show that $\overline{\mathcal{L}}$ factors through G_2 as well. It must even factor through P_β because of its block upper triangular shape. Then we are almost there, since the unipotent radical N_β of P_β has Ad^3 as a Jordan–Hölder constituent. This will mean that $*_3$ gives an extension of $\overline{\mathbb{Q}}_p$ by the appropriate twist of $(\mathrm{Ad}^3 \rho_F)^\vee$; a priori, $*_3$ gives an extension of $\overline{\mathbb{Q}}_p$ by

$$\rho_F(-(k-2)/2) \otimes (\mathrm{Ad}^2 \rho_F)^\vee \cong (\mathrm{Ad}^3 \rho_F)^\vee(k/2) \oplus \rho_F(-k/2).$$

But the factorization of $\overline{\mathcal{L}}$ through G_2 shows that the factor of this which is an extension of $\overline{\mathbb{Q}}_p$ by $\rho_F(-k/2)$ is trivial. So it is only a matter of showing that $*_3$ is crystalline.

At this point, we would like to use Kisin's lemma to $\overline{\mathcal{L}}$ to obtain a certain crystalline period with appropriate Frobenius eigenvalue. However, once again, the lemma gives the wrong period. But we make do with the period it does provide and show that it suffices to give the crystallinity of $*_3$ as long as this period does not occur in $\mathrm{Fil}^0(D_{\mathrm{crys}}(\mathrm{Ad}^2 \rho_F))$. But, if it does happen to occur here, there is the following trick: We can switch our choice of root α_p of the Hecke polynomial of F at p on which this entire construction depended, and we can repeat the construction and show that for the other choice of α_p, the crystalline period provided by Kisin's lemma does not occur in $\mathrm{Fil}^0(D_{\mathrm{crys}}(\mathrm{Ad}^2 \rho_F))$! From here it follows that $*_3$ is crystalline and we finish the construction of the appropriate element of our Bloch–Kato Selmer group.

At the beginning of each chapter below, we will summarize the contents of each section, and also the notation that will be in play throughout the chapter. We have tried to be as consistent as possible about the notation throughout each of the chapters, but the reader should keep in mind that the chapters are written to be more or less independent.

Chapter 1: Multiplicity of Eisenstein series in cohomology and applications to GSp_4 and G_2

This chapter is organized as follows. The first three chapters are devoted to a very general setup, working mostly for an arbitrary reductive group, and they will be used to make the main computations in Sections 1.4 and 1.5.

In Section 1.1, we review facts about Eisenstein series and the spaces they comprise, recalling the Franke–Schwermer decomposition and some facts about the Franke filtration. In Section 1.2, we compute the cohomology of some of these spaces of Eisenstein series. In Section 1.3, we explain what we mean when we say an automorphic representation has attached to it an ℓ-adic Galois representation.

Section 1.4 is then devoted to applying the tools set up in the first three sections to compute the cohomological multiplicity of certain Langlands quotients for GSp_4. Some of the results in this chapter were originally stated by Urban in [Urb11], Example 5.5.3. However, he made an error in that example which we take the opportunity to correct (see Remark 1.4.4.2 in this chapter).

Section 1.5 then makes the same kind of computations for G_2, and although a lot of the arguments there are completely analogous to the GSp_4 case, this section is written in such a way that the reader can read it without having read Section 1.4.

What is not completely analogous between these two sections is that for GSp_4, the CAP forms we need have been completely classified, and so the computation of the cuspidal multiplicity in the GSp_4 case is unconditional. As we mentioned in the introduction, this is not the case for G_2, and Section 1.6 is devoted to the computation of a particular Adams–Johnson packet which makes our conditional results reasonable.

Notation and conventions

We now set the notation that will be used throughout the rest of this chapter.

Groups and Lie algebras

In Sections 1.1 and 1.2, G will denote a reductive group over the field $\mathbb{Q}$ of rational numbers. In Section 1.3, G will furthermore be split over $\mathbb{Q}$. In Section 1.4, we will specialize to the group GSp_4 and, in Section 1.5, to G_2. In Section 1.6, we will be working primarily with a real reductive Lie group, and we will denote that group by $\mathbf{G}$.

In general, our convention is to use uppercase roman letters to denote groups over $\mathbb{Q}$, such as G, to use uppercase boldface letters to denote real Lie groups, such as $\mathbf{G}$, and to use the corresponding lowercase fraktur letters to denote complex Lie algebras. So for example, $\mathfrak{g}$ will always denote the complexified Lie algebra of either the $\mathbb{Q}$-group G or the real Lie group $\mathbf{G}$. There will be a few exceptions to this convention, however. For example, when we have fixed a reductive $\mathbb{Q}$-group G, unless otherwise noted, we will simply write $G(\mathbb{R})$ for the real Lie group consisting of its $\mathbb{R}$-points.

When working with the group G, we will often fix a parabolic subgroup P of G along with a Levi decomposition $P = MN$. In this decomposition, M will always denote the Levi factor and N the unipotent radical. If we have another parabolic subgroup with fixed Levi decomposition, then we use subscripts on the notation for its fixed Levi factor and its unipotent radical to distinguish them from those of P; so if Q is another parabolic subgroup, we will write $Q = M_Q N_Q$ for its Levi decomposition.

For any parabolic Q as above, the notation A_Q will denote the maximal $\mathbb{Q}$-split torus in the center of the Levi M_Q of Q. This applies in particular to P and G; we use A_G to denote the maximal $\mathbb{Q}$-split torus in the center of G, and A_P that of M.

Now we have the complexified Lie algebras $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{m}$, $\mathfrak{m}_Q$, $\mathfrak{n}$, $\mathfrak{n}_Q$, $\mathfrak{a}_P$, and $\mathfrak{a}_Q$ of, respectively, G, P, Q, M, M_Q, N, N_Q, A_P, and A_Q. We let $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$, the self-commutator of $\mathfrak{g}$, and more generally, we write $\mathfrak{m}_{Q,0} = [\mathfrak{m}_Q, \mathfrak{m}_Q]$, or $\mathfrak{m}_0 = [\mathfrak{m}, \mathfrak{m}]$. We also write $\mathfrak{q}_0 = \mathfrak{q} \cap \mathfrak{g}_0$ and $\mathfrak{a}_{Q,0} = \mathfrak{a}_Q \cap \mathfrak{g}_0$, and similarly for $\mathfrak{p}_0$ and $\mathfrak{a}_{P,0}$. Then there are decompositions

$$\mathfrak{q} = \mathfrak{m}_{Q,0} \oplus \mathfrak{a}_Q \oplus \mathfrak{n}_Q,$$

and

$$\mathfrak{q}_0 = \mathfrak{m}_{Q,0} \oplus \mathfrak{a}_{Q,0} \oplus \mathfrak{n}_Q.$$

When P and Q are fixed along with their respective Levi decompositions, we will write $W(P,Q)$ for the set equivalence classes of elements $w \in G(\mathbb{Q})$ such that $wMw^{-1} = M_Q$. where w and w' are considered equivalent if $w^{-1}w'$ centralizes M.

We will always write ρ_Q for the character $\rho_Q : \mathfrak{a}_{Q,0} \to \mathbb{C}$ given by

$$\rho_Q(X) = \mathrm{Tr}(\mathrm{ad}(X)|\mathfrak{n}_Q), \qquad X \in \mathfrak{a}_{Q,0},$$

and similarly for ρ_P.

Points of groups

When v is a place of $\mathbb{Q}$, we write $\mathbb{Q}_v$ for the completion of $\mathbb{Q}$ at v. Then $\mathbb{R} = \mathbb{Q}_\infty$. The group of $\mathbb{Q}_v$-points of any affine algebraic group over $\mathbb{Q}$ is always given the usual topology induced from $\mathbb{Q}_v$.

We write $\mathbb{A}$ for the adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ for the finite adeles. The groups of $\mathbb{A}$-points or $\mathbb{A}_f$-points of any affine algebraic group over $\mathbb{Q}$ are also given their standard topologies.

When $P = MN$ is fixed as above, we will often consider the associated height function H_P. This is a function

$$H_P : G(\mathbb{A}) \to \mathfrak{a}_{P,0}.$$

To define it, we must fix a maximal compact subgroup $K \subset G(\mathbb{A})$. We assume $K = K_f K_\infty$ where K_∞ is a fixed maximal compact subgroup of $G(\mathbb{R})$ and $K_f = \prod_{v<\infty} K_v$ is a maximal compact subgroup of $G(\mathbb{A}_f)$ which we assume to be in good position with respect to a fixed minimal parabolic inside P. (Here the groups K_v are maximal compact subgroups of $G(\mathbb{Q}_v)$.) In particular, the Iwasawa decomposition holds for $P(\mathbb{A})$ and K.

Write $\langle \cdot, \cdot \rangle$ for the natural pairing

$$\langle \cdot, \cdot \rangle : \mathfrak{a}_{P,0} \times \mathfrak{a}_{P,0}^\vee \to \mathbb{C}$$

given by evaluation, where $\mathfrak{a}_{P,0}^\vee = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}_{P,0}, \mathbb{C})$. Write $X^*(M)$ for the group of algebraic characters of M. Then H_P is defined first on the subgroup $M(\mathbb{A})$ by requiring

$$e^{\langle H_P(m), d\Lambda\rangle} = |\Lambda(m)|, \qquad m \in M(\mathbb{A}),\ \Lambda \in X^*(M),$$

where $d\Lambda$ denotes the restriction to $\mathfrak{a}_{P,0}$ of the differential at the identity of the restriction of Λ to $A_P(\mathbb{R})$, and $|\cdot|$ is usual the adelic absolute value. Then H_P is defined in general by declaring it to be left invariant with respect to $N(\mathbb{A})$ and right invariant with respect to K.

If R is one of the rings $\mathbb{Q}_v$, $\mathbb{A}$, or $\mathbb{A}_f$, we use the notation $\delta_{P(R)}$ to denote the modulus character of $P(R)$, and similarly for other parabolics.

Automorphic representations

We take the point of view that an "automorphic representation" of $G(\mathbb{A})$ is (among other things) an irreducible object in the category of admissible $G(\mathbb{A}_f) \times (\mathfrak{g}, K_\infty)$-modules. We often even view automorphic representations as $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules by restriction. We let $\mathcal{A}(G)$ denote the space of all automorphic forms on $G(\mathbb{A})$.

If Π is an automorphic representation of $G(\mathbb{A})$ and v is a place of $\mathbb{Q}$, we will denote by Π_v the local component of Π at v. If v is finite, then this is an irreducible admissible representation of $G(\mathbb{Q}_v)$, and if $v = \infty$, then this is an irreducible admissible $(\mathfrak{g}, K_\infty)$-module.

Galois theory

We will write $G_{\mathbb{Q}}$ for the absolute Galois group of $\mathbb{Q}$, and for any place v of $\mathbb{Q}$, we will similarly write $G_{\mathbb{Q}_v}$ for the absolute Galois group of $\mathbb{Q}_v$. If v is finite, we always view $G_{\mathbb{Q}_v}$ as a subgroup of $\mathbb{Q}$ via by fixing a decomposition group at v.

Galois representations for us will always be into the $\overline{\mathbb{Q}}_\ell$-points of a fixed algebraic group. We always identify $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ via a fixed isomorphism.

For p a prime, Frob_p always denotes a fixed geometric Frobenius element at p in $G_{\mathbb{Q}}$. If χ_{cyc} denotes the ℓ-adic cyclotomic character, then our conventions will be such that twists by $|\cdot|$ on the automorphic side correspond to twists by χ_{cyc} on the Galois side; $|\cdot|$ sends $p \in \mathbb{Q}_p^\times$ to p^{-1}, and χ_{cyc} also sends Frob_p to p^{-1}.

Duals

We use the symbol $(\cdot)^\vee$ in various ways. If $\mathfrak{a}$ is an abelian Lie algebra, $\mathfrak{a}^\vee$ will denote the characters of $\mathfrak{a}$. If R is a complex representation of a group, then $R^\vee$ is the usual dual representation over $\mathbb{C}$. Similarly, if ρ is an ℓ-adic Galois representation, then $\rho^\vee$ is the usual dual representation over $\overline{\mathbb{Q}}_\ell$. If G is our reductive $\mathbb{Q}$-group, then $G^\vee(\mathbb{C})$ or $G^\vee(\overline{\mathbb{Q}}_\ell)$ will denote the dual group over either of the algebraically closed fields $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$, respectively. Similarly, if $\mathbf{G}$ is a real reductive Lie group, $\mathbf{G}^\vee(\mathbb{C})$ will denote its dual group.

1.1 Eisenstein series and spaces of automorphic forms

This section will be devoted to studying spaces of automorphic forms in the style of Franke [Fra98] and Franke–Schwermer [FS98]. We will state the Franke–Schwermer decomposition and study the structure of its pieces using the Franke filtration. But first, we recall some of the theory of Eisenstein series.

1.1.1 Review of Eisenstein series

Let $P \subset G$ a parabolic $\mathbb{Q}$-subgroup of our reductive group G (see the section on notation in the introduction) with fixed Levi decomposition $P = MN$. In this section, we will recall how to use automorphic representations of $M(\mathbb{A})$ to construct Eisenstein series, and we will explain how to study these Eisenstein series using parabolically induced representations and intertwining operators.

Eisenstein series and their constant terms

We start with a cuspidal automorphic representation π of $M(\mathbb{A})$ with central character χ_π, and we assume χ_π is trivial on $A_G(\mathbb{R})^\circ$. So if

$$L^2(M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A}), \chi_\pi)$$

denotes the space of functions on $M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A})$ which are square integrable modulo center and which transform under the center with respect to χ_π, then π occurs in the cuspidal spectrum

$$L^2_{\text{cusp}}(M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A}), \chi_\pi) \subset L^2(M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A}), \chi_\pi).$$

Write $d\chi_\pi : \mathfrak{a}_{P,0} \to \mathbb{C}$ for the differential of the restriction of χ_π to $A_P(\mathbb{R})^\circ/A_G(\mathbb{R})^\circ$. The character $d\chi_\pi$ is an element of $\mathfrak{a}^\vee_{P,0}$. Then we consider the automorphic representation

$$\tilde{\pi} = \pi \otimes e^{-\langle H_P(\cdot), d\chi_\pi\rangle}.$$

The representation $\tilde{\pi}$ is a unitary automorphic representation. If π is realized on a space of functions

$$V_\pi \subset L^2_{\text{cusp}}(M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A}), \chi_\pi),$$

then $\tilde{\pi}$ is realized on the space

$$V_{\tilde{\pi}} = \{e^{-\langle H_P(\cdot), d\chi_\pi\rangle} f \mid f \in V_\pi\},$$

which is a subspace of $L^2_{\text{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash M(\mathbb{A}))$.

Now we let $W_{P,\tilde{\pi}}$ be the space of smooth, K-finite, $\mathbb{C}$-valued functions ϕ on

$$M(\mathbb{Q})N(\mathbb{A})A_P(\mathbb{R})^\circ\backslash G(\mathbb{A})$$

such that, for all $g \in G(\mathbb{A})$, the function

$$m \mapsto \phi(mg)$$

of $m \in M(\mathbb{A})$ lies in the space

$$L^2_{\text{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash M(\mathbb{A}))[\tilde{\pi}].$$

Here, the brackets denote an isotypic component.

The space $W_{P,\tilde{\pi}}$ lets us build Eisenstein series. In fact, let $\phi \in W_{P,\tilde{\pi}}$. We define, for $\lambda \in \mathfrak{a}_{P,0}^{\vee}$ and $g \in G(\mathbb{A})$, the Eisenstein series $E(\phi, \lambda)$ by

$$E(\phi, \lambda)(g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \phi(\gamma g) e^{\langle H_P(g), d\chi_\pi + \rho_P \rangle}.$$

This series only converges for λ sufficiently far inside a positive Weyl chamber, but it defines a holomorphic function there in the variable λ which continues meromorphically to all of $\mathfrak{a}_{P,0}^{\vee}$; see [Lan76], [MW95], or more recently [BL20], where the proof has been greatly simplified.

For each fixed ϕ and for each fixed λ at which $E(\phi, \lambda)$ does not have a pole, the Eisenstein series $E(\phi, \lambda)$ is an automorphic form on $G(\mathbb{A})$. It will be important for us in our examples of GSp_4 and G_2 to study when and how certain Eisenstein series have poles. The general theory which explains how to do this, as developed for instance in [Lan71] and [Sha10], goes through two steps. First, one reduces to studying the constant terms of Eisenstein series, and second, one computes the constant terms using local calculations involving intertwining operators.

This first step is relatively easy to explain. Let $Q \subset G$ be another parabolic subgroup, this time with Levi decomposition $Q = M_Q N_Q$. The *constant term* of $E(\phi, \lambda)$ along Q is, as usual, defined by

$$E_Q(\phi, \lambda)(g) = \int_{N_Q(\mathbb{Q}) \backslash N_Q(\mathbb{A})} E(\phi, \lambda)(ng)\, dn.$$

It is meromorphic in λ. Furthermore, the Eisenstein series $E(\phi, \lambda)$ has a pole at a point $\lambda = \mu$ if and only if there is a proper parabolic subgroup Q such that $E_Q(\phi, \lambda)$ has a pole at $\lambda = \mu$.

Next, to proceed and compute the constant terms of Eisenstein series using local computations, we first need to express the space $W_{P,\tilde{\pi}}$ in terms of local pieces.

Induced representations

The space $W_{P,\tilde{\pi}}$ is a parabolic induction space. In fact, let us view $\tilde{\pi}$ as acting on the subspace $V_{\tilde{\pi}}$ of $L^2_{\mathrm{disc}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ \backslash M(\mathbb{A}))$. The pair $(\tilde{\pi}, V_{\tilde{\pi}})$ is an $M(\mathbb{A}_f) \times (\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}))$-module, and we extend this structure to a $P(\mathbb{A}_f) \times (\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module structure via the trivial action by the unipotent radical. We consider the parabolic induction functor

$$\mathrm{Ind}_{P(\mathbb{A}_f)\times(\mathfrak{p}_0,K_\infty\cap P(\mathbb{R}))}^{G(\mathbb{A}_f)\times(\mathfrak{g}_0,K_\infty)}$$

and, for $\lambda \in \mathfrak{a}_{P,0}^\vee$, we write

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}\otimes e^{\langle H_P(\cdot),\lambda\rangle}) = \mathrm{Ind}_{P(\mathbb{A}_f)\times(\mathfrak{p}_0,K_\infty\cap P(\mathbb{R}))}^{G(\mathbb{A}_f)\times(\mathfrak{g}_0,K_\infty)}(\tilde{\pi}\otimes e^{\langle H_P(\cdot),\lambda\rangle}),$$

for short. The space above is an unnormalized induction, and we can normalize it by writing

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi},\lambda) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}\otimes e^{\langle H_P(\cdot),\lambda+\rho_P\rangle}).$$

Then there is an isomorphism of $G(\mathbb{A}_f)\times(\mathfrak{g}_0,K_\infty)$-modules

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(L^2_{\mathrm{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash M(\mathbb{A}))[\tilde{\pi}],\lambda) \cong e^{\langle H_P(\cdot),\lambda+\rho_P\rangle}W_{P,\tilde{\pi}},$$

where the space on the right hand side is just defined by

$$e^{\langle H_P(\cdot),\lambda+\rho_P\rangle}W_{P,\tilde{\pi}} = \{e^{\langle H_P(\cdot),\lambda+\rho_P\rangle}f \mid f\in W_{P,\tilde{\pi}}\}.$$

Therefore, elements of the induction $\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi},\lambda)$ can also be used to define Eisenstein series as above.

Intertwining operators

We now need to define the intertwining operators, which will let us access the constant terms of Eisenstein series.

Given another parabolic subgroup $Q = M_QN_Q$ of G, let given $w \in W(P,Q)$, let us identify w with an element of $G(\mathbb{Q})$. For $\lambda,\lambda' \in \mathfrak{a}_{P,0}^\vee$ and $\phi_\lambda \in \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi},\lambda)$, define a new element $\phi_{\lambda'} \in \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi},\lambda')$

$$\phi_{\lambda'} = \phi_\lambda e^{\langle H_P(\cdot),\lambda'-\lambda\rangle}.$$

We say that this assignment $\lambda \mapsto \phi_\lambda$ is a *flat section* of the induction.

Now define (formally) for ϕ_λ varying in a flat section, the *intertwining operator* $M(w,\cdot)$ by

$$M(w,\phi)_{w\lambda}(g) = \int_{(wNw^{-1}\cap N_Q)(\mathbb{A})\backslash N_Q(\mathbb{A})} \phi_\lambda(w^{-1}ng)\,dn.$$

When convergent, this defines a map of $G(\mathbb{A}_f)\times(\mathfrak{g}_0,K_\infty)$-modules,

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi},\lambda) \to \iota_{Q(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}^w, w\lambda),$$

where if σ is an automorphic representation of $M(\mathbb{A})$, then σ^w denotes the automorphic representation of $M_Q(\mathbb{A})$ defined by $\sigma^w(m)=\sigma(w^{-1}mw)$. It is a fact that the integral defining $M(w,\cdot)$ does converge for λ in a certain cone in $\mathfrak{a}_{P,0}^\vee$ and is holomorphic in λ there, and that it continues meromorphically to all of $\mathfrak{a}_{P,0}^\vee$.

We can use the intertwining operators to describe the constant term. The following theorem is due to Langlands. See Section 6.2 of the book by Shahidi [Sha10].

Theorem 1.1.1.1. *Let $Q=M_QN_Q$ be a parabolic $\mathbb{Q}$-subgroup of G. Then*

$$E_Q(\phi,\lambda) = \sum_{w\in W(P,Q)} M(w,\phi)_{w\lambda},$$

which is an equality of functions of $g\in G(\mathbb{A})$ varying meromorphically in λ.

Local study of intertwining operators

Now we make a local study of the intertwining operators in order to incorporate the theory of L-functions into our considerations. To do this, we first write the automorphic representation $\tilde{\pi}$ in terms of its local components as usual as

$$\tilde{\pi} \cong {\bigotimes_v}' \tilde{\pi}_v,$$

where the restricted tensor product is over all places v of $\mathbb{Q}$; the representation $\tilde{\pi}_v$ is a smooth, admissible representation of $M(\mathbb{Q}_v)$ if v is finite, and it is an admissible $(\mathfrak{m}_0, K_\infty\cap P(\mathbb{R}))$-module is $v=\infty$.

If v is finite, $\lambda \in \mathfrak{a}_{P,0}^\vee$, and σ is a smooth admissible representation of $M(\mathbb{Q}_v)$, let us write

$$\iota_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\sigma, \lambda) = \mathrm{Ind}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\sigma \otimes \lambda)$$

for the usual smooth, K_v-finite parabolic induction, where λ is being viewed as a character of $M(\mathbb{Q}_v)$ via the canonical identification $\mathfrak{a}_P^\vee \cong X^*(M) \otimes \mathbb{C}$ and the inclusion $\mathfrak{a}_{P,0}^\vee \hookrightarrow \mathfrak{a}_P^\vee$. Similarly if σ is instead an admissible $(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}))$-module, let us write

$$\iota_{P(\mathbb{R})}^{G(\mathbb{R})}(\sigma, \lambda) = \mathrm{Ind}_{(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))}^{(\mathfrak{g}_0, K_\infty)}(\sigma \otimes \lambda)$$

for the usual archimedean parabolic induction, where this time λ is being viewed as a character of $\mathfrak{p}_0$ by letting it act trivially on $\mathfrak{m}_0$ and $\mathfrak{n}$. Then via the decomposition of $\tilde{\pi}$ above, we have an isomorphism

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, \lambda) \cong \bigotimes_v{}' \iota_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\tilde{\pi}_v, \lambda).$$

Let $Q = M_Q N_Q$ again be another parabolic $\mathbb{Q}$-subgroup of G. For any $w \in W(P, Q)$ and any place v, there are also local intertwining operators

$$M_v(w, \cdot)_{w\lambda} : \iota_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\sigma, \lambda) \to \iota_{Q(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\sigma^w, w\lambda),$$

defined by integrals analogous to the global intertwining operator above (at least in the nonarchimedean case). Here σ^w is defined similarly as in the global case above.

If v is finite and σ is a smooth admissible representation of $M(\mathbb{Q}_v)$, then any $\phi_\lambda \in \iota_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\sigma, \lambda)$ can be made to vary with λ in a unique way such that $\phi_\lambda|_{K_v}$ is independent of λ, because of the Iwasawa decomposition. We say in this case that ϕ is a *flat section* of the induction.

If σ is furthermore irreducible and unramified, then $\iota_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\sigma, \lambda)$ has a unique up to scalar K_v-fixed vector; given a K_v-fixed vector v^{sph} in the space V_σ of σ, there is a unique $\phi_\lambda^{\mathrm{sph}} \in \iota_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\sigma, \lambda)$ such that $\phi_\lambda^{\mathrm{sph}}(k) = v^{\mathrm{sph}}$ for any $k \in K_v$. Then $\phi_\lambda^{\mathrm{sph}}$ is K_v-fixed and forms a flat section. If $w \in W(P, Q)$, then $M_v(w, \phi)_{w\lambda}$ is also K_v-fixed, and hence is a scalar multiple of the K_v-fixed vector $\phi_{w\lambda}^{w,\mathrm{sph}} \in \iota_{Q(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\sigma^w, w\lambda)$ given by the property that $\phi_{w\lambda}^{w,\mathrm{sph}}(k) = v^{\mathrm{sph}}$ for any $k \in K_v$ (recall that σ and σ^w act on the same space). If we let λ vary in a flat section, this scalar multiple will vary, and it is possible to say how in particular cases when P is maximal. In fact, there is a classical formula of Gindikin–Karpelevich which expresses this multiple in terms of local L-functions.

L-functions and intertwining operators

We will not need the local formula of Gindikin–Karpelevich here, but we will need a global consequence of it, which is at the heart of the Langlands–Shihidi method. We need to set up some notation before we can state it, however, and we do this now.

Assume for the rest of this section that G is split and P is maximal. Let $B \subset P$ be a Borel subgroup of G with Levi T, and fix a set Φ of positive simple roots for T in G that makes B standard. Assume P corresponds to the subset of Φ obtained by omitting a single simple root γ. Let w_0 be the unique element of the Weyl group of T in G which sends every root in $\Phi\backslash\{\gamma\}$ to positive simple roots, and which sends γ to a negative root. If P' is the standard maximal parabolic with Levi $w_0 M w_0$, then $w_0 \in W(P, P')$.

View γ as an element of $\mathfrak{a}_{P,0}^{\vee}$ and write

$$\tilde{\gamma} = \langle \rho_P, \gamma \rangle^{-1} \rho_P$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing on $\mathfrak{a}_{P,0}^{\vee}$ induced from the Killing form. Then $\mathfrak{a}_{P,0}^{\vee}$ is one dimensional, generated by $\tilde{\gamma}$.

Let $P^{\vee}$ be the parabolic subgroup of the dual group $G^{\vee}$ corresponding to the set of coroots associated with the simple roots in $\Phi\backslash\{\gamma\}$. The dual group $M^{\vee}$ is the Levi of $P^{\vee}$, and we let $N^{\vee}$ be the unipotent radical of $P^{\vee}$. The group $M^{\vee}$ acts on $\mathrm{Lie}(N^{\vee})$ via the adjoint action. For $i > 0$ an integer, let $V_i \subset \mathrm{Lie}(N^{\vee})$ generated by the coroots $\beta^{\vee}$ for which $\langle \tilde{\gamma}, \beta^{\vee} \rangle = i$. Then each V_i is a representation of $M^{\vee}$, and we denote the corresponding action of $M^{\vee}$ by R_i.

Theorem 1.1.1.2. *Let P be maximal and let w_0, P', $\tilde{\gamma}$, and R_i be as above. Let S be a set of places which includes all the ramified places for $\tilde{\pi}$ and the archimedean place. For $v \notin S$, fix v^{sph} a nonzero K_v-fixed vector in the space of $\tilde{\pi}_v$. Let $s \in \mathbb{C}$ and let $\phi_{v,s}^{\mathrm{sph}} \in \iota_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\tilde{\pi}_v, s\tilde{\gamma})$ and $\phi_{v,s}^{w_0,\mathrm{sph}} \in \iota_{P'(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\tilde{\pi}_v^{w_0}, s(w_0\tilde{\gamma}))$ be spherical sections defined as above so that $\phi_{v,s}^{\mathrm{sph}}(k) = v^{\mathrm{sph}} = \phi_{v,s}^{w_0,\mathrm{sph}}(k)$.*

Assume $\phi_s \in \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, s\tilde{\gamma})$ decomposes as $\otimes_v \phi_{v,s}$ where $\phi_{v,s} = \phi_{v,s}^{\mathrm{sph}}$ for $v \notin S$. Then we have the *formula*

$$M(\phi, w_0)_{s(w_0\tilde{\gamma})} = \prod_{j=1}^{m} \frac{L^S(js, \tilde{\pi}, R_i^\vee)}{L^S(js+1, \tilde{\pi}, R_i^\vee)} \bigotimes_{v \notin S} \phi_{v,s}^{w_0,\mathrm{sph}} \otimes \bigotimes_{v \in S} M_v(\phi_{v,s}, w_0)_{s(w_0\tilde{\gamma})},$$

where L^S denotes a partial L-function, away from the places of S.

Proof. See Shahidi [Sha10], Theorem 6.3.1. □

Thus the theorem above, in combination with Theorem 1.1.1.1, will later allow us to compute constant terms of maximal parabolically induced Eisenstein series along the maximal parabolics from which they are induced.

1.1.2 The Franke–Schwermer decomposition

Let E be a finite dimensional irreducible representation of $G(\mathbb{C})$. Then the annihilator of E in the center of the universal enveloping algebra of $\mathfrak{g}$ is an ideal, and we denote it by $\mathcal{J}_E$. Denote by $\mathcal{A}_E(G)$ the space of automorphic forms on $G(\mathbb{A})$ which are annihilated by a power of $\mathcal{J}_E$, and which transform trivially under $A_G(\mathbb{R})^\circ$. The forms in $\mathcal{A}_E(G)$ are the ones that can possibly contribute to the cohomology of E, as we will discuss later.

In [FS98], Franke and Schwermer wrote down a decomposition of $\mathcal{A}_E(G)$ into pieces defined by certain parabolic subgroups of G and cuspidal automorphic representations of their Levis. This decomposition is a direct sum decomposition of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules, and we describe it in this section.

First, given two parabolic subgroups of G defined over $\mathbb{Q}$, we say that they are *associate* if their Levis are conjugate by an element of $G(\mathbb{Q})$. Let $\mathcal{C}$ be the set of equivalence classes for this relation. It is a finite set. If P is a parabolic $\mathbb{Q}$-subgroup of G, let $[P]$ denote its equivalence class in $\mathcal{C}$.

Now fix P a parabolic $\mathbb{Q}$-subgroup of G with Levi decomposition $P = MN$. Given another parabolic $\mathbb{Q}$-subgroup $Q = M_Q N_Q$ of G, we say a function $f \in \mathcal{A}_E(G)$ is *negligible along* Q if for any $g \in G(\mathbb{A})$, the function given by

$$m \mapsto f(mg), \qquad m \in M_Q(\mathbb{Q}) A_G(\mathbb{R})^\circ \backslash M_Q(\mathbb{A}),$$

is orthogonal to the space of cuspidal functions on $M_Q(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M_Q(\mathbb{A})$. Let $\mathcal{A}_{E,[P]}(G)$ be the subspace of all functions in $\mathcal{A}_E(G)$ which are negligible along any parabolic subgroup $Q \notin [P]$. It is a theorem of Langlands that

$$\mathcal{A}_E(G) = \bigoplus_{C\in\mathcal{C}} \mathcal{A}_{E,C}(G)$$

as $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules. The summand $\mathcal{A}_{E,[G]}(G)$ is the space of cusp forms in $\mathcal{A}_E(G)$.

Franke and Schwermer refine this decomposition even further using cuspidal automorphic representations of the Levis of the parabolics in each class $C \in \mathcal{C}$. We briefly recall how.

Let φ be an *associate class of cuspidal automorphic representations of M.* We do not recall here the exact definition of this notion, referring instead to [FS98] Section 1.2, or [LS04] Section 1.3. Each φ is a collection of irreducible representations of the groups $M_{P'}(\mathbb{A})$ for each $P' \in [P]$ with Levi decomposition $P' = M_{P'}N_{P'}$, finitely many for each such P', and each such representation π must occur in $L^2_{\text{cusp}}(M_{P'}(\mathbb{Q})\backslash M_{P'}(\mathbb{A}), \chi_\pi)$ where χ_π is the central character of π. Conversely, any irreducible representation π of $M(\mathbb{A})$ with central character χ_π occurring in $L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A}), \chi_\pi)$ determines a unique φ. We let $\Phi_{E,[P]}$ denote the set of all associate classes of cuspidal automorphic representations of M.

Now given a $\varphi \in \Phi_{E,[P]}$, let π be one of the representations comprising φ; say π is a representation of the $\mathbb{A}$-points of a Levi $M_{P'}$ for P' a parabolic associate to P. Form the space $W_{P',\tilde{\pi}}$ as in Section 1.1.1. Let $d\chi_\pi$ be the differential of the central character of π at the archimedean place, viewed as an element of $\mathfrak{a}^\vee_{P',0}$. Then for any $\phi \in W_{P',\tilde{\pi}}$ we can form the Eisenstein series $E(\phi, \lambda)$, $\lambda \in \mathfrak{a}^\vee_{P',0}$.

Depending on the choice of ϕ, the Eisenstein series $E(\phi, \lambda)$ may have a pole at $\lambda = d\chi_\pi$. Nevertheless, one can still take residues of $E(\phi, \lambda)$ at $\lambda = d\chi_\pi$ to obtain residual Eisenstein series. We let $\mathcal{A}_{E,[P],\varphi}(G)$ to be the collection of all possible Eisenstein series, residual Eisenstein series, and partial derivatives of such with respect to λ, evaluated at $\lambda = d\chi_\pi$, built from any $\phi \in W_{P',\tilde{\pi}}$. (For a more precise description of this space, see [FS98], Section 1.3, or [LS04], Section 1.4. There is also a more intrinsic definition of this space, defined without reference to Eisenstein series, in [FS98], Section 1.2, or [LS04], Section 1.4, which is proved to be equivalent to this description in [FS98].) One can use the functional equation of Eisenstein series to show that the space $\mathcal{A}_{E,[P],\varphi}(G)$ is independent of the π in φ used to define it.

We can now state the Franke–Schwermer decomposition of $\mathcal{A}_E(G)$.

Theorem 1.1.2.1 (Franke–Schwermer [FS98])**.** *There is a direct sum decomposition of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules*

$$\mathcal{A}_E(G) = \bigoplus_{C \in \mathcal{C}} \bigoplus_{\varphi \in \Phi_{E,C}} \mathcal{A}_{E,C,\varphi}(G).$$

1.1.3 Structure of the pieces of the Franke–Schwermer decomposition

We introduce in this section certain $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules, whose structures as such modules are explicit, and explain how they can be related to the pieces of the Franke–Schwermer decomposition introduced just above. Almost everything in this section is done in Franke's paper [Fra98], pp. 218, 234, but without taking into consideration the associate classes φ.

We consider again a parabolic $\mathbb{Q}$-subgroup P of G with Levi decomposition $P = MN$. As before, let us fix π a cuspidal automorphic representation of $M(\mathbb{A})$, and let $\tilde{\pi}$ be its unitarization, as in Section 1.1.1. Then $\tilde{\pi}$ occurs in $L^2_{\mathrm{cusp}}(M(\mathbb{A})A_P(\mathbb{R})^\circ \backslash M(\mathbb{A}))$. For brevity, let us write $V[\tilde{\pi}]$ for the smooth, K-finite vectors in the $\tilde{\pi}$-isotypic component of $L^2_{\mathrm{cusp}}(M(\mathbb{A})A_P(\mathbb{R})^\circ \backslash M(\mathbb{A}))$. Then $V[\tilde{\pi}]$ is a $M(\mathbb{A}_f) \times (\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}))$-module, and we extend this structure to one of a $P(\mathbb{A}_f) \times (\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module by letting $\mathfrak{a}_{P,0}^\vee$ and $\mathfrak{n}$ act trivially, as well as $A_P(\mathbb{A}_f)$ and $N(\mathbb{A}_f)$.

Fix for the rest of this section a point $\mu \in \mathfrak{a}_{P,0}^\vee$. Let $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ be the symmetric algebra on the vector space $\mathfrak{a}_{P,0}$; we view this space as the space of differential operators on $\mathfrak{a}_{P,0}^\vee$ at the point μ. So if $H(\lambda)$ is a holomorphic function on $\mathfrak{a}_{P,0}^\vee$, then $D \in \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ acts on H by taking a sum of iterated partial derivatives of H and evaluating the result at the point μ. So in this way, every $D \in \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ can be viewed as a distribution on holomorphic functions on $\mathfrak{a}_{P,0}^\vee$ supported at the point μ.

With this point of view, these distributions can be multiplied by holomorphic functions on $\mathfrak{a}_{P,0}^\vee$; just multiply the test function by the given holomorphic function before evaluating the distribution. With this in mind, we can define an action of $\mathfrak{a}_{P,0}^\vee$ on $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ by

$$(XD)(f) = D(\langle X, \cdot \rangle f), \qquad X \in \mathfrak{a}_{P,0},\ D \in \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu.$$

We also let $\mathfrak{m}_0$ and $\mathfrak{n}$ act trivially on $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$, which gives us an action of $\mathfrak{p}_0$ on $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$.

We also let $K_\infty \cap P(\mathbb{R})$ act trivially on $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$. Since the Lie algebra of $K_\infty \cap P(\mathbb{R})$ lies in $\mathfrak{m}_0$, this is consistent with the $\mathfrak{p}_0$ action just defined and makes $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ a $(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module.

Finally, let $P(\mathbb{A}_f)$ act on $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ by the formula

$$(pD)(f) = D(e^{\langle H_P(p), \cdot \rangle} f), \qquad p \in P(\mathbb{A}_f),\ D \in \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu.$$

Then with the actions just defined, $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ gets the structure of a $P(\mathbb{A}_f) \times (\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module.

Now we form the tensor product $V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$, which carries a natural $P(\mathbb{A}_f) \times (\mathfrak{p}, K_\infty \cap P(\mathbb{R}))$-module structure coming from those on the two factors. We will consider in what follows the induced $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu).$$

This space turns out to be isomorphic to another $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module, which we now describe.

Let $W_{P,\tilde{\pi}}$ be the induction space introduced in Section 1.1.1; it is the unnormalized parabolic induction of the space $V[\tilde{\pi}]$ above. Form the tensor product

$$W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu.$$

While the first factor in this tensor product is a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module, the second is only a $P(\mathbb{A}_f) \times (\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module, and so we do not immediately get a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module structure on the tensor product. However, one can endow this space with a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module structure by viewing it as a space of distributions in a manner to be described now.

We first introduce the space of functions on which we will consider distributions. These will be functions on $G(\mathbb{A}) \times \mathfrak{a}_{P,0}^\vee$. Let us write g for a variable in $G(\mathbb{A})$ and λ for a variable in $\mathfrak{a}_{P,0}^\vee$. Let $\mathcal{S}$ be the space of functions $f(g, \lambda)$ on $G(\mathbb{A}) \times \mathfrak{a}_{P,0}^\vee$ which are smooth and compactly supported in the variable g when λ is fixed, and which are holomorphic in the variable λ when g is fixed. Then we consider the space $\mathcal{D}(\mathcal{S})$ of distributions on $\mathcal{S}$ which are compactly supported in the variable λ.

The space $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_\mu$ embeds naturally as a subspace of $\mathcal{D}(\mathcal{S})$. In fact, we can identify the simple tensor $\phi \otimes D$, where $\phi \in W_{P,\tilde{\pi}}$ and $D \in \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_\mu$, with the distribution given on functions $f \in \mathcal{S}$ by

$$(\phi \otimes D)(f) = D\left(\int_{G(\mathbb{A})} \phi(g) f(g, \cdot)\, dg\right).$$

Here, D is being viewed as a distribution on holomorphic functions on $\mathfrak{a}_{P,0}^\vee$ as described above, so indeed the right hand side of this equality is a complex number.

Now we describe a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module structure on the space $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ through formulas that make sense in $\mathcal{D}(\mathcal{S})$. Let us give these formulas and then make comments on them afterward. For $\phi \in W_{P,\tilde{\pi}}$ and $D \in \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$, we consider $\phi \otimes D$ as a distribution in the variables (g, λ) and we define:

$$(X(\phi \otimes D))(g, \lambda) = ((X\phi) \otimes D)(g, \lambda) + (\langle X H_P(g), \lambda \rangle (\phi \otimes D))(g, \lambda),$$

for $X \in \mathfrak{g}_0$,

$$(k(\phi \otimes D))(g, \lambda) = (\phi \otimes D)(gk, \lambda),$$

for $k \in K_\infty$, and

$$(h(\phi \otimes D))(g, \lambda) = (e^{\langle H_P(gh) - H_P(g), \lambda \rangle}(\phi \otimes D))(gh, \lambda),$$

for $h \in G(\mathbb{A}_f)$.

Now in the formulas defining the actions of $G(\mathbb{A}_f)$ and $\mathfrak{g}_0$, there are distributions on the right hand side that have been multiplied by functions depending on both g and λ. Therefore, it is not immediately obvious that these expressions define elements of the image of $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ in $\mathcal{D}(\mathcal{S})$; that is, it is not completely clear that these expressions can be written as a finite sum of simple tensors in $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$. However, using properties of the function H_P, this can be checked. We omit the verification here for sake of brevity.

Now we can relate the two $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules defined in this section. We have the following proposition, whose proof we again omit.

Proposition 1.1.3.1. *There is an isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules*

$$W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \cong \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu).$$

More generally, if E is a finite dimensional representation of $G(\mathbb{C})$, then we also have an isomorphism

$$W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes E \cong \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes E),$$

where on the left hand side, E is being viewed as a $(\mathfrak{g}_0, K_\infty)$-module, and on the right, it is viewed as a $(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module by restriction.

The reason we introduce the representation E in the second part of this proposition will become more apparent when we discuss cohomology later.

Now we come back to Eisenstein series. Assume π is such that there is an irreducible finite dimensional representation E of $G(\mathbb{C})$ such that the associate class φ containing π is in $\Phi_{E,[P]}$. Then we can construct elements of the piece $\mathcal{A}_{E,[P],\varphi}(G)$ of the Franke–Schwermer decomposition from Section 1.1.2 from elements of $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ using Eisenstein series as follows.

Recall that, in the notation of Section 1.1.1, we have

$$W_{P,\tilde{\pi}} \cong \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}]) = \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}], -\rho_P).$$

Elements $\phi \in \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}], -\rho_P)$ fit into flat sections $\phi_\lambda \in \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}], \lambda)$ where λ varies in $\mathfrak{a}_{P,0}^\vee$. Then for such ϕ we have $\phi = \phi_{-\rho_P}$. In what follows, we will identify elements of $W_{P,\tilde{\pi}}$ with elements of $\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}], -\rho_P)$, and then use this notation to vary them in flat sections.

Let $d\chi_\pi$ denote the differential of the archimedean component of the central character of π. Then as in Section 1.1.1, if we are given $\phi \in W_{P,\tilde{\pi}}$, we can form the Eisenstein series $E(\phi, \lambda)$ for λ varying in $\mathfrak{a}_{P,0}^\vee$. This is a family of automorphic forms which varies meromorphically in λ. Let h_0 be a holomorphic function on $\mathfrak{a}_{P,0}^\vee$ such that, for any $\phi \in W_{P,\tilde{\pi}}$, the product $h_0(\lambda)E(\phi, \lambda)$ is holomorphic near $\lambda = d\chi_\pi$. Then we define a map

$$\mathcal{E}_{h_0} : W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P} \to \mathcal{A}_{E,[P],\varphi}(G)$$

by

$$\phi \otimes D \mapsto D(h_0(\lambda)E(\phi, \lambda));$$

in other words, this map forms an Eisenstein series according to ϕ, multiplies it by $h_0(\lambda)$ in order to cancel any poles, and then differentiates the result at the point $\lambda = d\chi_\pi$ according to D.

The map $\mathcal{E}_{h_0}$ is surjective by our definition of $\mathcal{A}_{E,[P],\varphi}(G)$. If all the Eisenstein series $E(\phi, \lambda)$, for $\phi \in W_{P,\tilde{\pi}}$, are holomorphic at $\lambda = d\chi_\pi$, then we write $\mathcal{E} = \mathcal{E}_1$ for the map just defined with $h_0(\lambda) = 1$.

Proposition 1.1.3.2. *The map $\mathcal{E}_{h_0} : W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P} \to \mathcal{A}_{E,[P],\varphi}(G)$ defined just above is a surjective map of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules. Furthermore, if all the Eisenstein series $E(\phi, \lambda)$ arising from $\phi \in W_{P,\tilde{\pi}}$ are holomorphic at $\lambda = d\chi_\pi$, then the map $\mathcal{E}$ is an isomorphism.*

Proof. To check that $\mathcal{E}_{h_0}$ is a map of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules, one just needs to use the formulas defining the $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module structure on $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_\lambda$ and show they are preserved when forming Eisenstein series and taking derivatives; this can be checked when λ is in the region of convergence for the Eisenstein series, and then this extends to all λ by analytic continuation. We omit the precise details of this check.

For the second claim in the proposition, that $\mathcal{E}$ is an isomorphism, this follows essentially from Theorem 14 in Franke's paper [Fra98]; this theorem implies that $\mathcal{E}$ injective, since it equals the restriction of Franke's mean value map **MW** to $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P}$. Whence by surjectivity and the first part of the proposition, we are done. □

The spaces $\mathcal{A}_{E,[P],\varphi}(G)$ carry a filtration by $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules which is due to Franke. For our purposes, we will not need the precise definition of this filtration, but just a rough description of its graded pieces. This is described in the following theorem.

Theorem 1.1.3.3. *There is a decreasing filtration*

$$\cdots \supset \mathrm{Fil}^i \mathcal{A}_{E,C,\varphi}(G) \supset \mathrm{Fil}^{i+1} \mathcal{A}_{E,C,\varphi}(G) \supset \cdots$$

of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules on $\mathcal{A}_{E,C,\varphi}(G)$, for which we have

$$\mathrm{Fil}^0 \mathcal{A}_{E,C,\varphi}(G) = \mathcal{A}_{E,C,\varphi}(G)$$

and

$$\mathrm{Fil}^m \mathcal{A}_{E,C,\varphi}(G) = 0$$

for some $m > 0$ (depending on φ), and whose graded pieces have the property described below.

Fix π in φ, and say π is a representation of the $\mathbb{A}$-points of a Levi M of a parabolic P in C. Let $d\chi_\pi$ be the differential of the archimedean component of the central character of π. Let $\mathcal{M}$ be the set of quadruples (Q, ν, Π, μ) where:

- *Q is a parabolic subgroup of G which contains P;*
- *ν is an element of $(\mathfrak{a}_P \cap \mathfrak{m}_{Q,0})^\vee$;*
- *Π is an automorphic representation of $M(\mathbb{A})$ occurring in*

$$L^2_{\mathrm{disc}}(M_Q(\mathbb{Q})A_Q(\mathbb{R})^\circ \backslash M_Q(\mathbb{A}))$$

  and which is spanned by values at, or residues at, the point ν of Eisenstein series parabolically induced from $(P \cap M_Q)(\mathbb{A})$ to $M_Q(\mathbb{A})$ by representations in φ; and

- *μ is an element of $\mathfrak{a}^\vee_{Q,0}$ whose real part in $\mathrm{Lie}(A_G(\mathbb{R})\backslash A_{M_Q}(\mathbb{R}))$ is in the closure of the positive chamber, and such that the following relation between μ, ν and π holds: Let $\lambda_{\tilde{\pi}}$ be the infinitesimal character of the archimedean component of $\tilde{\pi}$. Then*

$$\lambda_{\tilde{\pi}} + \nu + \mu$$

  may be viewed as a collection of weights of a Cartan subalgebra of $\mathfrak{g}_0$, and the condition we impose is that these weights are in the support of the infinitesimal character of E.

For such a quadruple $(Q, \nu, \Pi, \mu) \in \mathcal{M}$, let $V[\Pi]$ denote the Π-isotypic component of the space

$$L^2_{\mathrm{disc}}(M_Q(\mathbb{Q})A_Q(\mathbb{R})^\circ \backslash M_Q(\mathbb{A})) \cap \mathcal{A}_{E,[P\cap M_Q],\varphi|_{M_Q}}(M_P).$$

Then the property of the graded pieces of the filtration above is that, for every i with $0 \leq i < m$, *there is a subset $\mathcal{M}^i_\varphi \subset \mathcal{M}$ and an isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules*

$$\operatorname{Fil}^i \mathcal{A}_{E,C,\varphi}(G)/\operatorname{Fil}^{i+1} \mathcal{A}_{E,C,\varphi}(G) \cong \bigoplus_{(Q,\nu,\Pi,\mu)\in\mathcal{M}^i_\varphi} \operatorname{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})}(V[\Pi] \otimes \operatorname{Sym}(\mathfrak{a}_{Q,0})_{\mu+\rho_Q}).$$

Proof. While this essentially follows again from the work of Franke [Fra98], in this form, this theorem is a consequence of Theorem 4 in the paper of Grobner [Gro13]; the latter paper takes into account the presence of the class φ while the former does not. □

Remark 1.1.3.4. In the context of Proposition 1.1.3.2 and Theorem 1.1.3.3, when all the Eisenstein series $E(\phi, \lambda)$ arising from $\phi \in W_{P,\tilde{\pi}}$ are holomorphic at $\lambda = d\chi_\pi$, what happens is that the filtration of Theorem 1.1.3.3 collapses to a single step. The nontrivial piece of this filtration is then given by $\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \operatorname{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi+\rho_P})$ through the map $\mathcal{E}$ along with the isomorphism of Proposition 1.1.3.1.

When P is a maximal parabolic, the filtration of Theorem 1.1.3.3 becomes particularly simple. To describe it, we set some notation.

Assume P is maximal. If $\tilde{\pi}$ is a unitary cuspidal automorphic representation of M and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, let us write

$$\mathcal{L}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, s)$$

for the Langlands quotient of

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, 2s\rho_P).$$

One definition of this is that it is the quotient of the induction above by the kernel of the intertwining operator

$$M(\cdot, w_0) : \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, s) \to \iota_{P'(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, -s)$$

of Section 1.1.1. Here, if we fix a minimal parabolic contained in P, then w_0 is the Weyl element that sends every simple root in M to another positive simple root, and which sends the positive simple root not in M to a negative root, and P' is the standard parabolic with Levi $w_0 M w_0$. Then we have

Theorem 1.1.3.5 (Grbac [Grb12])**.** *In the setting above, with P maximal and* $\operatorname{Re}(s) > 0$*, assume $\tilde{\pi}$ defines an associate class $\varphi \in \Phi_{E,[P]}$. If any of the Eisenstein series $E(\phi, \lambda)$ coming from $\phi \in W_{\tilde{\pi}}$*

have a pole at $\lambda = 2s_0\rho_P$, then there is an exact sequence of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules as follows:

$$0 \to \mathcal{L}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, s) \to \mathcal{A}_{E,[P],\varphi}(G) \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{(2s+1)\rho_P}) \to 0.$$

Proof. This follows from Theorem 3.1 in the paper of Grbac [Grb12]. □

1.2 Cohomology

We now would like to study the cohomology of the pieces of the Franke–Schwermer decomposition. We can reduce this to studying the parabolically induced representations introduced in the previous section and applying a classical argument involving the Kostant decomposition, as in [BW00], Theorem III.3.3. We start with a general discussion of cohomology.

1.2.1 The cohomology of the space of automorphic forms

We continue to use the notation set in the introduction, and in particular, we will resume working with our reductive $\mathbb{Q}$-group G. We have our maximal compact subgroup $K_\infty \subset G(\mathbb{R})$, and we fix an open subgroup K'_∞ of K_∞. Then we necessarily have $K_\infty^\circ \subset K'_\infty \subset K_\infty$.

We will be interested in the $(\mathfrak{g}_0, K'_\infty)$-cohomology of the space of automorphic forms on $G(\mathbb{A})$. By Franke's resolution of Borel's conjecture ([Fra98], Theorem 18), this cohomology space (for suitable K'_∞) computes the cohomology of certain locally symmetric spaces attached to G, and is therefore of arithmetic interest.

So as before, let E be an irreducible, finite dimensional, complex representation of $G(\mathbb{C})$. We view E as a $(\mathfrak{g}_0, K_\infty)$-module via its restriction to $G(\mathbb{R})$, and hence as a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module by giving it a trivial $G(\mathbb{A}_f)$ action. Our goal is to study the $(\mathfrak{g}_0, K'_\infty)$-cohomology space

$$H^i(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E)$$

for any i, which is naturally a $G(\mathbb{A}_f)$-module; see the standard reference by Borel–Wallach [BW00] for the definition of $(\mathfrak{g}_0, K'_\infty)$-cohomology and discussions of many of its most important properties.

Actually, the cohomology space above is smooth and admissible as a $G(\mathbb{A}_f)$-module, as can be seen by comparing it to the cohomology of certain local systems on the locally symmetric spaces attached to G. By the results recalled in Section 1.1.2 and the Franke–Schwermer decomposition (Theorem 1.1.2.1) we have a direct sum decomposition as $G(\mathbb{A}_f)$-modules

$$H^i(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E) = \bigoplus_{C \in \mathcal{C}} \bigoplus_{\varphi \in \Phi_{E,C}} H^i(\mathfrak{g}_0, K'_\infty; \mathcal{A}_{E,C,\varphi}(G) \otimes E).$$

Each summand in the decomposition above is therefore a smooth, admissible $G(\mathbb{A}_f)$-module, and although there may be infinitely summands on the right hand side which don't vanish, only finitely many of them have nonzero K'_f-invariants for any given open compact subgroup $K'_f \subset K_f$.

Let us write

$$H^i_{\mathrm{cusp}}(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E) = \bigoplus_{\varphi \in \Phi_{E,[G]}} H^i(\mathfrak{g}_0, K'_\infty; \mathcal{A}_{E,[G],\varphi}(G) \otimes E)$$

for the *cuspidal cohomology* of E. This is also the same as

$$H^i(\mathfrak{g}_0, K'_\infty; L^2_{\mathrm{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \backslash G(\mathbb{A})) \otimes E).$$

The natural complement to the cuspidal cohomology in the decomposition above is called the *Eisenstein cohomology,* i.e.,

$$H^i_{\mathrm{Eis}}(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E) = \bigoplus_{\substack{C \in \mathcal{C} \\ C \neq [G]}} \bigoplus_{\varphi \in \Phi_{E,C}} H^i(\mathfrak{g}_0, K'_\infty; \mathcal{A}_{E,C,\varphi}(G) \otimes E).$$

If P is a proper parabolic subgroup of G defined over $\mathbb{Q}$, let us define the $[P]$-*Eisenstein cohomology* to be the summand corresponding to the class $[P]$, so

$$H^i_{[P]}(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E) = \bigoplus_{\varphi \in \Phi_{E,[P]}} H^i(\mathfrak{g}_0, K'_\infty; \mathcal{A}_{E,[P],\varphi}(G) \otimes E).$$

Now let $\mathcal{H}_G$ be the Hecke algebra of smooth, compactly supported, complex-valued functions on $G(\mathbb{A}_f)$,

$$\mathcal{H}_G = C_c^\infty(G(\mathbb{A}_f)).$$

Then $\mathcal{H}_G$ acts on any smooth, admissible $G(\mathbb{A}_f)$-module (σ, V) via convolution. Furthermore, for any $f \in \mathcal{H}_G$ and any open compact subgroup $K'_f \subset K_f$ for which f is K'_f-biinvariant, we can consider the trace $\mathrm{Tr}(f|V^{K'_f})$ of f acting as a linear operator on the K'_f invariants of V. This is independent of the choice of K'_f and defines an association

$$f \mapsto J_\sigma(f) = \mathrm{Tr}(f|V^{K'_f}),$$

and we call J_σ the *character distribution* associated with σ. An irreducible admissible $G(\mathbb{A}_f)$-module is determined by its character distribution.

Definition 1.2.1.1. The *multiplicity* of an irreducible admissible $G(\mathbb{A}_f)$-module σ in the ith $(\mathfrak{g}_0, K'_\infty)$-cohomology of $\mathcal{A}_E(G)$ is the nonnegative integer $m^i(\sigma, K'_\infty, E)$ such that

$$\mathrm{Tr}(f|H^i(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E)^{K'_f}) = \sum_\sigma m^i(\sigma, K'_\infty, E) J_\sigma(f)$$

for any $f \in \mathcal{H}_G$ and any open compact subgroup $K'_f \subset K_f$ for which f is K'_f-biinvariant. Here, on the right hand side, the sum is over all irreducible admissible $G(\mathbb{A}_f)$-modules.

Similarly we define $m^i_{\mathrm{cusp}}(\sigma, K'_\infty, E)$, $m^i_{\mathrm{Eis}}(\sigma, K'_\infty, E)$, and $m^i_{[P]}(\sigma, K'_\infty, E)$ for a proper parabolic $\mathbb{Q}$-subgroup P of G, by formulas similar to the one above, namely:

$$\mathrm{Tr}(f|H^i_{\mathrm{cusp}}(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E)^{K'_f}) = \sum_\sigma m^i_{\mathrm{cusp}}(\sigma, K'_\infty, E) J_\sigma(f),$$

$$\mathrm{Tr}(f|H^i_{\mathrm{Eis}}(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E)^{K'_f}) = \sum_\sigma m^i_{\mathrm{Eis}}(\sigma, K'_\infty, E) J_\sigma(f),$$

and

$$\mathrm{Tr}(f|H^i_{[P]}(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E)^{K'_f}) = \sum_\sigma m^i_{[P]}(\sigma, K'_\infty, E) J_\sigma(f).$$

We call these numbers, respectively, the *cuspidal multiplicity,* the *Eisenstein multiplicity,* and the $[P]$-*Eisenstein multiplicity* of σ in the ith cohomology of E.

It follows immediately from the definitions that

$$m^i(\sigma, K'_\infty, E) = m^i_{\mathrm{cusp}}(\sigma, K'_\infty, E) + m^i_{\mathrm{Eis}}(\sigma, K'_\infty, E)$$

and

$$m^i_{\mathrm{Eis}}(\sigma, K_\infty, E) = \sum_{\substack{C\in\mathcal{C}\\ C\neq[G]}} m^i_C(\sigma, K'_\infty, E).$$

The goal in the following will be to precisely compute, for certain choices of G, the multiplicity of the Langlands quotients of certain induced representations, induced from maximal parabolic subgroups, in the cohomology of particular E's. These induced representations will show up in Eisenstein cohomology naturally, as we will explain in the next section. Perhaps more interestingly is that these Langlands quotients can also occur in cuspidal cohomology, and we will see examples of this in the cases of GSp_4 and G_2 later.

1.2.2 The cohomology of induced representations

We now calculate the cohomology of representations of G that are parabolically induced from automorphic representations of Levi subgroups, and hence compute the cohomology of the graded pieces of the Franke filtration described in Theorem 1.1.3.3. The computations done in this section were essentially carried out by Franke in [Fra98], Section 7.4, but not in so much detail. We fill in some of the details and give a sharper result, which we can give because we are focusing on one representation of the Levi at a time, and we can do this because we have access to the Franke–Schwermer decomposition, Theorem 1.1.2.1. The method is essentially that of the proof of Theorem III.3.3 in [BW00]. This method also appears in the computations of Grbac–Grobner [GG13] and Grbac–Schwermer [GS11].

Let $P \subset G$ be a parabolic subgroup defined over $\mathbb{Q}$ with Levi decomposition $P = MN$. Fix an automorphic representation π of $M(\mathbb{A})$ with central character χ_π, occurring in

$$L^2_{\mathrm{disc}}(M(\mathbb{Q})\backslash M(\mathbb{A}), \chi_\pi).$$

Then the unitarization $\tilde{\pi}$ occurs in

$$L^2_{\mathrm{disc}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash M(\mathbb{A})).$$

Let $d\chi_\pi$ denote the differential of the archimedean component of χ_π. Fix also an irreducible finite dimensional representation E of $G(\mathbb{C})$.

As before, fix a compact subgroup K'_∞ of $G(\mathbb{R})$ such that $K_\infty^\circ \subset K'_\infty \subset K_\infty$. We will compute the $(\mathfrak{g}_0, K'_\infty)$-cohomology space

$$H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P}) \otimes E)$$

in terms of $(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}))$-cohomology spaces attached to π. We will require the following lemma.

Lemma 1.2.2.1. *Let $\mu, \mu' \in \mathfrak{a}_{P,0}^\vee$. Let $\mathbb{C}_{\mu'}$ denote the one dimensional $\mathfrak{a}_{P,0}$-module on which $X \in \mathfrak{a}_{P,0}$ acts through multiplication by $\langle X, \mu' \rangle$. Then there is an isomorphism of $P(\mathbb{A}_f)$-modules*

$$H^i(\mathfrak{a}_{P,0}, \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes \mathbb{C}_{\mu'}) \cong \begin{cases} \mathbb{C}(e^{\langle H_P(\cdot), \mu \rangle}) & \text{if } \mu' = -\mu \text{ and } i = 0; \\ 0 & \text{if } \mu' \neq -\mu \text{ or } i > 0. \end{cases}$$

Here, $\mathbb{C}(e^{\langle H_P(\cdot), \mu \rangle})$ is just the one dimensional representation of $P(\mathbb{A}_f)$ on which $p \in P(\mathbb{A}_f)$ acts via $e^{\langle H_P(p), \mu \rangle}$.

Proof. It will be convenient to work in coordinates. So let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be coordinates on $\mathfrak{a}_{P,0}^\vee$; this is the same as fixing a basis of $\mathfrak{a}_{P,0}$. Then the elements of $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ may be viewed as polynomials in the variables $\lambda_1, \ldots, \lambda_r$.

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a multi-index. By definition, the monomial $\lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r}$ acts as a distribution on holomorphic functions f on $\mathfrak{a}_{P,0}^\vee$ via the formula

$$\lambda^\alpha f = \frac{\partial^\alpha}{\partial \lambda^\alpha} f(\lambda)|_{\lambda = \mu}.$$

Also by definition, if $X \in \mathfrak{a}_{P,0}$, then $X\lambda^\alpha$ acts as

$$(X\lambda^\alpha) f = \frac{\partial^\alpha}{\partial \lambda^\alpha} (\langle X, \lambda \rangle f(\lambda))|_{\lambda = \mu}.$$

Let $P(\lambda)$ be a polynomial in λ. Then a quick induction using the above formulas shows that

$X \in \mathfrak{a}_{P,0}$ acts on $P(\lambda)$ as

$$X(P(\lambda)) = \langle X, \mu \rangle P(\lambda) + \sum_{i=1}^{r} \frac{\partial}{\partial \lambda_i} P(\lambda).$$

Hence X acts on the element $P(\lambda) \otimes 1$ in $\mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_\mu \otimes \mathbb{C}_{\mu'}$ by

$$X(P(\lambda) \otimes 1) = \langle X, \mu + \mu' \rangle (P(\lambda) \otimes 1) + \sum_{i=1}^{r} \left(\frac{\partial}{\partial \lambda_i} P(\lambda) \otimes 1 \right).$$

It follows from this that if $X_1, \ldots, X_r$ is the basis of $\mathfrak{a}_{P,0}$ corresponding to the coordinates $\lambda_1, \ldots, \lambda_r$, then the decomposition

$$\mathfrak{a}_{P,0} = \mathbb{C}X_1 \oplus \cdots \oplus \mathbb{C}X_r$$

realizes $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes \mathbb{C}_{\mu'}$ as an exterior tensor product of analogous single-variable symmetric powers:

$$\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes \mathbb{C}_{\mu'} \cong (\mathrm{Sym}(\mathbb{C}X_1)_{\mu_1} \otimes \mathbb{C}_{\mu_1'}) \otimes \cdots \otimes (\mathrm{Sym}(\mathbb{C}X_r)_{\mu_r} \otimes \mathbb{C}_{\mu_r'}),$$

where $\mu_i, \mu_i' \in (\mathbb{C}X_i)^\vee$ are the ith components of μ, μ' in the dual basis of $\mathfrak{a}_{P,0}^\vee$ to $X_1, \ldots, X_r$. To be explicit, here the space $\mathrm{Sym}(\mathbb{C}X_i)_{\mu_i}$ can be identified as the space of polynomials in the variable λ_i with the structure of a module over the one-dimensional abelian Lie algebra $\mathbb{C}X_i$ given by

$$X_i(\lambda_i^n) = \langle X_i, \mu_i \rangle + n\lambda_i^{n-1}.$$

By the Künneth formula, if we ignore for now the $P(\mathbb{A}_f)$-action, we then reduce to checking the one-dimensional analog of the lemma, that

$$H^i(\mathbb{C}X_i, \mathrm{Sym}(\mathbb{C}X_i)_{\mu_i} \otimes \mathbb{C}_{\mu_i'}) \cong \begin{cases} \mathbb{C} & \text{if } \mu' = -\mu \text{ and } i = 0; \\ 0 & \text{if } \mu' \neq -\mu \text{ or } i > 0. \end{cases}$$

To check this formula, we first note that by definition of Lie algebra cohomology, the space $H^*(\mathbb{C}X_i, \mathrm{Sym}(\mathbb{C}X_i)_{\mu_i} \otimes \mathbb{C}_{\mu_i'})$ is the cohomology of the complex

$$\mathrm{Sym}(\mathbb{C}X_i)_{\mu_i} \otimes \mathbb{C}_{\mu_i'} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}X_i, \mathrm{Sym}(\mathbb{C}X_i)_{\mu_i} \otimes \mathbb{C}_{\mu_i'}) \to 0 \to \cdots,$$

where the map between the first two terms is given by

$$(P \otimes 1) \mapsto (X_i \mapsto X_i(P \otimes 1)).$$

If $\mu' \neq -\mu$, then this map is an isomorphism since the action of X_i on a polynomial preserves its degree. On the other hand, if $\mu' = -\mu$, then X_i decreases the degree of a polynomial by one exactly, and therefore this map is surjective with kernel consisting of constant polynomials. This therefore proves our formula, at least without taking into account the $P(\mathbb{A}_f)$ action, and shows in fact that $H^0(\mathfrak{a}_{P,0}, \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes \mathbb{C}_{-\mu})$ can be identified with subspace of $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ consisting of constants. By definition, this space has an action of $P(\mathbb{A}_f)$ given by the character $e^{\langle H_P(\cdot),\mu\rangle}$, which proves our lemma. $\square$

Another ingredient we need is a well-known theorem of Kostant. To state it, we need to set some notation.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and assume $\mathfrak{h} \subset \mathfrak{m}$. Fix an ordering on the roots of $\mathfrak{h}$ in $\mathfrak{g}$ which makes $\mathfrak{p}$ standard. If $W(\mathfrak{h}, \mathfrak{g})$ denotes the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$, then write

$$W^P = \{w \in W(\mathfrak{h}, \mathfrak{g}) \mid w^{-1}\alpha > 0 \text{ for all positive roots } \alpha \text{ in } \mathfrak{m}\}.$$

Write ρ for half the sum of the positive roots of $\mathfrak{h}$ in $\mathfrak{g}$.

If $\Lambda \in \mathfrak{h}^\vee$ is a dominant weight, write E_Λ for the representation of $\mathfrak{g}$ of highest weight Λ. If $\nu \in \mathfrak{h}^\vee$ is a weight which is dominant for $\mathfrak{m}$ we denote by F_ν the representation of $\mathfrak{m}$ of highest weight ν. In both cases, these weights may be nontrivial on the center, in which case these representations are considered to have central character given by the restriction of these weights to the respective centers. Then we have the following well-known theorem, whose proof we omit.

Theorem 1.2.2.2 (Kostant). *With notation as above, let $\Lambda \in \mathfrak{h}^\vee$ be a dominant weight. Then, as representations of $\mathfrak{m}$, we have an isomorphism*

$$H^i(\mathfrak{n}, E_\Lambda) \cong \bigoplus_{\substack{w \in W^P \\ \ell(w) = i}} F_{w(\Lambda+\rho)-\rho},$$

where $\ell(w)$ denotes the length of the Weyl group element w.

Now we are ready to state and prove the main theorem of this section. Its proof follows the strategy in Borel–Wallach [BW00], Theorem III.3.3.

Theorem 1.2.2.3. *Notation as above, let $\Lambda \in \mathfrak{h}^\vee$ be a dominant weight such that $E = E_\Lambda$. Assume that the cohomology space*

$$H^*(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P}) \otimes E)$$

is nontrivial. Then there is a unique $w \in W^P$ such that

$$-w(\Lambda + \rho)|_{\mathfrak{a}_{P,0}} = d\chi_\pi$$

and such that the infinitesimal character of the archimedean component of $\tilde{\pi}$ contains $-w(\Lambda + \rho)|_{\mathfrak{h}\cap\mathfrak{m}_0}$. Furthermore, if $\ell(w)$ is the length of such an element w, then for any i we have

$$\begin{aligned} &H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P}) \otimes E) \\ &\qquad\qquad \cong \iota_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}), \end{aligned}$$

where ι denotes a normalized parabolic induction functor, and $F_{w(\Lambda+\rho)-\rho,0}$ denotes the restriction to $\mathfrak{m}_0$ of the representation of $\mathfrak{m}$ of highest weight $w(\Lambda+\rho) - \rho$.

Proof. Let us first prove the uniqueness of the element w in the theorem. Note first that

$$\mathfrak{h} \cap \mathfrak{g}_0 = \mathfrak{a}_{P,0} \oplus (\mathfrak{h} \cap \mathfrak{m}_0).$$

Because Λ is dominant, we know $(\Lambda + \rho)$ is regular, and the conditions in the theorem therefore pin down the element $w(\Lambda+\rho)$ uniquely up to the Weyl group $W(\mathfrak{h}\cap\mathfrak{m}_0, \mathfrak{m}_0)$ of $\mathfrak{h}\cap\mathfrak{m}_0$ in $\mathfrak{m}_0$. But it is well known that W^P is a set of representatives for $W(\mathfrak{h}, \mathfrak{g})$ modulo $W(\mathfrak{h}\cap\mathfrak{m}_0, \mathfrak{m}_0)$. Therefore $w(\Lambda + \rho)$ lies in a unique Weyl chamber, and so w is determined.

Let i be an integer. We now begin to compute the cohomology space

$$H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P}) \otimes E).$$

First, Proposition 1.1.3.1 allows us to pull the tensor product with E inside the induction, whence by Frobenius reciprocity, we have

$$\begin{aligned} H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P}) \otimes E) \\ \cong \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(H^i(\mathfrak{p}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P} \otimes E)). \quad (1.2.2.1) \end{aligned}$$

It is our goal, therefore, to compute

$$H^i(\mathfrak{p}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P} \otimes E).$$

Now, as $(\mathfrak{p}_0, K'_\infty \cap P(\mathbb{R}))$-modules, the space $\tilde{\pi}$ comes from a $(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}))$-module and $\mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P}$ comes from an $\mathfrak{a}_{P,0}$-module. Thus, using

$$\mathfrak{p}_0 = (\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}) \oplus \mathfrak{n},$$

we get a spectral sequence whose E_2 page is

$$E_2^{j,k} = H^j(\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P} \otimes H^k(\mathfrak{n}; E))$$

and which degenerates to the cohomology space above with $i = j + k$. We will eventually be able to say that this spectral sequence degenerates on its E_2 page, but this will follow from the vanishing of enough of its terms. So we compute this page now.

By the Kostant decomposition (Theorem 1.2.2.2), the (j,k)-term on this E_2 page is

$$\bigoplus_{\substack{w' \in W^P \\ \ell(w) = k}} H^j(\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P} \otimes F_{w'(\Lambda+\rho)-\rho}).$$

Write $\nu(w') = (w'(\Lambda + \rho) - \rho)|_{\mathfrak{a}_{P.0}}$, As an $(\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0})$-module, the representation $F_{w'(\Lambda+\rho)-\rho}$ decomposes as

$$F_{w'(\Lambda+\rho)-\rho} = F_{w'(\Lambda+\rho)-\rho,0} \otimes \mathbb{C}_{\nu(w')},$$

as an exterior tensor product over the direct sum $\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}$. Thus by the Künneth formula, we get

$$\begin{aligned} &H^*(\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi + \rho_P} \otimes F_{w'(\Lambda+\rho)-\rho}) \\ &\qquad \cong H^*(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w'(\Lambda+\rho)-\rho,0}) \otimes H^*(\mathfrak{a}_{P,0}, \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi+\rho_P} \otimes \mathbb{C}_{\nu(w')}). \end{aligned}$$

By Lemma 1.2.2.1, the second factor here is nonvanishing if and only if

$$d\chi_\pi + \rho_P = -\nu(w'),$$

and the first factor is nonvanishing only if the infinitesimal character of $F_{w'(\Lambda+\rho)-\rho,0}$ matches the negative of that of the archimedean component of $\tilde{\pi}$. Since $\mathfrak{p}$ is standard, we have $\rho_P = \rho|_{\mathfrak{a}_{P,0}}$, which implies

$$\nu(w') = w'(\Lambda+\rho)|_{\mathfrak{a}_{P.0}} - \rho_P$$

and so this first nonvanishing condition is equivalent to

$$= w'(\Lambda+\rho)|_{\mathfrak{a}_{P.0}} = d\chi_\pi;$$

the second of these nonvanishing conditions is just that $-w'(\Lambda+\rho)$ occurs in the infinitesimal character of the archimedean component of $\tilde{\pi}$. As shown at the beginning of this proof, there is only one w' satisfying these two conditions, and we will denote it by w.

Thus, by Lemma 1.2.2.1, we get

$$\begin{aligned} &H^*(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes H^*(\mathfrak{a}_{P,0}, \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi+\rho_P} \otimes \mathbb{C}_{\nu(w)}) \\ &\qquad \cong H^*(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi_\pi+\rho_P\rangle}), \end{aligned}$$

where the factor $\mathbb{C}(e^{\langle H_P(\cdot), d\chi_\pi+\rho_P\rangle})$ is concentrated in degree zero.

Retracing our steps, we have thus computed the E_2 page of our spectral sequence. It is

$$E_2^{j,k} \cong \begin{cases} H^j(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi_\pi+\rho_P\rangle}) & \text{if } k = \ell(w); \\ 0 & \text{if } k \neq \ell(w). \end{cases}$$

The E_2 page therefore consists only of one row, and thus our spectral sequence degenerates. Hence we have shown

$$\begin{aligned} &H^i(\mathfrak{p}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi+\rho_P} \otimes E) \\ &\qquad\qquad \cong H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi_\pi+\rho_P\rangle}) \end{aligned}$$

Now we rewrite

$$\begin{aligned} &H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi_\pi+\rho_P\rangle}) \\ &\qquad \cong \tilde{\pi}_f \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi_\pi+\rho_P\rangle}) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}) \\ &\qquad \cong \pi_f \otimes \mathbb{C}(e^{\langle H_P(\cdot), \rho_P\rangle}) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}), \end{aligned}$$

so that

$$\begin{aligned} &H^i(\mathfrak{p}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi+\rho_P} \otimes E) \\ &\qquad\qquad \cong \pi_f \otimes \mathbb{C}(e^{\langle H_P(\cdot), \rho_P\rangle}) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}). \end{aligned}$$

We therefore have, by (1.2.2.1),

$$\begin{aligned} &H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi+\rho_P}) \otimes E) \\ &\qquad \cong \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f \otimes \mathbb{C}(e^{\langle H_P(\cdot), \rho_P\rangle})) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}), \end{aligned}$$

which is what we wanted to prove. □

The above theorem will allow us to produce Eisenstein cohomology classes. To distinguish the representations of $G(\mathbb{A}_f)$ generated by these classes, we will need to see what might correspond to them on the Galois side. We set up the tools to do this in the next section.

1.3 Galois representations

We now recall the facts we need about ℓ-adic Galois representations. The reason for introducing Galois representations into the picture is that they will allow us to distinguish the automorphic representations to which they will be attached.

Our notion of what it means for a Galois representation to be attached to an automorphic representation is relatively weak, but it will suffice for our purposes.

1.3.1 Galois representations attached to automorphic representations

We continue to use the notation set previously, and in particular we will continue working with our reductive $\mathbb{Q}$-group G, but with one modification: We now assume that G is split. This will simplify our discussion of Satake parameters, and it will also allow us to work only with the Galois group of $\mathbb{Q}$ instead of that of some finite extension.

We explain in this section what we mean when we say that an automorphic representation of $G(\mathbb{A})$ has attached to it a Galois representation. Our version of this notion will be a weak one, in the sense that it will only depend on the automorphic representation in question at all but finitely many of its unramified places. But this will suffice for our purposes.

So to get started, fix a prime p. We will recall some of the theory of unramified representations of $G(\mathbb{Q}_p)$ due to Langlands, Satake, and others.

First we fix a split maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ containing T. Write U for the unipotent radical of B. Let

$$W = N_G(T)/T$$

be the Weyl group of G. Let $\delta_{B(\mathbb{Q}_p)}$ denote the modulus character of $B(\mathbb{Q}_p)$.

Next, fix a model of G over $\mathbb{Z}_p$. Write $K_p = G(\mathbb{Z}_p)$; this is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. Let $\mathcal{H}(K_p)$ be the spherical Hecke algebra, defined as the convolution algebra of smooth, compactly supported, K_p-biinvariant, $\mathbb{C}$-valued functions on $G(\mathbb{Q}_p)$.

Fix an irreducible admissible representation σ of $G(\mathbb{Q}_p)$ which is spherical, i.e., which has a K_p-fixed vector. Then the K_p-invariant subspace σ^{K_p} is one dimensional. Thus we get a character of the Hecke algebra

$$\omega_\sigma^{\mathrm{H}} : \mathcal{H}(K_p) \to \mathrm{End}(\sigma^{K_p}) \cong \mathbb{C}.$$

On the other hand, we have the Satake transform $\mathcal{S}$, which is an isomorphism from $\mathcal{H}(K_p)$ to the Weyl group invariants of the analogously defined Hecke algebra $\mathcal{H}(T(\mathbb{Z}_p))$. In more detail, the Hecke algebra $\mathcal{H}(T(\mathbb{Z}_p))$ is defined to be the convolution algebra of smooth, compactly supported, $T(\mathbb{Z}_p)$-biinvariant, $\mathbb{C}$-valued functions on $T(\mathbb{Q}_p)$. Because T is abelian, this is the same as the group algebra $\mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]$. Of course, W acts on T and therefore gives compatible actions on both $\mathcal{H}(T(\mathbb{Z}_p))$ and $\mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]$.

The Satake transform

$$\mathcal{S} : \mathcal{H}(K_p) \to \mathcal{H}(T(\mathbb{Z}_p))$$

is defined by

$$\mathcal{S}(f)(t) = \delta_{B(\mathbb{Q}_p)}(t)^{1/2} \int_{U(\mathbb{Q}_p)} f(tu)\, du.$$

It is a theorem that the image of $\mathcal{S}$ is contained in the Weyl group invariants $\mathcal{H}(T(\mathbb{Z}_p))^W$ and, in fact, is an isomorphism when $\mathcal{H}(T(\mathbb{Z}_p))^W$ is considered at its target. Thus, through the identifications above, we get an isomorphism

$$\mathcal{H}(K_p) \cong \mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^W.$$

We can therefore transfer the character $\omega_\sigma^{\mathrm{H}}$ defined above to $\mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^W$ and obtain a character

$$\omega_\sigma^{\mathrm{S}} : \mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^W \to \mathbb{C}.$$

There is another construction that gives a character of $\mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^W$ starting from the representation σ, which we describe now. It is a theorem that σ, since it is spherical, occurs as a subquotient of a principal series representation

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\chi \cdot \delta_{B(\mathbb{Q}_p)})$$

for some character χ of $T(\mathbb{Q}_p)$ which is trivial on $T(\mathbb{Z}_p)$. The character χ with this property is unique only up to the action of W. But in any case, the character χ, when viewed as a character

$T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$, gives naturally a character

$$\tilde{\omega} : \mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)] \to \mathbb{C}.$$

The restriction of this character to the Weyl invariants will be written as

$$\omega_\sigma^{\mathrm{I}} : \mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^W \to \mathbb{C}.$$

While there is a choice involved in selecting the character χ, and hence in defining $\tilde{\omega}$, the character $\omega_\sigma^{\mathrm{I}}$ does not depend on this choice and is well defined.

We state the following well known result as a proposition.

Proposition 1.3.1.1. *In the setting above, the two characters*

$$\omega_\sigma^{\mathrm{S}}, \omega_\sigma^{\mathrm{I}} : \mathbb{C}[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^W \to \mathbb{C}$$

coincide.

Let us denote by ω_σ the common character $\omega_\sigma^{\mathrm{S}} = \omega_\sigma^{\mathrm{I}}$.

Now the group $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ can be naturally identified with the cocharacter group $X_*(T)$; the identification is given by evaluating a cocharacter $\lambda \in X_*(T)$ at a uniformizer in $\mathbb{Q}_p^\times$. Also, if we fix a maximal torus $T^\vee$ in the dual group $G^\vee$, we have a natural identification $X_*(T) = X^*(T^\vee)$ of the cocharacter group of T with the character group of $T^\vee$.

Therefore the character ω_σ just constructed may well be viewed as a character

$$\omega_\sigma : \mathbb{C}[X^*(T^\vee)]^W \to \mathbb{C}.$$

Now given a finite dimensional representation V of $G^\vee(\mathbb{C})$, we can view its character χ_V as an element of $\mathbb{C}[X^*(T^\vee)]^W$. Then the character ω_σ gives a conjugacy class $s(\sigma)$ in $G^\vee(\mathbb{C})$; it is the unique conjugacy class with the property that

$$\omega_\sigma(\chi_V) = \mathrm{Tr}(s(\sigma)|V)$$

for any finite dimensional representation V of $G^\vee(\mathbb{C})$. We call $s(\sigma)$ the *Satake parameter* or *Langlands parameter* attached to σ.

We now fix a prime ℓ different from p. Since $\overline{\mathbb{Q}}_\ell$ is isomorphic to $\mathbb{C}$, everything above could be done over $\overline{\mathbb{Q}}_\ell$ instead. In particular, we may view ω_σ as a character of $\overline{\mathbb{Q}}_\ell[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^W \cong \overline{\mathbb{Q}}_\ell[X^*(T^\vee)]^W$, and we may view the Satake parameter $s(\sigma)$ as a conjugacy class in $G^\vee(\overline{\mathbb{Q}}_\ell)$.

We need to make this change of field because our Galois representations will have as their target the group $G^\vee(\overline{\mathbb{Q}}_\ell)$. In fact, we are ready to give the following definition.

Definition 1.3.1.2. Let Π be an automorphic representation of $G(\mathbb{A})$. We will say that a continuous representation

$$\rho : G_{\mathbb{Q}} \to G^\vee(\overline{\mathbb{Q}}_\ell)$$

is *attached to* Π if there is a finite set S of places of $\mathbb{Q}$ containing ℓ, the archimedean place, and all the ramified places for Π, such that for any prime $p \notin S$, ρ is unramified at p and we have

$$\rho(\mathrm{Frob}_p)^{\mathrm{ss}} \in s(\Pi_p),$$

where Frob_p is any choice of (geometric) Frobenius element at p, the element $\rho(\mathrm{Frob}_p)^{\mathrm{ss}}$ is the semisimplification of $\rho(\mathrm{Frob}_p)$, and the Satake parameter $s(\Pi_p)$ of the local component of Π at p is viewed as a conjugacy class in $G^\vee(\overline{\mathbb{Q}}_\ell)$.

We remark that in the definition, the semisimplification $\rho(\mathrm{Frob}_p)^{\mathrm{ss}}$ may be defined to be the semisimple element of $G^\vee(\overline{\mathbb{Q}}_\ell)$ whose image in any finite dimensional representation of $G^\vee(\overline{\mathbb{Q}}_\ell)$ has the same characteristic polynomial as $\rho(\mathrm{Frob}_p)$.

Now in the case of the group GL_2 a lot is known about when such Galois representations exist. Let us recall some results in this direction.

Let F be a holomorphic cuspidal eigenform of weight $k \geq 1$, conductor $N \geq 1$, and nebentypus ω_F. Then F gives rise to a unitary automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A})$. This representation $\tilde{\pi}$ has central character given by the adelization of ω_F. Write

$$\pi = \tilde{\pi} \otimes |\det|^{(k-1)/2}$$

This normalization is necessary to recover the usual Galois representation attached to F. In fact, we have the following theorem.

Theorem 1.3.1.3. *With the setting as in the above paragraph, fix a prime ℓ not dividing N. Then there is a continuous Galois representation*

$$\rho_\pi : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

which is attached to π in the sense of Definition 1.3.1.2; in fact the set S in that definition can be taken to be the set of primes dividing N, ℓ, and ∞. This representation ρ_π is unique up to conjugation by elements of $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$, and it is irreducible. Furthermore, ρ_π is Hodge–Tate (in fact, de Rham) at ℓ with Hodge–Tate weights 0 and $-(k-1)$. (Here, our conventions are such that the ℓ-adic cyclotomic character has Hodge–Tate weight -1.)

Remark 1.3.1.4. The above is a classical theorem which (except for the final claim about ρ_π being Hodge–Tate) is due to Eichler–Shimura when $k = 2$, to Deligne when $k > 2$, and to Deligne–Serre when $k = 1$. Usually when recalling this theorem one states explicitly the properties that, for $p \notin S$ we have

$$\mathrm{Tr}(\rho_\pi(\mathrm{Frob}_p^{-1})) = a_p,$$

where a_p is the pth Hecke eigenvalue of F, and

$$\det(\rho_\pi) = \omega_F \chi_{\mathrm{cyc}}^{k-1},$$

where $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \mathbb{Z}_\ell^\times$ denotes the ℓ-adic cyclotomic character and ω_F is viewed as a finite order Galois character by class field theory. Actually, these assertions follow from our statement of the theorem once we know π_p explicitly enough to know the characteristic polynomial of $s(\pi_p)$ for $p \notin S$.

We conclude this section with a proposition which will be useful for us later when distinguishing between different automorphic representations. To state it, we recall the following definition.

Definition 1.3.1.5. Let Π, Π' be two automorphic representations of a reductive group G, with respective local components Π_v, Π'_v at places v. We say Π and Π' are *nearly equivalent* if, for all but finitely many places v, there is an isomorphism $\Pi_v \cong \Pi'_v$.

Proposition 1.3.1.6. *Let* Π, Π' *be two automorphic representations of* $G(\mathbb{A})$ *with respective Galois representations*

$$\rho, \rho' : G_{\mathbb{Q}} \to G^{\vee}(\overline{\mathbb{Q}}_{\ell}).$$

Assume Π *and* Π' *are nearly equivalent. Let*

$$R : G^{\vee} \to \mathrm{GL}_n$$

be a finite dimensional representation of $G^{\vee}$*. Then the semisimplified Galois representations*

$$(R \circ \rho)^{\mathrm{ss}},\ (R \circ \rho')^{\mathrm{ss}},$$

which are semisimple representations of $G_{\mathbb{Q}}$ *into* $\mathrm{GL}_n(\overline{\mathbb{Q}}_{\ell})$*, are equivalent.*

Proof. By the hypotheses, there is a finite set S of places, including ℓ and the archimedean place, such that for $p \notin S$, the local components Π_p and Π'_p of our automorphic representations at p are unramified and isomorphic. Therefore we have an equality of Satake parameters for $p \notin S$,

$$s(\Pi_p) = s(\Pi'_p).$$

After possibly enlarging S, we have then that for $p \notin S$, the semisimple elements

$$\rho(\mathrm{Frob}_p)^{\mathrm{ss}},\ \rho'(\mathrm{Frob}_p)^{\mathrm{ss}}$$

are conjugate in $G^{\vee}(\overline{\mathbb{Q}}_{\ell})$. Therefore we have an equality of traces

$$\mathrm{Tr}(R(\rho(\mathrm{Frob}_p))) = \mathrm{Tr}(R(\rho'(\mathrm{Frob}_p)))$$

By continuity and Chebotarev, this implies an equality of characters

$$\mathrm{Tr}(R \circ \rho) = \mathrm{Tr}(R \circ \rho'),$$

which in turn implies the conclusion of our proposition. □

Remark 1.3.1.7. The above proposition may be summarized as saying that $(R \circ \rho)^{\mathrm{ss}}$ is a near-equivalence invariant of automorphic representations (at least when ρ exists). It is therefore also an isomorphism invariant; that is, the proposition can be applied when $\Pi \cong \Pi'$. This is useful, since it is possible for an automorphic representation to have many Galois representations attached to it in the sense of our definition. This is especially possible when ρ is reducible (i.e., factors through a proper parabolic subgroup of $G^\vee(\overline{\mathbb{Q}}_\ell)$).

1.3.2 Galois representations and induced representations

In this section we explain how to attach Galois representations to subquotients of parabolically induced representations. This will therefore give us a way of attaching Galois representations to Eisenstein series.

We continue with the notation of the previous section, and in particular we will work with our split reductive $\mathbb{Q}$-group G and a choice of split maximal torus $T \subset G$ and Borel subgroup $B \subset G$ containing T. As we did before, we choose a split maximal torus $T^\vee$ in the dual group $G^\vee$ and a Borel $B^\vee$ containing $T^\vee$.

Now let $P \subset G$ be a parabolic subgroup containing B, and let M be its standard Levi. The parabolic P corresponds to a subset of the set of simple roots of T in G, and the set of corresponding coroots gives us a standard parabolic $P^\vee$ in $G^\vee$. Its standard Levi $M^\vee$ is, as this notation suggests, identified with the dual group of M.

Proposition 1.3.2.1. *Let π be an automorphic representation of $M(\mathbb{A})$. Assume that π has attached to it a Galois representation*

$$\rho_\pi : G_{\mathbb{Q}} \to M^\vee(\overline{\mathbb{Q}}_\ell),$$

in the sense of Definition 1.3.1.2. Let Π be an automorphic representation of $G(\mathbb{A})$ which is a subquotient of the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi \otimes \delta_{P(\mathbb{A})}^{1/2}),$$

where $\delta_{P(\mathbb{A})}$ is the modulus character of $P(\mathbb{A})$. Let i_M be the inclusion map

$$i_M : M^\vee(\overline{\mathbb{Q}}_\ell) \hookrightarrow G^\vee(\overline{\mathbb{Q}}_\ell).$$

Then the Galois representation

$$\rho_\Pi : G_\mathbb{Q} \to G^\vee(\overline{\mathbb{Q}}_\ell)$$

given by

$$\rho_\Pi = i_M \circ \rho_\pi$$

is attached to Π, again in the sense of Definition 1.3.1.2.

Proof. Decompose π and Π into their local components,

$$\pi \cong \bigotimes_v{}' \pi_v, \qquad \Pi \cong \bigotimes_v{}' \Pi_v.$$

Let S_0 be a finite set of places of $\mathbb{Q}$, which contains ℓ and the archimedean place, and such that for $p \notin S_0$, the condition

$$\rho(\mathrm{Frob}_p)^{\mathrm{ss}} \in s(\pi_p)$$

of Definition 1.3.1.2 is satisfied for π_p. Let S be the set of primes containing all those in S, as well as any place v for which Π_v is not spherical. We are to verify that

$$i_M(s(\pi_p)) \subset s(\Pi_p). \tag{1.3.2.1}$$

for $p \notin S$.

Let W_G be the Weyl group of T in G, and W_M that of T in M, and let

$$\omega_{\pi_p} : \overline{\mathbb{Q}}_\ell[X^*(T^\vee)]^{W_M} \to \overline{\mathbb{Q}}_\ell, \qquad \omega_{\Pi_p} : \overline{\mathbb{Q}}_\ell[X^*(T^\vee)]^{W_G} \to \overline{\mathbb{Q}}_\ell$$

be the characters constructed in Proposition 1.3.1.1. Let V be any finite dimensional representation of G, and let $V|_M$ be the same representation but viewed as a representation of M. By the characterizing property of the Satake parameter, checking (1.3.2.1) is the same as checking that

$$\omega_{\pi_p}(\chi_V) = \omega_{\Pi_p}(\chi_V)$$

where χ_V is the character of V. This will of course follow if we show ω_{Π_p} is the restriction of ω_{π_p} to the W_G-invariants $\overline{\mathbb{Q}}_\ell[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^{W_G}$.

Recall the construction of ω_{π_p} via normalized induction; the representation π_p occurs as the irreducible spherical subquotient of a Borel induction

$$\mathrm{Ind}_{(B\cap M)(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)}(\chi \cdot \delta_{(B\cap M)(\mathbb{Q}_p)}).$$

But by induction in stages, Π_p is the irreducible spherical subquotient of

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\chi \cdot \delta_{B(\mathbb{Q}_p)}).$$

This shows then that ω_{π_p} is the restriction of the character $\overline{\mathbb{Q}}_\ell[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)] \to \overline{\mathbb{Q}}_\ell$ induced from χ to the W_M-invariants, and similarly ω_{Π_p} is the restriction of the same character to $\overline{\mathbb{Q}}_\ell[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]^{W_G}$. Once we pass through the identification $T(\mathbb{Q}_p)/T(\mathbb{Z}_p) = X^*(T^\vee)$, this is exactly what we wanted to show. $\square$

1.4 The case of GSp_4

We now apply the theory of the previous three sections to the case when $G = \mathrm{GSp}_4$. We will define certain Langlands quotients of parabolically induced representations, induced from the Siegel parabolic, and study their multiplicities in Eisenstein and cuspidal cohomology.

1.4.1 The group GSp_4

We fix in this section some notation that will be used throughout this section.

Let J be the matrix

$$J = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}.$$

Define GSp_4 to be the group over $\mathbb{Q}$ defined matricially for $\mathbb{Q}$-algebras A by

$$\mathrm{GSp}_4(A) = \{g \in \mathrm{GL}_4(A) \mid {}^t gJg = \nu J \text{ for some } \nu = \nu(g) \in A^\times\}.$$

The group GSp_4 is reductive and split. In fact, a split maximal torus T is given by the subgroup of all diagonal matrices in GSp_4.

The assignment $g \mapsto \nu(g)$, where $\nu(g)$ is as in the definition above, defines a character of GSp_4, called the *similitude character,* and which we denote simply by ν. We also denote by the same letter the restriction of ν to the maximal torus T.

The group GSp_4 contains the subgroup Sp_4, defined as

$$\mathrm{Sp}_4 = \{g \in \mathrm{GSp}_4 \mid \nu(g) = 1\}.$$

The group Sp_4 is the split simple group of type C_2, with a choice of split maximal torus $T_0 = T \cap \mathrm{Sp}_4$, given again by diagonal matrices. Let us now study this group from the perspective of its root lattice.

The root lattice

The Dynkin diagram of Sp_4 is as in Figure 1.4.1. So we are writing α for the long simple root

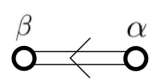


Figure 1.4.1: The Dynkin diagram of GSp_4

and β for the short simple root. This way of labelling the roots will be consistent with our notation for the simple roots of G_2 later.

Explicitly, any element of T_0 is a diagonal matrix of the form

$$\mathrm{diag}(a, b, a^{-1}, b^{-1}),$$

and the characters α and β act on these matrices by

$$\alpha(\mathrm{diag}(a, b, a^{-1}, b^{-1})) = b^2, \qquad \beta(\mathrm{diag}(a, b, a^{-1}, b^{-1})) = ab^{-1}.$$

The character α has an obvious square root, which we write additively as $\alpha/2$, which picks out the b entry of a diagonal matrix as above. Then $\alpha/2$ and β generate the character group $X^*(T_0)$.

The inner product space $X^*(T_0) \otimes \mathbb{R}$ is isometric to $\mathbb{R}^2$ with its usual inner product, and an isometry is given by $\alpha \mapsto (0, 2)$ and $\beta \mapsto (1, -1)$. Thus we get a picture of the root lattice as in Figure 1.4.2; there, the dominant chamber is shaded.

We can extend the characters α and β to characters of the torus $T \subset \mathrm{GSp}_4$ as follows. Every

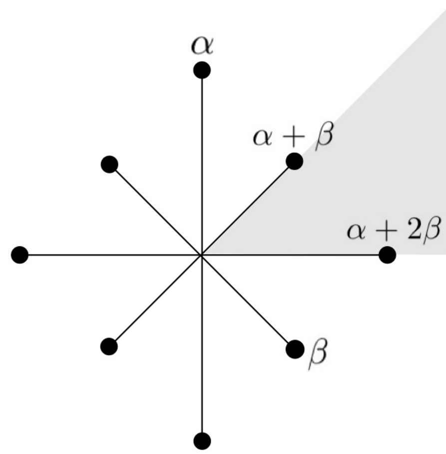


Figure 1.4.2: The root lattice of GSp_4

element of T can be written as a diagonal matrix of the form

$$\mathrm{diag}(a, b, ca^{-1}, cb^{-1}),$$

and for such matrices, we let

$$\alpha(\mathrm{diag}(a, b, ca^{-1}, cb^{-1})) = b^2c^{-1}, \qquad \beta(\mathrm{diag}(a, b, ca^{-1}, cb^{-1})) = ab^{-1}.$$

By its definition, the character ν acts on these matrices as

$$\nu(\mathrm{diag}(a, b, ca^{-1}, cb^{-1})) = c.$$

The characters α, β, ν generate an index 2 subgroup in the character group $X^*(T)$, and the character $\alpha + \nu$ (we write the group law in $X^*(T)$ additively) has a square root.

The center of GSp_4 is equal to the center of GL_4; it is just the subgroup of invertible multiples of the identity matrix I. The center of Sp_4 has order 2 and is equal to $\{\pm I\}$.

Let us write Δ for the set of roots of T in GSp_4 obtained above, or for the set of roots in T_0 in Sp_4. Write Δ^+ for the positive roots. So

$$\Delta^+ = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta\}.$$

Parabolic subgroups

For $\gamma \in \Delta$, write $\mathbf{x}_\gamma$ for the unipotent root group homomorphism

$$\mathbf{x}_\gamma : \mathbb{G}_a \to \mathrm{GSp}_4.$$

Here $\mathbb{G}_a$ denotes as usual the additive group scheme. Then we have the following matrix formulas for each $\mathbf{x}_\gamma$,

$$\begin{aligned}
\mathbf{x}_\alpha(a) &= \begin{pmatrix} 1 & & & \\ & 1 & & a \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad \mathbf{x}_\beta(a) = \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & -a & 1 \end{pmatrix}, \\
\mathbf{x}_{\alpha+\beta}(a) &= \begin{pmatrix} 1 & & & a \\ & 1 & a & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad \mathbf{x}_{\alpha+2\beta}(a) = \begin{pmatrix} 1 & & a & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},
\end{aligned} \tag{1.4.1.1}$$

and then

$$\mathbf{x}_{-\gamma}(a) = {}^t\mathbf{x}_\gamma(a)$$

for $\gamma \in \Delta$.

Let $P_\alpha \subset \mathrm{GSp}_4$ be the standard parabolic subgroup whose Levi contains the image of $\mathbf{x}_\alpha$. Write $P_\alpha = M_\alpha N_\alpha$ for its Levi decomposition. We similarly define P_β and write $P_\beta = M_\beta N_\beta$ for its Levi decomposition. We write B for the standard Borel and $B = TU$ for its Levi decomposition. Then by (1.4.1.1) it follows that B, P_α and P_β take the following forms:

$$B = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix} \right\}, \qquad P_\alpha = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & * & * & * \end{pmatrix} \right\}, \qquad P_\beta = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \right\},$$

Along with GSp_4 itself, these comprise all the standard parabolic subgroups of GSp_4. The parabolics P_α and P_β are both maximal and have Levis isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_1$. Explicit isomorphisms

$$i_\alpha : \mathrm{GL}_2 \times \mathrm{GL}_1 \to M_\alpha, \quad \text{and} \quad i_\beta : \mathrm{GL}_2 \times \mathrm{GL}_1 \to M_\beta$$

are given by, for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2$ and $t \in \mathrm{GL}_1$,

$$i_\alpha(A,t) = \begin{pmatrix} t^{-1}\det(A) & & & \\ & a & & b \\ & & t & \\ & c & & d \end{pmatrix} \in M_\alpha, \qquad i_\beta(A,t) = \begin{pmatrix} A & * \\ & t\,{}^t\!A^{-1} \end{pmatrix} \in M_\beta. \qquad (1.4.1.2)$$

As is often done, we call P_α the *Klingen parabolic* and P_β the *Siegel parabolic.*

Duality

The group GSp_4 is self dual. Identifying GSp_4 with its dual group switches the long and short simple roots. For us this will mean that certain data associated with the Siegel parabolic will become associated with the Klingen parabolic on the dual side, and vice-versa.

This can be made explicit as follows. There are isomorphisms $\mathrm{GL}_1 \cong \mathrm{GL}_1^\vee$, $\mathrm{GL}_2 \cong \mathrm{GL}_2^\vee$, and $\mathrm{GSp}_4 \cong \mathrm{GSp}_4^\vee$ such that the diagrams below commute. Identify M_α and M_β with $\mathrm{GL}_2 \times \mathrm{GL}_1$ via the maps i_α and i_β of (1.4.1.2). Then $M_\alpha^\vee$ and $M_\beta^\vee$ are identified with $\mathrm{GL}_2^\vee \times \mathrm{GL}_1^\vee$ as well, and these latter identifications fit into a commutative diagram as follows. We have

$$\begin{tikzcd} \mathrm{GL}_2^\vee \times \mathrm{GL}_1^\vee \arrow[rr, "\sim"] \arrow[d, "\sim"] & & M_\alpha^\vee \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{GSp}_4^\vee \arrow[d, "\sim"] \\ \mathrm{GL}_2 \times \mathrm{GL}_1 \arrow[r, "\varphi_\alpha"] & \mathrm{GL}_2 \times \mathrm{GL}_1 \arrow[r, "i_\beta"] & M_\beta \arrow[r, hook] & \mathrm{GSp}_4 \end{tikzcd} \qquad (1.4.1.3)$$

where the map φ_α is the map given by

$$\varphi_\alpha(A,t) = (A, \det(A)t). \qquad (1.4.1.4)$$

Similarly, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{GL}_2^\vee \times \mathrm{GL}_1^\vee \arrow[rr, "\sim"] \arrow[d, "\sim"] & & M_\beta^\vee \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{GSp}_4^\vee \arrow[d, "\sim"] \\
\mathrm{GL}_2 \times \mathrm{GL}_1 \arrow[r, "\varphi_\beta"] & \mathrm{GL}_2 \times \mathrm{GL}_1 \arrow[r, "i_\alpha"] & M_\alpha \arrow[r, hook] & \mathrm{GSp}_4
\end{tikzcd} \tag{1.4.1.5}$$

where the map φ_β given by

$$\varphi_\beta(A, t) = (tA, t). \tag{1.4.1.6}$$

Finally, for the Borel, the map $i_0 : \mathrm{GL}_1^3 \to T$ given by

$$i_0(a, b, c) = \mathrm{diag}(a, b, ca^{-1}, cb^{-1}) \tag{1.4.1.7}$$

is an isomorphism which identifies T with GL_1^3, and hence also $T^\vee$ and $(\mathrm{GL}_1^\vee)^3$. This latter identification fits into the commutative diagram

$$\begin{tikzcd}
(\mathrm{GL}_1^\vee)^3 \arrow[rr, "\sim"] \arrow[d, "\sim"] & & T^\vee \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{GSp}_4^\vee \arrow[d, "\sim"] \\
(\mathrm{GL}_1)^3 \arrow[r, "\varphi_0"] & (\mathrm{GL}_1)^3 \arrow[r, "i_0"] & T \arrow[r, hook] & \mathrm{GSp}_4
\end{tikzcd} \tag{1.4.1.8}$$

where the map φ_0 is given by

$$\varphi_0(t_1, t_2, t_3) = (t_1 t_2 t_3, t_1 t_3, t_1 t_2 t_3^2). \tag{1.4.1.9}$$

The Weyl group

Let $W = W(T, \mathrm{GSp}_4)$ be the Weyl group of GSp_4. The group W is isomorphic to the dihedral group D_4 with eight elements acting naturally on the root lattice.

For $\gamma \in \Delta$, let w_γ be the reflection about the line perpendicular to γ. Then W is generated by the simple reflections w_α and w_β. Let us amalgamate products of these reflections into a single notation: Write $w_{\alpha\beta} = w_\alpha w_\beta$, $w_{\alpha\beta\alpha} = w_\alpha w_\beta w_\alpha$, and so on. Then

$$W = \{1, w_\alpha, w_\beta, w_{\alpha\beta}, w_{\beta\alpha}, w_{\alpha\beta\alpha}, w_{\beta\alpha\beta}, w_{-1}\}.$$

The elements above are written minimally in terms of products of the simple reflections w_α and w_β, except for the final element w_{-1}. This element w_{-1} is the element that acts by negation on the root lattice, and it of length 4, equal to both $w_{\alpha\beta\alpha\beta}$ and $w_{\beta\alpha\beta\alpha}$.

For $P = MN$ one of the standard parabolic subgroups of GSp_4, let

$$W^P = \{w \in W \mid w^{-1}\gamma > 0 \text{ for all positive roots } \gamma \text{ in } M\}.$$

This is the set of minimal length representatives for the quotient $W(T, M)\backslash W$. Then

$$W^{P_\alpha} = \{1, w_\beta, w_{\beta\alpha}, w_{\beta\alpha\beta}\}, \qquad W^{P_\beta} = \{1, w_\alpha, w_{\alpha\beta}, w_{\alpha\beta\alpha}\},$$

and $W^B = W$.

Finally, we note for future reference that the action of W on T is given by

$$\begin{aligned} \mathrm{diag}(a, b, ca^{-1}, cb^{-1})^{w_\alpha} &= \mathrm{diag}(a, cb^{-1}, ca^{-1}, b), \\ \mathrm{diag}(a, b, ca^{-1}, cb^{-1})^{w_\beta} &= \mathrm{diag}(b, a, cb^{-1}, ca^{-1}). \end{aligned} \tag{1.4.1.10}$$

The group $\mathrm{GSp}_4(\mathbb{R})$

The real Lie group $\mathrm{GSp}_4(\mathbb{R})$ has discrete series representations, but is disconnected. However, $\mathrm{Sp}_4(\mathbb{R})$ is connected as a real Lie group. Therefore it will be easier to describe the classification of the discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ first and then use it to classify those of $\mathrm{GSp}_4(\mathbb{R})$. For a review of Harish-Chandra's classification of discrete series, the reader may jump ahead to Section 1.6.2, specifically Theorem 1.6.2.1.

Fix first a maximal compact subgroup K_∞ in $\mathrm{GSp}_4(\mathbb{R})$. Then the connected component K_∞° of the identity is a maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$.

The group K_∞° is isomorphic to the real unitary group $\mathrm{U}(2)$. Therefore any maximal torus in K_∞° is two dimensional. Fix $T_c \subset K_\infty^\circ$ such a maximal torus. Then T_c is also a maximal torus in $\mathrm{Sp}_4(\mathbb{R})$.

Let $\mathfrak{t}_c$ be the complexified Lie algebra of T_c and $\mathfrak{k}$ that of K_∞°. Abusing notation, we let $\Delta = \Delta(\mathfrak{t}_c, \mathfrak{sp}_4)$ be the roots of $\mathfrak{t}_c$ in $\mathfrak{sp}_4$, and let $\Delta_c = \Delta(\mathfrak{t}_c, \mathfrak{k}) \subset \Delta$ be the set of compact roots.

There are two roots in Δ_c and they are short. Pick one, and again by abuse of notation, call it β. Choose a long root α in Δ such that α and β are a pair of simple roots. The roots β and $\alpha/2$ generate the lattice of analytically integral weights in $\mathfrak{t}_c^\vee$.

The compact Weyl group $W_c = W(\mathfrak{t}_c, \mathfrak{k})$ has two elements and is generated by the simple reflection w_β across the line perpendicular to β. If we write $W = W(\mathfrak{t}_c, \mathfrak{sp}_4)$ for the Weyl group of Δ, then W_c has index 4 in W. Therefore the discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ are parametrized by analytically integral weights that lie far enough inside the four chambers below the line perpendicular to β.

The element w_{-1} is in the Weyl group W, and the element $w_\beta \circ w_{-1}$ is equal to the simple reflection $w_{\alpha+\beta}$ across the line perpendicular to $\alpha + \beta$. If a discrete series representation V has Harish-Chandra parameter λ, then the contragredient $V^\vee$ has Harish-Chandra parameter $-\lambda$; but if the weight λ is in one of the four chambers under the line perpendicular to β, then $-\lambda$ will lie above this line. Therefore we should choose $w_\beta(-\lambda) = w_{\alpha+\beta}\lambda$ as the parameter for $V^\vee$.

Now there is an element k_0 of order 2 in the nonidentity component of K_∞ such that the adjoint action of k_0 on K_∞° preserves T_c and acts as inversion there. Write $\mathrm{GSp}_4(\mathbb{R})^+$ for the subgroup of $\mathrm{GSp}_4(\mathbb{R})$ given by

$$\mathrm{GSp}_4(\mathbb{R})^+ = \{g \in \mathrm{GSp}_4(\mathbb{R}) \mid \nu(g) > 0\}.$$

Then

$$\mathrm{GSp}_4(\mathbb{R})^+ \cong \mathrm{Sp}_4(\mathbb{R}) \times \mathbb{R}_{>0},$$

and each discrete series representation V of $\mathrm{Sp}_4(\mathbb{R})$ can be extended to a representation V_+ of $\mathrm{GSp}_4(\mathbb{R})^+$ by letting the $\mathbb{R}_{>0}$ component act trivially. Then we can induce to $\mathrm{GSp}_4(\mathbb{R})$ to get a representation $\widetilde{V}$,

$$\widetilde{V} = \mathrm{Ind}_{\mathrm{GSp}_4(\mathbb{R})^+}^{\mathrm{GSp}_4(\mathbb{R})}(V_+).$$

As a representation of $\mathrm{GSp}_4(\mathbb{R})^+$, $\widetilde{V}$ splits as

$$\widetilde{V} \cong V_+ \oplus V_+^\vee,$$

with k_0 switching between the two summands. It follows that, up to twists, the discrete series representations of $\mathrm{GSp}_4(\mathbb{R})$ are parametrized by orbits of certain analytically integral weights under the action of the four element subgroup

$$\{1, w_\alpha, w_{\alpha+\beta}, w_{-1}\} \subset W,$$

and the discrete series representations obtained in the same manner as $\widetilde{V}$, without twisting, are self-dual.

1.4.2 Near equivalence and induced representations

In this section we introduce the induced representations whose Langlands quotients we will be interested in. These representations will be induced from the maximal parabolics of GSp_4, and when computing the Eisenstein multiplicity of their Langlands quotients it will be enough, by multiplicity one theorems, to distinguish them up to near equivalence.

Now by Theorem 1.2.2.3, the pieces of the Franke filtration that can contribute to Eisenstein cohomology are those which are induced from a cuspidal representation of a Levi subgroup which itself has cohomology. For the Levis of the maximal parabolic subgroups of GSp_4, which are both isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_1$, such representations are given by pairs (F, ψ), where F is a holomorphic cuspidal eigenform form of weight at least 2, and ψ is a Dirichlet character.

For such a pair (F, ψ), let us view ψ as a character of $\mathrm{GL}_1(\mathbb{A})$ in the usual way, and let us write $\tilde{\pi}_F$ for the unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ attached to F. Write k for the weight of F and ω_F for the nebentypus. Then ω_F is identified with the central character of $\tilde{\pi}_F$.

The archimedean component $\tilde{\pi}_{F,\infty}$ of $\tilde{\pi}_F$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight k with trivial character on the central $\mathbb{R}_{>0}$. This representation is the sum of the two discrete series representations of $\mathrm{SL}_2(\mathbb{R})$ with Harish-Chandra parameters $\pm(k-1)$, and occurs in the cohomology of the representation of GL_2 of highest weight $k-2$.

Now we have the representation $\tilde{\pi}_F \boxtimes \psi$ of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GL}_1(\mathbb{A})$; the symbol $\boxtimes$ here is meant to signal that this is an exterior tensor product. We identify the maximal Levis M_α and M_β with $\mathrm{GL}_2 \times \mathrm{GL}_1$ via the isomorphisms of (1.4.1.2). Let $\delta_{P_\alpha(\mathbb{A})}$ and $\delta_{P_\beta(\mathbb{A})}$ be the respective modulus characters of $P_\alpha(\mathbb{A})$ and $P_\beta(\mathbb{A})$. We note that for $A \in \mathrm{GL}_2(\mathbb{A})$ and $t \in \mathrm{GL}_1(\mathbb{A})$, we have

$$\delta_{P_\alpha(\mathbb{A})}(A, t) = |\det(A)|^2 |t|^{-4}, \qquad \delta_{P_\beta(\mathbb{A})}(A, t) = |\det(A)|^3 t^{-3}. \tag{1.4.2.1}$$

Now let $s \in \mathbb{C}$. We define the normalized induced representations

$$\iota_{P_\gamma(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_F \boxtimes \psi, s) = \mathrm{Ind}_{P_\gamma(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\tilde{\pi}_F \boxtimes \psi) \otimes \delta_{P_\gamma}^{s+1/2}), \qquad \gamma \in \{\alpha, \beta\}. \tag{1.4.2.2}$$

These representations are trivial on $A_{\mathrm{GSp}_4}(\mathbb{R})^\circ$.

Proposition 1.4.2.1. *Let $\gamma \in \{\alpha, \beta\}$ be one of the simple roots of GSp_4. Let F, F' be holomorphic cuspidal eigenforms, let ψ, ψ' be Dirichlet characters, and let $s, s' \in \mathbb{R}_{>0}$. If there are irreducible subquotients*

$$\Pi \text{ of } \iota_{P_\gamma(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_F \boxtimes \psi, s)$$

and

$$\Pi' \text{ of } \iota_{P_\gamma(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_{F'} \boxtimes \psi', s')$$

such that Π and Π' are nearly equivalent, then $\tilde{\pi}_F = \tilde{\pi}_{F'}$, $\psi = \psi'$, and $s = s'$.

Proof. We will prove this proposition for the Siegel parabolic P_β; the proof in the Klingen case is analogous.

Let S be a finite set of places, including the archimedean place, such that for $p \notin S$, the local components Π_p and Π'_p are unramified and isomorphic. Then for such p, we have in particular that $\tilde{\pi}_{F,p}$ and $\tilde{\pi}_{F',p}$ are unramified, and so are ψ_p and ψ'_p. Write T_2 for the standard diagonal torus of GL_2 and B_2 for the standard upper triangular Borel in GL_2. Then we know that there are unramified characters χ_1, χ_2 of $\mathbb{Q}_p^\times$ such that $\tilde{\pi}_{F,p}$ is the unramified subquotient of

$$\mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi_1 \boxtimes \chi_2) \otimes \delta_{B_2(\mathbb{Q}_p)}^{1/2}),$$

where $\chi_1 \boxtimes \chi_2$ is the character of $T_2(\mathbb{Q}_p)$ defined by

$$(\chi_1 \boxtimes \chi_2)(\mathrm{diag}(x, y)) = \chi_1(x)\chi_2(y)$$

and $\delta_{B_2(\mathbb{Q}_p)}$ is the usual modulus character of $B(\mathbb{Q}_p)$. Similarly, there are also unramified characters χ'_1, χ'_2 of $\mathbb{Q}_p^\times$ such that $\tilde{\pi}_{F',p}$ is the unramified subquotient of

$$\mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi'_1 \boxtimes \chi'_2) \otimes \delta_{B_2(\mathbb{Q}_p)}^{1/2}).$$

Furthermore, by temperedness, we know that $\chi_1, \chi_2, \chi_1', \chi_2'$ are all unitary.

For $x, y, t \in \mathbb{Q}_p^\times$, consider the element of $T(\mathbb{Q}_p)$ given in $M_\beta \cong \mathrm{GL}_2 \times \mathrm{GL}_1$ by $(\mathrm{diag}(x, y), t)$. Write $\chi_1 \boxtimes \chi_2 \boxtimes \psi_p$ for the character of $T(\mathbb{Q}_p)$ given on such elements by

$$(\chi_1 \boxtimes \chi_2 \boxtimes \psi_p)(\mathrm{diag}(x, y), t) = \chi_1(x)\chi_2(y)\psi_p(t).$$

By induction in stages, we have that Π_p is the unramified subquotient of

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GSp}_4(\mathbb{Q}_p)}((\chi_1 \boxtimes \chi_2 \boxtimes \psi_p) \otimes \delta_{P_\beta(\mathbb{Q}_p)}^s \otimes \delta_{B(\mathbb{Q}_p)}^{1/2}),$$

and similarly Π_p' is the unramified subquotient of

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GSp}_4(\mathbb{Q}_p)}((\chi_1' \boxtimes \chi_2' \boxtimes \psi_p') \otimes \delta_{P_\beta(\mathbb{Q}_p)}^{s'} \otimes \delta_{B(\mathbb{Q}_p)}^{1/2}).$$

By the theory of the Satake isomorphism recalled in Section 1.3.1, since $\Pi_p \cong \Pi_p'$, the characters

$$(\chi_1 \boxtimes \chi_2 \boxtimes \psi_p) \otimes \delta_{B(\mathbb{Q}_p)}^s \quad \text{and} \quad (\chi_1' \boxtimes \chi_2' \boxtimes \psi_p') \otimes \delta_{B(\mathbb{Q}_p)}^{s'}$$

are equal up to the Weyl group W; that is, there is a $w \in W$ such that for all $x, y, t \in \mathbb{Q}_p^\times$, we have

$$((\chi_1 \boxtimes \chi_2 \boxtimes \psi_p) \otimes \delta_{P_\beta(\mathbb{Q}_p)}^s)((\mathrm{diag}(x, y), t)^w) = ((\chi_1' \boxtimes \chi_2' \boxtimes \psi_p') \otimes \delta_{P_\beta(\mathbb{Q}_p)}^{s'})(\mathrm{diag}(x, y), t). \quad (1.4.2.3)$$

First, let us take the absolute value of both sides of (1.4.2.3). Since all characters involved except $\delta_{P_\beta(\mathbb{Q}_p)}$ are unitary, this gives

$$\delta_{P_\beta(\mathbb{Q}_p)}^s((\mathrm{diag}(x, y), t)^w) = \delta_{P_\beta(\mathbb{Q}_p)}^{s'}(\mathrm{diag}(x, y), t), \quad (1.4.2.4)$$

By the local analogue of (1.4.2.1), this becomes

$$\delta_{P_\beta(\mathbb{Q}_p)}^s((\mathrm{diag}(x, y), t)^w) = |xy|^{3s'}|t|^{-3s'}. \quad (1.4.2.5)$$

Now we compute, using (1.4.1.10), the following identities

$$(\mathrm{diag}(x,y),t)^{w_\alpha} = (\mathrm{diag}(x,ty^{-1}),t), \qquad (\mathrm{diag}(x,y),t)^{w_{\alpha\beta}} = (\mathrm{diag}(y,tx^{-1}),t),$$
$$(\mathrm{diag}(x,y),t)^{w_{\beta\alpha}} = (\mathrm{diag}(ty^{-1},x),t), \qquad (\mathrm{diag}(x,y),t)^{w_{\alpha\beta\alpha}} = (\mathrm{diag}(ty^{-1},tx^{-1}),t),$$
$$(\mathrm{diag}(x,y),t)^{w_{\beta\alpha\beta}} = (\mathrm{diag}(tx^{-1},y),t), \qquad (\mathrm{diag}(x,y),t)^{w_{-1}} = (\mathrm{diag}(tx^{-1},ty^{-1}),t).$$

From these identities follow

$$\delta^s_{P_\beta(\mathbb{Q}_p)}((\mathrm{diag}(x,y),t)^{w_\alpha}) = |xy^{-1}|^{3s}, \qquad \delta^s_{P_\beta(\mathbb{Q}_p)}((\mathrm{diag}(x,y),t)^{w_{\alpha\beta}}) = |x^{-1}y|^{3s},$$
$$\delta^s_{P_\beta(\mathbb{Q}_p)}((\mathrm{diag}(x,y),t)^{w_{\beta\alpha}}) = |xy^{-1}|^{3s}, \qquad \delta^s_{P_\beta(\mathbb{Q}_p)}((\mathrm{diag}(x,y),t)^{w_{\alpha\beta\alpha}}) = |xy|^{-3s}|t|^{3s},$$
$$\delta^s_{P_\beta(\mathbb{Q}_p)}((\mathrm{diag}(x,y),t)^{w_{\beta\alpha\beta}}) = |x^{-1}y|^{3s}, \qquad \delta^s_{P_\beta(\mathbb{Q}_p)}((\mathrm{diag}(x,y),t)^{w_{-1}}) = |xy|^{-3s}|t|^{3s}.$$

Letting $(x,y,t) = (1,1,p)$ in the equations above and using (1.4.2.5) then gives

$$p^{3s'} = \begin{cases} 1 & \text{if } w \in \{w_\alpha, w_{\alpha\beta}, w_{\beta\alpha}, w_{\alpha\beta\alpha}\}; \\ p^{-3s} & \text{if } w \in \{w_{\alpha\beta\alpha}, w_{-1}\}. \end{cases}$$

Since $s, s' > 0$, this is impossible, which forces $w = 1$ or $w = w_\beta$. In either case, an analogous computation as above then gives

$$p^{3s'} = p^{3s},$$

from which we conclude $s = s'$. Then we can cancel the modulus characters in (1.4.2.3) and get

$$(\chi_1 \boxtimes \chi_2 \boxtimes \psi_p)((\mathrm{diag}(x,y),t)^w) = (\chi_1' \boxtimes \chi_2' \boxtimes \psi_p')(\mathrm{diag}(x,y),t), \qquad \text{for some } w \in \{1, w_\beta\}.$$

In the case that $w = 1$, we conclude that $\chi_1 = \chi_1'$, $\chi_2 = \chi_2'$, and $\psi_p = \psi_p'$. If instead $w = w_\beta$, then

$$(\mathrm{diag}(x,y),t)^w = (\mathrm{diag}(y,x),t),$$

and we conclude $\chi_1 = \chi_2'$, $\chi_2 = \chi_1'$, and $\psi_p = \psi_p'$. In either case we have $\psi_p = \psi_p'$ and that

$$\mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi_1 \boxtimes \chi_2) \otimes \delta^{1/2}_{B_2(\mathbb{Q}_p)}) \quad \text{and} \quad \mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi_1' \boxtimes \chi_2') \otimes \delta^{1/2}_{B_2(\mathbb{Q}_p)})$$

have the same unramified subquotients, which means $\tilde{\pi}_{F,p} \cong \tilde{\pi}_{F',p}$. Since this is true for any $p \notin S$, strong multiplicity one for GL_2 (and GL_1) finishes the proof. □

Next we want to distinguish between representations induced from different standard parabolics, including the Borel. So let us first describe how we will induce characters from the torus T.

First identify T with $(\mathrm{GL}_1)^3$ via the map (1.4.1.7). Let ψ_1, ψ_2, ψ_3 be Dirichlet characters, viewed as characters of $\mathrm{GL}_1(\mathbb{A})$. Write $\psi_1 \boxtimes \psi_2 \boxtimes \psi_3$ for the character of $T(\mathbb{A})$ given by

$$(\psi_1 \boxtimes \psi_2 \boxtimes \psi_3)(t_1, t_2, t_3) = \psi_1(t_1)\psi_2(t_2)\psi_3(t_3).$$

Let $\delta_{B(\mathbb{A})}$ be the modulus character of $B(\mathbb{A})$. When restricted to $T(\mathbb{A})$, this gives

$$\delta_{B(\mathbb{A})}(t_1, t_2, t_3) = |t_1|^4 |t_2|^2 |t_3|^{-3}.$$

More generally, for $s_1, s_2 \in \mathbb{C}$, we will consider the character of $B(\mathbb{A})$ given by

$$e^{\langle H_B(\cdot), s_1\alpha + s_2\beta \rangle}.$$

If $\rho = \frac{1}{2}(3\alpha + 4\beta)$ is half the sum of the positive roots, then we have

$$\delta_{B(\mathbb{A})}^{1/2} = e^{\langle H_B(\cdot), \rho \rangle}.$$

We write

$$\iota_{B(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\psi_1 \boxtimes \psi_2 \boxtimes \psi_3; s_1, s_2) = \mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\psi_1 \boxtimes \psi_2 \boxtimes \psi_3) \otimes e^{\langle H_B(\cdot), s_1\alpha + s_2\beta + \rho \rangle}) \tag{1.4.2.6}$$

for the normalized induction.

To distinguish between representations induced from different parabolics, we will attach to them Galois representations and distinguish between those. The next three propositions will do this for B, M_α, and M_β, respectively.

Fix any prime ℓ and fix an isomorphism of $\mathbb{C}$ with $\overline{\mathbb{Q}}_\ell$.

Proposition 1.4.2.2. *Let ψ_1, ψ_2, ψ_3 be Dirichlet characters, and let $m_1, m_2 \in \mathbb{Z}$. Let Π be an* *irreducible subquotient of*

$$\iota_{B(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\psi_1 \boxtimes \psi_2 \boxtimes \psi_3; m_1/2, m_2).$$

Let j_{T,GSp_4} *be the inclusion* $T \hookrightarrow \mathrm{GSp}_4$*. Then* $\Pi \otimes |\nu|^{m_1/2}$ *has attached to it the Galois representation* $G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$ *given by*

$$j_{T,\mathrm{GSp}_4} \circ \left((\psi_1\psi_2\psi_3\chi_{\mathrm{cyc}}^{m_2}) \times (\psi_1\psi_3\chi_{\mathrm{cyc}}^{m_1-m_2}) \times (\psi_1\psi_2\psi_3^2)\right),$$

where we have viewed ψ_1, ψ_2, ψ_3 *as Galois characters via class field theory.*

Proof. Let p be a prime different from ℓ which is unramified for Π, and hence which not divide the conductors of the ψ_i's. Let $\lambda_i = \psi_i(p)$ for $i = 1, 2, 3$. (This is the Satake parameter of ψ_i at p.) Then the character

$$(\psi_1 \boxtimes \psi_2 \boxtimes \psi_3) \otimes e^{\langle H_B(\cdot),(m_1/2)\alpha+m_2\beta\rangle}$$

of $GL_1(\mathbb{A})^3$ has Satake parameter at p

$$(p^{-m_2}\lambda_1, p^{-(m_1-m_2)}\lambda_2, p^{m_1/2}\lambda_3) \in \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell)^3.$$

Therefore

$$(\psi_1 \boxtimes \psi_2 \boxtimes \psi_3) \otimes e^{\langle H_B(\cdot),(m_1/2)\alpha+m_2\beta\rangle} \otimes |\nu|^{m_1/2} \tag{1.4.2.7}$$

has Satake parameter at p

$$(p^{-m_2}\lambda_1, p^{-(m_1-m_2)}\lambda_2, \lambda_3).$$

When identifying $(\mathrm{GL}_1)^3$ with T on the dual side via the map φ_B of (1.4.1.8) and (1.4.1.9), this implies that the character (1.4.2.7) has attached to it the Galois representation into $T(\overline{\mathbb{Q}}_\ell)$ given by

$$(\psi_1\psi_2\psi_3\chi_{\mathrm{cyc}}^{m_2}) \times (\psi_1\psi_3\chi_{\mathrm{cyc}}^{m_1-m_2}) \times (\psi_1\psi_2\psi_3^2) : G_{\mathbb{Q}} \to T(\overline{\mathbb{Q}}_\ell)$$

Now we can pass the similitude twist inside the induction and get that $\Pi \otimes |\nu|^{m_1/2}$ is a subquotient of the normalized induction of the character (1.4.2.7), whence an appeal to Proposition 1.3.2.1 finishes the proof. $\square$

Proposition 1.4.2.3. *Let F be a holomorphic cuspidal eigenform of weight k with central character ω_F, and let ψ be a Dirichlet character. Let $m \in \mathbb{Z}$, and let Π be any irreducible subquotient of*

$$\iota_{P_\alpha(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_F \boxtimes \psi, m/4).$$

Let $j_{M_\beta,\mathrm{GSp}_4}$ be the inclusion $M_\beta \hookrightarrow \mathrm{GSp}_4$. Then $\Pi \otimes |\nu|^{(k-1-m)/2}$ has attached to it the Galois representation $G_\mathbb{Q} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$ given by

$$j_{M_\beta,\mathrm{GSp}_4} \circ (\rho_F \times \psi\omega_F\chi_{\mathrm{cyc}}^{k-1-m}),$$

where ρ_F is the Galois representation attached to F by Eichler–Shimura, Deligne, and Deligne–Serre (Theorem 1.3.1.3), and ω_F and ψ are identified with Galois characters via class field theory.

Proof. The proof will be similar to the previous proposition. Let p be a prime different from ℓ which is unramified for Π, and hence which is unramified for $\tilde{\pi}_F$ and ψ. Let $\mathrm{diag}(\lambda_1, \lambda_2) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a diagonal representative of the Satake parameter of $\tilde{\pi}_F$ at p, and let $\lambda_3 = \psi(p)$. Then the automorphic representation of $M_\alpha(\mathbb{A})$ given by

$$(\tilde{\pi}_F \boxtimes \psi) \otimes \delta_{P_\alpha(\mathbb{A})}^{m/4}$$

has Satake parameter at p represented by

$$(p^{-m/2}\,\mathrm{diag}(\lambda_1, \lambda_2), p^m\lambda_3) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \times \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell),$$

by (1.4.2.1). Thus

$$(\tilde{\pi}_F \boxtimes \psi) \otimes \delta_{P_\alpha(\mathbb{A})}^{m/4} \otimes |\nu|^{(k-1-m)/2} \tag{1.4.2.8}$$

has Satake parameter at p represented by

$$(p^{-(k-1)/2}\,\mathrm{diag}(\lambda_1, \lambda_2), p^m\lambda_3) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \times \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell),$$

because $|\nu(A,t)| = |\det(A)|$ for $(A,t) \in M_\alpha(\mathbb{A})$.

Now we pass through the map φ_α of (1.4.1.3) and (1.4.1.4) to get that the representation of

(1.4.2.8) has Satake parameter represented by

$$(p^{-(k-1)/2}\operatorname{diag}(\lambda_1,\lambda_2), p^{-(k-1-m)}\lambda_1\lambda_2\lambda_3) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)\times\mathrm{GL}_1(\overline{\mathbb{Q}}_\ell),$$

and therefore has the Galois representation $G_{\mathbb{Q}}\to M_\beta(\overline{\mathbb{Q}}_\ell)$ given by

$$\rho_F \times \psi\omega_F\chi_{\mathrm{cyc}}^{k-1-m}$$

attached to it. Thus we are done by Proposition 1.3.2.1. $\square$

Proposition 1.4.2.4. *Let F be a holomorphic cuspidal eigenform of weight k, and let ψ be a Dirichlet character. Let $m\in\mathbb{Z}$, and assume $m\equiv k-1 \pmod 2$. Let Π be any irreducible subquotient of*

$$\iota_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_F\boxtimes\psi, m/6).$$

Let $j_{M_\alpha,\mathrm{GSp}_4}$ be the inclusion $M_\alpha\hookrightarrow\mathrm{GSp}_4$. Then $\Pi\otimes|\nu|^{(k-1)/2}$ has attached to it the Galois representation $G_{\mathbb{Q}}\to\mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$ given by

$$j_{M_\alpha,\mathrm{GSp}_4}\circ((\rho_F\otimes\psi)\times\psi\chi_{\mathrm{cyc}}^{(k-1-m)/2}),$$

where ρ_F is the Galois representation attached to F by Eichler–Shimura, Deligne, and Deligne–Serre (Theorem 1.3.1.3), and ψ is identified with a Galois character via class field theory.

Proof. The proof will again be very similar to the previous two propositions. Let p be a prime different from ℓ which is unramified for Π, and hence which is unramified for $\tilde{\pi}_F$ and ψ. Let $\operatorname{diag}(\lambda_1,\lambda_2)\in\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a diagonal representative of the Satake parameter of $\tilde{\pi}_F$ at p, and let $\lambda_3=\psi(p)$. Then the automorphic representation of $M_\alpha(\mathbb{A})$ given by

$$(\tilde{\pi}_F\boxtimes\psi)\otimes\delta_{P_\beta(\mathbb{A})}^{m/6}$$

has Satake parameter at p represented by

$$(p^{-m/2}\operatorname{diag}(\lambda_1,\lambda_2), p^{m/2}\lambda_3)\in\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)\times\mathrm{GL}_1(\overline{\mathbb{Q}}_\ell),$$

by (1.4.2.1). Thus

$$(\tilde{\pi}_F \boxtimes \psi) \otimes \delta_{P_\alpha(\mathbb{A})}^{m/4} \otimes |\nu|^{(k-1)/2} \tag{1.4.2.9}$$

has Satake parameter at p represented by

$$(p^{-m/2} \operatorname{diag}(\lambda_1, \lambda_2), p^{-(k-1-m)/2}\lambda_3) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \times \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell),$$

because $|\nu(A,t)| = |t|$ for $(A,t) \in M_\beta(\mathbb{A})$.

Now we pass through the map φ_β of (1.4.1.5) and (1.4.1.6) to get that the representation of (1.4.2.9) has Satake parameter represented by

$$(p^{-(k-1)/2}\lambda_3 \operatorname{diag}(\lambda_1, \lambda_2), p^{-(k-1-m)}\lambda_3) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \times \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell),$$

and therefore has the Galois representation $G_\mathbb{Q} \to M_\alpha(\overline{\mathbb{Q}}_\ell)$ given by

$$\rho_F \times \psi\omega_F\chi_{\mathrm{cyc}}^{(k-1-m)/2}$$

attached to it. Thus we are done once again by Proposition 1.3.2.1. $\square$

Proposition 1.4.2.5. *Let F_α, F_β be two holomorphic cuspidal eigenforms of weights k_α and k_β, respectively. Let $\psi_\alpha, \psi_\beta, \psi_1, \psi_2, \psi_3$ be Dirichlet characters, and let $m_\alpha, m_\beta, m_1, m_2 \in \mathbb{Z}$. Assume that $m_\beta \equiv k_\beta - 1 \pmod 2$. Then given any irreducible subquotients*

$$\Pi_\alpha \ \textit{of} \ \iota_{P_\alpha(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_{F_\alpha} \boxtimes \psi_\alpha, m_\alpha/4)$$

and

$$\Pi_\beta \ \textit{of} \ \iota_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_{F_\beta} \boxtimes \psi_\beta, m_\beta/6)$$

and

$$\Pi_0 \ \textit{of} \ \iota_{B(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\psi_1 \boxtimes \psi_2 \boxtimes \psi_3; m_1/2, m_2),$$

we have that no two of Π_α, Π_β, and Π_0 are nearly equivalent.

Proof. We first prove the proposition in the case that

$$k_\alpha - 1 - m_\alpha \equiv k_\beta - 1 \equiv m_1 \pmod 2). \tag{1.4.2.10}$$

Assume moreover that the quantities of (1.4.2.10) are all even. Then Propositions 1.4.2.3, 1.4.2.4, and 1.4.2.2 attach to Π_α, Π_β, and Π_0, respectively, a Galois representation (which, by the parity assumption just made, are central twists by an integral power of the cyclotomic character of the Galois representations from those propositions). Let ρ_α, ρ_β, and ρ_0, respectively, be these Galois representations. Denote by Std the standard representation of GSp_4 into GL_4 which we used to define the group GSp_4. Then we have the following formulas for our Galois representations when composed with Std:

$$\mathrm{Std}(\rho_\alpha \otimes \chi_{\mathrm{cyc}}^{(k_\alpha-1-m_\alpha)/2}) = \rho_{F_\alpha} \oplus (\rho_{F_\alpha}^\vee \otimes (\psi_\alpha \omega_{F_\alpha} \chi_{\mathrm{cyc}}^{k_\alpha-1-m_\alpha}));$$

$$\mathrm{Std}(\rho_\beta \otimes \chi_{\mathrm{cyc}}^{(k_\beta-1}{)/2}) = (\rho_{F_\beta} \otimes \psi_\beta) \oplus (\omega_{F_\beta} \psi_\beta \chi_{\mathrm{cyc}}^{(k_\beta-1+m_\beta)}) \oplus (\psi_\beta \chi_{\mathrm{cyc}}^{(k_\beta-1-m_\beta)/2});$$

$$\mathrm{Std} \circ (\rho_0 \otimes \chi_{\mathrm{cyc}}^{m_1/2}) = (\psi_1\psi_2\psi_3\chi_{\mathrm{cyc}}^{m_2}) \oplus (\psi_1\psi_3\chi_{\mathrm{cyc}}^{m_1-m_2}) \oplus (\psi_3\chi_{\mathrm{cyc}}^{-m_2}) \oplus (\psi_2\psi_3^2\chi_{\mathrm{cyc}}^{m_2-m_1}).$$

These follow from the usual formulas we use to include the Levis M_α, M_β, and T into GSp_4; here, of course, ω_{F_α} is the nebentypus of F_α and all Dirichlet characters are identified with Galois characters via class field theory.

Now since ρ_{F_α} and ρ_{F_β} are irreducible, the three representations above (and any twist of them) are semisimple. In particular, $\mathrm{Std} \circ \rho_\alpha$ is the sum of two irreducible representations, $\mathrm{Std} \circ \rho_\beta$ is the sum of three, and $\mathrm{Std} \circ \rho_0$ is the sum of four. Therefore these representations are pairwise non-isomorphic, and we can now appeal to Proposition 1.3.1.6 to conclude when the quantities of (1.4.2.10) are all even.

On the other hand, when the quantities of (1.4.2.10) are all odd, we can just apply a completely analogous argument to attach Galois representations to the twisted representations $\Pi_\alpha \otimes |\nu|^{1/2}$, $\Pi_\beta \otimes |\nu|^{1/2}$, and $\Pi_0 \otimes |\nu|^{1/2}$, and we conclude in this case too.

Finally, assume that one of the quantities in (1.4.2.10) is even and another is odd. Let Π be the representation of Π_α, Π_β, and Π_0 corresponding to the even quantity, and let Π' be the one corresponding to the odd quantity. Then, as we just saw, Π has attached to it a Galois representation $\rho_\Pi : G_\mathbb{Q} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$ such that $\mathrm{Std} \circ \rho_\Pi$ is semisimple and Hodge-Tate. But Π' may not have a Galois representation attached to it; we only know that $\Pi' \otimes |\nu|^{1/2}$ does. If there is no such Galois representation, then we are done. Otherwise, it does have a Galois representation, call it $\rho_{\Pi'}$, and we may assume ℓ is odd. We then restrict to $G_{\mathbb{Q}(\zeta_\ell)}$, where ζ_ℓ is a primitive ℓth root of unity. Then χ_{cyc} has a square root, and the representation of $G_{\mathbb{Q}(\zeta_\ell)}$ given by $(\rho_{\Pi'} \otimes \chi_{\mathrm{cyc}}^{1/2})^{\mathrm{ss}}$ must be the restriction to $G_{\mathbb{Q}(\zeta_\ell)}$ of the Galois representation attached to $\Pi' \otimes |\nu|^{1/2}$; they are both semisimple and their traces agree on Frobenius elements Frob_p with $p \equiv 1 \pmod{\ell}$. But since the Galois representation attached to $\Pi' \otimes |\nu|^{1/2}$ is Hodge–Tate, $\rho_{\Pi'}$ cannot be, and this distinguishes Π' from Π, as desired. □

Remark 1.4.2.6. Some of the assumptions above on the parameters in the proof may look strange at first, but there is an explanation for them. If one computes exactly which representations induced from M_α, M_β, and T can have cohomology for a given representation E of $\mathrm{GSp}_4(\mathbb{C})$, their parameters will satisfy (1.4.2.10). More precisely, if E has highest weight Λ, and if the representations Π_α, Π_β, and Π_0 of the proposition appear in the cohomology of E, then the quantities of (1.4.2.10) are all even if $\Lambda + \rho$ is in the integral span of the root lattice, and they are all odd otherwise. In either of these cases, the quantity $k_\beta - 1 - m_\beta$ is always even.

This is to be expected for the following reason. In the case that $\Lambda + \rho$ is in the integral span of the root lattice, the Galois representations attached to the automorphic representations appearing in the cohomology of E should be de Rham with Hodge–Tate weights given by the cocharacter of $T^\vee$ corresponding to the infinitesimal character of these automorphic representations at infinity. This infinitesimal character must then match that of E, and is therefore given by the integral parameter $\Lambda + \rho$. In the case of Π_α, Π_β, and Π_0, these Galois representations are described up to a twist by a power of the cyclotomic character respectively by Propositions 1.4.2.3, 1.4.2.4, and 1.4.2.2, and because the quantities of (1.4.2.10) are all even, the power we are twisting by is integral. Are applying that twist, the Hodge–Tate weights of these Galois representations will match the cocharacter of $T^\vee$ given by $\Lambda + \rho$.

On the other hand, if $\Lambda + \rho$ is not in the integral span of the root lattice, then the automorphic representations appearing in the cohomology of E only have associated Galois representations (at least ones which are de Rham) after twisting by a half power of the similitude character. Correspondingly, since the quantities of (1.4.2.10) are all odd in this case, the representations Π_α, Π_β, and Π_0 must also be twisted by a half power of the similitude character to obtain nice Galois representations.

1.4.3 Eisenstein multiplicity of Langlands quotients

In this section we introduce the Langlands quotients we are interested in and compute their multiplicities in Eisenstein cohomology. Before we do that, however, let us compute the cohomology of certain induced representations of the kind considered in Theorem 1.2.2.3. We do this in the next proposition for representations induced from the Siegel parabolic of GSp_4.

In what follows, we will be considering the $(\mathfrak{g}_0, K_\infty^\circ)$-cohomology of representations when $G = \mathrm{GSp}_4$. In this case we have $\mathfrak{g}_0 = \mathfrak{sp}_4$, the complexified Lie algebra of Sp_4. As discussed in Section 1.4.1, the group K_∞ has two connected components, and so the cohomology spaces we obtain will be modules for the two element group of components of K_∞, as well as for the group $\mathrm{GSp}_4(\mathbb{A}_f)$.

We will also consider the normalized induction functors $\iota_{P(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}$, for P a standard parabolic, defined in (1.4.2.2) and (1.4.2.6), and also their finite adelic analogues $\iota_{P(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}$ which are defined similarly.

The following proposition is essentially proved by Grbac and Grobner in [GG13], Proposition 4.2, using the same techniques as the ones we use. The main differences are that Grbac and Grobner work with Sp_4 instead of GSp_4, which is not a serious difference, and that they also obtain results for totally real fields instead of just $\mathbb{Q}$. Actually, we have set things up so that it is possible to use the results in this section to obtain results over totally real fields as well, but we are content with working over $\mathbb{Q}$ for simplicity.

Proposition 1.4.3.1. *Let E be an irreducible, finite dimensional representation of $\mathrm{GSp}_4(\mathbb{C})$, and say that E has highest weight $\tilde{\Lambda}$. Let $\Lambda = \tilde{\Lambda}|_{T_0}$, so that there are $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ such that*

$$\Lambda = \frac{c_1}{2}(\alpha + 2\beta) + c_2(\alpha + \beta).$$

Let F be a holomorphic cuspidal eigenform of weight k and trivial nebentypus, and let $s \in \mathbb{C}$ with

$\mathrm{Re}(s) \geq 0$. *Assume*

$$H^i(\mathfrak{sp}_4, K_\infty^\circ; \mathrm{Ind}_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\tilde{\pi}_F \boxtimes 1) \otimes \mathrm{Sym}(\mathfrak{a}_{P_\beta,0})_{(2s+1)\rho_{P_\beta}}) \otimes E) \neq 0.$$

Then either:

(i) We have

$$i = 3, \qquad k = c_1 + 2c_2 + 4, \qquad s = \frac{c_1 + 1}{6},$$

and

$$H^3(\mathfrak{sp}_4, K_\infty^\circ; \mathrm{Ind}_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\tilde{\pi}_F \boxtimes 1) \otimes \mathrm{Sym}(\mathfrak{a}_{P_\beta,0})_{(2s+1)\rho_{P_\beta}}) \otimes E) \\ \cong \iota_{P_\beta(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}(\tilde{\pi}_{F,f} \boxtimes 1, (c_1 + 1)/6),$$

or,

(ii) We have

$$i = 4, \qquad k = c_1 + 2, \qquad s = \frac{c_1 + 2c_2 + 3}{6},$$

and

$$H^4(\mathfrak{sp}_4, K_\infty^\circ; \mathrm{Ind}_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\tilde{\pi}_F \boxtimes 1) \otimes \mathrm{Sym}(\mathfrak{a}_{P_\beta,0})_{(2s+1)\rho_{P_\beta}}) \otimes E) \\ \cong \iota_{P_\beta(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}(\tilde{\pi}_{F,f} \boxtimes 1, (c_1 + 2c_2 + 3)/6).$$

In both cases the cohomology spaces have the trivial action of the component group of K_∞.

Proof. We will apply Theorem 1.2.2.3 to our present situation with $\pi = (\tilde{\pi}_F \boxtimes 1) \otimes \delta_{P_\beta(\mathbb{A})}^s$ and $\mathfrak{h} = \mathfrak{t}$, the complexified Lie algebra of T. In fact, it suffices to do all our computations restricted to the complexified Lie algebra $\mathfrak{t}_0$ of T_0, which is a Cartan subalgebra of $\mathfrak{sp}_4$. We have

$$W^{P_\beta} = \{1, w_\alpha, w_{\alpha\beta}, w_{\alpha\beta\alpha}\},$$

and one readily computes

$$\begin{aligned} -(\Lambda+\rho) &= -(c_1+1)\frac{\beta}{2} - (c_1+2c_2+3)\frac{\alpha+\beta}{2}, \\ -w_\alpha(\Lambda+\rho) &= -(c_1+2c_2+3)\frac{\beta}{2} - (c_1+1)\frac{\alpha+\beta}{2}, \\ -w_{\alpha\beta}(\Lambda+\rho) &= -(c_1+2c_2+3)\frac{\beta}{2} + (c_1+1)\frac{\alpha+\beta}{2}, \\ -w_{\alpha\beta\alpha}(\Lambda+\rho) &= -(c_1+1)\frac{\beta}{2} + (c_1+2c_2+3)\frac{\alpha+\beta}{2}. \end{aligned}$$

Note that we have a decomposition

$$\mathfrak{t}_0 = (\mathfrak{m}_{\beta,0} \cap \mathfrak{t}_0) \oplus \mathfrak{a}_{P_\beta,0},$$

and note also that $(\alpha+\beta)$ acts as zero on the first summand, while β acts as zero on the second.

Now by Theorem 1.2.2.3, in order for our cohomology space to be nontrivial, we need there to be a $w \in W^{P_\beta}$ with

$$-w(\Lambda+\rho)|_{\mathfrak{a}_{P_\beta},0} = 2s\rho_{P_\beta} = 6s\frac{\alpha+\beta}{2},$$

and

$$-w(\Lambda+\rho)|_{\mathfrak{m}_{\beta,0}} = \pm(k-1)\frac{\beta}{2}.$$

Therefore, because $\mathrm{Re}(s) \geq 0$, we see from the formulas for $-w(\Lambda+\rho)|_{\mathfrak{a}_{P_\beta},0}$ that w can only equal $w_{\alpha\beta}$ or $w_{\alpha\beta\alpha}$.

In the case that $w = w_{\alpha\beta}$, we obtain by matching coefficients that

$$k-1 = +(c_1+2c_2+3),$$

with this choice of sign because $k-1 \geq 0$, and

$$6s = c_1+1.$$

We have that the length $\ell(w_{\alpha\beta})$ of $w_{\alpha\beta}$ is 2. Also, since

$$\rho = \frac{\beta}{2} + 3\frac{\alpha+\beta}{2},$$

we have

$$(w_{\alpha\beta}(\Lambda+\rho)-\rho)|_{\mathfrak{m}_{\beta,0}} = (c_1+2c_2+2)\frac{\beta}{2} = (k-2)\frac{\beta}{2}.$$

Therefore, the isomorphism of Theorem 1.2.2.3 in our case is

$$\begin{aligned} &H^i(\mathfrak{sp}_4, K_\infty^\circ; \mathrm{Ind}_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\tilde{\pi}_F \boxtimes 1)\otimes \mathrm{Sym}(\mathfrak{a}_{P_\beta,0})_{(2s+1)\rho_{P_\beta}})\otimes E) \\ &\qquad \cong \iota_{P_\beta(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}(\tilde{\pi}_{F,f}\boxtimes 1, (c_1+1)/6)\otimes H^{i-2}(\mathfrak{m}_{\beta,0}, K_\infty^\circ \cap P_\beta(\mathbb{R}); (\tilde{\pi}_{F,\infty}\boxtimes 1)\otimes F_{k-2}),\end{aligned}$$

where F_{k-2} is the representation of $\mathfrak{m}_{\beta,0}$ of highest weight $(k-2)(\beta/2)$.

Now, since $k-1 = c_1+2c_2+3 > 0$, the representation $\tilde{\pi}_{F,\infty}$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight k, and therefore has nontrivial cohomology when tensored with F_{k-2} in degree 1 and degree 1 only. Since $K_\infty^\circ \cap \mathrm{GL}_2(\mathbb{R})$ is a maximal compact subgroup of $\mathrm{GL}_2(\mathbb{R})$ (instead of just being its identity component) the cohomology of $\tilde{\pi}_{F,\infty}$ in degree 1 is 1 dimensional (instead of being 2 dimensional). The claim (i) of our proposition is now immediate.

The computation which uses $w_{\alpha\beta\alpha}$ and which proves the claim (ii) of the proposition is completely similar, and we omit it. If instead we decided to take $(\mathfrak{sp}_4, K_\infty)$-cohomology, rather than $(\mathfrak{sp}_4, K_\infty^\circ)$-cohomology, then we would obtain the same results. This is because we decided to induce the trivial character on the GL_1 component of M_β, and the maximal compact subgroup $\{\pm 1\}$ of $\mathrm{GL}_1(\mathbb{R})$ acts trivially via this character. It follows that the component group of K_∞ acts trivially on the cohomology, and this finishes the proof. $\square$

Now fix F a holomorphic cuspidal eigenform of weight $k \geq 2$ and trivial nebentypus. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, let us write

$$\mathcal{L}_\beta(\tilde{\pi}_F, s) = \text{Langlands quotient of } \iota_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_F \boxtimes 1, s).$$

This notion was introduced just before Theorem 1.1.3.5.

The Langlands quotient $\mathcal{L}_\beta(\tilde{\pi}_F, s)$ is irreducible, and under a vanishing assumption on the L-function of $\tilde{\pi}_F$, we will calculate the multiplicity of the finite part $\mathcal{L}_\beta(\tilde{\pi}_F, s)_f$ in the Eisenstein cohomology of GSp_4. The following lemma will be key to this.

Lemma 1.4.3.2. *For any flat section* $\phi_s \in \iota_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}_F \boxtimes 1, s)$, *the Eisenstein series* $E(\phi, 2s\rho_{P_\beta})$

does not have a pole for $\mathrm{Re}(s) > 0$ *except perhaps if* $s = 1/6$*. If furthermore*

$$L(\tilde{\pi}_F, 1/2) = 0,$$

then $E(\phi, 2s\rho_{P_\beta})$ *is also holomorphic at* $s = 1/6$*.*

Proof. This is an easy consequence of what is done in the paper of Kim [Kim95], but let us quickly explain how this is proved, since we have set up the tools to do so already.

It suffices to prove the lemma for $\phi = \bigotimes_v \phi_v$ decomposable into local sections. Write $E(\phi, s) = E(\phi, 2s\rho_{P_\beta})$. By Theorem 1.1.1.1, the constant term of $E(\phi, s)$ along P_α (and hence along B) is zero, and the constant term along P_β is

$$E_{P_\beta}(\phi, s) = \phi_s + M(\phi, w_{\alpha\beta\alpha})_{-2s\rho_{P_\beta}}.$$

Then we apply Theorem 1.1.1.2; in our current setting the root γ of that theorem is β, and $\tilde{\beta} = \rho_{P_\beta}/3$, and adjusting for this gives

$$M(\phi, w_{\alpha\beta\alpha})_{-2s\rho_{P_\beta}} = \prod_{j=1}^{m} \frac{L^S(3js, \tilde{\pi}_F, R_i^\vee)}{L^S(3js+1, \tilde{\pi}_F, R_i^\vee)} \bigotimes_{v \notin S} \phi_{v,s}^{w_{\alpha\beta\alpha},\mathrm{sph}} \otimes \bigotimes_{v \in S} M_v(\phi_{v,s}, w_{\alpha\beta\alpha})_{-2s\rho_{P_\beta}},$$

where S is a finite set of places such that for $v \notin S$, $\phi_{v,s}$ is spherical, and $\phi_{v,s}^{w_{\alpha\beta\alpha},\mathrm{sph}}$ are certain spherical vectors. Also, the representations R_i of $M_\beta^\vee$ can be determined from the action of the Levi of P_α on its unipotent radical; there are two of them, and R_1 is the standard representation of GL_2, and R_2 is the determinant. Thus the quotient of L-functions is

$$\frac{L^S(3s, \tilde{\pi}_F)\zeta^S(6s)}{L^S(3s+1, \tilde{\pi}_F)\zeta^S(6s+1)}.$$

Now by Harish-Chandra, the local intertwining operators are all holomorphic for $\mathrm{Re}(s) > 0$ since $\tilde{\pi}_F$ is tempered. So we only have to worry about the poles and zeros of the L-functions in the quotient above. Again since $\mathrm{Re}(s) > 0$, the L-functions in the denominator do not vanish as they are in the range of convergence, and the only pole in the numerator comes from the ζ-function at $s = 1/6$. But if $L(\tilde{\pi}_F, 1/2) = 0$, this zero cancels with the pole from the ζ-function.

Since the poles of $E(\phi, s)$ are determined by the poles of the constant term at all standard proper parabolics, we are done. $\square$

We are now ready to put everything together and compute the Eisenstein multiplicity of $\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)$ for $\mathrm{Re}(s) > 0$. See Definition 1.2.1.1 for the definition of this multiplicity.

Theorem 1.4.3.3. *Let E be an irreducible representation of $\mathrm{GSp}_4(\mathbb{C})$, and say that E has highest weight $\tilde{\Lambda}$. Let $\Lambda = \tilde{\Lambda}|_{T_0}$, so that there are $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ such that*

$$\Lambda = \frac{c_1}{2}(\alpha + 2\beta) + c_2(\alpha + \beta).$$

Let F be a holomorphic cuspidal eigenform of weight k and trivial nebentypus, and let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. If $c_1 = 0$ and $k = 2c_2 + 4$, also assume that

$$L(\tilde{\pi}_F, 1/2) = 0.$$

Then

$$m^i_{[P_\beta]}(\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)_f, K_\infty^\circ, E) = \begin{cases} 1 & \text{if } i = 3,\ k = c_1 + 2c_2 + 4,\ s = (c_1+1)/6 \\ & \text{or if } i = 4,\ k = c_1 + 2,\ s = (c_1 + 2c_2 + 3)/6; \\ 0 & \text{otherwise}; \end{cases}$$

and

$$m^i_{[P_\alpha]}(\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)_f, K_\infty^\circ, E) = m^i_{[B]}(\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)_f, K_\infty^\circ, E) = 0.$$

Therefore we also have

$$m^i_{\mathrm{Eis}}(\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)_f, K_\infty^\circ, E) = m^i_{[P_\beta]}(\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)_f, K_\infty^\circ, E).$$

Finally, all of these multiplicities are the same if we replace K_∞° by K_∞.

Proof. There are four associate classes of parabolics for GSp_4 and they are equal to the conjugacy classes of such. From the Franke–Schwermer decomposition (Theorem 1.1.2.1) we have that the

Eisenstein cohomology decomposes as

$$H^i_{\mathrm{Eis}}(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{A}_E(\mathrm{GSp}_4) \otimes E) = \bigoplus_{P \in \{P_\alpha, P_\beta, B\}} \bigoplus_{\varphi \in \Phi_{E,[P]}} H^i(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{A}_{E,[P],\varphi}(\mathrm{GSp}_4) \otimes E).$$

We will study the summands corresponding to P_β, P_α, and B in what follows. The strategy for the P_β summand will be to show that if the representation $\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)$ occurs as a subquotient of one of these summands, then the corresponding associate class in $\Phi_{E,[P_\beta]}$ is the unique one that contains $(\tilde{\pi} \boxtimes 1) \otimes \delta^s_{M_\beta(\mathbb{A})}$. Then Proposition 1.4.3.1 will allow us to deduce the $[P_\beta]$-Eisenstein multiplicity claimed. In the remaining cases of P_α and B, we just show that none of the summands of the cohomology corresponding to these parabolic subgroups can contain $\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)$ as a subquotient, the key input being Proposition 1.4.2.5.

Case of P_β. Let φ' be an associate class of cuspidal automorphic representations for E and $[P_\beta]$ as in Section 1.1.2. Then φ' contains a cuspidal automorphic representation of $M_\beta(\mathbb{A})$ which tranforms trivially under $A_{\mathrm{GSp}_4}(\mathbb{R})^\circ$, and which therefore must be of the form

$$(\tilde{\pi}' \boxtimes \psi') \otimes \delta^{s'}_{M_\beta(\mathbb{A})}$$

where $\tilde{\pi}'$ is a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, ψ' is a Dirichlet character, and $s' \in \mathbb{C}$. After possibly conjugating by $w_{\alpha\beta\alpha}$, we may even assume $\mathrm{Re}(s') \geq 0$.

We will study the piece $\mathcal{A}_{E,[P_\beta],\varphi'}(\mathrm{GSp}_4)$ of the Franke–Schwermer decomposition using Theorem 1.1.3.5 of Grbac. But first, we note that the infinitesimal character of $\mathcal{A}_{E,[P_\beta],\varphi'}(\mathrm{GSp}_4)$ as an $(\mathfrak{sp}_4, K_\infty)$-module must match that of E. The former is given in terms of the representations in φ' by the Weyl orbit of $\lambda_{\tilde{\pi}'} + 2s'\rho_{P_\beta}$, where $\lambda_{\tilde{\pi}'}$ is the infinitesimal character of $\tilde{\pi}'$, and the latter is given by $\Lambda + \rho$. But the weight $\Lambda + \rho$ is regular and real, and so since $\lambda_{\pi'}$ is a multiple of the root β and ρ_{P_β} is a multiple of the root $\alpha + \beta$, it follows that $\lambda_{\pi'}$ and s' are real and nonzero. In particular, $s' > 0$ since we assumed $\mathrm{Re}(s') \geq 0$.

Now we apply Theorem 1.1.3.5 and Proposition 1.1.3.2 to find that the cohomology space

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{A}_{E,[P_\beta],\varphi'}(\mathrm{GSp}_4) \otimes E),$$

if nontrivial, is made up of subquotients of the cohomology spaces

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{L}_\beta(\tilde{\pi}' \boxtimes \psi', s') \otimes E) \tag{1.4.3.1}$$

and

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \otimes \mathrm{Ind}_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\tilde{\pi}' \boxtimes \psi') \otimes \mathrm{Sym}(\mathfrak{a}_{P_\beta,0})_{(2s'+1)\rho_{P_\beta}}) \otimes E). \tag{1.4.3.2}$$

We claim that if (1.4.3.1) is nonzero, then $\tilde{\pi}'$ is cohomological. This will imply that $\tilde{\pi}'$ is attached to a cuspidal holomorphic eigenform of weight at least 2. To start, we split into two cases: Either $\tilde{\pi}'_\infty$ is tempered or nontempered. Of course, by Selberg's conjecture, the latter possibility should not occur, but we will use the following ad-hoc argument to bypass a dependence on this conjecture.

So assume now, for sake of contradiction, both that the cohomology space (1.4.3.1) is nontrivial and that $\tilde{\pi}'_\infty$ is nontempered. By the Langlands classification for real groups, $\tilde{\pi}'_\infty$ is the Langlands quotient of a representation induced from a character, say χ, of $T(\mathbb{R})$, and then $\mathcal{L}_\beta(\tilde{\pi}' \boxtimes \psi', s')_\infty$ is the Langlands quotient of a representation induced from $\chi\delta_{P_\beta(\mathbb{R})}^{s'}$. If $\mathcal{L}_\beta(\tilde{\pi}' \boxtimes \psi', s')_\infty \otimes E$ has nontrivial $(\mathfrak{sp}_4, K_\infty^\circ)$-cohomology, then by [BW00], Theorem VI.1.7 (iii) (or rather, the analogue of this theorem with twisted coefficients) so does the (normalized) induced representation

$$\iota_{B(\mathbb{R})}^{\mathrm{GSp}_4(\mathbb{R})}(\chi\delta_{P_\beta(\mathbb{R})}^{s'}).$$

By [BW00], Theorem III.3.3 and induction in stages, the induction

$$\iota_{(B\cap\mathrm{GL}_2)(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi\delta_{P_\beta(\mathbb{R})}^{s'})$$

has nontrivial $(\mathfrak{sl}_2, \mathrm{O}(2))$-cohomology when twisted by some finite dimensional representation of $\mathrm{GL}_2(\mathbb{C})$, and hence so does

$$\iota_{(B\cap\mathrm{GL}_2)(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi)$$

since $\delta_{P_\beta(\mathbb{R})}$ is trivial on $\mathrm{SL}_2(\mathbb{R})$. Thus by [BW00], Theorem VI.1.7 (ii), $\tilde{\pi}'_\infty$, which is the Langlands quotient of this induction, also has cohomology. But the cohomological cusp forms for GL_2 are the holomorphic modular forms, which are in particular tempered at infinity. This is a contradiction.

Therefore, still assuming (1.4.3.1) is nonzero, we must have $\tilde{\pi}'_\infty$ is tempered. Then by (the twisted version of) [BW00], Lemma VI.1.5,

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \iota_{P_\beta(\mathbb{R})}^{\mathrm{GSp}_4(\mathbb{R})}((\tilde{\pi}' \boxtimes \psi')_\infty, s') \otimes E) \neq 0.$$

But by [BW00], Theorem III.3.3, this is computed in terms of the cohomology of $\tilde{\pi}'_\infty$, and we conclude that $\tilde{\pi}'$ is cohomological, as desired.

If instead (1.4.3.2) is nonzero, then we can use Theorem 1.2.2.3 to conclude that $\tilde{\pi}'$ is cohomological. In any case, if

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{A}_{E,[P_\beta],\varphi'}(\mathrm{GSp}_4) \otimes E) \neq 0,$$

then $\tilde{\pi}' = \tilde{\pi}_{F'}$ for some cuspidal holomorphic eigenform F' of weight at least 2. Furthermore, any irreducible subquotient of this cohomology space must be an irreducible subquotient of either (1.4.3.2) or (1.4.3.1). The former, by Theorem 1.2.2.3 is a sum of copies of

$$\iota_{P_\beta(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}((\tilde{\pi}_{F'} \boxtimes \psi')_f, s'),$$

while the latter is a sum of copies of the Langlands quotient of this induction. In particular, they are all nearly equivalent and occur in this induction.

So if we now assume that

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{A}_{E,[P_\beta],\varphi'}(\mathrm{GSp}_4) \otimes E)$$

contains $\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes \psi, s)_f$ as a subquotient, then since we have shown $s' > 0$, by Proposition 1.4.2.1, $\tilde{\pi}' = \tilde{\pi}_F$, $\psi' = 1$, and $s = s'$.

Therefore we have just shown that φ' contains $(\tilde{\pi}_F \boxtimes 1) \otimes \delta^s_{P_\beta(\mathbb{A})}$. Since no two classes φ' overlap, this determines φ' uniquely. By Proposition 1.1.3.2, Proposition 1.4.3.2 and our vanishing assumption on the L-function of $\tilde{\pi}_F$, we have

$$\mathcal{A}_{E,[P_\beta],\varphi}(\mathrm{GSp}_4) \cong \mathrm{Ind}_{P_\beta(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\tilde{\pi}_F \boxtimes 1) \otimes \mathrm{Sym}(\mathfrak{a}_{P_\beta,0})_{(2s+1)\rho_{P_\beta}}),$$

and then Proposition 1.4.3.1 gives the $[P_\beta]$-Eisenstein multiplicities claimed.

Case of P_α. Let φ' this time be an associate class for E and P_α. Then φ' contains a representation of the form

$$(\tilde{\pi}' \boxtimes \psi') \otimes \delta_{M_\alpha(\mathbb{A})}^{s'}$$

with $\tilde{\pi}'$ a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, ψ' a Dirichlet character, and $s' \in \mathbb{C}$ with $\mathrm{Re}(s') \geq 0$. Then the same argument as in the P_β case shows that s' is real and positive.

Now we once again apply Theorem 1.1.3.5 and Proposition 1.1.3.2 to find that the cohomology space

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{A}_{E,[P_\alpha],\varphi'}(\mathrm{GSp}_4) \otimes E),$$

if nontrivial, is made up of subquotients of the cohomology spaces

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{L}_{P_\alpha(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}' \otimes \psi', s') \otimes E) \tag{1.4.3.3}$$

and

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}((\tilde{\pi}' \boxtimes \psi') \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0})_{(2s'+1)\rho_{P_\alpha}}) \otimes E). \tag{1.4.3.4}$$

Just as in the P_β case, the nonvanishing of either (1.4.3.3) or (1.4.3.4) implies that $\tilde{\pi}' = \tilde{\pi}_{F'}$ for a cuspidal holomorphic eigenform F' of weight at least 2, and that any irreducible subquotient of

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathcal{A}_{E,[P_\alpha],\varphi'}(\mathrm{GSp}_4) \otimes E)$$

is nearly equivalent to an irreducible subquotient of

$$\iota_{P_\alpha(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}((\tilde{\pi}' \boxtimes \psi')_f, s').$$

Now we use Proposition 1.4.2.5 to conclude that $\mathcal{L}_\beta(\tilde{\pi}_F \boxtimes 1, s)$ cannot also occur as a subquotient, which finishes the proof in the case of P_α.

Case of B. Now we let φ' be an associate class for E and $[B]$. So φ' contains a character of $T(\mathbb{A})$ of the form

$$(\psi_1' \boxtimes \psi_2' \boxtimes \psi_3') \otimes e^{\langle H_B(\cdot), s_1'\alpha + s_2'\beta \rangle},$$

where $\psi_1', \psi_2', \psi_3'$ are Dirichlet characters and $s_1', s_2' \in \mathbb{C}$. Let us write

$$\psi' = \psi_1' \boxtimes \psi_2' \boxtimes \psi_3'$$

for short.

We will study the piece $\mathcal{A}_{E,[B],\varphi'}(\mathrm{GSp}_4)$ of the Franke–Schwermer decomposition using the (Franke) filtration of Theorem 1.1.3.3. By that theorem, there is a filtration on the space $\mathcal{A}_{E,[B],\varphi'}(\mathrm{GSp}_4)$ whose graded pieces are parametrized by certain quadruples (Q, ν, Π, μ). For the convenience of the reader, we recall what these quadruples consist of now:

- Q is a standard parabolic subgroup of GSp_4;
- ν is an element of $(\mathfrak{t} \cap \mathfrak{m}_{Q,0})^\vee$;
- Π is an automorphic representation of $M_Q(\mathbb{A})$ occurring in

$$L^2_{\mathrm{disc}}(M_Q(\mathbb{Q})A_Q(\mathbb{R})^\circ \backslash M_Q(\mathbb{A}))$$

  and which is spanned by values at, or residues at, the point ν of Eisenstein series parabolically induced from $(B \cap M_Q)(\mathbb{A})$ to $M_Q(\mathbb{A})$ by representations in φ'; and
- μ is an element of $\mathfrak{a}_{Q,0}^\vee$ whose real part in $\mathrm{Lie}(A_{\mathrm{GSp}_4}(\mathbb{R}) \backslash A_{M_Q}(\mathbb{R}))$ is in the closure of the positive cone, and such that $\nu + \mu$ lies in the Weyl orbit of $\Lambda + \rho$.

Then the graded pieces of $\mathcal{A}_{E,[B],\varphi'}(\mathrm{GSp}_4)$ are isomorphic to direct sums of $\mathrm{GSp}_4(\mathbb{A}_f) \times (\mathfrak{sp}_4, K_\infty)$-modules of the form

$$\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\Pi \otimes \mathrm{Sym}(\mathfrak{a}_{Q,0})_{\mu+\rho_Q})$$

for certain quadruples (Q, ν, Π, μ) of the form just described.

For each of the four possible parabolic subgroups Q and any corresponding quadruple (Q, ν, Π, μ) as above, we will show using Proposition 1.4.2.5 that the cohomology

$$H^*(\mathfrak{sp}_4, K_\infty^\circ; \mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\Pi \otimes \mathrm{Sym}(\mathfrak{a}_{Q,0})_{\mu+\rho_Q})) \tag{1.4.3.5}$$

cannot have $\mathcal{L}_\beta((\tilde{\pi}_F \boxtimes 1)_f, s)$ as a subquotient, which will finish the proof.

So first assume we have a quadruple (Q, ν, Π, μ) as above where $Q = B$. Then $\mathfrak{m}_{Q,0} = 0$, forcing $\nu = 0$. The entry Π is the unitarization of a representation in φ', and thus must be a character ψ' of $T(\mathbb{A})$ conjugate to $\psi'_1 \boxtimes \psi'_2 \boxtimes \psi'_3$. Finally, we have μ is Weyl conjugate to $\Lambda + \rho$.

Therefore the cohomology (1.4.3.5) is isomorphic, by Theorem 1.2.2.3, to a finite sum of copies of

$$\iota_{B(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}(\psi'_f, \mu).$$

By Proposition 1.4.2.5, $\mathcal{L}_\beta((\tilde{\pi}_F \boxtimes 1)_f, s)$ cannot be a subquotient of this space, and we conclude in the case when $Q = B$.

If now we have a quadruple (Q, ν, Π, μ) where $Q = P_\alpha$, then ν is an integer multiple of $\alpha/2$ and μ is a multiple of $(\alpha + 2\beta)/2$, and $\nu + \mu$ is conjugate to $\Lambda + \rho$. We find that Π is a representation generated by residual Eisenstein series at the point ν and is therefore a subquotient of the normalized induction

$$\iota_{(B \cap M_\alpha)(\mathbb{A})}^{M_\alpha(\mathbb{A})}(\psi', \nu),$$

where ψ' is a character of $T(\mathbb{A})$ conjugate to $\psi'_1 \boxtimes \psi'_2 \boxtimes \psi'_3$. Then by 1.2.2.3 and induction in stages, (1.4.3.5) is isomorphic to a subquotient of a finite sum of copies of

$$\iota_{B(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}(\psi'_f, \nu + \mu).$$

We then conclude in this case as well using Proposition 1.4.2.5.

The case when $Q = P_\beta$ is completely similar, and we omit the details. When $Q = G$, it is once again similar, but easier since we do not need to use induction in stages. So we are done with the proof of the $[B]$-Eisenstein multiplicity.

Finally, if we instead used K_∞ instead of K_∞° to compute cohomology, then all the multiplicities that were zero remain zero. The multiplicities that were 1 remain 1 because they followed from Proposition 1.4.3.1, which gets the same answer in both cases. The final claim about the action of the component group of K_∞ follows. $\square$

1.4.4 Cuspidal multiplicity of Langlands quotients

Despite being nontempered quotients of induced representations, some of the Langlands quotients we studied in the previous section can be found in cuspidal cohomology as well as Eisenstein cohomology. The purpose of this section is to explain how this happens.

The occurrence of this phenomenon relies on the study of CAP representations, which were first studied in an automorphic context by Piatetski-Shapiro in [PS83]. These, by definition, are cuspidal automorphic representations which are nearly equivalent to an irreducible constituent of a parabolically induced representation. In our context, these show up in the proof of the following theorem.

Theorem 1.4.4.1. *Let F be a cuspidal holomorphic eigenform of even weight $k \geq 4$, and let ϵ be the sign of the functional equation for the L-function $L(\tilde{\pi}_F, s)$. Assume $L(\tilde{\pi}_F, 1/2) = 0$. Let E be the irreducible representation of $\mathrm{GSp}_4(\mathbb{C})$ of highest weight $\frac{k-4}{2}(\alpha+\beta)$. Then*

$$m^i_{\text{cusp}}(\mathcal{L}_\beta(\tilde{\pi} \boxtimes 1, 1/6)_f, K_\infty^\circ, E) = \begin{cases} 1 & \textit{if } \epsilon = 1 \textit{ and } i = 2 \textit{ or } 4; \\ 2 & \textit{if } \epsilon = -1 \textit{ and } i = 3; \\ 0 & \textit{otherwise.} \end{cases}$$

Consequently,

$$m^i(\mathcal{L}_\beta(\tilde{\pi} \boxtimes 1, 1/6)_f, K_\infty^\circ, E) = \begin{cases} 1 & \textit{if } \epsilon = 1 \textit{ and } i = 2, 3, \textit{ or } 4; \\ 3 & \textit{if } \epsilon = -1 \textit{ and } i = 3; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. In [PS83], Piatetski-Shapiro proves that all CAP representations which are nearly equivalent to $\mathcal{L}_\beta(\tilde{\pi} \boxtimes 1, 1/6)$ come from Saito–Kurokawa forms, and each Saito–Kurokawa form appears with multiplicity one. If $\epsilon = -1$, then the corresponding Saito–Kurokawa representation which, at finite places, is given by $\mathcal{L}_\beta(\tilde{\pi} \boxtimes 1, 1/6)_f$, is in the (holomorphic) discrete series at infinity with Harish-Chandra parameter $\frac{k-4}{2}(\alpha+\beta)+\rho$. The $(\mathfrak{sp}_4, K_\infty^\circ)$-cohomology of the archimedean component of this Saito–Kurokawa representation, with coefficients twisted by E, is therefore concentrated in middle degree 3 and is 2 dimensional. (See the remarks on discrete representations in Section 1.4.1; the discrete series representations of $\mathrm{GSp}_4(\mathbb{R})$ are sums to two such representations of $\mathrm{Sp}_4(\mathbb{R})$.)

If instead $\epsilon = 1$, then the Saito–Kurokawa representation in question has archimedean component isomorphic to $\mathcal{L}_\beta(\tilde{\pi} \boxtimes 1, 1/6)_\infty$. Its cohomology is therefore concentrated in degrees 2 and 4, and there it is isomorphic to the $(\mathfrak{sl}_2, \mathrm{O}(2))$-cohomology of $\tilde{\pi}_{F,\infty}$. (Note $P_\beta(\mathbb{R}) \cap K_\infty^\circ$ contains all of $\mathrm{O}(2)$.) Since $\tilde{\pi}_{F,\infty}$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight k, its cohomology is 1 dimensional. Therefore we have justified the cuspidal multiplicity of $\mathcal{L}_\beta(\tilde{\pi} \boxtimes 1, 1/6)_f$. The full multiplicity follows from adding the Eisenstein multiplicity computed in Theorem 1.4.3.3.

For a nice account of the facts we used about the CAP representations appearing here, see Gan [Gan08] □

We now make several remarks.

Remark 1.4.4.2. The above theorem corrects a computation made in the paper of Urban [Urb11], 5.5.3. There he obtains the same result except with the claim that

$$m^2_{[P_\beta]}(\mathcal{L}_\beta(\tilde{\pi} \boxtimes 1, 1/6)_f, K_\infty^\circ, E)$$

equals 1 instead of 0. When we factor in this correction, this shows that the Euler–Poincaré multiplicity (equivalent to the alternating sum of our multiplicities m^i) discussed there is 1 when $\epsilon = 1$ and is -3 when $\epsilon = -1$.

But this would seem to throw off the computation in [Urb11] of the cuspidal overconvergent multiplicity of the critical p-stabilization of $\mathcal{L}_\beta(\tilde{\pi}\boxtimes 1, 1/6)_f$. However, when we take into account the fact that $P_\beta(\mathbb{R}) \cap K_\infty^\circ$ contains the maximal compact subgroup $\mathrm{O}(2)$ of the $\mathrm{GL}_2(\mathbb{R})$ factor of $M_\beta(\mathbb{R})$, we see that all Eisenstein multiplicities there should be computed via $(\mathfrak{sl}_2, \mathrm{O}(2))$-cohomology, instead of $(\mathfrak{sl}_2, \mathrm{SO}(2))$-cohomology. Taking this into account makes Urban's overconvergent Eisenstein multiplicities equal to 1 instead of 2 when they are nonzero. The cuspidal overconvergent multiplicity is then still equal to $2(\epsilon - 1)$, which is what was claimed in [Urb11].

Remark 1.4.4.3. One could, in principle, use our methods to obtain analogous results as Theorems 1.4.3.3 and 1.4.4.1 in the case of P_α instead of P_β. To compute the cuspidal multiplicities for P_α, one would instead need to use results of Howe–Piatetski-Shapiro [HPS79] and Soudry [Sou88].

Remark 1.4.4.4. In the case of G_2, the results that allow us to compute the cuspidal multiplicity for Langlands quotients coming from the short root parabolic are contained in the work on Gan–Gurevich [GG06]. However, the corresponding results in the case of the long root parabolic are not known. There are partial results in another work of Gan and Gurevich [GG09], but it does not give all the results we need. In particular, they say nothing about the CAP representations they obtain at infinity, and so we compute what these representations should be explicitly assuming Arthur's conjectures in Section 1.6.

1.5 The case of G_2

In this section, we carry out computations analogous to those in the previous section for Langlands quotients coming from the long root parabolic in G_2. However, we note that it is not necessary to have read the previous section in order to read this one.

1.5.1 The group G_2

We define G_2 to be the split simple group over $\mathbb{Q}$ with Dynkin diagram as in Figure 1.5.1. Fixing a maximal $\mathbb{Q}$-split torus T in G_2, we choose a long simple root α and a short simple root β,

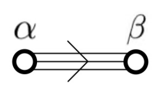


Figure 1.5.1: The Dynkin diagram of G_2

as notated in the Dynkin diagram.

The group G_2 has trivial center, so unlike Sp_4, there are no central extensions of it which are nontrivial.

Also different from Sp_4 is that G_2 does not have such a nice matricial definition. There is a faithful representation of G_2 into GL_7 that we will make some use of, and while it is possible to characterize the image of that representation in terms of the preservation of certain alternating 3-forms, it is hard to make that characterization explicit in terms of matrices. Consequently, we will study G_2 from the point of view of its root system, which we discuss now.

The root lattice

The root lattice of G_2 looks as in Figure 1.5.2. There, the dominant chamber is shaded. Write

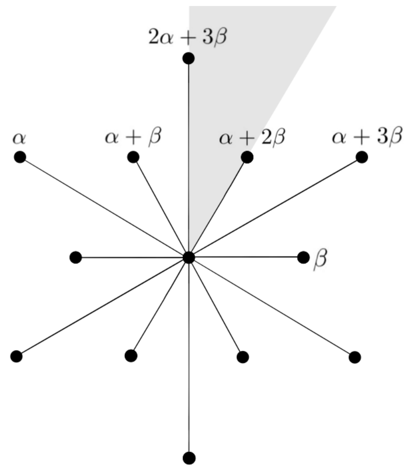


Figure 1.5.2: The root lattice of G_2

Δ for the set of roots of T in G_2, and write Δ^+ for the positive ones. So we have

$$\Delta^+ = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta\}.$$

One nice feature of G_2 is that the $\mathbb{Z}$-span of the root lattice equals the character group of T:

$$X^*(T) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta.$$

Since the Cartan matrix of G_2 has determinant 1, an analogous fact holds for the cocharacter group:

$$X_*(T) = \mathbb{Z}\alpha^\vee \oplus \mathbb{Z}\beta^\vee. \tag{1.5.1.1}$$

Parabolic subgroups

Let B denote the standard Borel subgroup of G_2 with respect to our positive system of roots Δ^+. We write $B = TU$ for its Levi decomposition. Besides B, there are two other proper standard parabolic subgroups, and they are maximal. Let P_α denote the standard parabolic subgroup whose Levi contains α, and write $P_\alpha = M_\alpha N_\alpha$ for its Levi decomposition. Similarly define $P_\beta = M_\beta N_\beta$.

The maximal torus T is of course isomorphic to $GL_1 \times GL_1$. In fact we fix an isomorphism

$i_0 : \mathrm{GL}_1 \times \mathrm{GL}_1 \to T$, defined by

$$i_0(t_1, t_2) = \alpha^\vee(t_1)\beta^\vee(t_2).$$

This is indeed an isomorphism by (1.5.1.1).

For $\gamma \in \Delta$ a root, write

$$\mathbf{x}_\gamma : \mathbb{G}_a \to \mathrm{G}_2$$

for the corresponding root group homomorphism, where $\mathbb{G}_a$ is the additive group scheme.

The Levis M_α and M_β are both isomorphic to GL_2. We write

$$i_\alpha : \mathrm{GL}_2 \to M_\alpha \quad \text{and} \quad i_\beta : \mathrm{GL}_2 \to M_\beta$$

for the isomorphisms which send the upper triangular matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ in GL_2 to the element $\mathbf{x}_\alpha(a)$ and $\mathbf{x}_\beta(a)$, respectively.

We then have the following relations among these isomorphisms:

$$i_\alpha^{-1}(i_0(t_1, t_2)) = \begin{pmatrix} t_1 t_2^{-1} & \\ & t_1^{-1} t_2^2 \end{pmatrix}, \qquad i_\beta^{-1}(i_0(t_1, t_2)) = \begin{pmatrix} t_2 & \\ & t_1 t_2^{-1} \end{pmatrix}.$$

We will often identify T with $\mathrm{GL}_1 \times \mathrm{GL}_1$ via i_0 and drop the notation from formulas. Similarly we will often identify M_α and M_β with GL_2 and drop i_α and i_β from notation when it causes no confusion.

The standard representation

The smallest fundamental weight of G_2 is $\alpha + 2\beta$, and the representation attached to it is seven dimensional. We denote it by R_7 and call it the *standard representation* of G_2; it is the representation one naturally gets when defining G_2 through its action on traceless split octonions.

Let V_7 be the space of R_7. This representation contains weight vectors for the seven weights given by the six short roots together with the zero weight; see Figure 1.5.3. For such a weight λ, choose a nonzero vector $v_\lambda \in V_7$ corresponding to that weight.

Let us order our weight vectors as follows:

$$v_{-\alpha-2\beta}, v_{-\alpha-\beta}, v_{-\beta}, v_0, v_\beta, v_{\alpha+\beta}, v_{\alpha+2\beta}.$$

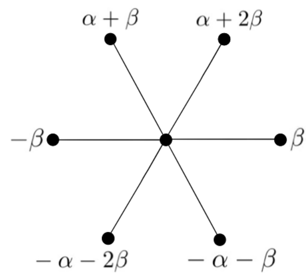


Figure 1.5.3: The weights of R_7

Then using the above list as an ordered basis represents G_2 as 7×7 matrices acting on the linear span of these seven weight vectors. We then have the following matrix representations of the standard Levi subgroups of G_2. For T we have

$$R_7(i_0(t_1,t_2)) = \mathrm{diag}(t_2^{-1}, t_1^{-1}t_2, t_1t_2^{-2}, 1, t_1^{-1}t_2^2, t_1t_2^{-1}, t_2), \tag{1.5.1.2}$$

and for M_α and M_β we have

$$R_7\circ i_\alpha = \begin{pmatrix} \det^{-1} & & & & \\ & \mathrm{Std}^\vee & & & \\ & & 1 & & \\ & & & \mathrm{Std} & \\ & & & & \det \end{pmatrix}, \tag{1.5.1.3}$$

where Std is the standard representation of GL_2, and

$$R_7\circ i_\beta = \begin{pmatrix} \mathrm{Std}^\vee & & \\ & \mathrm{Ad} & \\ & & \mathrm{Std} \end{pmatrix}, \tag{1.5.1.4}$$

where $\mathrm{Ad} = \mathrm{Sym}^2 \otimes \det^{-1}$ is the adjoint representation of GL_2. These can be seen by looking at strings in the directions of α and β in the weight diagram as in Figure 1.5.4.

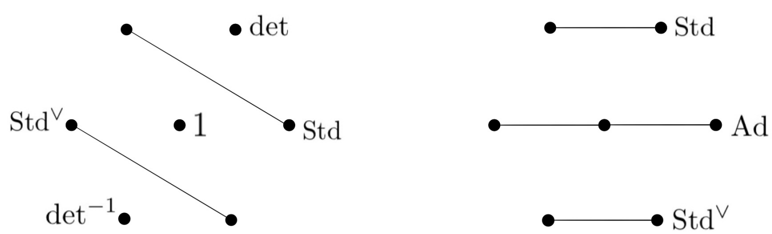


Figure 1.5.4: The standard Levis of G_2 under R_7

Duality

As in the case of GSp_4, the group G_2 is self dual, and identifying G_2 with its dual group switches the long and short simple roots.

More explicitly, fix identifications $\mathrm{GL}_2^\vee \cong \mathrm{GL}_2$ and $\mathrm{G}_2 \cong \mathrm{G}_2^\vee$ so that positive coroots correspond on the dual side to positive roots. Identify M_α and M_β with GL_2 via the maps i_α and i_β introduced above. Then $M_\alpha^\vee$ and $M_\beta^\vee$ are identified with $\mathrm{GL}_2^\vee$, and we have commuting diagrams

$$\begin{tikzcd} \mathrm{GL}_2^\vee \arrow[r, "\sim"] \arrow[d, "\sim"] & M_\alpha^\vee \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{G}_2^\vee \arrow[d, "\sim"] \\ \mathrm{GL}_2 \arrow[r, "i_\beta"] & M_\beta \arrow[r, hook] & \mathrm{G}_2, \end{tikzcd} \tag{1.5.1.5}$$

and

$$\begin{tikzcd} \mathrm{GL}_2^\vee \arrow[r, "\sim"] \arrow[d, "\sim"] & M_\beta^\vee \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{G}_2^\vee \arrow[d, "\sim"] \\ \mathrm{GL}_2 \arrow[r, "i_\alpha"] & M_\alpha \arrow[r, hook] & \mathrm{G}_2. \end{tikzcd} \tag{1.5.1.6}$$

This is simpler than in the GSp_4 case; the obvious identifications are the correct ones. However, the situation for the Borel is still a little bit complicated. Identifying $\mathrm{GL}_1 \cong \mathrm{GL}_1^\vee$ and $T \cong T^\vee$, we have a commuting diagram

$$\begin{tikzcd} \mathrm{GL}_1^\vee \times \mathrm{GL}_1^\vee \arrow[rr, "\sim"] \arrow[d, "\sim"] & & T^\vee \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{G}_2^\vee \arrow[d, "\sim"] \\ \mathrm{GL}_1 \times \mathrm{GL}_1 \arrow[r, "\varphi_0"] & \mathrm{GL}_1 \times \mathrm{GL}_1 \arrow[r, "i_0"] & T \arrow[r, hook] & \mathrm{G}_2, \end{tikzcd} \tag{1.5.1.7}$$

where φ_0 is given by

$$\varphi_0(t_1, t_2) = (t_1^3 t_2^2, t_1^2 t_2). \tag{1.5.1.8}$$

The Weyl group

Let $W = W(T, \mathrm{G}_2)$ be the Weyl group of G_2. The group W is isomorphic to the dihedral group D_6 with 12 elements acting naturally on the root lattice.

For $\gamma \in \Delta$, let w_γ be the reflection about the line perpendicular to γ. Then W is generated by the simple reflections w_α and w_β. As before, we use the following notation: Write $w_{\alpha\beta} = w_\alpha w_\beta$,

$w_{\alpha\beta\alpha} = w_\alpha w_\beta w_\alpha$, and so on. Then

$$W = \{1, w_\alpha, w_\beta, w_{\alpha\beta}, w_{\beta\alpha}, w_{\alpha\beta\alpha}, w_{\beta\alpha\beta}, w_{\alpha\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\alpha\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}, w_{-1}\}.$$

The elements above are written minimally in terms of products of the simple reflections w_α and w_β, except for the final element w_{-1}. This is the element that acts by negation on the root lattice, and it of length 6, equal to both $w_{\alpha\beta\alpha\beta\alpha\beta}$ and $w_{\beta\alpha\beta\alpha\beta\alpha}$.

For $P = MN$ one of the standard parabolic subgroups of GSp_4, we write as usual

$$W^P = \{w \in W \mid w^{-1}\gamma > 0 \text{ for all positive roots } \gamma \text{ in } M\}$$

for the set of representatives for the quotient $W(T, M)\backslash W$ of minimal length. Then

$$W^{P_\alpha} = \{1, w_\beta, w_{\beta\alpha}, w_{\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}\}, \qquad W^{P_\beta} = \{1, w_\alpha, w_{\alpha\beta}, w_{\alpha\beta\alpha}, w_{\alpha\beta\alpha\beta}, w_{\alpha\beta\alpha\beta\alpha}\},$$

and $W^B = W$.

We note for later use that

$$i_0(t_1, t_2)^{w_\alpha} = i_0(t_1^{-1}t_2^3, t_2), \qquad i_0(t_1, t_2)^{w_\beta} = i_0(t_1, t_1 t_2^{-1}). \tag{1.5.1.9}$$

The group $\mathrm{G}_2(\mathbb{R})$

The real Lie group $\mathrm{G}_2(\mathbb{R})$ is connected and has discrete series (see Section 1.6.2 for a review of the classification of discrete series, particularly Theorem 1.6.2.1).

Fix a maximal compact torus T_c in $\mathrm{G}_2(\mathbb{R})$. Then T_c is two dimensional and lies in a maximal compact subgroup of $\mathrm{G}_2(\mathbb{R})$, which we denote by K_∞. Then K_∞ is connected and 6 dimensional. In fact

$$K_\infty \cong \mathrm{SU}(2) \times \mathrm{SU}(2)/\mu_2,$$

where $\mu_2 = \{\pm 1\}$ is diagonally embedded in $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Let $\mathfrak{t}_c$ be the complexified Lie algebra of T_c, and $\mathfrak{k}$ that of K_∞. We abuse notation and write $\Delta = \Delta(\mathfrak{t}_c, \mathfrak{g}_2)$ for the roots of $\mathfrak{t}_c$ in $\mathfrak{g}_2$. Let $\Delta_c = \Delta(\mathfrak{t}_c, \mathfrak{k})$ denote the set of compact roots. There are four roots in Δ_c consisting of a pair of short roots and a pair of long roots. The short compact roots are orthogonal to the long ones.

Again, abusing notation, choose two simple roots α, β of $\mathfrak{t}_c$ in $\mathfrak{g}_2$ with α long and β short, and choose them so that β is compact. Then

$$\Delta_c = \{\pm\beta, \pm(2\alpha + 3\beta)\}.$$

The compact Weyl group $W_c = W(\mathfrak{t}_c, \mathfrak{k})$ has four elements and is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. We in fact have

$$W_c = \{1, w_\beta, w_{\alpha\beta\alpha\beta\alpha}, w_{-1}\},$$

and $w_{\alpha\beta\alpha\beta\alpha}$ equals the reflection across the line perpendicular to $2\alpha + 3\beta$. It follows that the discrete series representations of $\mathrm{G}_2(\mathbb{R})$ are parameterized by integral weights in the union of the three chambers between β and $2\alpha + 3\beta$ which are far enough from the walls of those chambers.

1.5.2 Near equivalence and induced representations

In this section we will study the parabolically induced representations whose Langlands quotients we will try to locate in cohomology later.

Let F be a cuspidal holomorphic eigenform, and let $\tilde{\pi}$ be the unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated with it. We can then view $\tilde{\pi}$ as an automorphic representation of either $M_\alpha(\mathbb{A})$ or $M_\beta(\mathbb{A})$.

Let $\delta_{M_\alpha(\mathbb{A})}$ be the modulus character of $M_\alpha(\mathbb{A})$, and $\delta_{M_\beta(\mathbb{A})}$ that of $M_\beta(\mathbb{A})$. Then for $A \in \mathrm{GL}_2(\mathbb{A})$, we have

$$\delta_{M_\alpha(\mathbb{A})}(A) = |\det A|^5, \qquad \delta_{M_\beta(\mathbb{A})}(A) = |\det A|^3.$$

If $s \in \mathbb{C}$, we define the normalized parabolic inductions

$$\iota_{P_\gamma(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F, s) = \mathrm{Ind}_{P_\gamma(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F \otimes \delta_{P_\gamma}^{s+1/2}), \qquad \gamma \in \{\alpha, \beta\}. \tag{1.5.2.1}$$

We then have the following analogue of Proposition 1.4.2.1.

Proposition 1.5.2.1. *Let $\gamma \in \{\alpha, \beta\}$ be one of the simple roots of G_2. Let F, F' be cuspidal* *holomorphic eigenforms, and let* $s, s' \in \mathbb{R}_{>0}$*. If there are irreducible subquotients*

$$\Pi \text{ of } \iota_{P_\gamma(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F, s)$$

and

$$\Pi' \text{ of } \iota_{P_\gamma(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_{F'}, s')$$

such that Π *and* Π' *are nearly equivalent, then* $\tilde{\pi}_F = \tilde{\pi}_{F'}$ *and* $s = s'$*.*

Proof. We prove this for the short root parabolic P_β since the proof in the case of P_α is completely analogous.

Let p be a prime where the local components Π_p and Π'_p are unramified and isomorphic. Then $\tilde{\pi}_{F,p}$ and $\tilde{\pi}_{F',p}$ are unramified.

Write T_2 for the standard diagonal torus of GL_2 and B_2 for the standard upper triangular Borel in GL_2. Let $\delta_{B_2(\mathbb{Q}_p)}$ be the usual modulus character of $B_2(\mathbb{Q}_p)$. Then by the results recalled in Section 1.3.1, there are characters $\chi_1, \chi_2, \chi'_1, \chi'_2$ of $\mathbb{Q}_p^\times$ such that $\tilde{\pi}_{F,p}$ is the unramified subquotient of

$$\mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi_1 \boxtimes \chi_2) \otimes \delta_{B_2(\mathbb{Q}_p)}^{1/2}),$$

and $\tilde{\pi}_{F',p}$ is the unramified subquotient of

$$\mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi'_1 \boxtimes \chi'_2) \otimes \delta_{B_2(\mathbb{Q}_p)}^{1/2}).$$

Here, $\chi_1 \boxtimes \chi_2$ is the character of T_2 which evaluated at $\mathrm{diag}(x, y) \in T_2(\mathbb{Q}_p)$ gives the product $\chi_1(x)\chi_2(y)$, and similarly for $\chi'_1 \boxtimes \chi'_2$. By temperedness, the characters $\chi_1, \chi_2, \chi'_1, \chi_2$ are unitary.

By induction in stages, Π is the unramified subquotient of

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi \delta_{P_\beta(\mathbb{Q}_p)}^{s} \delta_{B(\mathbb{Q}_p)}^{1/2}),$$

where χ is the character of T given by $\chi = (\chi_1 \boxtimes \chi_2) \circ i_\beta \circ i_0^{-1}$ (see the subsection on parabolic subgroups in Section 1.5.1) and similarly Π'_p is the unramified subquotient of

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi' \delta_{P_\beta(\mathbb{Q}_p)}^{s'} \delta_{B(\mathbb{Q}_p)}^{1/2}),$$

where $\chi' = (\chi'_1 \boxtimes \chi'_2) \circ i_\beta \circ i_0^{-1}$. The characters χ and χ' are unitary. Since $\Pi \cong \Pi'$, the characters

$$\chi\delta^s_{P_\beta(\mathbb{Q}_p)} \quad \text{and} \quad \chi'\delta^{s'}_{P_\beta(\mathbb{Q}_p)}$$

are equal up to the Weyl group W; there is a $w \in W$ such that for all $x, y \in \mathbb{Q}_p^\times$, we have

$$\chi\delta^s_{P_\beta(\mathbb{Q}_p)}(i_0(x,y)^w) = \chi'\delta^{s'}_{P_\beta(\mathbb{Q}_p)}(i_0(x,y)). \tag{1.5.2.2}$$

Now let $t \in \mathbb{Q}_p^\times$ and let

$$T = i_0(t^2, t).$$

Then we compute, using (1.5.1.9), that

$$\begin{aligned} T = T^{w_\beta} &= i_0(t^2, t), \\ T^{w_\alpha} = T^{w_{\alpha\beta}} &= i_0(t, t), \\ T^{w_{\beta\alpha}} = T^{w_{\beta\alpha\beta}} &= i_0(t, 1), \\ T^{w_{\alpha\beta\alpha}} = T^{w_{\alpha\beta\alpha\beta}} &= i_0(t^{-1}, 1), \\ T^{w_{\beta\alpha\beta\alpha}} = T^{w_{\beta\alpha\beta\alpha\beta}} &= i_0(t^{-1}, t^{-1}), \\ T^{w_{\alpha\beta\alpha\beta\alpha}} = T^{w_{-1}} &= i_0(t^{-2}, t^{-1}). \end{aligned}$$

Since $\det(i_\beta^{-1}(i_0(x,y))) = x$, the above gives

$$|\chi\delta^s_{P_\beta(\mathbb{Q}_p)}(T^w)| = \begin{cases} p^{6s} & \text{if } w \in \{1, w_\beta\}; \\ p^{3s} & \text{if } w \in \{w_\alpha, w_{\alpha\beta}, w_{\beta\alpha}, w_{\beta\alpha\beta}\}; \\ p^{-3s} & \text{if } w \in \{w_{\alpha\beta\alpha}, w_{\alpha\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}\}; \\ p^{-6s} & \text{if } w \in \{w_{\alpha\beta\alpha\beta\alpha}, w_{-1}\}. \end{cases}$$

Comparing this to

$$|\chi'\delta^{s'}_{P_\beta(\mathbb{Q}_p)}(T)| = |t|^{6s'}$$

via (1.5.2.2) gives, since $s, s' > 0$,

$$s = s' \text{ and } w \in \{1, w_\beta\}, \quad \text{or} \quad s = 2s' \text{ and } w \in \{w_\alpha, w_{\alpha\beta}, w_{\beta\alpha}, w_{\beta\alpha\beta}\}.$$

But in this latter case we can then repeat the calculation with $T = i_0(t, 1)$ instead. In this case we then find

$$T^{w_\alpha} = T^{w_{\alpha\beta}} = (t^{-1}, 1),$$
$$T^{w_{\beta\alpha}} = T^{w_{\beta\alpha\beta}} = (t^{-1}, t^{-1}),$$

and the same argument then rules out $w \in \{w_\alpha, w_{\alpha\beta}, w_{\beta\alpha}, w_{\beta\alpha\beta}\}$.

Therefore $w \in \{1, w_\beta\}$ and $s = s'$. Then (1.5.2.2) implies $\chi_1 = \chi_1'$ and $\chi_2 = \chi_2'$ if $w = 1$, or $\chi_1 = \chi_2'$ and $\chi_2 = \chi_1'$ if $w = w_\beta$. In either case we have

$$\mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi_1 \boxtimes \chi_2) \otimes \delta_{B_2(\mathbb{Q}_p)}^{1/2}) \text{ and } \mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi_1' \boxtimes \chi_2') \otimes \delta_{B_2(\mathbb{Q}_p)}^{1/2})$$

have the same unramified subquotients, which means $\tilde{\pi}_{F,p} \cong \tilde{\pi}_{F',p}$.

Now letting p vary over all unramified primes for which $\Pi_p \cong \Pi_p'$ and applying strong multiplicity one for GL_2 finishes the proof. □

Let ψ_1, ψ_2 be Dirichlet characters, and consider the character $\psi_1 \boxtimes \psi_2$ of $T(\mathbb{A})$ given by

$$(\psi_1 \boxtimes \psi_2)(i_0(t_1, t_2)) = \psi_1(t_1)\psi_2(t_2).$$

Let $\delta_{B(\mathbb{A})}$ be the modulus character of $B(\mathbb{A})$. We have

$$\delta_{B(\mathbb{A})}^{1/2} = e^{\langle H_B(\cdot), \rho \rangle},$$

where $\rho = 3\alpha + 5\beta$ is half the sum of positive roots. If $s_1, s_2 \in \mathbb{C}$, write

$$\iota_{B(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\psi_1 \boxtimes \psi_2; s_1, s_2) = \mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}((\psi_1 \boxtimes \psi_2) \otimes e^{\langle H_B(\cdot), s_1\alpha + s_2\beta + \rho \rangle}) \tag{1.5.2.3}$$

for the normalized induction.

For the following we fix any prime ℓ and identify $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$ via a fixed isomorphism.

Proposition 1.5.2.2. *Let ψ_1, ψ_2 be Dirichlet characters, and let $m_1, m_2 \in \mathbb{Z}$. Let Π be an irreducible subquotient of*

$$\iota_{B(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\psi_1 \boxtimes \psi_2; m_1, m_2).$$

Let j_{T,G_2} be the inclusion of T into G_2. Then Π has attached to it the Galois representation $G_{\mathbb{Q}} \to \mathrm{G}_2(\overline{\mathbb{Q}}_\ell)$ given by

$$j_{T,G_2} \circ i_0 \circ \left((\chi_{\mathrm{cyc}}^{m_2}\psi_1^3\psi_2^2) \times (\chi_{\mathrm{cyc}}^{m_1}\psi_1^2\psi_2)\right),$$

where we have viewed ψ_1, ψ_2 as Galois characters via class field theory.

Proof. Let p be a prime different from ℓ which is unramified for Π, and hence which not divide the conductors of the ψ_i's. Let $\lambda_i = \psi_i(p)$ for $i = 1, 2$. Then the character

$$(\psi_1 \boxtimes \psi_2) \otimes e^{\langle H_B(\cdot), m_1\alpha + m_2\beta\rangle} \tag{1.5.2.4}$$

of $\mathrm{GL}_1(\mathbb{A})^2$ has Satake parameter at p

$$(p^{-(2m_1-m_2)}\lambda_1, p^{-(2m_2-3m_1)}\lambda_2) \in \mathrm{GL}_1(\overline{\mathbb{Q}}_\ell)^2.$$

Identifying $(\mathrm{GL}_1)^2$ with T on the dual side via the map φ_0 of (1.5.1.7) and (1.5.1.8) gives that the character (1.5.2.4) has attached to it the Galois representation

$$i_0 \circ \left((\chi_{\mathrm{cyc}}^{m_2}\psi_1^3\psi_2^2) \times (\chi_{\mathrm{cyc}}^{m_1}\psi_1^2\psi_2)\right).$$

Then we appeal to Proposition 1.3.2.1 to finish the proof. $\square$

Proposition 1.5.2.3. *Let F be a holomorphic cuspidal eigenform of weight k and let $m \in \mathbb{Z}$ with $m \equiv k-1 \pmod 2$. Let Π be any irreducible subquotient of*

$$\iota_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F, m/10).$$

Let j_{M_β,G_2} be the inclusion $M_\beta \hookrightarrow \mathrm{G}_2$. Then Π has attached to it the Galois representation $G_{\mathbb{Q}} \to \mathrm{G}_2(\overline{\mathbb{Q}}_\ell)$ given by

$$j_{M_\beta,\mathrm{G}_2} \circ i_\beta \circ (\rho_F \otimes \chi_{\mathrm{cyc}}^{(m-k+1)/2}),$$

where ρ_F is the Galois representation attached to F by Eichler–Shimura, Deligne, and Deligne–Serre (Theorem 1.3.1.3).

Proof. Let p be a prime different from ℓ which is unramified for Π, and hence which is unramified for $\tilde{\pi}_F$. Let $\mathrm{diag}(\lambda_1, \lambda_2) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a diagonal representative of the Satake parameter of $\tilde{\pi}_F$ at p. Then

$$\tilde{\pi}_F \otimes \delta_{P_\alpha(\mathbb{A})}^{m/10}$$

has Satake parameter at p represented by

$$p^{-m/2}\,\mathrm{diag}(\lambda_1, \lambda_2) \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell),$$

because $\delta_{P_\alpha(\mathbb{A})}$ acts as $|\det|^5$. Now we use the commutativity of (1.5.1.5) and Proposition 1.3.2.1 to conclude. ☐

Proposition 1.5.2.4. *Let F be a holomorphic cuspidal eigenform of weight k and let $m \in \mathbb{Z}$ with $m \equiv k - 1 \pmod 2$. Let Π be any irreducible subquotient of*

$$\iota_{P_\beta(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F, m/6).$$

Let $j_{M_\alpha,\mathrm{G}_2}$ be the inclusion $M_\alpha \hookrightarrow \mathrm{G}_2$. Then Π has attached to it the Galois representation $G_{\mathbb{Q}} \to \mathrm{G}_2(\overline{\mathbb{Q}}_\ell)$ given by

$$j_{M_\alpha,\mathrm{G}_2} \circ i_\alpha \circ (\rho_F \otimes \chi_{\mathrm{cyc}}^{(m-k+1)/2}),$$

where ρ_F is the Galois representation attached to F by Eichler–Shimura, Deligne, and Deligne–Serre (Theorem 1.3.1.3).

Proof. The proof is completely similar to that of 1.5.2.3 above; switch α and β and appeal to (1.5.1.6) instead of (1.5.1.5). ☐

Before giving the analogue of Proposition 1.4.2.5, we need to prove a lemma about Galois representations attached to modular forms. The analogue of this lemma in the GSp_4 case was not necessary because of the nice shape of the Levis of the standard parabolic subgroups in the standard representation of GSp_4. Here in the G_2 case, however, the blocks of M_β in the standard representation R_7 include a symmetric square representation of GL_2, and we will need the following lemma to distinguish representations factoring through it and those factoring through M_α.

Lemma 1.5.2.5. *Let F be a holomorphic cuspidal eigenform of weight $k \geq 2$, and let ρ_F be its Galois representation into $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ (Theorem 1.3.1.3). Then $\mathrm{Sym}^2 \rho_F$ is either irreducible, or is the direct sum of two irreducible representations.*

Proof. We separate the proof into two cases, first when F does not have CM and second when it does.

Assume F does not have CM. By results of Momose [Mom81] (See also [Loe17]) we know then that the image of ρ_F in $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ can be conjugated to be either:

- an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_\ell)$, or
- an open subgroup of $B^\times$, where B is a certain quaternion algebra over $\mathbb{Q}_\ell$.

In either case the image of ρ_F is large enough for $\mathrm{Sym}^2 \rho_F$ to be irreducible.

Now assume F has CM by an imaginary quadratic field K. Then ρ_F is the induction

$$\rho_F \cong \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\chi),$$

where χ is a Hecke character of G_K. Thus, if $c \in G_\mathbb{Q}$ is a complex conjugation, then writing V for the space of ρ_F, there are linearly independent vectors $u, v \in V$ such that

$$gu = \chi(g)u, \qquad gv = \chi(cgc)v, \qquad \text{for } g \in G_K,$$

and

$$cu = v, \qquad cv = u.$$

Let us write χ_c for the character given by

$$\chi_c(g) = \chi(cgc)$$

for $g \in G_{\mathbb{Q}}$. If $v_1, v_2 \in V$, write $v_1 \otimes v_2 = v_2 \otimes v_1$ for the corresponding product in $\mathrm{Sym}^2(V)$.

Now we have

$$c(u \otimes v) = u \otimes v,$$

and

$$g(u \otimes v) = (\chi\chi_c)(g) u \otimes v,$$

for $g \in G_K$. So the space spanned by $u \otimes v$ is invariant and gives the character χ' of $G_{\mathbb{Q}}$ which is given by $\chi\chi_c$ on G_K and is trivial on c. Also,

$$c(u \otimes u) = v \otimes v, \qquad c(v \otimes v) = u \otimes u,$$

and

$$g(u \otimes u) = \chi^2(g) u \otimes u, \qquad g(v \otimes v) = \chi_c^2(g) v \otimes v,$$

for $g \in G_K$. Therefore the space spanned by $u \otimes u$ and $v \otimes v$ is also invariant, and $G_{\mathbb{Q}}$ acts on it as $\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi^2)$. Thus

$$\mathrm{Sym}^2 \rho_F \cong \chi' \oplus \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi^2).$$

It now suffices to prove that $\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi^2)$ is irreducible.

To this end, we first note that

$$\mathrm{Sym}^2 \rho_F|_{G_K} = \chi^2 \oplus \chi_c^2 \oplus \chi\chi_c.$$

Since $\mathrm{Sym}^2 \rho_F$ is Hodge–Tate with Hodge–Tate weights 0, $k-1$, and $2k-2$, it follows that either χ^2 or χ_c^2 is finite order, and the other is a finite order character times $\chi_{\mathrm{cyc}}^{2k-2}|_{G_K}$. Therefore χ^2 and χ_c^2 are distinct, because the evaluation of either character on a Frobenius element Frob_p in G_K gives p-Weil numbers of different weights (since $k > 1$) and by Chebotarev, there are infinitely many such Frobenius elements in G_K.

Now assume that the space of $\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi^2)$, spanned by $u \otimes u$ and $v \otimes v$, has an invariant vector

$$a(u \otimes u) + b(v \otimes v)$$

for some scalars a, b. We will show that this implies $a = b = 0$, which will prove that $\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi^2)$ is irreducible. Choose $g \in G_K$ with $\chi^2(g) \neq \chi_c^2(g)$. Then we have

$$g(a(u \otimes u) + b(v \otimes v)) = a\chi^2(g)(u \otimes u) + b\chi_c^2(g)(v \otimes v),$$

which cannot be in the span of $a(u \otimes u) + b(v \otimes v)$ unless $a = 0$ or $b = 0$. Since c switches $u \otimes u$ and $v \otimes v$, we must have both $a = 0$ and $b = 0$, which finishes the proof. $\square$

Remark 1.5.2.6. In our applications, we actually only need this lemma for one single ℓ, but it was essentially no harder to write down the proof for all ℓ.

Remark 1.5.2.7. We thank Shuai Wang for bringing the following to our attention. There are examples of irreducible, two dimensional representations of finite groups whose symmetric squares do actually decompose as sums of three characters. It seems they tend to come from certain representations of dihedral groups of order divisible by 8, though they can also come from other groups of order divisible by 8 as well.

Therefore we cannot hope to get by on the irreducibility of ρ_F alone in proving the above lemma. Also, this shows that the hypothesis that weight $k \geq 2$ is essential, otherwise ρ_F may factor through one of the aforementioned dihedral representations (for example if ρ_F has image precisely D_4).

Proposition 1.5.2.8. *Let F_α, F_β be two holomorphic cuspidal eigenforms of weights k_α and k_β, respectively, and assume $k_\beta \geq 2$. Let ψ_1 and ψ_2 be Dirichlet characters, and let $m_\alpha, m_\beta, m_1, m_2 \in \mathbb{Z}$. Assume that $m_\alpha \equiv k_\alpha - 1 \pmod 2$ and $m_\beta \equiv k_\beta - 1 \pmod 2$. Then given any irreducible subquotients*

$$\Pi_\alpha \textit{ of } \iota_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_{F_\alpha}, m_\alpha/10)$$

and

$$\Pi_\beta \textit{ of } \iota_{P_\beta(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_{F_\beta}, m_\beta/6)$$

and

$$\Pi_0 \textit{ of } \iota_{B(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\psi_1 \boxtimes \psi_2; m_1, m_2),$$

we have that no two of Π_α, Π_β, and Π_0 are nearly equivalent.

Proof. Let ρ_α, ρ_β, and ρ_0 be, respectively, the Galois representations attached to Π_α, Π_β, and Π_0 by Propositions 1.5.2.3, 1.5.2.4, and 1.5.2.2. We compose these with the standard representation R_7 and obtain, using (1.5.1.4), (1.5.1.3), and (1.5.1.2),

$$R_7 \circ \rho_\alpha = (\rho_{F_\alpha} \otimes \chi_{\mathrm{cyc}}^{(m_\alpha - k_\alpha + 1)/2}) \oplus (\rho_{F_\alpha} \otimes \chi_{\mathrm{cyc}}^{(m_\alpha - k_\alpha + 1)/2})^\vee \oplus \mathrm{Ad}(\rho_{F_\alpha} \otimes \chi_{\mathrm{cyc}}^{(m_\alpha - k_\alpha + 1)/2}),$$

$$R_7 \circ \rho_\beta = 1 \oplus (\omega_{F_\beta} \chi_{\mathrm{cyc}}^{m_\beta}) \oplus (\omega_{F_\beta}^{-1} \chi_{\mathrm{cyc}}^{-m_\beta}) \oplus (\rho_{F_\beta} \otimes \chi_{\mathrm{cyc}}^{(m_\beta - k_\beta + 1)/2}) \oplus (\rho_{F_\beta} \otimes \chi_{\mathrm{cyc}}^{(m_\beta - k_\beta + 1)/2})^\vee,$$

$$\begin{aligned} R_7 \circ \rho_0 = 1 \oplus (\chi_{\mathrm{cyc}}^{m_1} \psi_1^2 \psi_2) \oplus (\chi_{\mathrm{cyc}}^{-m_1} \psi_1^{-2} \psi_2^{-1}) \oplus (\psi_1 \psi_2 \chi_{\mathrm{cyc}}^{m_2 - m_1}) \oplus (\psi_1^{-1} \psi_2^{-1} \chi_{\mathrm{cyc}}^{m_1 - m_2}) \\ \oplus (\psi_1 \chi_{\mathrm{cyc}}^{2m_1 - m_2}) \oplus (\psi_1^{-1} \chi_{\mathrm{cyc}}^{m_2 - 2m_1}). \end{aligned}$$

Here ω_{F_β} is the nebentypus of F_β. We see that the first of these representations is either the sum of 3 or 4 irreducible representations by Lemma 1.5.2.5, that the second is the sum of 5 irreducible representations, and the last is the sum of 7. Therefore we are done by invoking Proposition 1.3.1.6. □

1.5.3 Eisenstein multiplicity of Langlands quotients

We compute in this section the Eisenstein multiplicity of Langlands quotients coming from the long root parabolic P_α. What follows will be highly analogous to the content of Section 1.4.3 where we computed the Eisenstein multiplicity of Langlands quotients coming from the Siegel (short root) parabolic of GSp_4. It is interesting to note that the roles of the long and short root parabolics switch when passing from GSp_4 to G_2.

For standard parabolics P in G_2, we will make use of the normalized parabolic induction functors $\iota_{P(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}$ defined in (1.5.2.1) and (1.5.2.3), and their similarly defined finite adelic analogues $\iota_{P(\mathbb{A}_f)}^{\mathrm{G}_2(\mathbb{A}_f)}$.

Proposition 1.5.3.1. *Let E be an irreducible, finite dimensional representation of $\mathrm{G}_2(\mathbb{C})$, and say that E has highest weight Λ. Write*

$$\Lambda = c_1(2\alpha + 3\beta) + c_2(\alpha + 2\beta)$$

with $c_1, c_2 \in \mathbb{Z}_{\geq 0}$. Let F be a holomorphic cuspidal eigenform of weight k and trivial nebentypus, *and let* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) \geq 0$. *Assume*

$$H^i(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E) \neq 0.$$

Then either:

(i) We have

$$i = 4, \qquad k = 2c_1 + c_2 + 4, \qquad s = \frac{c_2+1}{10},$$

and

$$H^4(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E) \cong \iota_{P_\alpha(\mathbb{A}_f)}^{\mathrm{G}_2(\mathbb{A}_f)}(\tilde{\pi}_{F,f}, (c_2+1)/10),$$

or,

(ii) We have

$$i = 5, \qquad k = c_1 + c_2 + 3, \qquad s = \frac{3c_1 + c_2 + 4}{10},$$

and

$$H^5(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E)$$
$$\cong \iota_{P_\alpha(\mathbb{A}_f)}^{\mathrm{G}_2(\mathbb{A}_f)}(\tilde{\pi}_{F,f}, (3c_1 + c_2 + 4)/10),$$

or,

(iii) We have

$$i = 6, \qquad k = c_1 + 2, \qquad s = \frac{3c_1 + 2c_2 + 5}{10},$$

and

$$H^6(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E)$$
$$\cong \iota_{P_\alpha(\mathbb{A}_f)}^{\mathrm{G}_2(\mathbb{A}_f)}(\tilde{\pi}_{F,f}, (3c_1 + 2c_2 + 5)/10).$$

Proof. Let $\mathfrak{t}$ be the complexified Lie algebra of T. Note that we have a decomposition

$$\mathfrak{t} = (\mathfrak{m}_{\alpha,0} \cap \mathfrak{t}) \oplus \mathfrak{a}_{P_\alpha,0},$$

and $(\alpha + 2\beta)$ acts by zero on the first component while α acts by zero on the second. We also have

$$W^{P_\alpha} = \{1, w_\beta, w_{\beta\alpha}, w_{\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}\},$$

and one computes

$$\begin{aligned}
-(\Lambda+\rho) &= -(c_1+1)\frac{\beta}{2} - (3c_1+2c_2+5)\frac{\alpha+2\beta}{2},\\
-w_\beta(\Lambda+\rho) &= -(c_1+c_2+2)\frac{\beta}{2} - (3c_1+c_2+4)\frac{\alpha+2\beta}{2},\\
-w_{\beta\alpha}(\Lambda+\rho) &= -(2c_1+c_2+3)\frac{\beta}{2} - (c_2+1)\frac{\alpha+2\beta}{2},\\
-w_{\beta\alpha\beta}(\Lambda+\rho) &= -(2c_1+c_2+3)\frac{\beta}{2} + (c_2+1)\frac{\alpha+2\beta}{2},\\
-w_{\beta\alpha\beta\alpha}(\Lambda+\rho) &= -(c_1+c_2+2)\frac{\beta}{2} + (3c_1+c_2+4)\frac{\alpha+2\beta}{2},\\
-w_{\beta\alpha\beta\alpha\beta}(\Lambda+\rho) &= -(c_1+1)\frac{\beta}{2} + (3c_1+2c_2+5)\frac{\alpha+2\beta}{2}.
\end{aligned}$$

Now by Theorem 1.2.2.3, in order for our cohomology space to be nontrivial, there needs to be a $w \in W^{P_\alpha}$ with

$$-w(\Lambda+\rho)|_{\mathfrak{a}_{P_\alpha,0}} = 2s\rho_{P_\alpha} = 10s\frac{\alpha+2\beta}{2},$$

and

$$-w(\Lambda+\rho)|_{\mathfrak{m}_{\alpha,0}} = \pm(k-1)\frac{\alpha}{2}.$$

Therefore, since $\mathrm{Re}(s) \geq 0$, we see from the formulas for each $-w(\Lambda+\rho)|_{\mathfrak{a}_{P_\beta},0}$ that w can only equal $w_{\beta\alpha\beta}$, $w_{\beta\alpha\beta\alpha}$, or $w_{\beta\alpha\beta\alpha\beta}$.

In the case that $w = w_{\beta\alpha\beta}$, we get that

$$k-1 = +(2c_1+c_2+3),$$

with this choice of sign because $k-1 \geq 0$, and

$$6s = c_2 + 1.$$

We also have that the length $\ell(w_{\beta\alpha\beta})$ of $w_{\beta\alpha\beta}$ is 3. Finally, since

$$\rho = \frac{\alpha}{2} + 5\frac{\alpha + 2\beta}{2},$$

we have

$$(w_{\beta\alpha\beta}(\Lambda + \rho) - \rho)|_{\mathfrak{m}_{\alpha,0}} = (2c_1 + c_2 + 2)\frac{\alpha}{2} = (k-2)\frac{\alpha}{2}.$$

Therefore, the isomorphism of Theorem 1.2.2.3 in our case is

$$\begin{aligned} H^i(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E) \\ \cong \iota_{P_\alpha(\mathbb{A}_f)}^{\mathrm{G}_2(\mathbb{A}_f)}(\tilde{\pi}_{F,f}, (c_2+1)/10) \otimes H^{i-3}(\mathfrak{m}_{\alpha,0}, K_\infty \cap P_\alpha(\mathbb{R}); \tilde{\pi}_{F,\infty} \otimes F_{k-2}), \end{aligned}$$

where F_{k-2} is the representation of $\mathfrak{m}_{\alpha,0}$ of highest weight $(k-2)(\alpha/2)$.

Now, since $k - 1 = 2c_1 + c_2 + 3 > 0$, the representation $\tilde{\pi}_{F,\infty}$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight k, and therefore has nontrivial cohomology when tensored with F_{k-2} in degree 1 and degree 1 only. Since $K_\infty \cap \mathrm{GL}_2(\mathbb{R})$ is a maximal compact subgroup of $\mathrm{GL}_2(\mathbb{R})$, the cohomology of $\tilde{\pi}_{F,\infty}$ in degree 1 is 1 dimensional. The claim (i) of our proposition is now immediate.

The claims (ii) and (iii) are completely similar, using instead the length 4 element $w_{\beta\alpha\beta\alpha}$ and the length 5 element $w_{\beta\alpha\beta\alpha\beta}$, respectively; we omit the details. □

We now prove an analogue of Lemma 1.4.3.2 in our context.

Lemma 1.5.3.2. *Let F be a holomorphic cuspidal eigenform of weight $k \geq 2$ and trivial nebentypus. For any flat section $\phi_s \in \iota_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F, s)$, the Eisenstein series $E(\phi, 2s\rho_{P_\alpha})$ does not have a pole for* $\mathrm{Re}(s) > 0$ *except perhaps if $s = 1/10$. If furthermore*

$$L(1/2, \tilde{\pi}_F, \mathrm{Sym}^3) = 0,$$

then $E(\phi, 2s\rho_{P_\alpha})$ is also holomorphic at $s = 1/10$.

Proof. This is an easy consequence of what is done in the paper of Žampera [Ž97], but let us quickly explain how this is proved, since we have set up the tools to do so already.

It suffices to prove the lemma for $\phi = \bigotimes_v \phi_v$ decomposable into local sections. Write $E(\phi, s) = E(\phi, 2s\rho_{P_\beta})$. By Theorem 1.1.1.1, the constant term of $E(\phi, s)$ along P_β (and hence along B) is zero, and the constant term along P_α is

$$E_{P_\alpha}(\phi, s) = \phi_s + M(\phi, w_{\beta\alpha\beta\alpha\beta})_{-2s\rho_{P_\beta}}.$$

Then we apply Theorem 1.1.1.2; in our current setting the root γ of that theorem is α, and $\tilde{\beta} = \rho_{P_\alpha}/5$, and adjusting for this gives

$$M(\phi, w_{\beta\alpha\beta\alpha\beta})_{-2s\rho_{P_\alpha}} = \prod_{j=1}^{m} \frac{L^S(5js, \tilde{\pi}_F, R_i^\vee)}{L^S(5js+1, \tilde{\pi}_F, R_i^\vee)} \bigotimes_{v \notin S} \phi_{v,s}^{w_{\beta\alpha\beta\alpha\beta},\mathrm{sph}} \otimes \bigotimes_{v \in S} M_v(\phi_{v,s}, w_{\beta\alpha\beta\alpha\beta})_{-2s\rho_{P_\alpha}},$$

where S is a finite set of places such that for $v \notin S$, $\phi_{v,s}$ is spherical, and $\phi_{v,s}^{w_{\beta\alpha\beta\alpha\beta},\mathrm{sph}}$ are certain spherical vectors. Also, the representations R_i of $M_\beta^\vee$ can be determined from the action of the Levi of P_α on its unipotent radical; there are two of them, and R_1 is the representation $\mathrm{Sym}^3 \otimes \det^{-1}$ of GL_2, and R_2 is the determinant. Thus the quotient of L-functions is

$$\frac{L^S(5s, \tilde{\pi}_F, \mathrm{Sym}^3)\zeta^S(10s)}{L^S(5s+1, \tilde{\pi}_F, \mathrm{Sym}^3)\zeta^S(10s+1)}.$$

Now by Harish-Chandra, the local intertwining operators are all holomorphic for $\mathrm{Re}(s) > 0$ since $\tilde{\pi}_F$ is tempered. So we only have to worry about the poles and zeros of the L-functions in the quotient above. Again since $\mathrm{Re}(s) > 0$, the L-functions in the denominator do not vanish as they are in the range of convergence. By a result of Kim and Shahidi [KS99], the symmetric cube L-function is entire, and so the only pole in the numerator comes from the ζ-function at $s = 1/10$. But if $L(1/2, \tilde{\pi}_F, \mathrm{Sym}^3) = 0$, this zero cancels with the pole from the ζ-function.

Since the poles of $E(\phi, s)$ are determined by the poles of the constant term at all standard proper parabolics, we are done. □

Now fix F a holomorphic cuspidal eigenform of weight $k \geq 2$ and trivial nebentypus. For $s \in \mathbb{C}$

with $\mathrm{Re}(s) > 0$, let us write

$$\mathcal{L}_\alpha(\tilde{\pi}_F, s) = \text{Langlands quotient of } \iota_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F, s).$$

This notion was introduced just before Theorem 1.1.3.5.

We now compute the Eisenstein multiplicity of this Langlands quotient (see Definition 1.2.1.1).

Theorem 1.5.3.3. *Let E be an irreducible, finite dimensional representation of $\mathrm{G}_2(\mathbb{C})$, and say that E has highest weight Λ. Write*

$$\Lambda = c_1(2\alpha + 3\beta) + c_2(\alpha + 2\beta)$$

with $c_1, c_2 \in \mathbb{Z}_{\geq 0}$. Let F be a holomorphic cuspidal eigenform of weight k and trivial nebentypus, and let $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 0$. If $c_2 = 0$ and $k = 2c_1 + 4$, also assume that

$$L(1/2, \tilde{\pi}_F, \mathrm{Sym}^3) = 0.$$

Then

$$m^i_{[P_\alpha]}(\mathcal{L}_\alpha(\tilde{\pi}_F, s), K_\infty, E) = \begin{cases} 1 & \textit{if } i = 4,\ k = 2c_1 + c_2 + 4,\ s = (c_2 + 1)/10 \\ & \textit{or if } i = 5,\ k = c_1 + c_2 + 3,\ s = (3c_1 + c_2 + 4)/10 \\ & \textit{or if } i = 6,\ k = c_1 + 2,\ s = (3c_1 + 2c_2 + 5)/10; \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$m^i_{[P_\beta]}(\mathcal{L}_\alpha(\tilde{\pi}_F, s), K_\infty, E) = m^i_{[B]}(\mathcal{L}_\alpha(\tilde{\pi}_F, s), K_\infty, E) = 0.$$

Therefore we also have

$$m^i_{\mathrm{Eis}}(\mathcal{L}_\alpha(\tilde{\pi}_F, s), K_\infty, E) = m^i_{[P_\alpha]}(\mathcal{L}_\alpha(\tilde{\pi}_F, s), K_\infty, E).$$

Proof. From the Franke–Schwermer decomposition (Theorem 1.1.2.1) we have that the Eisenstein cohomology decomposes as

$$H^i_{\mathrm{Eis}}(\mathfrak{g}_2, K_\infty; \mathcal{A}_E(\mathrm{G}_2) \otimes E) = \bigoplus_{P \in \{P_\alpha, P_\beta, B\}} \bigoplus_{\varphi \in \Phi_{E,[P]}} H^i(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E,[P],\varphi}(\mathrm{G}_2) \otimes E).$$

We will study the summands corresponding to P_α, P_β, and B in what follows.

Case of P_α. Let φ' be an associate class of cuspidal automorphic representations for E and $[P_\alpha]$ as in Section 1.1.2. Then φ' contains a cuspidal automorphic representation of $M_\alpha(\mathbb{A}) \cong \mathrm{GL}_2(\mathbb{A})$, and which therefore must be of the form

$$\tilde{\pi}' \otimes \delta^{s'}_{M_\beta(\mathbb{A})}$$

where $\tilde{\pi}'$ is a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ and $s' \in \mathbb{C}$. After possibly conjugating by $w_{\beta\alpha\beta\alpha\beta}$, we may even assume $\mathrm{Re}(s') \geq 0$.

First, we note that the infinitesimal character of $\mathcal{A}_{E,[P_\alpha],\varphi'}(\mathrm{G}_2)$ as a $(\mathfrak{g}_2, K_\infty)$-module must match that of E. The former is given by the Weyl orbit of $\lambda_{\tilde{\pi}'} + 2s'\rho_{P_\alpha}$, where $\lambda_{\tilde{\pi}'}$ is the infinitesimal character of $\tilde{\pi}'$, and the latter is given by the Weyl orbit of $\Lambda + \rho$. But the weight $\Lambda + \rho$ is regular and real, and so since $\lambda_{\pi'}$ is a multiple of the root α and ρ_{P_α} is a multiple of the root $\alpha + 2\beta$, it follows that $\lambda_{\pi'}$ and s' are real and nonzero. In particular, $s' > 0$ since we assumed $\mathrm{Re}(s') \geq 0$.

Now we apply Theorem 1.1.3.5 and Proposition 1.1.3.2 to find that the cohomology space

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E,[P_\alpha],\varphi'}(\mathrm{G}_2) \otimes E),$$

if nontrivial, is made up of subquotients of the cohomology spaces

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{L}_\alpha(\tilde{\pi}', s') \otimes E) \tag{1.5.3.1}$$

and

$$H^*(\mathfrak{g}_2, K_\infty; \otimes \mathrm{Ind}^{\mathrm{G}_2(\mathbb{A})}_{P_\alpha(\mathbb{A})}(\tilde{\pi}' \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0})_{(2s'+1)\rho_{P_\alpha}}) \otimes E). \tag{1.5.3.2}$$

We claim that if (1.5.3.1) is nonzero, then $\tilde{\pi}'$ is cohomological. This will imply that $\tilde{\pi}'$ is attached to a cuspidal holomorphic eigenform of weight at least 2. To start, we split into two cases:

Either $\tilde{\pi}'_\infty$ is tempered or nontempered. Of course, by Selberg's conjecture, the latter possibility should not occur, but we will use the following ad-hoc argument to bypass a dependence on this conjecture.

So assume now, for sake of contradiction, both that the cohomology space (1.5.3.3) is nontrivial and that $\tilde{\pi}'_\infty$ is nontempered. By the Langlands classification for real groups, $\tilde{\pi}'_\infty$ is the Langlands quotient of a representation induced from a character, say χ, of $T(\mathbb{R})$, and then $\mathcal{L}_\alpha(\tilde{\pi}', s')_\infty$ is the Langlands quotient of a representation induced from $\chi\delta^{s'}_{P_\alpha(\mathbb{R})}$. If $\mathcal{L}_\alpha(\tilde{\pi}', s')_\infty \otimes E$ has nontrivial $(\mathfrak{g}_2, K_\infty)$-cohomology, then by [BW00], Theorem VI.1.7 (iii) (or rather, the analogue of this theorem with twisted coefficients) so does the (normalized) induced representation

$$\iota^{\mathrm{GSp}_4(\mathbb{R})}_{B(\mathbb{R})}(\chi\delta^{s'}_{P_\alpha(\mathbb{R})}).$$

By [BW00], Theorem III.3.3 and induction in stages, the induction

$$\iota^{\mathrm{GL}_2(\mathbb{R})}_{(B\cap\mathrm{GL}_2)(\mathbb{R})}(\chi\delta^{s'}_{P_\alpha(\mathbb{R})})$$

has nontrivial $(\mathfrak{sl}_2, \mathrm{O}(2))$-cohomology when twisted by some finite dimensional representation of $\mathrm{GL}_2(\mathbb{C})$, and hence so does

$$\iota^{\mathrm{GL}_2(\mathbb{R})}_{(B\cap\mathrm{GL}_2)(\mathbb{R})}(\chi)$$

since $\delta_{P_\beta(\mathbb{R})}$ is trivial on $\mathrm{SL}_2(\mathbb{R})$. Thus by [BW00], Theorem VI.1.7 (ii), $\tilde{\pi}'_\infty$, which is the Langlands quotient of this induction, also has cohomology. But the cohomological cusp forms for GL_2 are the holomorphic modular forms, which are in particular tempered at infinity. This is a contradiction.

Therefore, still assuming (1.5.3.1) is nonzero, we must have $\tilde{\pi}'_\infty$ is tempered. Then by (the twisted version of) [BW00], Lemma VI.1.5,

$$H^*(\mathfrak{g}_2, K_\infty; \iota^{\mathrm{G}_2(\mathbb{R})}_{P_\alpha(\mathbb{R})}(\tilde{\pi}'_\infty, s') \otimes E) \neq 0.$$

But by [BW00], Theorem III.3.3, this is computed in terms of the cohomology of $\tilde{\pi}'_\infty$ itself, and we conclude that $\tilde{\pi}'$ is cohomological, as desired.

If instead (1.5.3.2) is nonzero, then we can use Theorem 1.2.2.3 to conclude that $\tilde{\pi}'$ is cohomological. In any case, if

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E,[P_\alpha],\varphi'}(\mathrm{G}_2) \otimes E) \neq 0,$$

then $\tilde{\pi}' = \tilde{\pi}_{F'}$ for some cuspidal holomorphic eigenform F' of weight at least 2. Furthermore, any irreducible subquotient of this cohomology space must be an irreducible subquotient of either (1.5.3.2) or (1.4.3.1). The former, by Theorem 1.2.2.3 is a sum of copies of

$$\iota_{P_\beta(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}((\tilde{\pi}_{F'} \boxtimes \psi')_f, s'),$$

while the latter is a sum of copies of the Langlands quotient of this induction. In particular, they are all nearly equivalent and occur in this induction.

So if we now assume that

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E,[P_\alpha],\varphi'}(\mathrm{G}_2) \otimes E)$$

contains $\mathcal{L}_\alpha(\tilde{\pi}_F, s)_f$ as a subquotient, then since we have shown $s' > 0$, by Proposition 1.5.2.1, $\tilde{\pi}' = \tilde{\pi}_F$ and $s = s'$.

Therefore we have just shown that φ' contains $\tilde{\pi}_F \otimes \delta^s_{P_\alpha(\mathbb{A})}$. Since no two classes φ' overlap, this determines φ' uniquely. By Proposition 1.1.3.2, Proposition 1.5.3.2 and our vanishing assumption on the symmetric cube L-function of $\tilde{\pi}_F$, we have

$$\mathcal{A}_{E,[P_\alpha],\varphi}(\mathrm{G}_2) \cong \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}_F \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}),$$

and then Proposition 1.5.3.1 gives the $[P_\beta]$-Eisenstein multiplicities claimed.

Case of P_β. Let φ' this time be an associate class for E and P_β. Then φ' contains a representation of the form

$$\tilde{\pi}' \otimes \delta^{s'}_{M_\alpha(\mathbb{A})}$$

with $\tilde{\pi}'$ a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ and $s' \in \mathbb{C}$ with $\mathrm{Re}(s') \geq 0$. Then the same argument as in the P_α case shows that s' is real and positive.

Now we once again apply Theorem 1.1.3.5 and Proposition 1.1.3.2 to find that the cohomology space

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E,[P_\beta],\varphi'}(\mathrm{G}_2) \otimes E),$$

if nontrivial, is made up of subquotients of the cohomology spaces

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{L}_{P_\beta(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\tilde{\pi}' \otimes \psi', s') \otimes E) \tag{1.5.3.3}$$

and

$$H^*(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{\mathrm{GSp}_4(\mathbb{A})}(\tilde{\pi}' \otimes \mathrm{Sym}(\mathfrak{a}_{P_\beta,0})_{(2s'+1)\rho_{P_\beta}}) \otimes E). \tag{1.5.3.4}$$

Just as in the P_α case, the nonvanishing of either (1.5.3.3) or (1.5.3.4) implies that $\tilde{\pi}' = \tilde{\pi}_{F'}$ for a cuspidal holomorphic eigenform F' of weight at least 2, and that any irreducible subquotient of

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E,[P_\beta],\varphi'}(\mathrm{G}_2) \otimes E)$$

is nearly equivalent to an irreducible subquotient of

$$\iota_{P_\beta(\mathbb{A}_f)}^{\mathrm{G}_2(\mathbb{A}_f)}(\tilde{\pi}'_f, s').$$

Now we use Proposition 1.5.2.8 to conclude that $\mathcal{L}_\beta(\tilde{\pi}_F, s)$ cannot also occur as a subquotient, which finishes the proof in the case of P_β.

Case of B. Now we let φ' be an associate class for E and $[B]$. So φ' contains a character of $T(\mathbb{A})$ of the form

$$(\psi'_1 \boxtimes \psi'_2) \otimes e^{\langle H_B(\cdot), s'_1\alpha + s'_2\beta\rangle},$$

where ψ'_1, ψ'_2 are Dirichlet characters and $s'_1, s'_2 \in \mathbb{C}$. Let us write

$$\psi' = \psi'_1 \boxtimes \psi'_2 \boxtimes \psi'_3$$

for short.

We will study the piece $\mathcal{A}_{E,[B],\varphi'}(\mathrm{G}_2)$ of the Franke–Schwermer decomposition using the (Franke) filtration of Theorem 1.1.3.3. By that theorem, there is a filtration on the space $\mathcal{A}_{E,[B],\varphi'}(\mathrm{G}_2)$ whose graded pieces are parametrized by certain quadruples (Q, ν, Π, μ). For the convenience of the reader, we recall what these quadruples consist of now:

- Q is a standard parabolic subgroup of G_2;
- ν is an element of $(\mathfrak{t} \cap \mathfrak{m}_{Q,0})^\vee$;
- Π is an automorphic representation of $M_Q(\mathbb{A})$ occurring in

$$L^2_{\mathrm{disc}}(M_Q(\mathbb{Q})A_Q(\mathbb{R})^\circ \backslash M_Q(\mathbb{A}))$$

  and which is spanned by values at, or residues at, the point ν of Eisenstein series parabolically induced from $(B \cap M_Q)(\mathbb{A})$ to $M_Q(\mathbb{A})$ by representations in φ'; and
- μ is an element of $\mathfrak{a}^\vee_{Q,0}$ whose real part in $\mathrm{Lie}(A_{M_Q}(\mathbb{R}))$ is in the closure of the positive cone, and such that $\nu + \mu$ lies in the Weyl orbit of $\Lambda + \rho$.

Then the graded pieces of $\mathcal{A}_{E,[B],\varphi'}(\mathrm{G}_2)$ are isomorphic to direct sums of $\mathrm{G}_2(\mathbb{A}_f) \times (\mathfrak{g}_2, K_\infty)$-modules of the form

$$\mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\Pi \otimes \mathrm{Sym}(\mathfrak{a}_{Q,0})_{\mu+\rho_Q})$$

for certain quadruples (Q, ν, Π, μ) of the form just described.

For each of the four possible parabolic subgroups Q and any corresponding quadruple (Q, ν, Π, μ) as above, we will show using Proposition 1.5.2.8 that the cohomology

$$H^*(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\Pi \otimes \mathrm{Sym}(\mathfrak{a}_{Q,0})_{\mu+\rho_Q})) \tag{1.5.3.5}$$

cannot have $\mathcal{L}_\alpha(\tilde{\pi}_{F,f}, s)$ as a subquotient, which will finish the proof.

So first assume we have a quadruple (Q, ν, Π, μ) as above where $Q = B$. Then $\mathfrak{m}_{Q,0} = 0$, forcing $\nu = 0$. The entry Π is the unitarization of a representation in φ', and thus must be a character ψ' of $T(\mathbb{A})$ conjugate to $\psi'_1 \boxtimes \psi'_2$. Finally, we have μ is Weyl conjugate to $\Lambda + \rho$.

Therefore the cohomology (1.4.3.5) is isomorphic, by Theorem 1.2.2.3, to a finite sum of copies of

$$\iota_{B(\mathbb{A}_f)}^{\mathrm{GSp}_4(\mathbb{A}_f)}(\psi'_f, \mu).$$

By Proposition 1.5.2.8, $\mathcal{L}_\beta((\tilde{\pi}_F \boxtimes 1)_f, s)$ cannot be a subquotient of this space, and we conclude in the case when $Q = B$.

If now we have a quadruple (Q, ν, Π, μ) where $Q = P_\alpha$, and $\nu + \mu$ is an integral weight because it is conjugate to $\Lambda + \rho$. We find that Π is a representation generated by residual Eisenstein series at the point ν and is therefore a subquotient of the normalized induction

$$\iota_{(B\cap M_\alpha)(\mathbb{A})}^{M_\alpha(\mathbb{A})}(\psi', \nu),$$

where ψ' is a character of $T(\mathbb{A})$ conjugate to $\psi_1' \boxtimes \psi_2'$. Then by 1.2.2.3 and induction in stages, (1.5.3.5) is isomorphic to a subquotient of a finite sum of copies of

$$\iota_{B(\mathbb{A}_f)}^{\mathrm{G}_2(\mathbb{A}_f)}(\psi_f', \nu + \mu).$$

We then conclude in this case as well using Proposition 1.5.2.8.

The case when $Q = P_\beta$ is completely similar, and we omit the details. When $Q = G$, it is once again similar, but easier since we do not need to use induction in stages. So we are done. $\square$

1.5.4 Arthur's conjectures and the cuspidal multiplicity of Langlands quotients

We would like now to determine the cuspidal multiplicity of the Langlands quotient we studied in Theorem 1.5.3.3. Unfortunately, not enough information is known about the CAP representations which can occur in the cuspidal spectrum of G_2. So our computation will have to rely on some conjectures.

Recall that a cuspidal automorphic representation is CAP if it is nearly equivalent to an irreducible subquotient of a parabolically induced representation. A point of view put forth by Gan and others is that CAP representations should be studied through the lens of Arthur's conjectures, as we explain now.

In his celebrated work [Art84], Arthur introduced a series of conjectures which, for a reductive $\mathbb{Q}$-group G, classify the representations occurring in the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The data involved in this classification decomposes into local data, and so part of this classification is to build packets of representations of $G(\mathbb{Q}_v)$ for every place v. Of particular importance for us will be the shape of these local packets at $v = \infty$, and so we start (as Arthur did in [Art84]) by reviewing these conjectures for real groups.

Arthur's conjecture for real groups

Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$. Recall that $W_{\mathbb{R}}$ is the union $\mathbb{C}^{\times} \cup \mathbb{C}^{\times} j$ where the element j has the properties that $j^2 = -1$ and

$$jzj^{-1} = \overline{z}, \qquad z \in \mathbb{C}^{\times}.$$

The group $W_{\mathbb{R}}$ comes equipped with a natural multiplicative map

$$|\cdot| : W_{\mathbb{R}} \to \mathbb{R}_{>0}$$

extending the usual absolute value on $\mathbb{C}^{\times}$ and for which $|j| = 1$.

Now let $\mathbf{G}$ be a real reductive group. Attached to $\mathbf{G}$ we have the complex dual group $\mathbf{G}^{\vee}(\mathbb{C})$ and the L-group

$${}^{L}\mathbf{G} = \mathbf{G}^{\vee}(\mathbb{C}) \rtimes W_{\mathbb{R}};$$

we will not need to recall how the action of $W_{\mathbb{R}}$ on $\mathbf{G}^{\vee}(\mathbb{C})$ is defined here, but we will remark that it is trivial if $\mathbf{G}$ is split.

Langlands classified the irreducible admissible representations of $\mathbf{G}$ in terms of certain homomorphisms $\psi : W_{\mathbb{R}} \to {}^{L}\mathbf{G}$, viewed up to conjugacy under $\mathbf{G}^{\vee}(\mathbb{C})$, called *Langlands parameters.* The classification is finite-to-one from representations to parameters, the preimage of any parameter under the classification being called an *L-packet.* Certain properties of parameters correspond to certain properties of the representations in the corresponding L-packets; for example, if the projection of the image of a parameter ϕ onto $\mathbf{G}^{\vee}(\mathbb{C})$ is bounded, then ϕ is called *tempered* because all of the representations in the corresponding L-packet are tempered.

In formulating his conjectures, Arthur needed to define a new kind of parameter; the goal of his definition of parameters is not to classify representations of $\mathbf{G}$, but rather to define the local (in our case, archimedean) components of a classification of certain representations of the adelic points of a group which has $\mathbf{G}$ as its real factor. An *Arthur parameter,* as we will call it, is a homomorphism

$$\psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G},$$

viewed up to conjugacy under $\mathbf{G}^\vee(\mathbb{C})$, whose restriction to $W_{\mathbb{R}}$ is a tempered Langlands parameter.

There are at least two ways to obtain a Langlands parameter from an Arthur parameter ψ, and the correct way perhaps is not the one suggested by the definition. Instead, given an Arthur parameter ψ, we define the attached Langlands parameter $\phi_\psi : W_{\mathbb{R}} \to {}^L\mathbf{G}$ to be given by

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}\right).$$

The statement of Arthur's conjecture for $\mathbf{G}$ will involve the L-packet attached to the parameter ϕ_ψ.

It would be unreasonable for us to recall here all of the ingredients necessary to completely define everything that appears in the statement of Arthur's conjecture, but we will recall some of these ingredients now before stating the conjecture, albeit a minimal amount.

Fix now an Arthur parameter ψ for $\mathbf{G}$. Write

$$\widetilde{C}_\psi = Z(\mathrm{Im}(\psi), \mathbf{G}^\vee(\mathbb{C}))$$

for the centralizer of the image of ψ in $\mathbf{G}^\vee(\mathbb{C})$, and define the finite group

$$C_\psi = \widetilde{C}_\psi / \widetilde{C}_\psi^\circ Z({}^L\mathbf{G}, \mathbf{G}^\vee(\mathbb{C})).$$

We can make the same definition for the Langlands parameter ϕ_ψ to get a group $\widetilde{C}_{\phi_\psi}$ and a finite group C_{ϕ_ψ}, and we get a natural map

$$C_\psi \to C_{\phi_\psi},$$

which is surjective. Hence we get an injective map

$$\widehat{C}_{\phi_\psi} \to \widehat{C}_\psi,$$

where the hat denotes the set of irreducible characters of the group which it decorates.

Let ψ be an Arthur parameter for $\mathbf{G}$. Arthur's conjecture asserts that there is a unique triple $(A_\psi, \epsilon_\psi, \langle\cdot,\cdot\rangle)$ where

- A_ψ is a finite set of irreducible representations of $\mathbf{G}$,
- $\epsilon_\psi : A_\psi \to \{\pm 1\}$ is a function, and
- $\pi \mapsto \langle\cdot,\pi\rangle$ is a function $A_\psi \to \widehat{C}_\psi$,

satisfying certain properties. Among these are that A_ψ contains the L-packet for ϕ_ψ, ϵ_ψ equals 1 on this L-packet, and that for $\pi \in A_\psi$, we have that π appears in the L-packet for ϕ_ψ if and only if $\langle\cdot,\pi\rangle$ is in $\widehat{C}_{\phi_\psi}$. There are two more properties that these triples are expected to satisfy (labelled (ii) and (iii) in [Art84], Conjecture 1.3.3). The property (ii) is that a certain distribution built out of the triple and ψ is stable, and the property (iii) is an identity involving these triples for endoscopic groups of $\mathbf{G}$; it asserts that the distributions constructed in (ii) for $\mathbf{G}$ and its endoscopic groups are related by transfer.

In any case, we do not actually need the precise statement of this conjecture; we will check this conjecture in Section 1.6 for a candidate triple attached to a particular Arthur parameter for $\mathrm{G}_2(\mathbb{R})$ by proving that our triple is an instance of a general construction of Adams–Johnson [AJ87], who prove that their construction satisfies the properties asserted by Arthur's conjecture.

Given a triple $(\Pi_\psi, \epsilon_\psi, \langle\cdot,\cdot\rangle)$ as described above, let us call the component Π_ψ the *Arthur packet* attached to ψ.

Arthur's global conjecture

Arthur's archimedean conjecture discussed above also has an analogue for nonarchimedean local fields, as long as one replaces the Weil group with the Weil–Deligne group. There is also a global conjecture which Arthur formulates (at least for split groups) in Section 2 of [Art84].

So let G be a split reductive group over $\mathbb{Q}$. To make a global conjecture, one must replace the Weil–Deligne group from the local situation with the conjectural Langlands group $L_\mathbb{Q}$. For any place v, the group $L_\mathbb{Q}$ should come equipped with embeddings $W_v \to L_\mathbb{Q}$, where we use W_v to denote the Weil–Deligne group of $\mathbb{Q}_v$ unless v is archimedean, in which case we use it to denote the

Weil group.

An *Arthur parameter* is then a $G^\vee(\mathbb{C})$-conjugacy class of maps

$$\psi : L_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

satisfying certain properties. Here ${}^L G$ is the global L-group of G. Restriction of such a parameter ψ to W_v then gives a local Arthur parameter ψ_v at v.

One can also make definitions of $\widetilde{C}_\psi$ and C_ψ in the global setting, analogous to those made in the local setting. Then there are maps

$$\widetilde{C}_\psi \to \widetilde{C}_{\psi_v}, \qquad C_\psi \to C_{\psi_v}.$$

We consider the set A_ψ to be the set of all representations of the form $\pi = \otimes'_v \pi_v$ for $\pi_v \in A_{\psi_v}$. For such a π, we define $\langle \cdot, \cdot \rangle$ by

$$\langle s, \pi \rangle = \prod_v \langle s_v, \pi_v \rangle,$$

where $s \in C_\psi$ and s_v is its image in C_{ψ_v}, and $\langle \cdot, \pi_v \rangle$ is the function appearing in Arthur's local conjecture.

Then Arthur conjectures the following. First of all, the representations occurring in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ all occur in some A_ψ, and if ψ is such that $\widetilde{C}_\psi$ is finite, then the representations in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ which lie in A_ψ all occur in the discrete spectrum $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

Furthermore, he gives a formula for the multiplicity with which these representations occur in the discrete spectrum: There should be an integer $d_\psi > 0$ and a homomorphism $\xi_\psi : C_\psi \to \{\pm 1\}$ such that the multiplicity m_π for which any $\pi \in A_\psi$ occurs in $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is given by

$$m_\pi = \frac{d_\psi}{\# C_\psi} \sum_{s \in C_\psi} \langle s, \pi \rangle \xi_\psi(s).$$

If ψ is such that $\widetilde{C}_\psi$ is finite, let us call A_ψ the *Arthur packet* attached to ψ.

An Arthur packet for G_2

In [GG09], Gan and Gurevich made a study of certain automorphic representations of $\mathrm{G}_2(\mathbb{A})$ which are CAP with respect to the long root parabolic P_α, and in Section 13 of that paper, they interpret what Arthur's conjectures would mean in terms of those CAP representations. More precisely, they define a certain Arthur parameter for G_2 and explain the shape of the corresponding Arthur packets, both globally and locally. We now recall their work.

First, let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character. Then π can be viewed as a representation of $\mathrm{PGL}_2(\mathbb{A})$ and there should correspond to π a global Langlands parameter

$$\phi_\pi : L_{\mathbb{Q}} \to \mathrm{SL}_2(\mathbb{C}).$$

We remark that $\mathrm{SL}_2(\mathbb{C})$ is the dual group of PGL_2, and that since PGL_2 is split, we may replace the L-group of PGL_2 with just the dual group in the definition of Langlands parameter.

Now from ϕ_π Gan and Gurevich construct an Arthur parameter for G_2 as follows. Let γ and γ' be two orthogonal roots of G_2. Assume γ is short, so that γ' is long. Let SL_γ be the SL_2 subgroup of G_2 corresponding to the $\mathfrak{sl}_2$-triple coming from γ, and similarly for $\mathrm{SL}_{\gamma'}$. Then because γ and γ' are orthogonal, SL_γ and $\mathrm{SL}_{\gamma'}$ centralize each other. Let j_γ be the inclusion $\mathrm{SL}_\gamma \hookrightarrow \mathrm{G}_2$, and similarly define $j_{\gamma'}$. Then we can make the following composition, which we take to be our Arthur parameter ψ.

$$L_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{(\phi_\pi,\mathrm{id})} \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sim} \mathrm{SL}_\gamma(\mathbb{C}) \times \mathrm{SL}_{\gamma'}(\mathbb{C}) \xrightarrow{j_\gamma \times j_{\gamma'}} \mathrm{G}_2(\mathbb{C}).$$

The last map in this composition is well defined because SL_γ and $\mathrm{SL}_{\gamma'}$ centralize each other, and in fact its kernel is $\mu_2 = \{\pm 1\}$ diagonally embedded. Since G_2 is split and self dual, we may view Arthur parameters for G_2 as maps into $\mathrm{G}_2(\mathbb{C})$.

Now we can start to look at the multiplicity formula. According to [GG09], we have

$$\widetilde{C}_\psi = C_\psi \cong \mathbb{Z}/2\mathbb{Z}.$$

Therefore the representations occurring in $L^2(\mathrm{G}_2(\mathbb{Q})\backslash\mathrm{G}_2(\mathbb{A}))$ and A_ψ are discrete. We also have $d_\psi = 1$ and

$$\xi_\psi(c) = \begin{cases} 1 & \text{if } \epsilon(\pi, \mathrm{Sym}^3, 1/2) = 1; \\ (-1)^c & \text{if } \epsilon(\pi, \mathrm{Sym}^3, 1/2) = -1, \end{cases}$$

for $c \in \mathbb{Z}/2\mathbb{Z}$. Here $\epsilon(\pi, \mathrm{Sym}^3, 1/2)$ is the sign of the functional equation for the symmetric cube L-function of π.

Of course, to get further information, we have to inspect the local situation. Write $\pi = \otimes'_v \pi_v$. The Langlands parameter ϕ_π decomposes into local parameters ϕ_{π_v} which are the parameters attached by the local Langlands correspondence for GL_2 to the representations π_v. The local Arthur parameter ψ_v then equals the composition

$$W_v \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{(\phi_{\pi_v}, \mathrm{id})} \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\sim} \mathrm{SL}_\gamma(\mathbb{C}) \times \mathrm{SL}_{\gamma'}(\mathbb{C}) \xrightarrow{j_\gamma \times j_{\gamma'}} \mathrm{G}_2(\mathbb{C}).$$

Again according to [GG09], the local component group C_{ψ_v} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if π_v is discrete series, and is trivial otherwise. The local Arthur packet A_{ψ_v} should have two elements if π_v is discrete series, and should have one element otherwise. Let us write

$$A_{\psi_v} = \{\Pi_v^+, \Pi_v^-\}, \qquad \text{if } \pi_v \text{ is discrete series},$$

and otherwise

$$A_{\psi_v} = \{\Pi_v^+\}, \qquad \text{if } \pi_v \text{ is not discrete series}.$$

Here Π_v^+ should be the representation which is attached to ϕ_{ψ_v} by the local Langlands correspondence. Hence

$$\Pi_v^+ = \mathcal{L}_\alpha(\pi_v, 1/10),$$

which is the Langlands quotient of the unitary induction of $\pi_v \otimes |\det|^{1/2}$ from $M_\alpha(\mathbb{Q}_v)$. (We will explain later in more detail why the parameter ψ_v corresponds to this representation, at least in the archimedean case.) Then if π_v is discrete series, we have

$$\langle c, \Pi_v^+ \rangle = 1, \qquad \langle c, \Pi_v^- \rangle = (-1)^c.$$

Feeding all this back into the multiplicity formula above gives the following. If $\Pi \in \tilde{A}_\psi$ with $\Pi = \otimes'_v \Pi_v$ and with each Π_v in A_{ψ_v}, then we have $m(\Pi) = 1$ if and only if $\epsilon(\pi, \mathrm{Sym}^3, 1/2) = 1$ and $\Pi_v = \Pi_v^-$ for an even number of v, or $\epsilon(\pi, \mathrm{Sym}^3, 1/2) = -1$ and $\Pi_v = \Pi_v^-$ for an odd number of v. Otherwise $m(\Pi) = 0$.

Based on what we have seen, we feel it is reasonable to make the following conjecture.

Conjecture 1.5.4.1. *Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Write $\pi = \otimes'_v \pi_v$ and write $\mathcal{L}_\alpha(\pi_v, 1/10)$ for the Langlands quotient of the unitary induction of $\pi_v \otimes |\det|^{1/2}$ from $M_\alpha(\mathbb{Q}_v)$ to $\mathrm{G}_2(\mathbb{Q}_v)$.*

(a) Let S be the set of places v for which π_v is discrete series. For every $v \in S$, there is a representation Π_v^- of $\mathrm{G}_2(\mathbb{Q}_v)$, different from $\mathcal{L}_\alpha(\pi_v, 1/10)$, such that the following holds. Let $S' \subset S$ be a subset. Then

$$\Pi = \bigotimes_{v \in S'} \Pi_v^- \otimes \bigotimes_{v \notin S'}{}' \mathcal{L}_\alpha(\pi_v, 1/10)$$

occurs in $L^2_{\mathrm{disc}}(\mathrm{G}_2(\mathbb{Q}) \backslash \mathrm{G}_2(\mathbb{A}))$ with either multiplicity zero or one, and it does so with multiplicity one if and only if either $\epsilon(\pi, \mathrm{Sym}^3, 1/2) = 1$ and $\#S$ is even, or $\epsilon(\pi, \mathrm{Sym}^3, 1/2) = -1$ and $\#S$ is odd.

(b) If $L(\pi, \mathrm{Sym}^3, 1/2) = 0$, then the representations Π above which occur in the discrete spectrum are cuspidal.

(c) If π_∞ is the discrete series of $\mathrm{GL}_2(\mathbb{R})$ of even weight $k \geq 4$, then Π_∞^- is the discrete series representation of $\mathrm{G}_2(\mathbb{R})$ with Harish-Chandra parameter $\frac{k-4}{2}(2\alpha + 3\beta) + \rho$.

Of course, part (a) of this conjecture is just a slight reformulation of what was said above, and what was expected in [GG09]. Part (b) was also expected by [GG09], and is more generally reflective of the expected behavior for CAP forms. Part (c), on the other hand, will require some explanation, and Section 1.6 will be devoted to justifying it. Essentially, Adams and Johnson [AJ87] have made a general construction of packets corresponding to a certain type of archimedean Arthur parameters. What we will show is that the Arthur parameter ψ_∞ constructed just above is of this type, and that the corresponding Adams–Johnson construction yields a packet of two representations. We will explicitly compute these two representations and show that one is the

Langlands quotient $\mathcal{L}_\alpha(\pi_v, 1/10)$ while the other is the discrete series representation with Harish-Chandra parameter $\frac{k-4}{2}(2\alpha+3\beta)+\rho$ from our conjecture.

We remark that this discrete series representation is the one which is called "quaternionic of weight $k/2$" by Gan–Gross–Savin [GGS02]. This class of quaternionic discrete series is an analogue of the holomorphic discrete series for groups such as GSp_4. In fact, the analogue of our conjecture holds for GSp_4 and its Siegel parabolic (as partially discussed in the proof of Theorem 1.4.4.1) and one even gets holomorphic discrete series in that case.

Back to cohomology

We can now state what consequences Conjecture 1.5.4.1 has for cohomology. We consider again the Langlands quotients $\mathcal{L}_\alpha(\tilde{\pi}, 1/10)$ from Section 1.5.3.

Theorem 1.5.4.2. *Let F be a cuspidal holomorphic eigenform of even weight $k \geq 4$. Assume $L(\tilde{\pi}_F, \mathrm{Sym}^3, 1/2) = 0$. Let E be the irreducible representation of $\mathrm{G}_2(\mathbb{C})$ of highest weight $\frac{k-4}{2}(2\alpha+3\beta)$. Assume Conjecture 1.5.4.1. Then*

$$m^i_{\mathrm{cusp}}(\mathcal{L}_\alpha(\tilde{\pi}, 1/10)_f, K_\infty, E) = \begin{cases} 1 & \textit{if } \epsilon(\tilde{\pi}_F, \mathrm{Sym}^3, 1/2) = 1 \textit{ and } i = 3 \textit{ or } 5, \\ & \textit{or if } \epsilon(\tilde{\pi}_F, \mathrm{Sym}^3, 1/2) = -1 \textit{ and } i = 4; \\ 0 & \textit{otherwise.} \end{cases}$$

Consequently, under Conjecture 1.5.4.1, we have

$$m^i(\mathcal{L}_\alpha(\tilde{\pi}, 1/10)_f, K_\infty, E) = \begin{cases} 1 & \textit{if } \epsilon(\tilde{\pi}_F, \mathrm{Sym}^3, 1/2) = 1 \textit{ and } i = 3, 4, \textit{ or } 5; \\ 2 & \textit{if } \epsilon(\tilde{\pi}_F, \mathrm{Sym}^3, 1/2) = -1 \textit{ and } i = 4; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The theorem just follows from the description of the archimedean components of the representations Π appearing in Conjecture 1.5.4.1. Indeed, the discrete series representation appearing there must be cohomological in middle degree 4, and the Langlands quotient is cohomological in one degree above and below middle. The multiplicities are 1 and not higher because K_∞ is connected. $\square$

1.6 The archimedean Arthur packet for G_2

In this section, we compute what should be the archimedean Arthur packet discussed in Section 1.5.4 above. We are indebted to Jeffrey Adams, who suggested to us that this packet might be constructed via cohomological induction.

1.6.1 Cohomological induction

In this section we recall a few facts about the cohomological induction functors of Zuckerman. Everything we discuss in this section is contained in the reference of Knapp–Vogan [KV95]. We will not actually need give the definition of the cohomological induction functors because we will be able to study them explicitly enough using certain properties which we will give instead. The interested reader may refer to Chapter V of [KV95].

We now set some notation that will be in play throughout this section. Let $\mathbf{G}$ be a real reductive Lie group with complexified Lie algebra $\mathfrak{g}$. We fix $\mathbf{K}$ a maximal compact subgroup of $\mathbf{G}$ and θ a Cartan involution which gives $\mathbf{K}$. Let $\mathfrak{k}$ be the complexified Lie algebra of $\mathbf{K}$.

We fix a θ-stable Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Let $\langle \cdot, \cdot \rangle$ be the pairing on $\mathfrak{t}^\vee$ induced by the Killing form. We also fix a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ containing $\mathfrak{t}$, and we let $\mathfrak{l}$ be the Levi subalgebra of $\mathfrak{q}$ containing $\mathfrak{t}$, and $\mathfrak{u}$ its nilpotent radical. Then $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Let $\mathbf{L}$ be the Levi subgroup of $\mathbf{G}$ corresponding to $\mathfrak{l}$. Note that $\mathbf{L} \cap \mathbf{K}$ is a maximal compact subgroup of $\mathbf{L}$.

Finally, if $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra which is stable under the adjoint action of the Cartan subalgebra $\mathfrak{t}$, we write $\rho(\mathfrak{h}) \in \mathfrak{t}^\vee$ for half the sum of the roots of $\mathfrak{t}$ in $\mathfrak{h}$.

For $i \geq 0$, we consider the cohomological induction functors $\mathcal{R}^i$ from $(\mathfrak{l}, \mathbf{L} \cap \mathbf{K})$-modules to $(\mathfrak{g}, \mathbf{K})$-modules as defined in Section V.1 of [KV95]. These are normalized so that if Z is an $(\mathfrak{l}, \mathbf{L} \cap \mathbf{K})$-module with infinitesimal character given by $\Lambda \in \mathfrak{t}^\vee$, the $\mathcal{R}^i(Z)$ has infinitesimal character given by $\Lambda + \rho(\mathfrak{u})$ ([KV95], Corollary 5.25). We recall the following fact about the functors $\mathcal{R}^i$.

Theorem 1.6.1.1. *Let Z be an irreducible $(\mathfrak{l}, \mathbf{L} \cap \mathbf{K})$-module with infinitesimal character given by $\Lambda \in \mathfrak{t}^\vee$. Write $S = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$. Assume*

$$\mathrm{Re}\langle \Lambda + \rho(\mathfrak{u}), \gamma \rangle > 0$$

for all roots γ of $\mathfrak{t}$ in $\mathfrak{u}$. Then $\mathcal{R}^i(Z) = 0$ for $i \neq S$ and $\mathcal{R}^S(Z)$ is nonzero and irreducible.

Proof. This is part of Theorem 0.50 in [KV95]. $\square$

If $\Lambda \in \mathfrak{t}^\vee$ is a weight such that

$$\mathrm{Re}\langle \Lambda + \rho(\mathfrak{u}), \gamma \rangle > 0$$

for all roots γ of $\mathfrak{t}$ in $\mathfrak{u}$, like in the theorem above, then we say Λ is in the *good range.* Modules whose infinitesimal characters are sufficiently far in the good range are nice for us because they will make the spectral sequence we are about to discuss degenerate.

Now this spectral sequence will be the one for cohomological induction in stages. It is slightly tricky to state with our current notation because the functors $\mathcal{R}^i$ have a normalization built into them which will need to be undone when writing down this spectral sequence.

In the following, we will consider another θ-stable parabolic subalgebra $\mathfrak{q}'$ contained in $\mathfrak{q}$ and containing $\mathfrak{t}$. Write $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$ for the Levi decomposition, and let $\mathbf{L}'$ be the Levi subgroup of $\mathbf{G}$ corresponding to $\mathfrak{l}'$. We will also view the weight $-2\rho(\mathfrak{u}')$ as a character of $\mathbf{L}'$, and we will let $\mathbb{C}_{-2\rho(\mathfrak{u}')}$ be the associated one dimensional $(\mathfrak{l}', \mathbf{L}' \cap \mathbf{K})$-module. Similarly, we consider the one dimensional $(\mathfrak{l}', \mathbf{L}' \cap \mathbf{K})$-module $\mathbb{C}_{-2\rho(\mathfrak{u}' \cap \mathfrak{l})}$, and the one dimensional $(\mathfrak{l}, \mathbf{L} \cap \mathbf{K})$-module $\mathbb{C}_{-2\rho(\mathfrak{u})}$, both similarly defined.

Theorem 1.6.1.2. *With the notation as above, for $(\mathfrak{l}', \mathbf{L}' \cap \mathbf{K})$-modules Z, there is a convergent, first-quadrant spectral sequence*

$$\mathcal{R}^i(\mathcal{R}^j(Z \otimes \mathbb{C}_{-2\rho(\mathfrak{u}' \cap \mathfrak{l})}) \otimes \mathbb{C}_{-2\rho(\mathfrak{u})}) \Longrightarrow \mathcal{R}^{i+j}(Z \otimes \mathbb{C}_{-2\rho(\mathfrak{u}')}).$$

Proof. This is Theorem 11.77 of [KV95]. (See also the Formula (11.73) there for the discrepancy in notation which forced us to twist by characters in each step.) $\square$

Note that nothing about cohomologically induced modules is made explicit by the two theorems in this section; no result here tells us how to actually compute a given cohomologically induced module. The results of the next section will begin to do this, and can be combined with the spectral sequence above to obtain even more information.

1.6.2 Discrete series and Harish-Chandra's classification

In this section we classify discrete series representations in a manner which is classical, and then recast this classification using cohomolgical induction. Let us begin by setting some notation that will be used throughout this section.

Like the previous section we fix a $\mathbf{G}$ a real reductive Lie group with complexified Lie algebra $\mathfrak{g}$. However, now we assume that $\mathbf{G}$ contains a compact Cartan subgroup, say $\mathbf{T}_c$. By results of Harish-Chandra, this assumption is equivalent to the assumption that $\mathbf{G}$ has discrete series representations.

Let $\mathbf{K}$ be a maximal compact subgroup containing $\mathbf{T}_c$. We furthermore assume that $\mathbf{K}$ is connected, and hence so is $\mathbf{G}$. Let $\mathfrak{k}$ and $\mathfrak{t}_c$ denote, respectively, the complexified Lie algebras of $\mathbf{K}$ and $\mathbf{T}_c$. If θ is the Cartan involution of $\mathbf{G}$ which gives $\mathbf{K}$, then everything we have just defined is θ-stable. Finally, let us write $W = W(\mathfrak{t}_c, \mathfrak{g})$ for the Weyl group of $\mathfrak{t}_c$ in $\mathfrak{g}$ and $W_c = W(\mathfrak{t}_c, \mathfrak{k})$ for the compact Weyl group.

We call a weight of $\mathfrak{t}_c$ *analytically integral* if it is the differential of a character $\mathbf{T}_c \to \mathbb{C}^\times$. Harish-Chandra classified the discrete series representations of $\mathbf{G}$ in terms of certain analytically integral weights of $\mathfrak{t}_c$. Here is his classification.

Theorem 1.6.2.1 (Harish-Chandra)**.** *Let Λ be a regular weight of $\mathfrak{t}_c$. So the weight Λ determines a dominant Weyl chamber in $\mathfrak{t}_c^\vee$ and hence also an ordering on the roots of $\mathfrak{t}_c$ in $\mathfrak{g}$, and we let ρ_Λ denote half the sum of the roots which are positive with respect to this ordering.*

Then there is a bijection between W_c-orbits of regular weights Λ of $\mathfrak{t}_c$ such that $\Lambda - \rho_\Lambda$ is dominant and analytically integral, and discrete series representations of $\mathbf{G}$ with trivial central character. Let π_Λ be the discrete series representation corresponding to such a weight Λ. Then this bijection is determined by the following property.

The ordering determined by Λ also determines an ordering on the compact roots (that is, the roots of $\mathfrak{t}_c$ in $\mathfrak{k}$). Let

$$\pi_\Lambda|_K = \bigoplus_{\Lambda'} V_{\Lambda'}^{m_{\Lambda'}}$$

be the decomposition of π_Λ into its $\mathbf{K}$-types, where the sum is over all analytically integral weights Λ' of $\mathfrak{t}_c$ which are dominant with respect to the positive compact roots, and $V_{\Lambda'}$ is the irreducible *representation of $\mathfrak{k}$ with highest weight Λ'. Let $\rho_{\Lambda,c}$ denote half the sum of the positive compact roots. Then the property determining π_Λ in terms of Λ is that the smallest Λ' for which $m_{\Lambda'}$ is nonzero is*

$$\Lambda' = \Lambda + \rho_\Lambda - 2\rho_{\Lambda,c}.$$

For this particular Λ' we have $m_{\Lambda'} = 1$.

Finally, any discrete series representation can be obtained from one of the ones above by a central twist.

In the setting of this theorem, we call (the W_c orbit of) Λ the *Harish-Chandra parameter* of the discrete series representation π_Λ (or any of its central twists) and we call the representation $V_{\Lambda'}$ with

$$\Lambda' = \Lambda + \rho_\Lambda - 2\rho_{\Lambda,c}.$$

the *lowest* $\mathbf{K}$*-type* of π_Λ.

Now we explain how to get discrete series representations by cohomologically inducing characters. Let Λ be a regular weight of $\mathfrak{t}_c$, and as in Theorem 1.6.2.1, order the roots of $\mathfrak{t}_c$ in $\mathfrak{g}$ so that Λ is dominant and let ρ_Λ be half the sum of positive roots. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ which is determined by the positive roots in this ordering, and let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}_0$ be its Levi decomposition. Then $\mathfrak{t}$ is just the sum of $\mathfrak{t}_c$ and the center of $\mathfrak{g}$. Let $\mathbf{T}$ be the corresponding maximal torus in $\mathbf{G}$.

We will consider the cohomological induction from $(\mathfrak{t}, \mathbf{T} \cap \mathbf{K})$-modules to $(\mathfrak{g}, \mathbf{K})$-modules with respect to the parabolic subalgebra $\mathfrak{b}$. If $\Lambda - \rho_\Lambda$ is analytically integral, we view it as a character of $\mathbf{T}$ which has a trivial action of the center of $\mathbf{G}$, and also as a $(\mathfrak{t}, \mathbf{T} \cap \mathbf{K})$-module.

Theorem 1.6.2.2. *With notation as above, assume $\Lambda - \rho_\Lambda$ is dominant and analytically integral. Then the cohomologically induced module $\mathcal{R}(\Lambda - \rho_\Lambda)$ is isomorphic to (the $(\mathfrak{g}, \mathbf{K})$-module of) the discrete series representation of $\mathbf{G}$ with Harish-Chandra parameter Λ.*

Proof. This is Theorem 11.178(a) in [KV95]. □

We remark that the weight $\Lambda - \rho_\Lambda$ in the theorem is by definition in the good range, in the sense of Theorem 1.6.1.1. Therefore it makes sense to drop the cohomological degree from the notation $\mathcal{R}$.

1.6.3 The Adams–Johnson construction

We will now recall the main results of the work of Adams–Johnson [AJ87] as interpreted in terms of Arthur parameters. Section 3 of [AJ87] explains the connection between these results and Arthur's conjectures quickly, but there is also an article of Arthur [Art89] which explains this in more detail, and which is very explicit about the parameters involved. We mostly follow this latter article.

Most of this section is devoted to explaining the construction of the parameters that are relevant to the Adams–Johnson construction. After constructing these parameters, the construction of the associated packet by Adams–Johnson will be easy to describe using cohomological induction.

We keep the notation of the previous section, and in particular we will be working with the objects $\mathbf{G}$, $\mathbf{T}_c$, $\mathbf{K}$, $\mathfrak{g}$, $\mathfrak{t}_c$, $\mathfrak{k}$, θ, W, and W_c defined there. As before, we also write $\mathbf{T}$ for the maximal torus of $\mathbf{G}$ containing $\mathbf{T}_c$, and $\mathfrak{t}$ for its Lie algebra. Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra with Levi factor $\mathfrak{l}$ containing $\mathfrak{t}$, and let $\mathbf{L}$ be the corresponding Levi subgroup of $\mathbf{G}$. Let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{q}$.

We will consider, in what follows, the L-group of $\mathbf{G}$,

$$^L\mathbf{G} = \mathbf{G}^\vee(\mathbb{C}) \rtimes W_\mathbb{R}$$

and also that of $\mathbf{L}$. Here $W_\mathbb{R}$ is the Weil group of $\mathbb{R}$,

$$W_\mathbb{R} = \mathbb{C}^\times \cup \mathbb{C}^\times j,$$

where $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbb{C}^\times$.

Choose a maximal torus in $\mathbf{G}^\vee(\mathbb{C})$ and identify it with $\mathbf{T}^\vee(\mathbb{C})$. Then $\mathbf{L}^\vee(\mathbb{C})$ is identified with a Levi subgroup of $\mathbf{G}^\vee(\mathbb{C})$ containing $\mathbf{T}^\vee(\mathbb{C})$.

Construction of ξ

The first order of business is to construct an embedding ξ of $^L\mathbf{L}$ into $^L\mathbf{G}$. This is done on pp. 30-31 of [Art89] and we recall here the process.

We already have an embedding $\mathbf{L}^\vee(\mathbb{C}) \hookrightarrow {}^L\mathbf{G}$ because we have embedded $\mathbf{L}^\vee(\mathbb{C})$ into $\mathbf{G}^\vee(\mathbb{C})$.

So to get the embedding ξ of L-groups, we only need to describe where to send elements of $W_{\mathbb{R}}$. We will describe $\xi(z)$ for $z \in \mathbb{C}^\times$ and $\xi(j)$ separately.

First, for $z \in \mathbb{C}^\times$, let t_z be the unique element of $\mathbf{T}^\vee(\mathbb{C})$ such that

$$\Lambda^\vee(t_z) = z^{\langle \Lambda^\vee, \rho(\mathfrak{u})\rangle} \overline{z}^{-\langle \Lambda^\vee, \rho(\mathfrak{u})\rangle},$$

for any character $\Lambda^\vee$ of $\mathbf{T}^\vee(\mathbb{C})$ (equivalently, $\Lambda^\vee$ is a cocharacter of $\mathbf{T}(\mathbb{C})$). Here, as before, $\rho(\mathfrak{u})$ is half the sum of the roots of $\mathfrak{t}$ in $\mathfrak{u}$. Then the map $z \mapsto t_z$ is a homomorphism $\mathbb{C}^\times \to \mathbf{T}^\vee(\mathbb{C})$, and we set

$$\xi(z) = t_z \rtimes z.$$

We now describe $\xi(j)$. Let n_L be any element of the derived group of $\mathbf{L}^\vee(\mathbb{C})$ normalizing $\mathbf{T}^\vee(\mathbb{C})$ and such that $\mathrm{Ad}(n_L)$ sends the positive roots of $\mathrm{Lie}(\mathbf{T}^\vee(\mathbb{C}))$ in $\mathrm{Lie}(\mathbf{L}^\vee(\mathbb{C}))$ to negative ones. Similarly define the element n_G in the derived group of $\mathbf{G}^\vee(\mathbb{C})$. Then we declare

$$\xi(j) = n_G n_L^{-1} \rtimes j.$$

Thus we have defined the embedding $\xi : {}^L\mathbf{L} \to {}^L\mathbf{G}$. It depends on certain choices, but only up to conjugation in ${}^L\mathbf{G}$.

Construction of ψ

Now we construct an Arthur parameter ψ. Fix a character $\Lambda : \mathbf{L} \to \mathbb{C}^\times$. We also denote by Λ the restriction of this character to $\mathbf{T}$, and the weight of $\mathfrak{t}$ which that gives. We assume that Λ is dominant with respect to the roots of $\mathfrak{t}$ in $\mathfrak{u}$.

This character Λ, when viewed as a one dimensional representation of $\mathbf{L}$, determines by the archimedean local Langlands correspondence, a Langlands parameter

$$\phi_\Lambda : W_{\mathbb{R}} \to {}^L\mathbf{L}$$

whose image lies in $Z(\mathbf{L}^\vee(\mathbb{C})) \rtimes W_\mathbb{R}$, where we have written $Z(\mathbf{L}^\vee(\mathbb{C}))$ for the center of $\mathbf{L}^\vee(\mathbb{C})$.

Now let

$$\psi_L : W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{L}$$

be the Arthur parameter for $\mathbf{L}$ determined by the requirements that

$$\psi_L|_{W_\mathbb{R}} = \phi_\Lambda$$

and that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ maps to a principal unipotent element in $\mathbf{L}^\vee(\mathbb{C})$. Then we define the Arthur parameter ψ for $\mathbf{G}$ by

$$\psi = \xi \circ \psi_L,$$

where ξ is the embedding above.

Construction of the Adams–Johnson packet

Let $w \in W$ be a Weyl group element. We use w to twist our parabolic subalgebra $\mathfrak{q}$ in the following way. If $\Delta(\mathfrak{q})$ denotes the set of roots of $\mathfrak{t}$ in $\mathfrak{q}$, we let $\mathfrak{q}_w$ be the parabolic subalgebra of $\mathfrak{g}$ containing precisely all the roots $w\gamma$ for $\gamma \in \Delta(\mathfrak{q})$, along with $\mathfrak{t}$. Let $\mathfrak{l}_w$ be the Levi factor containing $\mathfrak{t}$ and let $\mathbf{L}_w$ be the Levi subgroup of $\mathbf{G}$ corresponding to $\mathfrak{l}_w$.

Now Lemma 2.5 (1) of [AJ87] states that all the Levis $\mathbf{L}_w$ for $w \in W$ are inner forms of each other. Therefore they have the same L-groups. So let $\phi_{\Lambda,w} : W_\mathbb{R} \to {}^L\mathbf{L}_w$ be the Langlands parameter given by ϕ_Λ, but viewed as a parameter for $\mathbf{L}_w$. Then $\phi_{\Lambda,w}$ corresponds to a one dimensional representation of $\mathbf{L}_w$, which we denote by Λ_w.

We may now define the Adams–Johnson packet. Let $\mathcal{R} = \mathcal{R}^S$ be the cohomological induction functor of Section 1.6.1 (see in particular Theorem 1.6.1.1).

Definition 1.6.3.1. The *Adams–Johnson* packet attached to the Arthur parameter ψ constructed above is the set

$$\mathrm{AJ}_\psi = \{\mathcal{R}(\Lambda_w) \mid w \in W\},$$

where, for $w \in W$, the cohomological induction of Λ_w is taken with respect to the parabolic subalgebra $\mathfrak{q}_w$.

Actually, it can be that a lot of the representations in AJ_ψ corresponding to different elements w are equal. In fact, it is noted in [AJ87] that $w, w' \in W$ lie in the same double coset in $W_c \backslash W/W(\mathfrak{t},\mathfrak{l})$ if and only if $\mathcal{R}(\Lambda_w) \cong \mathcal{R}(\Lambda_{w'})$. Arthur notes in [Art89] that this set $W_c \backslash W/W(\mathfrak{t},\mathfrak{l})$ of double cosets is in bijection with the component group C_ψ attached to ψ.

Now we have the following theorem, which is the main result of [AJ87] as interpreted by Arthur [Art89].

Theorem 1.6.3.2 (Adams–Johnson)**.** *For each ψ as above, there is a function $\epsilon_\psi : \mathrm{AJ}_\psi \to \{\pm 1\}$ and a pairing $\langle \cdot, \cdot \rangle$ between C_ψ and AJ_ψ, such that the triples $(\mathrm{AJ}_\psi, \epsilon_\psi, \langle \cdot, \cdot \rangle)$ satisfy the conclusion of Arthur's conjecture ([Art84], Conjecture 1.3.3).*

Proof. This is the main result of [AJ87]; See Theorems 2.13 and 2.21 there. Arthur [Art89], Section 5, also describes how to get the objects ϵ_ψ and $\langle \cdot, \cdot \rangle$ from the objects appearing in [AJ87]. $\square$

Remark 1.6.3.3. Strictly speaking, although it is mentioned in [AJ87] and [Art89] that AJ_ψ contains the L-packet attached to the Langlands parameter ϕ_ψ associated with ψ, a proof of this is written down in neither of these references. We will be able to check this directly, however, for the packet that we obtain for $\mathrm{G}_2(\mathbb{R})$ in the next section.

1.6.4 Determination of the packet for $\mathrm{G}_2(\mathbb{R})$

Recall that in Section 1.5.4 we constructed a global Arthur parameter for G_2 whose associated packet should contain the CAP forms that are nearly equivalent to the Eisenstein series considered there. This Arthur parameter has a local archimedean component, and in this section we will recall how it is constructed and denote it ψ'.

This notation suggests that there will be another Arthur parameter in play, and indeed, we will construct one via the process of the previous section. This other parameter will be denoted ψ. But the parameters ψ and ψ' will turn out to be equal, which means that we can apply the methods of the previous section and obtain an Adams–Johnson packet for ψ, or equivalently, for ψ'. We then determine the representations in this packet explicitly. They will turn out to be the Langlands quotient and the discrete series representation discussed in Conjecture 1.5.4.1.

To be consistent with the rest of this section, we change some of the notation used in Section 1.5.1. In particular, let us write $\mathbf{G}_2 = \mathrm{G}_2(\mathbb{R})$ for the real split G_2, $\mathbf{K}$ for a fixed maximal compact subgroup of $\mathbf{G}_2$, and $\mathbf{T}_c$ for a fixed maximal torus contained in $\mathbf{K}$. Let $\mathfrak{g}_2$, $\mathfrak{k}$, and $\mathfrak{t}_c$ be the respective complexified Lie algebras. We still write α and β, respectively, for fixed long and short simple roots of $\mathfrak{t}_c$ in $\mathfrak{g}_2$, and we assume we have chosen $\mathbf{K}$ so that $\pm\beta$ and $\pm(2\alpha+3\beta)$ are the compact roots. We fix θ the Cartan involution giving $\mathbf{K}$. Write W for the Weyl group of $\mathfrak{t}_c$ in $\mathfrak{g}_2$ and W_c for the Weyl group of $\mathfrak{t}_c$ in $\mathfrak{k}$.

On the dual side of things, we have $\mathbf{G}_2^\vee(\mathbb{C}) = \mathrm{G}_2(\mathbb{C})$, and we fix a maximal torus in $\mathrm{G}_2(\mathbb{C})$, identifying it with $\mathbf{T}_c^\vee(\mathbb{C})$. Passage to the dual side switches the long and short simple roots, so $\alpha^\vee$ becomes a short simple root for $\mathbf{T}_c^\vee(\mathbb{C})$ in $\mathrm{G}_2(\mathbb{C})$, and $\beta^\vee$ becomes a long simple root. In order to distinguish when we are on the dual side and when we are not, we will denote roots of $\mathbf{T}_c^\vee(\mathbb{C})$ in $\mathrm{G}_2(\mathbb{C})$ with a prime and thus write $\beta' = \alpha^\vee$ and $\alpha' = \beta^\vee$. Then we have

$$(\alpha+\beta)^\vee = \alpha' + 3\beta', \qquad (\alpha+2\beta)^\vee = 2\alpha' + 3\beta',$$
$$(\alpha+3\beta)^\vee = \alpha' + \beta', \qquad (2\alpha+3\beta)^\vee = \alpha' + 2\beta'.$$

The parameter ψ'

Fix throughout this section an even integer $k \geq 4$. We denote by π the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with trivial central character. Then π may be viewed as a representation of $\mathrm{PGL}_2(\mathbb{R})$. Let

$$\phi : W_{\mathbb{R}} \times {}^L\mathrm{PGL}_2(\mathbb{R})$$

be its Langlands parameter. This can be made explicit. For one thing, the L-group of $\mathrm{PGL}_2(\mathbb{R})$ is just $\mathrm{SL}_2(\mathbb{C}) \times W_{\mathbb{R}}$, and then ϕ takes the following form. For $z \in \mathbb{C}^\times$, we have

$$\phi(z) = \begin{pmatrix} (z/\overline{z})^{(k-1)/2} & 0 \\ 0 & (z/\overline{z})^{-(k-1)/2} \end{pmatrix} \times z, \tag{1.6.4.1}$$

which is an element of $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times \subset \mathrm{SL}_2(\mathbb{C}) \times W_{\mathbb{R}}$, and

$$\phi(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times j. \tag{1.6.4.2}$$

(See, for example [Pra18], Proposition 2.) Note that the quantity $(z/\overline{z})^{(k-1)/2}$ should be interpreted as

$$(z/\overline{z})^{(k-1)/2} = |z|^{k-1}\overline{z}^{-(k-1)}.$$

For γ' a root of $\mathbf{T}_c^\vee(\mathbb{C})$ in $\mathrm{G}_2(\mathbb{C})$, let $\mathrm{SL}_{\gamma'}(\mathbb{C}) \subset \mathrm{G}_2(\mathbb{C})$ be the $\mathrm{SL}_2(\mathbb{C})$ associated with γ'. It is generated by the images of the unipotent root group homomorphisms $\mathbf{x}_{\pm\gamma'}$ corresponding to $\pm\gamma'$. If γ_1' and γ_2' are orthogonal roots, then the elements in the image of $\mathbf{x}_{\gamma_1'}$ and $\mathbf{x}_{\gamma_2'}$ commute. Hence $\mathrm{SL}_{\gamma_1'}(\mathbb{C})$ and $\mathrm{SL}_{\gamma_2'}(\mathbb{C})$ centralize each other. We thus get a map

$$\mathrm{SL}_{\gamma_1'}(\mathbb{C}) \times \mathrm{SL}_{\gamma_2'}(\mathbb{C}) \to \mathrm{G}_2(\mathbb{C}). \tag{1.6.4.3}$$

Now the maximal torus $\mathbf{T}_c^\vee(\mathbb{C})$ in $\mathrm{G}_2(\mathbb{C})$ is just the image under this map of the product of the diagonal tori in $\mathrm{SL}_{\gamma_1'}(\mathbb{C})$ and $\mathrm{SL}_{\gamma_2'}(\mathbb{C})$. The character group of $\mathbf{T}_c^\vee(\mathbb{C})$ is generated by its root lattice, and it is visible from the root lattice of G_2 that the characters γ_1' and γ_2' generate an index 2 subgroup of the character group of $\mathbf{T}_c^\vee(\mathbb{C})$. It follows that the map above has a kernel of order 2. The character $(\gamma_1' + \gamma_2')/2$ is a root of $\mathbf{T}_c^\vee(\mathbb{C})$, and it generates the whole character group along with γ_1' and γ_2'. All three of these characters, when lifted to $\mathrm{SL}_{\gamma_1'}(\mathbb{C}) \times \mathrm{SL}_{\gamma_2'}(\mathbb{C})$, are trivial on the diagonally embedded $\mu_2 = \{\pm 1\}$, and so in fact the kernel of the above map is this μ_2. Thus we identify

$$\mathrm{SL}_{\gamma_1'}(\mathbb{C}) \times \mathrm{SL}_{\gamma_2'}(\mathbb{C})/\mu_2$$

as a subgroup of $\mathrm{G}_2(\mathbb{C})$ in this way. It contains $\mathbf{T}_c^\vee(\mathbb{C})$ and is simply the subgroup generated by the images of the unipotent root group homomorphisms $\mathbf{x}_{\pm\gamma_1'}$ and $\mathbf{x}_{\pm\gamma_2'}$.

Now we can define the Arthur parameter ψ'. It is the composition

$$\begin{aligned} W_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{(\phi,\mathrm{id})} (\mathrm{SL}_2(\mathbb{C}) \times W_\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C}) &\xrightarrow{\sim} \mathrm{SL}_{\beta'}(\mathbb{C}) \times \mathrm{SL}_{2\alpha'+3\beta'}(\mathbb{C}) \times W_\mathbb{R} \\ &\to \mathrm{G}_2(\mathbb{C}) \times W_\mathbb{R} = {}^L\mathbf{G}_2, \end{aligned}$$

where middle map leaves the order of the SL_2's the same and only rearranges the placement of $W_\mathbb{R}$, and the last map is the product of the map from (1.6.4.3) with the identity map of $W_\mathbb{R}$. To be clear, ϕ is mapping into the subgroup $\mathrm{SL}_{\beta'}(\mathbb{C}) \times W_\mathbb{R}$ of ${}^L\mathbf{G}_2$, so the image of the restriction of ψ'

to $W_{\mathbb{R}}$ lands in that subgroup.

The choice of the pair $(\beta', 2\alpha' + 3\beta')$ of orthogonal roots doesn't really matter, as long as ϕ is mapping to the short root $\mathrm{SL}_2(\mathbb{C})$. Any other choice of orthogonal roots would lead to an Arthur parameter which is conjugate to ψ'.

The Levi $\mathbf{L}_{1,1}$

We now begin working towards constructing a parameter ψ via the constructions from Section 1.6.3, and therefore we must start by constructing a Levi subgroup of $\mathbf{G}_2$.

First, let $\mathfrak{q}_{1,1}$ be the parabolic subalgebra of $\mathfrak{g}_2$ whose Levi $\mathfrak{l}_{1,1}$ contains the roots $\pm(\alpha + 2\beta)$ along with $\mathfrak{t}_c$, and whose nilpotent radical $\mathfrak{u}_{1,1}$ contains the roots $-(\alpha + 3\beta)$, $-\beta$, α, $\alpha + \beta$, and $2\alpha + 3\beta$. (These five roots are the ones lying above the line containing $\alpha + 2\beta$ in the root diagram.) Then let $\mathbf{L}_{1,1}$ be the Levi subgroup of $\mathbf{G}_2$ containing $\mathbf{T}_c$ and corresponding to $\mathfrak{l}_{1,1}$. The notation is justified by the following lemma.

Lemma 1.6.4.1. *The Levi* $\mathbf{L}_{1,1}$ *is isomorphic to* $\mathrm{U}(1,1)$

Proof. First we look at the complexified situation. The group $\mathbf{L}_{1,1}(\mathbb{C})$ is the Levi subgroup of $\mathrm{G}_2(\mathbb{C})$ containing $\mathbf{T}_c(\mathbb{C})$ and the images of the unipotent root group homomorphisms $\mathbf{x}_{\pm(\alpha+2\beta)}$. Therefore, as α is orthogonal to $\alpha + 2\beta$, $\mathbf{L}_{1,1}(\mathbb{C})$ is the subgroup of

$$\mathrm{SL}_{\alpha+2\beta}(\mathbb{C}) \times \mathrm{SL}_{\alpha}(\mathbb{C})/\mu_2$$

generated by the first factor and the diagonal torus from the second factor.

Now we take real points. The group of real points in the diagonal torus of $\mathrm{SL}_{\alpha}(\mathbb{C})$ is a one dimensional subtorus of $\mathbf{T}_c$, hence is a circle $\mathrm{U}(1)$, and the group of real points of $\mathrm{SL}_{\alpha+2\beta}(\mathbb{C})$ is a form of $\mathrm{SL}_2(\mathbb{R})$. Since the root $\alpha + 2\beta$ is noncompact, this form is noncompact and is thus $\mathrm{SL}_2(\mathbb{R})$ itself.

We conclude that

$$\mathbf{L}_{1,1} \cong \mathrm{SL}_2(\mathbb{R}) \times \mathrm{U}(1)/\mu_2,$$

and we are done since this latter group is $\mathrm{U}(1,1)$. $\square$

The embedding ξ

We now describe the embedding ξ constructed in Section 1.6.3 in our current context of $\mathbf{L}_{1,1}$. The complexification $\mathbf{L}_{1,1}(\mathbb{C})$ is the Levi subgroup of $\mathrm{G}_2(\mathbb{C})$ containing $\alpha+2\beta$ and the torus $\mathbf{T}_c(\mathbb{C})$, and therefore the dual group $\mathbf{L}_{1,1}^\vee(\mathbb{C})$ is the Levi containing $(\alpha+2\beta)^\vee = 2\alpha'+3\beta'$ and $\mathbf{T}_c^\vee(\mathbb{C})$. The group $\mathbf{L}_{1,1}^\vee(\mathbb{C})$ is therefore the subgroup of

$$\mathrm{SL}_{\beta'}(\mathbb{C}) \times \mathrm{SL}_{2\alpha'+3\beta'}(\mathbb{C})/\mu_2$$

containing the factor $\mathrm{SL}_{2\alpha'+3\beta'}(\mathbb{C})$ and the diagonal torus in the factor $\mathrm{SL}_{\beta'}(\mathbb{C})$. For

$$(A,B) \in \mathrm{SL}_{\beta'}(\mathbb{C}) \times \mathrm{SL}_{2\alpha'+3\beta'}(\mathbb{C}),$$

let $[A,B]$ denote the image of (A,B) modulo μ_2, viewed as an element of $\mathrm{G}_2(\mathbb{C})$.

To describe explicitly the embedding

$$\xi : {}^L\mathbf{L}_{1,1} \hookrightarrow {}^L\mathbf{G}_2,$$

we need to describe explicitly the elements t_z, $n_{L_{1,1}}$, and n_{G_2} from Section 1.6.3. Recall that for $z \in \mathbb{C}^\times$, $t_z \in \mathbf{T}_c^\vee(\mathbb{C})$ was defined by the requirement that for any character $\Lambda^\vee$ of $\mathbf{T}_c^\vee(\mathbb{C})$, we have

$$\Lambda^\vee(t_z) = z^{\langle \Lambda^\vee, \rho(\mathfrak{u}_{1,1})\rangle} \overline{z}^{-\langle \Lambda^\vee, \rho(\mathfrak{u}_{1,1})\rangle},$$

where $\rho(\mathfrak{u}_{1,1})$ is half the sum of roots of $\mathfrak{t}_c$ in $\mathfrak{u}_{1,1}$. This half sum equals $3\alpha/2$, and so we are requiring

$$\Lambda^\vee(t_z) = (z/\overline{z})^{\langle \Lambda^\vee, 3\alpha/2\rangle}.$$

Since $\alpha^\vee = \beta'$, it follows that

$$t_z = \left[\begin{pmatrix} (z/\overline{z})^{3/2} & 0 \\ 0 & (z/\overline{z})^{-3/2} \end{pmatrix}, 1\right] \in \mathrm{SL}_{\beta'}(\mathbb{C}) \times \mathrm{SL}_{2\alpha'+3\beta'}(\mathbb{C})/\mu_2.$$

Next we have the element n_{G_2}, which is any element of $\mathrm{G}_2(\mathbb{C})$ that normalizes $\mathbf{T}_c^\vee(\mathbb{C})$ and whose adjoint action negates every positive root. Thus we can take

$$n_{G_2} = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right].$$

The adjoint action of this element negates the orthogonal roots β' and $2\alpha' + 3\beta'$ and therefore acts as negation on the whole root lattice.

Finally, we have the element $n_{L_{1,1}}$, which is any element of the derived group of $\mathbf{L}_{1,1}^\vee(\mathbb{C})$ which normalizes $\mathbf{T}_c^\vee(\mathbb{C})$ and whose adjoint action negates every positive root of $\mathbf{L}_{1,1}^\vee(\mathbb{C})$. The derived group of $\mathbf{L}_{1,1}^\vee(\mathbb{C})$ is just the subgroup $\mathrm{SL}_{2\alpha'+3\beta'}(\mathbb{C})$, and so we may take

$$n_{L_{1,1}} = \left[1, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right].$$

We thus have

$$n_{G_2} n_{L_{1,1}}^{-1} = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right].$$

The embedding ξ is then just defined to be the usual inclusion

$$\mathbf{L}_{1,1}^\vee(\mathbb{C}) \hookrightarrow \mathrm{G}_2(\mathbb{C}) \subset {}^L\mathbf{G}_2$$

on the subgroup $\mathbf{L}_{1,1}^\vee(\mathbb{C})$ of ${}^L\mathbf{L}_{1,1}$, and on the Weil group it is defined by the rules $\xi(z) = t_z \rtimes z$ for $z \in \mathbb{C}^\times$, and $\xi(j) = n_{G_2} n_{L_{1,1}}^{-1} \rtimes j$. Thus

$$\xi(z) = \left[\begin{pmatrix} (z/\overline{z})^{3/2} & 0 \\ 0 & (z/\overline{z})^{-3/2} \end{pmatrix}, 1\right] \rtimes z, \qquad z \in \mathbb{C}^\times, \tag{1.6.4.4}$$

and

$$\xi(j) = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right] \rtimes j.$$

The parameter ψ

To construct our Arthur parameter ψ, we must start with a character Λ of $\mathbf{L}_{1,1}$. Such characters can be specified by weights of $\mathbf{T}_c$ that are multiples of $\alpha/2$. The weight α itself corresponds to the determinant character of $\mathbf{L}_{1,1} \cong \mathrm{U}(1,1)$. Thus we set

$$\Lambda = \frac{k-4}{2}\alpha.$$

The archimedean local Langlands correspondence attaches to this character the Langlands parameter

$$\phi_\Lambda : W_{\mathbb{R}} \to {}^L\mathbf{L}_{1,1}$$

given by

$$\phi_\Lambda(z) = \left[\begin{pmatrix} (z/\overline{z})^{(k-4)/2} & 0 \\ 0 & (z/\overline{z})^{-(k-4)/2} \end{pmatrix}, 1\right] \rtimes z \in \mathbf{L}_{1,1}^\vee(\mathbb{C})$$

for $z \in \mathbb{C}^\times$ and

$$\phi_\Lambda(j) = 1 \rtimes j.$$

We now define a parameter $\psi_{L_{1,1}}$ for the Levi, as in Section 1.6.3. This requires us to choose a principal unipotent element in $\mathbf{L}_{1,1}^\vee(\mathbb{C})$, and we choose $[1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)]$. Thus $\psi_{L_{1,1}}$ is defined to be the Arthur parameter

$$\psi_{L_{1,1}} : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{L}_{1,1}$$

given by

$$\psi_{L_{1,1}}(w, 1) = \phi_\Lambda(w), \qquad w \in W_{\mathbb{R}}$$

and

$$\psi_{L_{1,1}}\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \left[1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] \rtimes 1.$$

It follows that the restriction of $\psi_{L_{1,1}}$ to $\mathrm{SL}_2(\mathbb{C})$ is just the identification of $\mathrm{SL}_2(\mathbb{C})$ with $\mathrm{SL}_{2\alpha'+3\beta'}(\mathbb{C})$.

Finally, we define

$$\psi = \xi \circ \psi_{L_{1,1}}.$$

We have the following lemma, which is now not so difficult.

Lemma 1.6.4.2. *The parameters ψ and ψ' are equal.*

Proof. We check that ψ and ψ' coincide on $\mathbb{C}^\times$, j, and $\mathrm{SL}_2(\mathbb{C})$. We have, for $z \in \mathbb{C}^\times$,

$$\begin{aligned}
\psi(z) = \xi(\psi_{L_{1,1}}(z)) &= \xi\left(\left[\begin{pmatrix} (z/\bar{z})^{(k-4)/2} & 0 \\ 0 & (z/\bar{z})^{-(k-4)/2} \end{pmatrix}, 1\right] \rtimes z\right) \\
&= \left(\left[\begin{pmatrix} (z/\bar{z})^{(k-4)/2} & 0 \\ 0 & (z/\bar{z})^{-(k-4)/2} \end{pmatrix}, 1\right] \cdot t_z\right) \rtimes z \\
&= \left[\begin{pmatrix} (z/\bar{z})^{(k-1)/2} & 0 \\ 0 & (z/\bar{z})^{-(k-1)/2} \end{pmatrix}, 1\right] \rtimes z \qquad \text{(by (1.6.4.4))} \\
&= \psi'(z),
\end{aligned}$$

where the last equality is just the definition of ψ', along with (1.6.4.1). Also,

$$\xi(\psi_{L_{1,1}}(j)) = \xi(1 \rtimes j) = n_{G_2} n_{L_{1,1}}^{-1} \rtimes j = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right) \rtimes j = \psi'(j)$$

by (1.6.4.2). Finally, ψ and ψ' coincide on $\mathrm{SL}_2(\mathbb{C})$, because when restricted to $\mathrm{SL}_2(\mathbb{C})$, both become the inclusion of $\mathrm{SL}_{2\alpha'+3\beta'}(\mathbb{C})$ into ${}^L\mathbf{G}_2$. Therefore we have $\psi = \psi'$, as desired. $\square$

The character Λ_w

Consider the set of double cosets

$$W_c \backslash W / W(\mathfrak{t}_c, \mathfrak{l}_{1,1}).$$

This set has two elements and a representative for the nontrivial coset is given by the Weyl group element that rotates the root lattice by $\pi/3$ clockwise. Let w be this element.

We consider the parabolic subalgebra $\mathfrak{q}_2$ which contains $w\gamma$ for every root γ of $\mathfrak{t}_c$ in $\mathfrak{q}_{1,1}$. Thus $\mathfrak{q}_2$ contains the all the positive roots along with $-\beta$. The Levi subalgebra of $\mathfrak{q}_2$ containing $\mathfrak{t}_c$ contains the roots $\pm\beta$.

Let $\mathbf{L}_2$ be the Levi subgroup of $\mathbf{G}_2$ containing $\mathfrak{t}_c$ and corresponding to $\mathfrak{l}_2$. Again, the notation is suggestive of the following lemma.

Lemma 1.6.4.3. *The Levi $\mathbf{L}_2$ is isomorphic to* $\mathrm{U}(2)$.

Proof. The proof is completely similar to the proof of Lemma 1.6.4.1, except we replace the root $\alpha + 2\beta$ there by β. Then β is compact, so the form of $\mathrm{SL}_2(\mathbb{R})$ that appears here is $\mathrm{SU}(2)$. $\square$

Similar to what was discussed above for $\mathbf{L}_{1,1}$, the dual group $\mathbf{L}_2^\vee(\mathbb{C})$ of $\mathbf{L}_2$ is the subgroup of

$$\mathrm{SL}_{\alpha'+2\beta'}(\mathbb{C}) \times \mathrm{SL}_{\alpha'}(\mathbb{C})/\mu_2$$

generated by the factor $\mathrm{SL}_{\alpha'}$ along with the diagonal torus in $\mathrm{SL}_{\alpha'+2\beta'}(\mathbb{C})$. The Langlands parameter ϕ can be conjugated to give a parameter that sends $z \in \mathbb{C}^\times$ to

$$\begin{pmatrix} (z/\overline{z})^{(k-4)/2} & 0 \\ 0 & (z/\overline{z})^{-(k-4)/2} \end{pmatrix} \in \mathrm{SL}_{\alpha'+2\beta'}(\mathbb{C})$$

and which sends j to $1 \rtimes j$. This is how we view the parameter ϕ as a Langlands parameter for $\mathbf{L}_2$.

Corresponding to this parameter via the archimedean local Langlands correspondence is the character we call Λ_w; it is the character of $\mathbf{L}_2$ which acts on $\mathbf{T}_c$ via the weight $\frac{k-4}{2}(2\alpha + 3\beta)$.

The packet AJ_ψ

We can now construct our packet AJ_ψ. By definition, it consists of the cohomologically induced representations

$$\mathcal{R}(\Lambda) \text{ and } \mathcal{R}(\Lambda_w).$$

The following is the main result of this section.

Theorem 1.6.4.4. *We have that*

$$\mathcal{R}(\Lambda) \cong \mathcal{L}_\alpha(\pi, 1/10),$$

the Langlands quotient of the parabolic induction of π *from the long root parabolic, where* π *is the discrete series representation of* $\mathrm{GL}_2(\mathbb{R})$ *of weight* k*, and we have that* $\mathcal{R}(\Lambda_w)$ *is the discrete series representation of* $\mathrm{G}_2(\mathbb{R})$ *with Harish-Chandra parameter* $\frac{k-4}{2}(2\alpha + 3\beta) + \rho$*, where* $\rho = 3\alpha + 5\beta$ *is half the sum of positive roots.*

Thus the Adams–Johnson packet attached to $\psi = \psi'$ *consists of* $\mathcal{L}_\alpha(\pi, 1/10)$ *and this discrete series representation.*

Proof. We study $\mathcal{R}(\Lambda_w)$ first, using the spectral sequence of Theorem 1.6.1.2. Let $\mathfrak{b}$ be the standard Borel subalgebra of $\mathfrak{g}_2$ containing $\mathfrak{t}_c$, and $\mathfrak{u}$ its unipotent radical. Let Z be the one dimensional

$(\mathfrak{t}_c, \mathbf{T}_c)$-module given by the character

$$\frac{k-4}{2}(2\alpha+3\beta)+(6\alpha+10\beta).$$

We will induce first from $\mathfrak{t}_c$ to $\mathfrak{l}_2$, and then from $\mathfrak{l}_2$ to $\mathfrak{g}_2$. The relevant degrees S for the cohomological inductions (see Theorem 1.6.1.1) are both $S=1$; since $\mathfrak{l}_2$ is compact, the degree S for the induction from $\mathbf{T}_c$ to $\mathbf{L}_2$ just equals the dimension of the unipotent radical of the Borel subalgebra of $\mathfrak{l}_2$, and for $\mathbf{L}_2$ to $\mathbf{G}_2$, the degree is the number of compact roots not in $\mathfrak{l}_2$. Both of these numbers are 1.

Now the first step is to induce $Z \otimes \mathbb{C}_{-2\rho(\mathfrak{u}\cap\mathfrak{l}_2)}$ to $\mathbf{L}_2$. The weight $2\rho(\mathfrak{u}\cap\mathfrak{l}_2)$ equals β, and hence

$$\mathcal{R}^1(Z \otimes \mathbb{C}_{-2\rho(\mathfrak{u}\cap\mathfrak{l}_2)})$$

is the discrete series representation of $\mathbf{L}_2$ with Harish-Chandra parameter

$$\frac{k-4}{2}(2\alpha+3\beta)+(6\alpha+9\beta)+\frac{1}{2}\beta,$$

by Theorem 1.6.2.2. Since $\mathbf{L}_2$ is compact by Lemma 1.6.4.3, this is just the character of $\mathbf{L}_2$ given by

$$\frac{k-4}{2}(2\alpha+3\beta)+(6\alpha+9\beta).$$

Now $2\rho(\mathfrak{u}_2)=6\alpha+9\beta$, so

$$\mathcal{R}^1(Z \otimes \mathbb{C}_{-2\rho(\mathfrak{u}\cap\mathfrak{l}_2)}) \otimes \mathbb{C}_{-2\rho(\mathfrak{u}_2)}$$

is the one dimensional representation of $\mathbf{L}_2$ given by Λ_w, and therefore

$$\mathcal{R}^1(\mathcal{R}^1(Z \otimes \mathbb{C}_{-2\rho(\mathfrak{u}\cap\mathfrak{l}_2)}) \otimes \mathbb{C}_{-2\rho(\mathfrak{u}_2)}) = \mathcal{R}(\Lambda_w).$$

By Theorem 1.6.1.2, $\mathcal{R}(\Lambda_w)$ is the term $E_2^{1,1}$ of a spectral sequence converging to $\mathcal{R}(Z\otimes\mathbb{C}_{-2\rho(\mathfrak{u})})$. The other terms in this spectral sequence vanish by Theorem 1.6.1.1, so we have

$$\mathcal{R}(\Lambda_w) = \mathcal{R}(Z \otimes \mathbb{C}_{-2\rho(\mathfrak{u})}).$$

But $2\rho(\mathfrak{u}) = 6\alpha + 10\beta$, so $Z \otimes \mathbb{C}_{-2\rho(\mathfrak{u})}$ is the character $\frac{k-4}{2}(2\alpha + 3\beta)$. Thus by Theorem 1.6.2.2 again, $\mathcal{R}(\Lambda_w)$ is just the discrete series representation of $\mathbf{G}_2$ with Harish-Chandra parameter

$$\frac{k-4}{2}(2\alpha + 3\beta) + \rho,$$

as desired.

Now we show that $\mathcal{R}(\Lambda)$ is the Langlands quotient claimed. The key to this is a theorem in Vogan's book [Vog81], Theorem 6.6.15, which links the composition of ordinary parabolic induction with cohomological induction with the composition in the opposite order. Instead of recalling the theorem in general, we explain what it means in our special case. It requires three types of data as input: We need what Vogan calls θ-stable data, character data, and cuspidal data, which are defined in general in Definitions 6.5.1, 6.6.1, and 6.6.11, respectively, in Vogan's book. Moreover, there is a bijection between these first two kind of data (Proposition 6.6.2 in [Vog81]) and a surjective map from pieces of character data to pieces of cuspidal data (Proposition 6.6.12 in [Vog81]). Pieces of θ-stable data are used to construct cohomological inductions of parabolically induced representations and in our case will be used to realize the representation $\mathcal{R}(\Lambda)$. On the other hand, cuspidal data are used to construct parabolic inductions of discrete series representations and will be used to realize $\mathcal{L}_\alpha(\pi, 1/10)$. Theorem 6.6.15 in Vogan's book will then state that these two constructions coincide. We note that this theorem is stated in terms of Langlands subrepresentations instead of Langlands quotients, so we have to make a few minor adjustments.

To build the θ-stable data we need, we first construct a certain θ-stable maximal torus of $\mathbf{G}_2$. Let $\mathbf{T}_0$ be the center of $\mathbf{L}_{1,1}$. Let $\mathbf{A}$ be the θ-stable maximal split torus in the derived group of $\mathbf{L}_{1,1}$. Then $\mathbf{H} = \mathbf{T}_0\mathbf{A}$ is a maximal torus in $\mathbf{G}_2$. It is neither split nor compact. Let $\mu : \mathbf{T}_0 \to \mathbb{C}^\times$ be given by

$$\mu = \Lambda|_{\mathbf{T}_0} = \frac{k-4}{2}\alpha|_{\mathbf{T}_0}.$$

Fix a minimal parabolic subgroup $\mathbf{B}_{1,1}$ in $\mathbf{L}_{1,1}$ containing $\mathbf{H}$, and let $\nu : \mathbf{A} \to \mathbb{C}^\times$ be the character given by

$$\nu = \delta_{\mathbf{B}_{1,1}}^{-1/2}|_{\mathbf{A}}.$$

Then the quadruple $(\mathfrak{q}_{1,1}, \mathbf{H}, \mu, \nu)$ is a piece of θ-stable data in the sense of [Vog81]. We write $\mu \otimes \nu$

for the character of $\mathbf{H}$ given by μ on $\mathbf{T}_0$ and by ν on $\mathbf{A}$, and we construct the representation (called a *standard module* for our data) given by

$$\mathcal{R}(\mathrm{Ind}_{\mathbf{B}_{1,1}}^{\mathbf{L}_{1,1}}((\mu \otimes \nu) \otimes \delta_{\mathbf{B}_{1,1}}^{1/2})). \tag{1.6.4.5}$$

Of course, in the parabolic induction, the characters ν and $\delta_{\mathbf{B}_{1,1}}^{1/2}$ cancel, and the parabolic induction thus becomes

$$\mathrm{Ind}_{\mathbf{B}_{1,1}}^{\mathbf{L}_{1,1}}(\mu \otimes 1).$$

By definition of μ, this contains Λ as its unique subrepresentation. Since cohomological induction is exact in the good range, we see that $\mathcal{R}(\Lambda)$ is a subrepresentation of (1.6.4.5).

Now we construct a piece of character data from $(\mathfrak{q}_{1,1}, \mathbf{H}, \mu, \nu)$ as in [Vog81]. For us this will be a pair $(\mathbf{H}, \Gamma)$ where $\Gamma : \mathbf{H} \to \mathbb{C}^\times$ is a character satisfying certain properties. (Actually, Vogan's definition contains also the data of a character of the complexified Lie algebra $\mathfrak{h}$ of $\mathbf{H}$, but that character is determined from the differential of Γ.) We set $\Gamma|_{\mathbf{A}} = \nu$, and we let $\Gamma|_{\mathbf{T}_0}$ be the product of μ with the restriction to $\mathbf{T}_0$ of the character $\det(\mathfrak{g}_2^{\theta=-1} \cap \mathfrak{u}_{1,1})$. This latter character is equal to the sum of noncompact roots in $\mathfrak{u}_{1,1}$, and is therefore given by

$$\alpha + (\alpha + \beta) - (\alpha - 3\beta) = \alpha - 2\beta = 2\alpha - (\alpha + 2\beta).$$

Its restriction to $\mathbf{T}_0$ is therefore given by 2α, and thus

$$\Gamma|_{\mathbf{T}_0} = \frac{k}{2}\alpha|_{\mathbf{T}_0}.$$

From $(\mathbf{H}, \Gamma)$ we construct another piece of data, which Vogan calls cuspidal data. Consider the centralizer of $\mathbf{A}$ in $\mathbf{G}_2$; this is a Levi subgroup of $\mathbf{G}_2$, and we write $\mathbf{MA}$ for its Langlands decomposition. The torus $\mathbf{A}$ was a maximal split torus in a short root $\mathrm{SL}_2(\mathbb{R})$, and it follows that $\mathbf{M}$ is a long root $\mathrm{SL}_2(\mathbb{R})$ in $\mathbf{G}_2$. Therefore there is a long root parabolic $\mathbf{P} = \mathbf{MAN}$ in $\mathbf{G}_2$.

A piece of cuspidal data constructed from $(\mathbf{H}, \Gamma)$ will consist of the Levi $\mathbf{MA}$, along with a character of $\mathbf{A}$, which will is given by $\Gamma|_{\mathbf{A}} = \nu$, and also a discrete series representation π_0 of $\mathbf{MA}$. This latter representation is given as the cohomological induction of $\mu' \otimes \nu$ from $\mathbf{H}$ to $\mathbf{MA}$, where

$(\mathfrak{q}', \mathbf{H}, \mu', \nu)$ is the θ-stable data for $\mathbf{MA}$ obtained from the restriction of the character data $(\mathbf{H}, \Gamma)$ to $\mathbf{MA}$. In this data, the θ-stable parabolic $\mathfrak{q}'$ is the intersection of $\mathfrak{q}_{1,1}$ with the complexified Lie algebra $\mathfrak{m} \oplus \mathfrak{a}$ of $\mathbf{MA}$. It contains the noncompact root α in its radical. The character μ' is the restriction of Γ to $\mathbf{T}_0$ multiplied by the inverse of the sum of the noncompact roots in the radical of $\mathfrak{q}'$. Thus it is equal to $\frac{k-2}{2}\alpha|_{\mathbf{T}_0}$.

The Theorem 6.6.15 in [Vog81] then asserts that (1.6.4.5) is isomorphic to

$$\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}_2}(\mathcal{R}(\mu' \otimes \nu) \otimes \delta_{\mathbf{P}}^{1/2}). \tag{1.6.4.6}$$

The cohomological induction in this expression is, by Theorem 1.6.2.2, the twist of the discrete series representation of $\mathbf{MA}$ of weight k by the character $\det^{-1/2}$. Since $\mathbf{P}$ is a long root parabolic, $\det^{-1/2} = \delta_{\mathbf{P}}^{-1/10}|_{\mathbf{MA}}$, and we get that (1.6.4.6), and also hence (1.6.4.5), are isomorphic to the normalized induction

$$\iota_{M_\alpha(\mathbb{R})}^{\mathbf{G}_2}(\pi, -1/10).$$

Since π is self dual, the unique irreducible subrepresentation of this is, by dualizing, isomorphic to $\mathcal{L}_\alpha(\pi, 1/10)$, and this is isomorphic to $\mathcal{R}(\Lambda)$ by above. This is what we wanted to prove. $\square$

Remark 1.6.4.5. We make one more comparison to the GSp_4 case. For GSp_4, the normalized induced representation $\iota_{M_\beta(\mathbb{R})}^{\mathrm{GSp}_4(\mathbb{R})}(\pi \boxtimes 1, 1/6)$ from the Siegel parabolic contains as a subrepresentation a member of the large discrete series. This large discrete series representation has Harish-Chandra parameter is related, by the Weyl group element which rotates the root lattice clockwise by an angle of $\pi/2$, to the holomorphic discrete series which are the archimedean components of the CAP forms appearing in the proof of Theorem 1.4.4.1.

For G_2, on the other hand, it is possible to make a (somewhat lengthy) computation using the work of Blank [Bla85] to show that the induced representation $\iota_{M_\alpha(\mathbb{R})}^{\mathrm{G}_2(\mathbb{R})}(\pi, 1/10)$ contains as a subrepresentation the discrete series representation of $\mathrm{G}_2(\mathbb{R})$ with Harish-Chandra parameters

$$\frac{k-4}{2}(\alpha + 3\beta) + (\alpha + 4\beta) \quad \text{and} \quad \frac{k-4}{2}(\alpha + 3\beta) + (2\alpha + 5\beta).$$

These parameters are obtained from the Harish-Chandra parameter of $\mathcal{R}(\Lambda_w)$ by applying various

Weyl group elements.

However, in forthcoming work of R. Dalal, this same result is proved using our determination of the Arthur packet above. With this input, the result becomes much easier to prove, and in particular, Dalal circumvents the use of the results of Blank.

The Harish-Chandra parameter of $\mathcal{R}(\Lambda_w)$, as discussed before, is the one called quaternionic of weight $k/2$ in [GGS02]. These quaternionic discrete series are supposed to be analogous to the holomorphic ones, and so we once again see that the theory of representations induced from the long root parabolic of G_2 corresponds well to the theory of those induced from the Siegel parabolic in GSp_4.

Chapter 2: p-adic deformation of certain critical Eisenstein series for G_2

This paper is organized at follows. We first briefly recall some facts about the structure of the group G_2 in Section 1.5.1. Then in Section 2.2, we study certain unramified principal series representations of G_2 at p and obtain both qualitative and quantitative information about their p-stabilizations for later use. In Section 2.3 we carry out the p-adic deformation described in the introduction of this thesis.

In the appendix to this thesis, we take some time to recall the results from [Urb11] that we need, and we also take the opportunity to correct a mistake in that paper.

Notation and conventions

The ring of adeles of $\mathbb{Q}$ will be denoted $\mathbb{A}$, the ring of finite adeles will be denoted $\mathbb{A}_f$, and the ring of finite adeles away from a prime p will be denoted $\mathbb{A}_f^p$. The archimedean place of $\mathbb{Q}$ will be denoted ∞.

If G is a reductive group, then given an automorphic representation Π of $G(\mathbb{A})$, we let Π_f denote the representation of $G(\mathbb{A}_f)$ which is the finite component of Π. Similarly define the representation Π_f^p of $G(\mathbb{A}_f^p)$.

Some notation and conventions about the group G_2 will be introduced in Section 2.1 and will be used throughout this chapter. A significant amount of notation will also be introduced in the appendix and will be used in Section 2.3.

2.1 The group G_2

We collect in this section some facts about the group G_2 which we will use throughout this chapter.

By definition, G_2 will be the split simple group over $\mathbb{Q}$ of type G_2. We fix a split maximal torus T in G_2, and we also fix a pair of simple roots for T in G_2, the longer of which we denote by α and the shorter of which we denote by β. The Dynkin diagram is pictured here. The group G_2 has

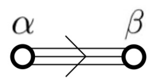


Figure 2.1.1: The Dynkin diagram of G_2

trivial center.

The set of positive roots for G_2 consists of the following six roots:

$$\alpha,\ \beta,\ \alpha+\beta,\ \alpha+2\beta,\ \alpha+3\beta,\ 2\alpha+3\beta.$$

The root lattice is shown in Figure 2.1.2. There, the dominant chamber is shaded.

The group G_2 has three proper standard parabolic subgroups. One is the Borel, which we

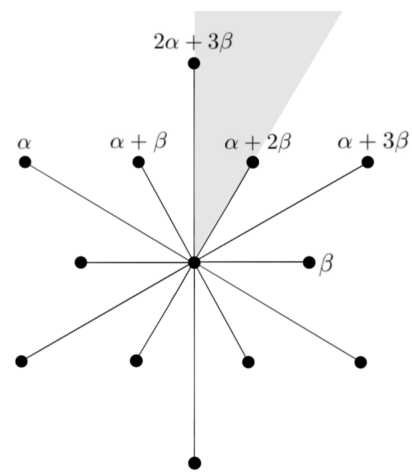


Figure 2.1.2: The root lattice of G_2

denote by B. Let us write U for its unipotent radical, so that $B = TU$. We also have the long root parabolic, which is the one whose Levi contains α. We write P_α for it, and denote its Levi by M_α and its unipotent radical by N_α. Similarly, there is the short root parabolic P_β whose Levi M_β contains β. Write N_β for its unipotent radical.

Both M_α and M_β are isomorphic to GL_2. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2$, we let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\alpha \in M_\alpha, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}_\beta \in M_\beta$$

be the corresponding matrices in M_α and M_β, respectively. Here, this correspondence is fixed so that if $x_\alpha : \mathbb{G}_a \to \mathrm{G}_2$ is the root group homomorphism for α, then $x_\alpha(a) = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)_\alpha$, and similarly for

β.

We note here that the root $\alpha + 2\beta$ acts on elements of T the same way as the determinant character for M_α. That is,

$$(\alpha + 2\beta)\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}_\alpha = t_1 t_2.$$

Similarly,

$$(2\alpha + 3\beta)\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}_\beta = t_1 t_2.$$

We will write W for the Weyl group of G_2. It has 12 elements and is generated by the simple reflections w_α and w_β across the lines perpendicular to, respectively, α and β. We amalgamate these notations to express the corresponding products. So, for example, we write $w_{\alpha\beta} = w_\alpha w_\beta$, and $w_{\alpha\beta\alpha} = w_\alpha w_\beta w_\alpha$, and so on. Then

$$W = \{1, w_\alpha, w_\beta, w_{\alpha\beta}, w_{\beta\alpha}, w_{\alpha\beta\alpha}, w_{\beta\alpha\beta}, w_{\alpha\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\alpha\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}, w_{\alpha\beta\alpha\beta\alpha\beta}\}.$$

For this last element we have $w_{\alpha\beta\alpha\beta\alpha\beta} = w_{\beta\alpha\beta\alpha\beta\alpha}$. This element acts on the root system by multiplication by -1.

The group $\mathrm{G}_2(\mathbb{R})$ is connected and has discrete series. Its maximal compact, which we will denote by K_∞, is 6 dimensional. Since G_2 is 14 dimensional, the symmetric space $\mathrm{G}_2(\mathbb{R})/K_\infty$ is 8-dimensional. In particular, the middle degree for the cohomology of the locally symmetric spaces attached to G_2 is 4.

Finally, we fix a maximal compact subgroup K_f of $\mathrm{G}_2(\mathbb{A}_f)$ which is hyperspecial at all places.

2.2 Unramified principal series and their p-stabilizations

In this section we study the unramified principal series representations of G_2 which give rise to the components at p of the functorial lifts we will be interested in. We will obtain information in this section about when certain unramified principal series are irreducible, as well as information about the slopes of unramified principal series representations.

We remark that, although we only prove results for G_2 here, a lot of the theory introduced in this section is not specific to G_2 and could in principal be done for any split reductive group.

2.2.1 Reducibility of certain principal series

In this chapter, we will be concerned with the representations obtained by unitary parabolic induction from the long root parabolic P_α in G_2, induced from automorphic representations of the form $\pi \otimes |\det|^{1/2}$, where π is the unitary automorphic representation of $\mathrm{GL}_2(\mathbb{A}) \cong M_\alpha(\mathbb{A})$ coming from a cuspidal holomorphic eigenform which is unramified at p and has trivial nebentypus. More precisely, we will be interested in the unique irreducible (Langlands) quotient of such an induction, and since the goal will be to p-adically deform this quotient, we will need to examine this quotient at p.

Now the p-component of this Langlands quotient is the Langlands quotient of the unitary parabolic induction

$$\iota_{P_\alpha(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\pi_p \otimes |\det|^{1/2}).$$

If we knew that this induced representation were irreducible, then of course it would equal its Langlands quotient and we would obtain a good description our global Langlands quotient at p. However, and rather interestingly, this will never be the case!

The reason for this can actually be seen from the root lattice for G_2. First, let us write π_p itself as an unramified principal series,

$$\pi_p \cong \iota_{B_\alpha(\mathbb{Q}_p)}^{M_\alpha(\mathbb{Q}_p)}(\chi) \otimes |\det|^{1/2},$$

where $B_\alpha = B \cap M_\alpha$ is the standard Borel of M_α, and $\tilde{\chi}$ is an unramified character of $T(\mathbb{Q}_p)$. By unitarity, $\tilde{\chi}$ has the form

$$\tilde{\chi}(\mathrm{diag}(t_1, t_2)) = \chi(t_1)\chi^{-1}(t_2), \qquad t_1, t_2 \in \mathbb{Q}_p^\times$$

for some unramified unitary character χ of $\mathbb{Q}_p^\times$, and this can be rewritten as

$$\tilde{\chi}(t) = \chi(\alpha(t)), \qquad t \in T(\mathbb{Q}_p).$$

The determinant factor of π_p can be moved into the induction, and using $(\alpha+2\beta)(t) = \det(t)$, we can rewrite our principal series as

$$\pi_p \cong \iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}((\chi \circ \alpha) \cdot (|\cdot|^{1/2} \circ (\alpha + 2\beta))).$$

Now, on the other hand, there is a element w_1 in the Weyl group of G_2 which rotates the root lattice clockwise by 60 degrees. This element takes the ordered pair $(\alpha, \alpha+2\beta)$ to the pair $(2\alpha+3\beta, \beta)$. Now the pair $(2\alpha+3\beta, \beta)$ plays the same role for the short root Levi M_β that $(\alpha, \alpha+2\beta)$ plays for M_α, except that the entries of this pair are reversed. It follows that

$$\iota_{B_\beta(\mathbb{Q}_p)}^{M_\beta(\mathbb{Q}_p)}((\chi \circ w_1\alpha) \cdot (|\cdot|^{1/2} \circ w_1(\alpha+2\beta))),$$

which is the principal series for M_β induced from the twist by w_1 same character as above, is isomorphic to

$$\iota_{B_\beta(\mathbb{Q}_p)}^{M_\beta(\mathbb{Q}_p)}(|\cdot|^{1/2} \otimes |\cdot|^{-1/2}) \otimes \chi.$$

(Here, of course, $B_\beta = B \cap M_\beta$ is the standard Borel of M_β.) Now the above representation is obviously reducible by the theory of principal series for $\mathrm{GL}_2(\mathbb{Q}_p)$, and so if we induced it further to $\mathrm{G}_2(\mathbb{Q}_p)$, we would obtain a reducible representation. By induction in stages, and the fact that the irreducible constituents of a principal series do not depend on the Weyl twist of the representation being induced, we find that $\iota_{P_\alpha(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q})}(\pi_p)$ is also reducible.

Nevertheless, this argument actually suggests what the Langlands quotient of this induction should be, and it turns out we will be able to write it as a parabolic induction of a character, neither from $M_\alpha(\mathbb{Q}_p)$ nor $T(\mathbb{Q}_p)$, but rather from $M_\beta(\mathbb{Q}_p)$. We make all this precise in the following proposition.

Proposition 2.2.1.1. *Let χ be an unramified unitary character of $\mathbb{Q}_p^\times$, and let $\tilde{\chi}$ be the character of $T(\mathbb{Q}_p)$ defined by*

$$\tilde{\chi}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}_\alpha = \chi(t_1)\chi^{-1}(t_2), \qquad t_1, t_2 \in \mathbb{Q}_p^\times.$$

Then the principal series representation

$$\iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\tilde{\chi} \cdot |\det_\alpha|^{1/2})$$

is reducible. Its Langlands quotient is isomorphic to

$$\iota_{P_\beta(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi \circ \det_\beta),$$

which is irreducible.

Proof. Let $w = w_\beta w_\alpha$, so that $w\alpha = 2\alpha + 3\beta$ and $w(\alpha + 2\beta) = \beta$. Rewrite our principal series representation as

$$\iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\tilde{\chi} \cdot |\det_\alpha|^{1/2}) = \iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}((\chi \circ \alpha) \cdot (|\cdot|^{1/2} \circ (\alpha + 2\beta))).$$

Because $\alpha + 2\beta$ is dominant, the Langlands quotient of this is the unique unramified irreducible constituent of it, and is therefore isomorphic to the unique unramified irreducible constituent of the twist by w,

$$\begin{aligned}\iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}((\chi \circ (w\alpha)) \cdot (|\cdot|^{1/2} \circ (w(\alpha + 2\beta)))) &= \iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}((\chi \circ (2\alpha + 3\beta)) \cdot (|\cdot|^{1/2} \circ \beta)) \\ &= \iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\delta_{B_\beta(\mathbb{Q}_p)}^{1/2} \cdot (\chi \circ \det_\beta)).\end{aligned}$$

By induction in stages, this is

$$\iota_{P_\beta(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\iota_{B_\beta(\mathbb{Q}_p)}^{M_\beta(\mathbb{Q}_p)}(\delta_{B_\beta(\mathbb{Q}_p)}^{1/2} \cdot (\chi \circ \det_\beta))).$$

The inner parabolic induction is reducible, hence so is the whole one, and also our original principal series (before applying the twist by w).

Now the unique unramified constituent of the inner induction

$$\iota_{B_\beta(\mathbb{Q}_p)}^{M_\beta(\mathbb{Q}_p)}(\delta_{B_\beta(\mathbb{Q}_p)}^{1/2} \cdot (\chi \circ \det_\beta))$$

is the unramified character $\chi \circ \det_\beta$ of $M_\beta(\mathbb{Q}_p)$, which is a quotient. Therefore the Langlands quotient we are interested in is the unique unramified irreducible constituent of

$$\iota_{P_\beta(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi \circ \det_\beta).$$

So to prove the proposition, it suffices to prove this is irreducible. But this was already done (for unitary characters) in the paper of Jantzen [Jan98]; See section 6 of that paper. So we are done. $\square$

We prove now a proposition which will be helpful later in this chapter. For this proposition, it is important for us to view the coefficient field of our representations as $\overline{\mathbb{Q}}_p$ rather than $\mathbb{C}$.

Proposition 2.2.1.2. *Let $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_p^\times$ be an unramified character. Assume that for all roots γ for G_2 we have*

$$v_p(\chi(\gamma^\vee(p))) \neq \pm 1.$$

Then the principal series representation

$$\iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi)$$

is irreducible.

Proof. We will use Proposition 2.4 of the paper of Jantzen [Jan98]. In our case, this says that if

$$\iota_{(B\cap M)(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)}(w\chi) \tag{2.2.1.1}$$

is irreducible for the maximal Levis $M = M_\alpha, M_\beta$ and for all w in the Weyl group of G_2, then for any w' in the Weyl group of G_2, we have

$$\iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi) \cong \iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(w'\chi) \tag{2.2.1.2}$$

So let us first check that the principal series representations (2.2.1.1) are irreducible for $M = M_\alpha, M_\beta$.

Let γ_0 denote the positive root in M, so $\gamma_0 = \alpha$ if $M = M_\alpha$, or $\gamma_0 = \beta$ if $M = M_\beta$. Then

(2.2.1.1) is reducible if and only if $(w\chi)(\gamma_0^\vee(t)) = |t|^{\pm 1}$ for all $t \in \mathbb{Q}_p^\times$. But we have

$$(w\chi)(\gamma_0^\vee(p)) = \chi((w^{-1}\gamma_0)^\vee(p)),$$

and the right hand side of the above equation is not equal to $p^{\pm 1}$ by hypothesis. Therefore the representations (2.2.1.1) are irreducible for both Levis M, and so (2.2.1.2) holds for all w'.

Now by the theory of Satake parameters, there are Weyl group elements w_1 and w_2 so that the irreducible spherical constituent of

$$\iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(w_1\chi)$$

is the unique irreducible quotient, and so that the spherical constituent of

$$\iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(w_2\chi)$$

is the unique irreducible subrepresentation. By (2.2.1.2), this subrepresentation and this quotient must coincide, forcing

$$\iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi)$$

to be irreducible, as desired. □

2.2.2 The algebra $\mathcal{U}_p$

Let us start by fixing a *Chevalley basis* for $\mathfrak{g}_2$. This means we fix root vectors E_γ for every root γ such that

$$[E_\gamma, E_{\gamma'}] = u_{\gamma,\gamma'} E_{\gamma+\gamma'}$$

whenever γ and γ' are roots, where $u_{\gamma,\gamma'} = \pm\langle\gamma, (\gamma')^\vee\rangle$ unless $\gamma+\gamma'$ is not a root, in which case $u_{\gamma,\gamma'} = 0$.

We can then define root group maps $x_\gamma : \mathbb{G}_a \to \mathrm{G}_2$ for any root γ, which are defined over $\mathbb{Z}$ for some model of G_2 over $\mathbb{Z}$. They are defined on points by

$$x_\gamma(a) = \exp(aE_\gamma).$$

Then over $\mathbb{Z}_p$, we have the group $\mathrm{G}_2(\mathbb{Z}_p)$ which is generated inside of $\mathrm{G}_2(\mathbb{Q}_p)$ by the images of $\mathbb{Z}_p$ under these root group homomorphisms, along with the images of $\mathbb{Z}_p^\times$ under all coroots $\gamma^\vee$. It is a hyperspecial maximal compact subgroup in $\mathrm{G}_2(\mathbb{Q}_p)$.

We note the following identity for future use:

$$x_\gamma(a)x_{\gamma'}(a') = x_{\gamma+\gamma'}(u_{\gamma,\gamma'}aa')x_{\gamma'}(a')x_\gamma(a). \tag{2.2.2.1}$$

For an integer $m \geq 1$, let $p_m : \mathrm{G}_2(\mathbb{Z}_p) \to \mathrm{G}_2(\mathbb{Z}/p^m\mathbb{Z})$ be the reduction modulo p^m map. We consider in this section the Iwahori subgroup I_m of depth m, defined by

$$I_m = \{g \in \mathrm{G}_2(\mathbb{Z}_p) \mid p_m(g) \in B(\mathbb{Z}/p^m\mathbb{Z})\}.$$

We then have the Iwahori decomposition, which says the following. Let $\overline{U}$ be the unipotent radical of the parabolic opposite to B, and let $U_m^- = \overline{U}(\mathbb{Z}_p) \cap I_m$. Then

$$I_m = U(\mathbb{Z}_p)T(\mathbb{Z}_p)U_m^- = U_m^- T(\mathbb{Z}_p)U(\mathbb{Z}_p).$$

As in [Urb11] and also the appendix of this thesis, we will use the Iwahori subgroup I_m to define a certain Hecke algebra $\mathcal{U}_p$ which will act on smooth admissible representations of $\mathrm{G}_2(\mathbb{Q}_p)$ with an I_m-fixed vector. To define the algebra $\mathcal{U}_p$, first we define a monoid T^- by

$$T^- = \{t \in T(\mathbb{Q}_p) \mid tU(\mathbb{Z}_p)t^{-1} \subset U(\mathbb{Z}_p)\}.$$

For a given $t \in T^-$, we write u_t for the element of $\mathcal{U}_p$ given by

$$u_t = \frac{1}{\mathrm{Vol}(I_m)} \mathrm{char}(I_m t I_m).$$

Then we define $\mathcal{U}_p$ to be the convolution algebra over $\mathbb{Z}_p$ generated by the u_t's for $t \in T^-$. The Hecke algebra $\mathcal{U}_p$ acts on smooth admissible representations of $\mathrm{G}_2(\mathbb{Q}_p)$ with an I_m fixed vector by convolution. It turns out that neither the structure of $\mathcal{U}_p$, nor its action on smooth admissible representations σ, depend on the choice of m, as long as $\sigma^{I_m} \neq 0$. Indeed, we have the relation

$u_t u_{t'} = u_{tt'}$ for $t, t' \in T^-$ (which can be checked using the Iwahori decomposition) which implies that $\mathcal{U}_p$ is isomorphic to the monoid algebra over $\mathbb{Z}_p$ generated by $T^-/T(\mathbb{Z}_p)$, and thus it is independent of m. In particular, $\mathcal{U}_p$ is commutative. That the action of $\mathcal{U}_p$ on smooth admissible representations is independent of m follows from writing the double coset $I_m t I_m$ in terms of right cosets for I_m and showing that there is a description of those cosets which is independent of m. Actually, this is more or less a corollary of the following lemma, which will be useful to us in the next section.

Lemma 2.2.2.1. *Let $t \in T^-$. Then*

$$I_m t I_m = t(t^{-1}U(\mathbb{Z}_p)t)T(\mathbb{Z}_p)U_m^-.$$

Proof. We use the Iwahori decomposition. We have

$$I_m t I_m = t(t^{-1}U(\mathbb{Z}_p)t)(t^{-1}T(\mathbb{Z}_p)t)(t^{-1}U_m^- t)I_m.$$

Since $t \in T^-$, $t^{-1}U_m^- t \subset U_m^-$, and $t^{-1}T(\mathbb{Z}_p)t = T(\mathbb{Z}_p)$. So

$$(t^{-1}T(\mathbb{Z}_p)t)(t^{-1}U_m^- t) \subset T(\mathbb{Z}_p)U_m^- \subset I_m.$$

Therefore we can absorb this factor into the I_m on the right and get

$$I_m t I_m = t(t^{-1}U(\mathbb{Z}_p)t)I_m.$$

Now we use the Iwahori decomposition again:

$$I_m t I_m = t(t^{-1}U(\mathbb{Z}_p)t)U(\mathbb{Z}_p)T(\mathbb{Z}_p)U_m^-.$$

Since $t \in T^-$, we have $t^{-1}U(\mathbb{Z}_p)t \supset U(\mathbb{Z}_p)$. So we get

$$I_m t I_m = t(t^{-1}U(\mathbb{Z}_p)t)T(\mathbb{Z}_p)U_m^-,$$

whence the lemma. $\square$

2.2.3 p-stabilization of unramified principal series

Let us first recall the following definition from [Urb11].

Definition 2.2.3.1. Let $m \geq 1$, and let σ be a smooth admissible representation of $\mathrm{G}_2(\mathbb{Q}_p)$ with $\sigma^{I_m} \neq 0$. An irreducible constituent of σ^{I_m} for the action of $\mathcal{U}_p$ will be called a *p-stabilization* of σ.

Actually, we note that the definition given in [Urb11] is more general, and will be recalled in the appendix.

Now any p-stabilization of a representation σ as in this definition is one dimensional. Indeed, $\mathcal{U}_p$ is generated by finitely many commuting operators. Since σ^{I_m} is finite dimensional, so is any p-stabilization of it, and therefore these operators must share a common eigenvector. This eigenvector must then generate the whole p-stabilization by the assumption that it is irreducible.

We now study the action of $\mathcal{U}_p$ on certain unramified principal series, and determine their p-stabilizations. In what follows, we take $m = 1$, and denote $I = I_1$ and $U^- = U_1^-$. We consider a proper standard parabolic subgroup P of G_2 and write $P = MN$ for its Levi decomposition. So $P \in \{B, P_\alpha, P_\beta\}$. If $P = P_\alpha, P_\beta$, then $M \cong \mathrm{GL}_2$, and by an unramified character of $M(\mathbb{Q}_p)$ we mean a character which is the composition of an unramified character of $\mathbb{Q}_p^\times$ with the determinant.

In the following proposition we will consider the set W^P of minimal length representatives in the Weyl group W of G_2 of the quotient $W_M \backslash W$, where W_M is the Weyl group of M. One computes that $W^M = W$ if $M = T$, and otherwise has 6 elements.

Proposition 2.2.3.2. *Let P be as above and let χ be an unramified character of $M(\mathbb{Q}_p)$. Then the (non-normalized) principal series representation*

$$\mathrm{Ind}_{P(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi)$$

has $|W^P|$ different p-stabilizations, one corresponding to each $w \in W^P$, and the action of u_t on each is given by multiplication by

$$\chi(t^w) \prod_{\substack{\gamma>0 \\ w\gamma<0 \text{ or } w\gamma\in\Delta_M}} |\gamma(t)|^{-1},$$

where Δ_M denotes the set of roots in M and the absolute value in the expression is the usual p-adic *absolute value.*

We will prove this proposition momentarily. The key will be to use the well-known decomposition

$$\mathrm{G}_2(\mathbb{Q}_p) = \coprod_{w \in W^P} P(\mathbb{Q}_p) w I, \tag{2.2.3.1}$$

which will allow us to write down an explicit basis for the I-invariants of our principal series representation. We will show that the operators u_t for $t \in T^-$ are lower triangular in this basis. A similar argument for GSp_4 appears in [SU06a], Proposition 4.2.2, but not in so much detail.

We will need a couple of lemmas. We consider the root group homomorphisms x_γ as defined in the beginning of Section 2.2.2. For $\gamma \in \{\alpha, \beta\}$ a simple root, let w_γ denote the corresponding representative element in $\mathrm{G}_2(\mathbb{Q}_p)$ as in the lemma above. For longer elements of the Weyl group, which are amalgamations of w_α and w_β, we consider representatives in $\mathrm{G}_2(\mathbb{Q}_p)$ formed by amalgamating the representatives just chosen. So for example, $w_{\alpha\beta\alpha} = w_\alpha w_\beta w_\alpha$, and so on.

Next we recall the fact that, given any ordering $\gamma_1, \ldots, \gamma_6$ on the positive roots of G_2, there is a way to write any $x \in U(\mathbb{Q}_p)$ as

$$x = x_{\gamma_1}(a_1) \ldots x_{\gamma_6}(a_6),$$

for some unique $a_1, \ldots, a_6 \in \mathbb{Q}_p$. This follows from, for example, [Spr09], Proposition 8.2.1.

Lemma 2.2.3.3. *Choose any order $\gamma_1, \ldots, \gamma_6$ on the roots of G_2. Let $x \in U(\mathbb{Z}_p)$. Then if we write*

$$x = x_{\gamma_1}(a_1) \ldots x_{\gamma_6}(a_6),$$

for some $a_1, \ldots, a_6 \in \mathbb{Q}_p$, then actually $a_1, \ldots, a_6 \in \mathbb{Z}_p$.

Proof. Assume for sake of contradiction that one of the a_i's is in $\mathbb{Q}_p$ but not in $\mathbb{Z}_p$. Let i be so that $a_i \notin \mathbb{Z}_p$ and γ_i is as small as possible. Since the only relations in $U(\mathbb{Q}_p)$ are

$$x_\gamma(a) x_\gamma(a') = x_\gamma(a + a')$$

and the commutation relation (2.2.2.1), then no matter how we expand x as a word in elements $x_\gamma(a)$, there will be one component of that word of the form $x_{\gamma_i}(a)$ with $a \notin \mathbb{Z}_p$. (Here we must use that the constants $u_{\gamma,\gamma'}$ in (2.2.2.1) are integers.) This is a contradiction, since $U(\mathbb{Z}_p)$ can be generated by $x_\gamma(a)$ for roots γ and $a \in \mathbb{Z}_p$. $\square$

Proof (of Proposition 2.2.3.2). From the decomposition (2.2.3.1), it follows that the space of Iwahori invariants,

$$\mathrm{Ind}_{P(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi)^I,$$

has a basis of $|W^P|$ elements, given by functions Φ^w for $w \in W^P$ defined uniquely by the conditions that

$$\Phi^w(w) = 1, \qquad \Phi^w(w') = 0 \text{ if } w' \neq w.$$

Now let $t \in T^-$ and $w, w' \in W^P$. Let us normalize the Haar measure on $\mathrm{G}_2(\mathbb{Q}_p)$ so that I has measure 1. Then we have by definition,

$$(u_t \Phi^w)(w') = \int_{ItI} \Phi^w(w'z)\, dz.$$

We invoke Lemma 2.2.2.1 to write this as

$$\int_{t^{-1}U(\mathbb{Z}_p)t} \int_{T(\mathbb{Z}_p)U^-} \Phi^w(w'txy)\, dy\, dx,$$

where the Haar measure on $t^{-1}U(\mathbb{Z}_p)t$ gives the subset $U(\mathbb{Z}_p)$ measure 1, and that on $T(\mathbb{Z}_p)U^-$ gives it measure 1. Then we use the fact that Φ^w is in the induced representation to write this as

$$\chi(t^{w'}) \int_{t^{-1}U(\mathbb{Z}_p)t} \int_{T(\mathbb{Z}_p)U^-} \Phi^w(w'xy)\, dy\, dx.$$

Since Φ^w is also I-invariant, this simplifies to

$$\chi(t^{w'}) \int_{t^{-1}U(\mathbb{Z}_p)t} \Phi^w(w'x)\, dx.$$

Then we make a change of variable to write this as

$$\chi(t^{w'}) \prod_{\gamma > 0} |\gamma(t)|^{-1} \int_{U(\mathbb{Z}_p)} \Phi^w(w't^{-1}xt)\, dx.$$

Now we order the positive roots of G_2 as follows: If $P = B$, order $\gamma_1, \ldots, \gamma_6$ so that $w'\gamma_i > 0$ for $i \leq 6 - l(w')$ and $w'\gamma_i < 0$ for $i > 6 - l(w')$. If $P = P_\alpha$, then let $\gamma_1 = \alpha$ and order the rest of the γ_i's so that $w'\gamma_i > 0$ for $1 < i \leq 6 - l(w')$ and $w'\gamma_i < 0$ for $i > 6 - l(w')$. Similarly if $P = P_\beta$, order the roots in an analogous way except that $\gamma_1 = \beta$. Then let $U_{w',+}$ be the set of all elements in $U(\mathbb{Z}_p)$ of the form

$$x_{\gamma_1}(a_1) \cdots x_{\gamma_{6-l(w')}}(a_{6-l(w')}),$$

where the a_i's are in $\mathbb{Z}_p$, and let $U_{w',-}$ be the set of all elements in $U(\mathbb{Z}_p)$ of the form

$$x_{\gamma_{6-l(w')+1}}(a_{6-l(w')+1}) \cdots x_{\gamma_6}(a_6)$$

where again the a_i's are in $\mathbb{Z}_p$. By Lemma 2.2.3.3, $U(\mathbb{Z}_p) = U_{w',+}U_{w',-}$. Then we rewrite

$$\chi(t^{w'}) \prod_{\gamma>0} |\gamma(t)|^{-1} \int_{U(\mathbb{Z}_p)} \Phi^w(w't^{-1}xt)\, dx$$
$$= \chi(t^{w'}) \prod_{\gamma>0} |\gamma(t)|^{-1} \int_{U_{w',-}} \int_{U_{w',+}} \Phi^w(w'(t^{-1}x't)(t^{-1}x''t))\, dx'\, dx'',$$

where $U_{w',-}$ and $U_{w',+}$ get measure 1.

Now the only way for $w'(tx't^{-1})(tx''t^{-1})$ to be in $P(\mathbb{Q}_p)wI$ is if $w' \geq w$ in the Bruhat order (see, for example, Proposition 6 in [BN11], where this statement is proved for $P = B$; the cases of the other two parabolic subgroups then follow from this one). Thus the integral above vanishes unless $w = w'$ or $l(w') > l(w)$. This implies that if we order the basis $\{\Phi^w\}$ increasingly in the length of w, then u_t is lower triangular. Hence there are $|W^P|$ p-stabilizations of our principal series representation, and to prove the proposition, we only need to compute the above integral when $w' = w$.

Setting $w' = w$, our integral becomes

$$\chi(t^w) \prod_{\gamma>0} |\gamma(t)|^{-1} \int_{U_{w,-}} \int_{U_{w,+}} \Phi^w(w(t^{-1}x't)(t^{-1}x''t))\, dx'\, dx''.$$

By definition of $U_{w',-}$, $w(tx't^{-1})w^{-1} \in P(\mathbb{Q}_p)$ and is a product of unipotent elements. Thus we may factor it out and we get

$$\chi(t^w)\prod_{\gamma>0}|\gamma(t)|^{-1}\int_{U_{w,+}}\Phi^w(w(t^{-1}x''t))\,dx''.$$

The only way for $w(tx''t^{-1})$ to be in $P(\mathbb{Q}_p)wI$ is if $(tx''t^{-1})\in U(\mathbb{Z}_p)$ (see [BN11], Proposition 8) and so we may shrink the region of integration to $tU_{w,+}t^{-1}$. Hence we get

$$\chi(t^w)\prod_{\gamma>0}|\gamma(t)|^{-1}\int_{tU_{w,+}t^{-1}}\Phi^w(w(t^{-1}x''t))\,dx''.$$

But then $(t^{-1}x''t)\in I$ for every $x''\in(tU_{w,+}t^{-1})$, so this is just

$$\chi(t^w)\prod_{\gamma>0}|\gamma(t)|^{-1}\int_{tU_{w,+}t^{-1}}\Phi^w(w)\,dx''=\chi(t^w)\prod_{\gamma>0}|\gamma(t)|^{-1}\operatorname{Vol}(tU_{w,+}t^{-1}).$$

Of course,

$$\operatorname{Vol}(tU_{w,+}t^{-1})=\prod_{\substack{\gamma>0,w\gamma<0\\ w\gamma\notin\Delta_M}}|\gamma(t)|.$$

Combining, we finally get

$$(u_t\Phi^w)(w)=\chi(t^w)\prod_{\substack{\gamma>0\\ w\gamma<0\text{ or }w\gamma\in\Delta_M}}|\gamma(t)|^{-1},$$

which yields the proposition. □

2.2.4 Slopes

Fix an integer $m\geq 1$. Let σ be an irreducible admissible representation of $\mathrm{G}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{Q}}_p$ which has a vector fixed by I_m. Fix a p-stabilization $\sigma_{(p)}$ of σ. Then $\sigma_{(p)}$ is one dimensional and induces a character $\theta:\mathcal{U}_p\to\overline{\mathbb{Q}}_p$. We define a rational character $\mu_\theta\in X^*(T)\otimes\mathbb{Q}$ by the condition that, for any algebraic cocharacter $\mu^\vee$ of T, we have

$$\langle\mu_\theta,\mu^\vee\rangle=v_p(\theta(\mu^\vee(p))).$$

This character μ_θ is called the *slope* of $\sigma_{(p)}$ or of θ. If μ_θ is not identically zero, we say that θ or $\sigma_{(p)}$ is of *finite slope.*

We finish this section with a proposition which will be useful to us later. In its hypotheses we use the terminology "sufficiently regular" in reference to a character μ of T, and by this we mean there is a regular character μ_0 of T such that, for any $\mu > \mu_0$ in the order induced by the Weyl chamber containing μ_0, the conclusion of the proposition holds.

Proposition 2.2.4.1. *Let $\chi : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_p$ be an unramified character. Let $\sigma_{(p)}$ be a p-stabilization of the principal series representation*

$$\sigma = \iota_{B(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi).$$

If the slope μ of $\sigma_{(p)}$ is sufficiently regular, then σ is irreducible.

Proof. By Proposition 2.2.3.2, σ has a p-stabilization for each $w \in W$, and the slope of the wth p-stabilization is given by the rational character μ such that, for any root γ, we have

$$\begin{aligned} \langle \mu, \gamma^\vee \rangle &= v_p(\chi \delta_{B(\mathbb{Q}_p)}^{1/2}((w\gamma)^\vee(p))) + \sum_{\substack{\gamma' > 0 \\ w\gamma' < 0}} \langle \gamma', \gamma^\vee \rangle \\ &= v_p(\chi((w\gamma)^\vee(p))) + \langle \rho, (w\gamma)^\vee \rangle + \sum_{\substack{\gamma' > 0 \\ w\gamma' < 0}} \langle \gamma', \gamma^\vee \rangle. \end{aligned}$$

If μ is sufficiently regular, then the quantity $v_p(\chi((w\gamma)^\vee(p)))$ on the right hand side is sufficiently far away from zero, as the other quantities on the right hand side can be explicitly bounded independently of γ. In particular,

$$v_p(\chi((w\gamma)^\vee(p))) \neq \pm 1,$$

and since this holds for any γ, we are done by Proposition 2.2.1.2. □

2.3 p-adic deformation of certain Eisenstein series

We now introduce the functorial lift Π, which is the "Eisenstein series" alluded to in the title of this section. After recalling some relevant facts about it from Chapter 1, we will prove that under certain conditions that it can be p-adically deformed in a generically cuspidal family. The main result in this direction here is Theorem 2.3.2.3.

2.3.1 The overconvergent multiplicity

Let F be a cuspidal holomorphic eigenform with level N, trivial nebentypus, and even weight $k \geq 4$. Assume $p \nmid N$. Let π be the unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ attached to F. We will be interested in deforming p-adically the Langlands quotient, which we will denote by Π, of the unitary induction

$$\iota_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\pi \otimes \delta_{P_\alpha(\mathbb{A})}^{1/10}).$$

We will explain under what circumstances this representation can be p-adically deformed in a generically cuspidal family of automorphic representations of $\mathrm{G}_2(\mathbb{A})$.

To get started, we invite the reader to read the appendix of this thesis, which explains the general tools we will use to make these p-adic deformations. We will assume familiarity with the content and notation of first three sections of this appendix in this section. In particular, we will be working with the cohomology of X_{G_2} and its overconvergent counterparts, and the associated multiplicities.

We first state what we know about how Π appears in the cohomology of X_{G_2}, using results of Chapter 1. Let λ_0 be the dominant weight

$$\lambda_0 = \frac{k-4}{2}(2\alpha + 3\beta).$$

The cohomology $\mathrm{G}_2(\mathbb{A}_f)$-module $H^i(X_{\mathrm{G}_2}, V_{\lambda_0})$, as usual, splits $\mathrm{G}_2(\mathbb{A}_f)$-equivariantly into a direct sum of two $\mathrm{G}_2(\mathbb{A}_f)$ submodules given by the cuspidal and Eisenstein cohomology:

$$H^i(X_{\mathrm{G}_2}, V_{\lambda_0}) = H^i_{\mathrm{cusp}}(X_{\mathrm{G}_2}, V_{\lambda_0}) \oplus H^i_{\mathrm{Eis}}(X_{\mathrm{G}_2}, V_{\lambda_0}).$$

We have the following result about the appearance of Π in Eisenstein cohomology. To state it, we write $\Pi = \bigotimes_v' \Pi_v$ for the decomposition over all places v of Π into representations of $\mathrm{G}_2(\mathbb{Q}_v)$, and we let Π_f be the finite part of Π.

Theorem 2.3.1.1. *Assume that* $L(1/2, \pi, \mathrm{Sym}^3) = 0$. *Then the representation* Π_f *appears in exactly one way as a subquotient of the cohomology space* $H^i_{\mathrm{Eis}}(X_{\mathrm{G}_2}, V_{\lambda_0})$ *when* $i = 4$, *and does not appear as a subquotient of this space for* $i \neq 4$.

Moreover, if Π'_f *if another irreducible admissible* $\mathrm{G}_2(\mathbb{A}_f)$*-module which at every finite place except* p *is isomorphic to* Π_f, *then* Π'_f *appears as a subquotient of* $H^i_{\mathrm{Eis}}(X_{\mathrm{G}_2}, V_{\lambda_0})$ *if and only if* $i = 4$ *and the local component* Π'_p *of* Π'_f *at* p *is an irreducible subquotient of the unitary induction*

$$\iota_{P_\alpha(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\pi_p \otimes \delta_{P_\alpha(\mathbb{Q}_p)}^{1/10}).$$

Moreover, in this case, any such subquotient of Π'_f *appears uniquely as a subquotient of the Eisenstein cohomology.*

Proof. For the representation Π_f, this is a consequence of Theorem 1.5.3.3. Strictly speaking, the statement of that theorem does not apply directly to Π'_f, but its proof works for this representation in exactly the same way because the methods used there for distinguishing subquotients of different parabolic inductions work by distinguishing them at all but finitely many places. In a little more detail, these methods show that any such Π'_f appearing in the Eisenstein cohomology of V_{λ_0} actually appears in the space

$$H^i(\mathfrak{g}_2, K_\infty; \mathcal{A}_{\lambda_0,[P_\alpha],\varphi}(\mathrm{G}_2) \otimes V_{\lambda_0}),$$

which is shown to be concentrated in degree 4 and isomorphic to the unitary parabolic induction

$$\iota_{P_\alpha(\mathbb{A}_f)}^{\mathrm{G}_2(\mathbb{A}_f)}(\pi \otimes \delta_{P_\alpha(\mathbb{A}_f)}^{1/10})$$

by Proposition 1.5.3.1, because of the assumption on the vanishing of the symmetric cube L-function. □

As for the appearance in cuspidal cohomology, the results in Chapter 1 explain how Π_f appears in cuspidal cohomology assuming conjectures about CAP representations. These conjectures are given as Conjecture 1.5.4.1, and it is explained there why they should be reasonable. We state a consequence of these conjectures here, which is Theorem 1.5.4.2 (whose proof assumes Conjecture 1.5.4.1).

Conjecture 2.3.1.2. *Assume $\epsilon(1/2, \pi, \mathrm{Sym}^3) = -1$ (so in pariticular, $L(1/2, \pi, \mathrm{Sym}^3) = 0$). Then Π_f appears exactly once as a summand of the cuspidal cohomology $H^i_{\mathrm{cusp}}(X_{\mathrm{G}_2}, V_{\lambda_0})$. Moreover, if Π'_f is another summand of this cuspidal cohomology which is equivalent to Π_f at every finite place except perhaps p, then $\Pi'_f \cong \Pi_f$.*

See Section 1.5.4 for a discussion of how this conjecture essentially reduces to considerations surrounding Arthur's conjectures.

Now we consider the representation Π_f as a module for the Hecke algebra $\mathcal{H}_p$ introduced in Section A.1.2 (considered here for $G = \mathrm{G}_2$). Also let Π_f^p be the irreducible admissible $\mathrm{G}_2(\mathbb{A}_f^p)$-module which is given by Π_v at every finite place $v \neq p$. We also consider the $\mathrm{G}_2(\mathbb{A}_f)$-module

$$\widetilde{\Pi}_f = \Pi_f^p \otimes \iota_{P_\alpha(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\pi_p \otimes \delta_{P_\alpha(\mathbb{Q}_p)}^{1/10}).$$

Then Π_f is the unique irreducible quotient of $\widetilde{\Pi}_f$.

Let I be the Iwahori subgroup of $\mathrm{G}_2(\mathbb{Z}_p)$ of depth 1 as considered previously. By Proposition 2.2.3.2, $\widetilde{\Pi}_f^I$ is exactly 12 dimensional, and Π_f^I is exactly 6 dimensional by the same proposition along with Proposition 2.2.1.1. However, it seems we cannot rule out the possibility that a given p-stabilization of either Π_f or $\widetilde{\Pi}_f$ occurs more than once in the respective space Iwahori fixed vectors. This could happen if, for example, the Hecke polynomial of F at p has a double root (although it seems this is expect never to occur, but this is not known). This will not matter for our applications, however.

As a matter of notation, given a p-stabilization σ of Π_f, let us write $m_p(\sigma)$ for the dimension of the σ-isotypic component of Π_f as a $\mathcal{U}_p$-module. Similarly define $\widetilde{m}_p(\tilde{\sigma})$ for a p-stabilization $\tilde{\sigma}$ of $\widetilde{\Pi}_f$. We note that any p-stabilization of Π_f is also a p-stabilization of $\widetilde{\Pi}_f$.

We will now fix a p-stabilization of Π_f which will be denoted $\sigma(\Pi)_{\mathrm{un}}$. (The decoration $(\cdot)_{\mathrm{un}}$ is there to signify that this p-stabilization will not yet have been normalized properly.) To do this, fix α_p a root of the Hecke polynomial of F at p. Then we can write the local component π_p of π at p as

$$\pi_p \cong \iota_{B_\alpha(\mathbb{Q}_p)}^{M_\alpha(\mathbb{Q}_p)}(\chi' \otimes |\det|^{1/2}),$$

where $B_\alpha = B \cap M_\alpha$, and χ' is an unramified character of $T(\mathbb{Q}_p)$ such that

$$\tilde{\chi}'(\mathrm{diag}(t_1, t_2)_\alpha) = (p^{(k-1)/2}\alpha_p)^{v_p(t_1)}(p^{(k-1)/2}\alpha_p)^{-v_p(t_2)}, \qquad t_1, t_2 \in \mathbb{Q}_p^\times.$$

Let χ be the unramified character of $\mathbb{Q}_p^\times$ such that

$$\chi(t) = (p^{(k-1)/2}\alpha_p)^{v_p(t)}, \qquad t \in \mathbb{Q}_p^\times.$$

Then by Proposition 2.2.1.1,

$$\Pi_p \cong \mathrm{Ind}_{P_\beta(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}((\chi \circ \det_\beta) \cdot \delta_{P_\beta(\mathbb{Q}_p)}^{1/2}).$$

By Proposition 2.2.3.2, Π_p^I has a p-stabilization corresponding to $w = 1 \in W^{P_\beta}$ whose u_t-eigenvalue is

$$\begin{aligned}\chi(\det_\beta(t))\delta_{P_\beta(\mathbb{Q}_p)}^{1/2}(t)|\beta(t)|^{-1} &= (p^{(k-1)/2}\alpha_p)^{v_p((2\alpha+3\beta)(t))}p^{(3/2)v_p((2\alpha+3\beta)(t))}p^{v_p(\beta(t))} \\ &= \alpha_p^{v_p((2\alpha+3\beta)(t))}p^{((k-4)/2)v_p((2\alpha+3\beta)(t))}p^{v_p(\beta(t))},\end{aligned}$$

for $t \in T^-$. The p-stabilization of Π_f with these $\mathcal{U}_p$-eigenvalues is the p-stabilization that we denote by $\sigma(\Pi)_{\mathrm{un}}$. If we write $s_p = v_p(\alpha_p)$, then the slope of this p-stabilization is

$$\left(s_p + \frac{k-4}{2}\right)(2\alpha + 3\beta) + \beta.$$

We normalize this representation $\sigma(\Pi)_{\mathrm{un}}$ in a way which is consistent with the definition of the character distribution $I_{\mathrm{G}_2}^{\mathrm{cl}}(\cdot, \lambda_0; K_\infty)$ made in Section A.1.2; to do this we simply define, for $f = u_t \otimes f^p$ with $t \in T^-$ and $f^p \in C_c^\infty(\mathrm{G}_2(\mathbb{A}_f^p))$,

$$\sigma(\Pi)(f) = |\lambda_0(t)|^{-1}\sigma(\Pi)_{\mathrm{un}}(f).$$

Then it follows that $\sigma(\Pi)$ has slope

$$s_p(2\alpha + 3\beta) + \beta.$$

We state the next proposition as a culmination of all the considerations made so far in this section. Recall the definitions of multiplicities that were made in Section A.1.3.

Proposition 2.3.1.3. *Assume* $\epsilon(1/2, \pi, \mathrm{Sym}^3) = -1$. *Then under Conjecture 2.3.1.2, we have*

$$m_0^{\mathrm{cl}}(\sigma(\Pi), \lambda_0; K_\infty) = m_p(\sigma(\Pi))$$

and

$$m^{\mathrm{cl}}(\sigma(\Pi), \lambda_0; K_\infty) = \tilde{m}_p(\sigma(\Pi)_{\mathrm{un}}) + m_p(\sigma(\Pi)_{\mathrm{un}}).$$

In particular (still assuming this conjecture) we have

$$m^{\mathrm{cl}}(\sigma(\Pi), \lambda_0; K_\infty) \geq 2.$$

The classical multiplicity in the above proposition is exactly 2 (under Conjecture 2.3.1.2) if all 12 of the p-stabilizations of $\widetilde{\Pi}_f$ are distinct.

Now we express the classical multiplicity just studied in terms of overconvergent multiplicities. For $w \in W$, let $\sigma(\Pi)^{w,\lambda_0}$ be defined as in Section A.1.3. So $\sigma(\Pi)^{w,\lambda_0}(f) = \sigma(\Pi)(f)$, and $\sigma(\Pi)^{w,\lambda_0}$ has slope

$$s_p(2\alpha + 3\beta) + \beta - \lambda_0 + w * \lambda_0,$$

where, we recall, $w * \lambda = w(\lambda + \rho) - \rho$, and $\rho = 3\alpha + 5\beta$ is half the sum of the positive roots.

Proposition 2.3.1.4. *Assume* $s_p < \frac{k-2}{4}$. *Then we have*

$$m^{\mathrm{cl}}(\sigma(\Pi), \lambda_0; K_\infty) = m^\dagger(\sigma(\Pi), \lambda_0; K_\infty) - m^\dagger(\sigma(\Pi)^{w_\beta,\lambda_0}, w_\beta * \lambda_0; K_\infty).$$

Proof. The proposition will follow from Theorem A.1.3.1 and Theorem A.1.3.3 if we can show that

$$s_p(2\alpha + 3\beta) + \beta - \lambda_0 + w * \lambda_0$$

is in the cone $X^*(T)_{\mathbb{Q},+}$ generated over $\mathbb{Q}_{\geq 0}$ by the positive simple roots only for $w = 1, w_\beta$. For $w = 1$ this is clear because this quantity is just the slope of $\sigma(\Pi)$. For $w = w_\beta$, we have that

$w_\beta(2\alpha+3\beta) = 2\alpha+3\beta$ and $w_\beta(\rho) = 3\alpha+4\beta$, so

$$s_p(2\alpha+3\beta) + \beta - \lambda_0 + w * \lambda_0 = s_p(2\alpha+3\beta) \in X^*(T)_{\mathbb{Q},+}.$$

For any other w, we have $w(2\alpha+3\beta) \neq 2\alpha+3\beta$, and therefore the coefficient of α in $w(2\alpha+3\beta)$, expressed as a linear combination of α and β, is at most 1. Therefore the coefficient of α in $w\lambda_0 - \lambda_0$ is at most $-\frac{k-4}{2}$. Furthermore, the coefficient of α in $w\rho - \rho$ must smaller than 0 because, as $w \neq \mathrm{id}, w_\beta$, w must negate some positive root whose α-coefficient is nonzero. Thus the coefficient of α in

$$\beta - \lambda_0 + w\lambda_0 + w\rho - \rho + s_p(2\alpha+3\beta)$$

is at most $-\frac{k-2}{2}$.

Now by assumption, $s_p < \frac{k-2}{4}$, so the coefficient of α in

$$s_p(2\alpha+3\beta) + \beta - \lambda_0 + w\lambda_0 + w\rho - \rho + s_p(2\alpha+3\beta)$$

is negative. Therefore it is not in the cone $X^*(T)_{\mathbb{Q},+}$, as desired. □

For $P = MN$ a standard parabolic in G_2 and $w \in W^P_{\mathrm{Eis}}$ (See Section A.1.1) let $m^\dagger_{\mathrm{G}_2,M,w}(\sigma(\Pi), \lambda; K_\infty)$ be the multiplicity of $\sigma(\Pi)$ in the character distribution $I^\dagger_{\mathrm{G}_2,M,w}(\cdot, \lambda; K_\infty)$ as defined in Section A.1.2. This is well defined and nonnegative by Theorem A.1.3.4. Similarly define $m^\dagger_{\mathrm{G}_2,M,w}(\sigma(\Pi)^{\lambda_0,w_\beta}, \lambda_0 - \beta; K_\infty)$. (Note that $w_\beta * \lambda_0 = \lambda_0 - \beta$.) We have, by definition of the cuspidal overconvergent character distribution,

$$\begin{aligned} m^\dagger(\sigma(\Pi), \lambda_0; K_\infty) &= m^\dagger_0(\sigma(\Pi), \lambda_0; K_\infty) \\ &\quad + \sum_{P = P_\alpha, P_\beta, B} \sum_{w \in W^P_{\mathrm{Eis}}} (-1)^{l(w)+\dim(N)} m^\dagger_{\mathrm{G}_2,M,w}(\sigma(\Pi), \lambda_0; K_\infty), \end{aligned} \quad (2.3.1.1)$$

and similarly,

$$m^{\dagger}(\sigma(\Pi)^{\lambda_0-\beta,w_\beta},\lambda_0;K_\infty)=m_0^{\dagger}(\sigma(\Pi)^{\lambda_0,w_\beta},\lambda_0-\beta;K_\infty)$$
$$+\sum_{P=P_\alpha,P_\beta,B}\sum_{w\in W^P_{\mathrm{Eis}}}(-1)^{l(w)+\dim(N)}m^{\dagger}_{\mathrm{G}_2,M,w}(\sigma(\Pi)^{\lambda_0,w_\beta},\lambda_0-\beta;K_\infty). \quad (2.3.1.2)$$

We now compute the sums in the right hand sides of these formulas in pieces. This will allow us to express the overconvergent multiplicities in terms of the cuspidal ones. We note for the following that the slope of $\sigma(\Pi)^{\lambda_0,w_\beta}$ is $s_p(2\alpha+3\beta)$.

Lemma 2.3.1.5. *We have*

$$m^{\dagger}_{\mathrm{G}_2,M_\beta,1}(\sigma(\Pi),\lambda_0;K_\infty)=0,$$

and

$$m^{\dagger}_{\mathrm{G}_2,M_\beta,1}(\sigma(\Pi)^{\lambda_0,w_\beta},\lambda_0-\beta;K_\infty)=0.$$

Proof. By definition, we have

$$I^{\dagger}_{\mathrm{G}_2,M_\beta,1}(f,\lambda_0;K_\infty)=I^{\dagger}_{M_\beta,0}(f^{\mathrm{reg}}_{M_\beta,1},\lambda_0+2\rho_{P_\beta};K_\infty\cap P_\beta(\mathbb{R})),$$

for $f\in\mathcal{H}'_p$. We note that

$$\lambda_0+2\rho_{P_\beta}=\frac{k+2}{2}(2\alpha+3\beta).$$

First let τ be a constituent of the character distribution $I^{\dagger}_{M_\beta,0}(\cdot,\frac{k+2}{2}(2\alpha+3\beta);K_\infty\cap P_\beta(\mathbb{R}))$, viewed as an irreducible representation of the corresponding Hecke algebra for $M_\beta\cong\mathrm{GL}_2$, which we will denote by $\mathcal{H}_{p,\beta}$. We claim that if $t=(2\alpha+3\beta)^\vee(p)$, then u_{t,M_β} acts on τ with eigenvalue 1.

To see this, put τ in a generically cuspidal p-adic family varying in the weight λ (which we can do by the machinery of Urban's eigenvariety). Then for $\lambda=\frac{k+2}{2}(2\alpha+3\beta)+p^n\beta$ where n is a sufficiently large integer, the family deforms to a classical cuspidal automorphic representation τ_n of cohomological weight λ. Because of the presence of $\frac{k+2}{2}(2\alpha+3\beta)$ in the weight, such a τ_n is a twist by $|\det|^{(k+2)/2}$ at all places different from p of a cuspidal representation of GL_2 with trivial central character; but at p, this twist cancels with the normalization of $|\lambda(t)|^{-1}$ used to define the classical character distribution. The element t is central in $M_\beta(\mathbb{Q}_p)$, so on any such τ_n, u_{t,M_β} acts trivially. By continuity, u_{t,M_β} also acts trivially on τ, which proves our claim.

Now we let u_t be the corresponding element of the Hecke algebra of G_2, for the same value of t just considered. Then for any $f^p \in C_c^\infty(\mathrm{G}_2(\mathbb{A}_f^p), \mathbb{Q}_p)$, consider the operator $f = u_t \otimes f^p \in \mathcal{H}_p$. Then since $f_{M_\beta,1}^{\mathrm{reg}} = u_t \otimes f_{M_\beta}^p$, the operator f acts via $I_{\mathrm{G}_2,M_\beta,1}^\dagger(\cdot, \lambda_0; K_\infty)$ by a sum of traces of the form

$$\mathrm{Tr}(f^p | \operatorname{Ind}_{M_\beta(\mathbb{A}_f^p)}^{\mathrm{G}_2(\mathbb{A}_f^p)}(\tau^p))$$

for representations τ as above, by our claim. (Here τ^p is the component of τ away from p.) Therefore the slope of any constituent of $I_{\mathrm{G}_2,M_\beta,1}^\dagger(\cdot, \lambda_0; K_\infty)$ is orthogonal to $2\alpha + 3\beta$. But the slope of $\sigma(\Pi)_{\mathrm{char}}$ is $s_p(2\alpha + 3\beta) + \beta$, so if we have that $s_p > 0$, then $\sigma(\Pi)$ is not a constituent of $I_{\mathrm{G}_2,M_\beta,1}^\dagger(\cdot, \lambda_0; K_\infty)$.

If $s_p = 0$, we must argue differently. But in this case, the representation τ is noncritical, and therefore classical. But now we appeal to Proposition 1.5.2.8, which says that such an irreducible subquotient representation induced from P_β cannot be equivalent to one induced from P_α at all but finitely many places. We see once again then that $\sigma(\Pi)$ cannot be a constituent of $I_{\mathrm{G}_2,M_\beta,1}^\dagger(\cdot, \lambda_0; K_\infty)$ in this case.

This proves the first equality in the statement of the proposition, and the second one is completely analogous using the fact that the slope of $\sigma(\Pi)^{\lambda_0,w_\beta}$ is $s_p(2\alpha + 3\beta)$. $\square$

Lemma 2.3.1.6. *Let $w \in \{w_\alpha, w_{\alpha\beta}\}$. Then if k is sufficiently large with respect to s_p, then we have*

$$m_{\mathrm{G}_2,M_\beta,w}^\dagger(\sigma(\Pi), \lambda_0; K_\infty) = 0,$$

and

$$m_{\mathrm{G}_2,M_\beta,w}^\dagger(\sigma(\Pi)^{\lambda_0,w_\beta}, \lambda_0 - \beta; K_\infty) = 0.$$

Proof. We will view all finite slope representations of $\mathcal{H}_p'$ as modules for the algebra

$$\mathcal{U}_p[u_t^{-1},\, t \in T^-] \otimes_{\mathbb{Z}_p} C_c^\infty(\mathrm{G}_2(\mathbb{A}_f^p), \mathbb{Q}_p)$$

in the natural way.

First we look at $m_{\mathrm{G}_2,M_\beta,w_\alpha}^\dagger(\sigma(\Pi), \lambda_0; K_\infty)$. Assume for sake of contradiction that there is a constituent τ of $I^{\dagger}_{M_\beta,0}(\cdot, w_\alpha * \lambda_0 + 2\rho_{P_\beta}, K_\infty \cap P_\beta(\mathbb{R}))$ such that $\sigma(\Pi)$ appears in

$$f \mapsto \mathrm{Tr}(f^{\mathrm{reg}}_{M_\beta,w_\alpha}|\tau).$$

We note that

$$w_\alpha * \lambda_0 + 2\rho_{P_\beta} = \left(\frac{k}{4}+\frac{3}{2}\right)(2\alpha+3\beta) + \left(\frac{k}{4}+\frac{1}{2}\right)\beta,$$

so τ has this cohomological weight.

Now we let $t = (\alpha+\beta)^\vee(p)$ so that

$$u_{w_\alpha(t),M_\beta} = u_{\beta^\vee(p),M_\beta}.$$

Let c_τ be the eigenvalue of $u_{\beta^\vee(p),M_\beta}$ on τ, and c_σ be the eigenvalue of u_t on $\sigma(\Pi)$. Then we have, if $f^p \in C_c^\infty(\mathrm{G}_2(\mathbb{A}_f^p), \mathbb{Q}_p)$ and $f = u_t \otimes f^p$, that

$$\mathrm{Tr}(f^{\mathrm{reg}}_{M_\beta,w_\alpha}|\tau) = \mathrm{Tr}(u_{\beta^\vee(p),M_\beta} \otimes f^p_{M_\beta}|\tau) = c_\tau \,\mathrm{Tr}(1 \otimes f^p_{M_\beta}|\tau).$$

But we also have

$$\mathrm{Tr}(f|\sigma(\Pi)) = c_\sigma \,\mathrm{Tr}(1 \otimes f^p|\sigma(\Pi)).$$

It follows that $c_\sigma = c_\tau$ and that

$$f^p \mapsto \mathrm{Tr}(1 \otimes f^p|\sigma(\Pi))$$

is a constituent of

$$f^p \mapsto \mathrm{Tr}(1 \otimes f^p_{M_\beta}|\tau).$$

This means, first of all, that $v_p(c_\sigma) = v_p(c_\tau)$, and since the slope of $\sigma(\Pi)$ is $s_p(2\alpha+3\beta)+\beta$, we find that

$$v_p(c_\tau) = v_p((s_p(2\alpha+3\beta)+\beta)(\beta^\vee(p))) = 3s_p + 1.$$

Hence τ has slope

$$\frac{3s_p+1}{2}\beta + a(2\alpha+3\beta)$$

for some $a \in \mathbb{Q}^\times$. But by looking at the weight, this implies that τ is noncritical and hence classical since k is sufficiently large with respect to s_p.

Second, by above we must have that $f^p \mapsto \mathrm{Tr}(1 \otimes f^p | \sigma(\Pi))$ is a constituent of

$$\mathrm{Ind}_{P_\beta(\mathbb{A}_f^p)}^{\mathrm{G}_2(\mathbb{A}_f^p)}(\tau^p \otimes |\det_\beta|^b)$$

for some half integer b of the same parity as the weight of τ; here τ is being viewed as a p-stabilization of the finite part of a cuspidal, cohomological automorphic representation of $M_\beta \cong \mathrm{GL}_2$ and τ^p is the corresponding representation of $\mathrm{GL}_2(\mathbb{A}_f^p)$.

But away from p, $\sigma(\Pi)$ is induced from a cuspidal, cohomological automorphic representation along P_α. So now we appeal to Proposition 1.5.2.8 which says that if we have such a representation induced from P_β, then it cannot be equivalent to one induced from P_α at all but finitely many places. This is the contradiction sought.

The proof for $\sigma(\Pi)^{\lambda_0, w_\beta}$ and $w = w_\alpha$ is completely similar, and so are the proofs when $w = w_{\alpha\beta}$; in the latter case, we just need to replace $(\alpha+\beta)^\vee(p)$ with $(\alpha+2\beta)^\vee(p)$. □

One computes easily that $W_{\mathrm{Eis}}^{P_\beta} = \{1, w_\alpha, w_{\alpha\beta}\}$. Thus the previous two lemmas have ruled out any contribution to (2.3.1.1) and (2.3.1.2) from multiplicities coming from P_β, at least when k is sufficiently large with respect to s_p. We now do the same for the multiplicities coming from B. We note here that $W_{\mathrm{Eis}}^B = W$.

Lemma 2.3.1.7. *Let $w \in W$. Then we have*

$$m^\dagger_{\mathrm{G}_2,T,w}(\sigma(\Pi), \lambda_0; K_\infty) = 0,$$

and

$$m^\dagger_{\mathrm{G}_2,T,w}(\sigma(\Pi)^{\lambda_0,w_\beta}, \lambda_0 - \beta; K_\infty) = 0.$$

Proof. The proof is very similar to that of Lemma 2.3.1.6, except that we do not need to examine the slope to prove the classicality of a representation of T, as such representations are already classical. So we can just appeal to Proposition 1.5.2.8 again, which considers representations induced from B as well as P_α and P_β. We omit the precise details. □

Now we examine multiplicities coming from P_α itself. We note that $W_{\mathrm{Eis}}^{P_\alpha} = \{1, w_\beta, w_{\beta\alpha}\}$.

Lemma 2.3.1.8. *Let $w \in \{1, w_\beta\}$. Then if k is sufficiently large with respect to s_p, then we have*

$$m^{\dagger}_{\mathrm{G}_2, M_\alpha, w}(\sigma(\Pi), \lambda_0; K_\infty) = 0,$$

and

$$m^{\dagger}_{\mathrm{G}_2, M_\alpha, w}(\sigma(\Pi)^{\lambda_0, w_\beta}, \lambda_0 - \beta; K_\infty) = 0.$$

Proof. We compute

$$\lambda_0 + 2\rho_{P_\beta} = w_\beta * (\lambda_0 - \beta) + 2\rho_{P_\beta} = \frac{k-4}{4}\alpha + \left(\frac{3(k-4)}{4} + 5\right)(\alpha + 2\beta)$$

and

$$w_\beta * \lambda_0 + 2\rho_{P_\beta} = \lambda_0 - \beta + 2\rho_{P_\beta} = \left(\frac{k-4}{4} + \frac{1}{2}\right)\alpha + \left(\frac{3(k-4)}{4} + \frac{9}{2}\right)(\alpha + 2\beta).$$

We can argue exactly as in Lemma 2.3.1.6 to see that if any of the multiplicities in the statement of our lemma is nonzero, then there exists τ, a p-stabilization of the finite part of a classical cohomological cuspidal automorphic representation of M_α, such that

$$\Pi_f^p = \mathrm{Ind}_{P_\alpha(\mathbb{A}_f^p)}^{\mathrm{G}_2(\mathbb{A}_f^p)}(\tau^p \otimes |\det{}_\alpha|^a)$$

for a either $\frac{3(k-4)}{4} + 5$ or $\frac{3(k-4)}{4} + \frac{9}{2}$ (depending on w). Here, Π_f^p and τ^p are the components away from p. Furthermore, this τ would have weight $\frac{k-4}{2}$ or $\frac{k-4}{2} + 1$, depending again on w. This would imply that the component π_f^p of π away from p and ∞ is isomorphic to τ by Proposition 1.5.2.1, which tells us when constituents of two parabolic inductions from P_α can be equivalent at almost all places. But just by comparing the weights of π and τ we obtain a contradiction. Thus the multiplicities in the statement of our lemma are all zero. □

Lemma 2.3.1.9. *We have*

$$m^{\dagger}_{\mathrm{G}_2, M_\alpha, w_{\beta\alpha}}(\sigma(\Pi), \lambda_0; K_\infty) = 0.$$

Proof. We can argue just as in Lemma 2.3.1.5 that any irreducible constituent τ of $I^{\dagger}_{M_\alpha, 0}(\cdot, w_{\beta\alpha} * \lambda_0 + 2\rho_{P_\alpha}; K_\infty \cap P_\alpha(\mathbb{R}))$ must have slope a multiple of α; the slope in the direction of $\alpha + 2\beta$ must be trivial.

So now assume for sake of contradiction that there is such a τ so that $\sigma(\Pi)$ appears in

$$f \mapsto \mathrm{Tr}(f^{\mathrm{reg}}_{M_\alpha, w_{\beta\alpha}} | \tau).$$

Let $t = \beta^\vee(p)$. Then $w(t) = (\alpha + 2\beta)^\vee(p)$. We have that u_t acts via a scalar of p-adic valuation 2 on $\sigma(\Pi)$ because $(s_p(2\alpha + 3\beta) + \beta)(\beta^\vee(p)) = p^2$. But by what we just said above, $u_{w(t)}$ must act on τ by a scalar with p-adic valuation 0. This is a contradiction. □

Lemma 2.3.1.10. *Assume $s_p < \frac{k-1}{2}$. Then*

$$m^\dagger_{\mathrm{G}_2, M_\alpha, w_{\beta\alpha}}(\sigma(\Pi)^{w_\beta, \lambda_0}, \lambda_0; K_\infty) = 1.$$

Proof. Note that

$$w_{\beta\alpha} * (\lambda_0 - \beta) + 2\rho_{P_\beta} = \frac{k-2}{2}\alpha + 2(\alpha + 2\beta).$$

In the character distribution $I^\dagger_{M_\alpha, 0}(\cdot, \frac{k-2}{2}\alpha + 2(\alpha + 2\beta); K_\infty \cap P_\alpha(\mathbb{R}))$, the representation given by the normalized p-stabilization of π_f with smaller slope appears, and it does so exactly once (because $K_\infty \cap P_\alpha(\mathbb{R})$ is a maximal compact subgroup of $M_\alpha(\mathbb{R})$ and the Hecke polynomial for F at p does not have a double root by the assumption of s_p). It follows that $\sigma(\Pi)^{w_{\beta\alpha}, \lambda_0}$ appears exactly once in the character distribution

$$f \mapsto \mathrm{Tr}(f^{\mathrm{reg}}_{M_\alpha, w_{\beta\alpha}} | \tau).$$

It cannot appear in any character distribution

$$f \mapsto \mathrm{Tr}(f^{\mathrm{reg}}_{M_\alpha, w_{\beta\alpha}} | \tau')$$

for any other τ' in $I^\dagger_{M_\alpha, 0}(\cdot, \frac{k-2}{2}\alpha + 2(\alpha + 2\beta); K_\infty \cap P_\alpha(\mathbb{R}))$ by Proposition 1.5.2.1. □

With all this preparation, we can now state the main result about multiplicities.

Theorem 2.3.1.11. *Let k be sufficiently large with respect to s_p. Then*

$$m^\dagger_0(\sigma(\Pi), \lambda_0; K_\infty) \geq m^{\mathrm{cl}}(\sigma(\Pi), \lambda_0; K_\infty) + 1.$$

In particular, under Conjecture 2.3.1.2, if $\epsilon(1/2, \pi, \mathrm{Sym}^3) = -1$*, then*

$$m_0^\dagger(\sigma(\Pi), \lambda_0; K_\infty) \geq 3.$$

Proof. Combining all the previous lemmas, Proposition 2.3.1.4, and (2.3.1.1) and (2.3.1.2), we find

$$m^{\mathrm{cl}}(\sigma(\Pi), \lambda_0; K_\infty) = m_0^\dagger(\sigma(\Pi), \lambda_0; K_\infty) - m_0^\dagger(\sigma(\Pi)^{w_\beta, \lambda_0}, \lambda_0 - \beta; K_\infty) - 1.$$

But $m_0^\dagger(\sigma(\Pi)^{w_\beta,\lambda_0}, \lambda_0 - \beta; K_\infty) \geq 0$ by Theorem A.1.3.4. This gives the first statement in the theorem, and the second follows from Proposition 2.3.1.3. □

Remark 2.3.1.12. Theorem 2.3.1.11 shows that, at least conjecturally, the cuspidal overconvergent multiplicity of $\sigma(\Pi)$ is at least 3 when the sign of the symmetric cube functional equation of F is -1 (at least under some conditions on s_p and k). In fact, in this case, it seems we should expect this multiplicity to be exactly 3. This is because the number of elements in a discrete series L-packet of $\mathrm{G}_2(\mathbb{R})$ is exactly three, from which it should follow that the multiplicity of a generic member of the cuspidal family in which $\sigma(\Pi)$ deforms should be 3.

The analogous assertion for GSp_4 is true for certain representations induced from the Siegel parabolic; when the corresponding multiplicities are computed with respect to the connected component of the maximal compact subgroup at infinity, one gets that the cuspidal overconvergent multiplicity is -4 when $\epsilon(1/2, \pi) = -1$. See Example 5.5.3 in [Urb11] and Section 1.4.4.

Now, when $\epsilon(1/2, \pi, \mathrm{Sym}^3) = 1$, we expect to always have $m_0^\dagger(\sigma(\Pi), \lambda_0; K_\infty) = 0$, even if $L(1/2, \pi, \mathrm{Sym}^3) = 0$. In this case, the CAP representation equivalent to Π at all finite places appears in cohomology in degrees 3 and 5, at least conjecturally, by the results in Chapter 1. This means that the classical multiplicity for $\sigma(\Pi)$ will differ by 3 from the case when $\epsilon(1/2, \pi, \mathrm{Sym}^3) = -1$, making the cuspidal overconvergent multiplicity zero. Again, the analogous fact is true for GSp_4.

2.3.2 The p-adic families

We retain the setting of the previous section; in particular, we will continue working with our modular form F of weight k, its associated automorphic representation π, the Langlands quotient Π, and the normalized p-stabilization $\sigma(\Pi)$ of Π. We continue to write s_p for the slope of a chosen

p-stabilization of F. Then $\sigma(\Pi)$ has slope $s_p(2\alpha+3\beta)+\beta$, and it has weight $\lambda_0 = \frac{k-4}{2}(2\alpha+3\beta)$.

We assume furthermore that F has level 1 throughout this section. In this case, the triple product L-function of π splits as

$$L(s,\pi\times\pi\times\pi) = L(s,\pi,\mathrm{Sym}^3)L(s,\pi)^2.$$

It is well known that the triple product L-function above has functional equation with sign -1 because π is unramified at all finite places and the triple (F,F,F) is in the balanced range. Therefore $\epsilon(1/2,\pi,\mathrm{Sym}^3) = -1$ if F has level 1.

The Langlands quotient Π has full level

$$K_{f,\max} = \prod_{\ell\neq\infty} \mathrm{G}_2(\mathbb{Z}_\ell)$$

when F has level 1. Write $K_f = K_{f,\max}$ for short, and

$$K_f^p = \prod_{\ell\neq p,\infty} \mathrm{G}_2(\mathbb{Z}_\ell)$$

Let $\mathfrak{X}$ be the weight space for G_2 as introduced in Section A.1.2. We will consider the eigenvariety $\mathfrak{E}_{J,K_f^p}$ of tame level K_f^p, introduced in Section A.1.4, for the $\mathfrak{X}$-family of cuspidal overconvergent character distributions J whose specializations at points $\lambda\in\mathfrak{X}(\overline{\mathbb{Q}}_p)$ are given by

$$J_\lambda = I^\dagger_{\mathrm{G}_2,0}(\cdot,\lambda;K_\infty).$$

We also consider the Hecke algebra

$$R_p = \mathcal{U}_p\otimes_{\mathbb{Z}_p} C_c^\infty(K_f^p\backslash \mathrm{G}_2(\mathbb{A}_f^p)/K_f^p,\mathbb{Z}_p).$$

This is the algebra called $R_{S,p}$ in Section A.1.4 when S is empty.

The first order of business is use the eigenvariety to get rid of the hypothesis on the slope in Theorem 2.3.1.11.

Theorem 2.3.2.1. *Assume F has level* 1*. Then under Conjecture 2.3.1.2, we have*

$$m_0^{\dagger}(\sigma(\Pi), \lambda_0; K_\infty) > 0.$$

Proof. We begin by deforming the chosen p-stabilization of F in a Coleman family $\mathcal{F}$. Choose an increasing sequence k_n of integers such that $k_n \to k$ p-adically, and let F_n be the specialization of $\mathcal{F}$ to weight k_n. The form F_n is p-stabilized of slope s_p.

Let $R_p(\mathrm{GL}_2)$ be the Hecke algebra for GL_2 as defined as R_p; so

$$R_p(\mathrm{GL}_2) = \mathcal{U}_p(\mathrm{GL}_2) \otimes_{\mathbb{Z}_p} C_c^\infty(K_f^p(\mathrm{GL}_2)\backslash \mathrm{GL}_2(\mathbb{A}_f^p)/K_f^p(\mathrm{GL}_2), \mathbb{Z}_p),$$

where $\mathcal{U}_p(\mathrm{GL}_2)$ is the $\mathcal{U}_p$ algebra for GL_2 and

$$K_f^p(\mathrm{GL}_2) = \prod_{\ell \neq p, \infty} \mathrm{GL}_2(\mathbb{Z}_\ell).$$

Then each F_n defines a character $\theta_{F_n} : R_p(\mathrm{GL}_2) \to \overline{\mathbb{Q}}_p$. Similarly, our p-stabilization of F defines a character $\theta_F : R_p(\mathrm{GL}_2) \to \overline{\mathbb{Q}}_p$ and we have

$$\lim_{n\to\infty} \sup_{f \in R_p(\mathrm{GL}_2)} |\theta_{F_n}(f) - \theta_F(f)| = 0. \tag{2.3.2.1}$$

Furthermore, letting

$$\lambda_n = \frac{k_n - 4}{2}(2\alpha + 3\beta),$$

we have $\lambda_n \to \lambda_0$ in $\mathfrak{X}(\overline{\mathbb{Q}}_p)$.

Now let π_n, Π_n and $\sigma(\Pi_n)$ be defined from F_n as π, Π and $\sigma(\Pi)$ were defined from our chosen p-stabilization of F. In particular, the p-stabilization of Π_n chosen corresponds to $w = 1$, and $\sigma(\Pi_n)$ has slope $s_p(2\alpha + 3\beta)$. Then for n sufficiently large, say greater than some n_0, we have by Theorem 2.3.1.11 that the multiplicity $m_{J_{\lambda_n}}(\sigma(\Pi), \lambda_n, K_\infty)$ is greater than 0.

Let θ_n be the character of R_p defined by $\sigma(\Pi)$. On $\mathcal{U}_p$, $\theta_n|_{\mathcal{U}_p}$ is obtained from $\theta_{F_n}|_{\mathcal{U}_p(\mathrm{GL}_2)}$ by the same process for each n. Away from p, on $C_c^\infty(K_f^p\backslash \mathrm{G}_2(\mathbb{A}_f^p)/K_f^p, \mathbb{Z}_p)$, θ_n is obtained from θ_{F_n} by restriction; the spherical Hecke algebra for G_2 is the subalgebra of Weyl invariants of that of GL_2.

So by Theorem A.1.4.2, the point $x_n = (\theta_n, \lambda_n)$ lies on the eigenvariety $\mathfrak{E}_{J,K_f^p}(\overline{\mathbb{Q}}_p)$. By what we just said, along with (2.3.2.1), we have $x_n \to x_0$ in the space $\mathfrak{R}_{\emptyset,p}(\overline{\mathbb{Q}}_p) \times \mathfrak{X}(\overline{\mathbb{Q}}_p)$, where $x_0 = (\theta_0, \lambda_0)$ and θ_0 is the character of R_p corresponding to $\sigma(\Pi)$, and $\mathfrak{R}_{\emptyset,p}$ is the space introduced in Section A.1.4. Therefore, by the local finiteness of the eigenvariety over weight space, x_0 is also a point on the eigenvariety, and so $m_J(\theta_0, \lambda_0) > 0$.

This implies that there is an irreducible finite slope representation σ' of $\mathcal{H}_p$ occurring in $J_{\lambda_0} = I^{\dagger}_{\mathrm{G}_2,0}(\cdot, \lambda_0; K_\infty)$ such that the character of R_p corresponding to σ' is θ_0. But, because a spherical representation is determined by the corresponding character of the spherical Hecke algebra, such a σ' is determined completely by θ_0 as θ_0 has full tame level K_f^p. Therefore σ' must equal $\sigma(\Pi)$. We conclude that $\sigma(\Pi)$ occurs in J_{λ_0}, and hence

$$m_0^{\dagger}(\sigma(\Pi), \lambda_0; K_\infty) > 0,$$

as desired. $\square$

We will apply this Theorem momentarily to obtain our main result. First, however, we need a lemma.

Lemma 2.3.2.2. *Let $t \in T^{--}$ (see Section A.1.2). Then for $m > 1$, we have*

$$I_{m-1} t I_m = I_m t I_m,$$

where I_m is the Iwahori subgroup of depth m.

Proof. The proof is very similar to that of Lemma 2.2.2.1. First we use the Iwahori decomposition to write

$$I_{m-1} t I_m = t(t^{-1} U(\mathbb{Z}_p) t)(t^{-1} T(\mathbb{Z}_p) t)(t^{-1} U^-_{m-1} t) I_m.$$

Since $t \in T^{--}$, we have $t^{-1} U^-_{m-1} t \subset U_m$, so

$$(t^{-1} T(\mathbb{Z}_p) t)(t^{-1} U^-_{m-1} t) \subset T(\mathbb{Z}_p) U^-_m \subset I_m.$$

Therefore we can absorb this factor into the I_m on the right and get

$$I_{m-1}tI_m = t(t^{-1}U(\mathbb{Z}_p)t)I_m.$$

Now we use the Iwahori decomposition again:

$$I_{m-1}tI_m = t(t^{-1}U(\mathbb{Z}_p)t)U(\mathbb{Z}_p)T(\mathbb{Z}_p)U_m^-.$$

Since $t \in T^-$, we have $t^{-1}U(\mathbb{Z}_p)t \supset U(\mathbb{Z}_p)$. So we get

$$I_{m-1}tI_m = t(t^{-1}U(\mathbb{Z}_p)t)T(\mathbb{Z}_p)U_m^-.$$

By Lemma 2.2.2.1, this is exactly I_mtI_m. □

We are now ready to prove our main theorem.

Theorem 2.3.2.3. *Assume the level of F is 1. Then under conjecture 2.3.1.2 there are*

- *an open affinoid subdomain $\mathfrak{U} \subset \mathfrak{X}$,*
- *a finite cover $\mathbf{w} : \mathfrak{V} \to \mathfrak{U}$,*
- *a point $y_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ with $\mathbf{w}(y_0) = \lambda_0$,*
- *a Zariski dense subset $\Sigma \subset \mathfrak{V}(\overline{\mathbb{Q}}_p)$ with $\mathbf{w}(y)$ regular algebraic for every $y \in \Sigma$,*
- *for each $y \in \Sigma$, a nonempty finite set Π_y of finite slope p-stabilizations of irreducible, cohomological, cuspidal automorphic representations of G_2 of weight $\mathbf{w}(y)$ and full tame level K_f^p,*
- *a $\mathbb{Z}_p$-algebra homomorphism $\theta_{\mathfrak{V}} : R_p \to \mathcal{O}(\mathfrak{V})$,*
- *a nontrivial $\mathbb{Q}_p$-linear map $I_{\mathfrak{V}} : \mathcal{H}_p(K_f^p) \to \mathcal{O}(\mathfrak{V})$,*

satisfying the following properties:

- *The specialization of $\theta_{\mathfrak{V}}$ at the point y_0 is the character of R_p coming from $\sigma(\Pi)$;*

- *The representation $\sigma(\Pi)$ is an irreducible component of the specialization of $I_{\mathfrak{Y}}$ at y_0;*

- *For each $y \in \Sigma$ and each $\sigma \in \Pi_y$, the specialization of $\theta_{\mathfrak{Y}}$ at y occurs in the representation of R_p on $\sigma^{K_f^p}$;*

- *For each $y \in \Sigma$, the specialization I_y of $I_{\mathfrak{Y}}$ at y satisfies*

$$I_y(f) = \sum_{\sigma \in \Pi_y} m^{\mathrm{cl}}(\sigma, \mathbf{w}(y); K_\infty) \operatorname{Tr}(f|\sigma),$$

  for $f \in \mathcal{H}_p(K_f^p)$;

- *The set $\mathbf{w}(\Sigma)$ contains all sufficiently regular dominant algebraic weights in $\mathfrak{X}(\overline{\mathbb{Q}}_p)$;*

- *There is a Zariski closed subset of $\mathfrak{U}$ such that for $y \in \Sigma$ with $\mathbf{w}(y)$ not in this closed subset, Π_y only contains one representation and this representation is the (normalized) p-stabilization of an everywhere unramified representation.*

Proof. This follows immediately from Theorem A.1.4.3 upon using Theorem 2.3.2.1 to show that the hypothesis that $m_0^\dagger(\sigma(\Pi), \lambda_0; K_\infty) > 0$ is satisfied, except for the very last claim about how the representations in Σ away from a Zariski closed set are unramified. To see this last point, we only need to show these representations are unramified at p.

Let $\sigma \in \Pi_y$ for such a $y \in \Sigma$. Let $t \in T^{--}$. Then since σ is finite slope p-stabilization of an automorphic representation of weight $\mathbf{w}(y)$ which we are assuming to be algebraic, we have that there is a constant $c_t \in \overline{\mathbb{Q}}_p^\times$ such that

$$u_t v = c_t v$$

for all $v \in \sigma$. Write Π_σ for the automorphic representation of which σ is a p-stabilization. Then there is an integer m such that $\sigma \subset \Pi_\sigma^{I_m}$.

Now let $v \in \sigma$, so that v is I_m-fixed. Let $g \in I_{m-1}$ We compute

$$c_t g v = g(u_t v) = g \int_{I_m t I_m} hv\, dh = \int_{I_{m-1} t I_m} ghv\, dv = \int_{I_{m-1} t I_m} hv\, dv = u_t v = c_t v,$$

where we used Lemma 2.3.2.2 in the third equality as well as the second-to-last. Since $c_t \neq 0$, this shows that v is I_{m-1}-fixed.

Repeating this argument sufficiently many times shows that Π_σ has an I_1-fixed vector. Therefore the local component of Π_σ at p is a subquotient of an unramified principal series representation, call it I_p^σ.

Now after possibly shrinking Σ to contain points lying over only very regular weights that are close p-adically to λ_0, for $y \in \Sigma$, the slope of $\sigma \in \Pi_y$ is very close to the slope μ_0 of $\sigma(\Pi)$. Thus, for such σ (because the slopes of representations occurring in the cuspidal overconvergent distribution differ from their automorphic counterparts by a normalization factor equal to the weight) Π_σ has a p-stabilization whose slope is very close to $\mu_0 + \mathbf{w}(y)$, and hence is very regular. By Proposition 2.2.4.1, this implies that the principal series representation I_p^σ is irreducible, hence equal to the local component of Π_σ at p. So Π_σ is unramified for such σ, as desired. $\square$

Chapter 3: Galois representations into G_2 and the symmetric cube Bloch–Kato conjecture

This chapter is organized as follows. In Section 3.1, we recall the facts about G_2 and its 7-dimensional representation we need. In Section 3.2, we review the main result of Chapter 2 and state a conjectural global Langlands correspondence for cohomological representations of G_2. We then prove give some p-adic Hodge theoretic results in Section 3.3 that we will need to use later on in this chapter.

Section 3.4 then constructs families of G_2-Galois representations using the theory of pseudocharacters. Finally, in Section 3.5, we construct our lattice and the nontrivial Selmer class whose existence proves the main theorem of this thesis.

Notation and conventions

Throughout this chapter, p is a fixed prime number, and we work relative to a fixed isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$. This means, for example, that automorphic representations will be viewed with coefficients in $\overline{\mathbb{Q}}_p$.

Given a reductive group G over a field k and A a k-algebra, we let $G(A)$ denote the A-points of G. When k is of characteristic zero, we let $G^\vee$ be the dual reductive group of G over $\overline{k}$.

Given a maximal torus T in such a reductive group G, we have the group $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ of weights of T and the group $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ of coweights of T. If G is split, this latter group can be viewed as the group of weights of a fixed maximal torus in $G^\vee$. We let $\langle \cdot, \cdot \rangle$ denote the natural pairing between weights and coweights given by composition.

We let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$. We also let $\mathbb{A}_f$ denote the ring of finite adeles, and $\mathbb{A}_f^p$ the ring of adeles away from p and ∞. Similar conventions are used for automorphic representations of a reductive group G over $\mathbb{Q}$; given such an automorphic representation π, we let π_f be the associated admissible representation of $G(\mathbb{A}_f)$, and π_f^p that of $G(\mathbb{A}_f^p)$.

Given a rigid analytic space $\mathfrak{V}$ over $\mathbb{Q}_p$ or a finite extension thereof, we let $\mathcal{O}(\mathfrak{V})$ denote the ring of analytic functions on $\mathfrak{V}$. If $\mathfrak{V}$ is affinoid, we let $\mathcal{O}(\mathfrak{V})^\circ$ denote the subring of $\mathcal{O}(\mathfrak{V})$ of analytic functions whose evaluations at all points in $\mathfrak{V}$ are bounded above in absolute value by 1. If $\mathfrak{V}$ is affinoid and reduced, we view $\mathcal{O}(\mathfrak{V})$ with its usual $\mathbb{Q}_p$-Banach space topology. Then $\mathcal{O}(\mathfrak{V})^\circ$ is an open ball in $\mathcal{O}(\mathfrak{V})$.

We always write $G_{\mathbb{Q}}$ for the absolute Galois group of $\mathbb{Q}$. We fix a decomposition group $G_{\mathbb{Q}_p}$ in $G_{\mathbb{Q}}$, viewed as the absolute Galois group of $\mathbb{Q}_p$.

All representations of the Galois group $G_{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ will be continuous. For simplicity, we will consider any representation of $G_{\mathbb{Q}}$ that, a priori, has coefficients in a finite extension of $\mathbb{Q}_p$, as instead having coefficients base changed up to $\overline{\mathbb{Q}}_p$. So for example, the p-adic Galois representation attached to a modular eigenform will always be considered to have coefficients in $\overline{\mathbb{Q}}_p$.

We will consider Fontaine's functors D_{dR}, D_{st}, and D_{crys} of, respectively, de Rham, semistable, and crystalline periods. Correspondingly to the above convention about Galois representations, all p-adic Hodge theoretic constructions we consider in this chapter will be considered as $\overline{\mathbb{Q}}_p$-linear objects. Therefore, given a Galois representation V of $G_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_p$, the spaces $D_{\mathrm{dR}}(V)$, $D_{\mathrm{st}}(V)$, and $D_{\mathrm{crys}}(V)$ will be considered as $\overline{\mathbb{Q}}_p$-vector spaces with extra structure.

The conventions we use in this chapter are geometric. So for a prime ℓ, Frob_ℓ denotes a geometric Frobenius element of the Galois group $G_{\mathbb{Q}}$. The Hodge–Tate weight of the cyclotomic character is -1. Given a semistable representation V of $G_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_p$, the crystalline Frobenius ϕ on $D_{\mathrm{st}}(V)$ will also be geometric. The Filtrations on $D_{\mathrm{dR}}(V)$, $D_{\mathrm{st}}(V)$, and $D_{\mathrm{crys}}(V)$ do not change.

For example, if V is the 2-dimensional Galois representation attached to a modular eigenform of weight k which is unramified at p, then V is crystalline at p and $D_{\mathrm{crys}}(V)$ has Hodge–Tate weights 0 and $-(k-1)$. The filtration Fil^i for $D_{\mathrm{dR}}(V)$ falls at $i = 0$ and $i = k - 1$. Both the Newton and Hodge polygons lie on or below the horizontal axis.

Finally, given $n > 1$ an integer, we let Ad^n denote the representation of the reductive group GL_2 given by

$$\mathrm{Ad}^n = \mathrm{Sym}^n(\mathrm{Std}) \otimes \det{}^{-1},$$

where Std denotes the 2-dimensional standard representation of GL_2. We will use this notation both for $n = 2$ and $n = 3$ in this chapter.

3.1 The group G_2

We begin by recalling some basic facts we need about the exceptional group G_2. The group G_2 will carry an isomorphism class of 7-dimensional representations. It is not so easy to describe the image under any of these representations matricially in a very explicit way. However, the image will be characterized by the preservation of a certain kind of alternating trilinear form. If this form is chosen to be simple and explicit enough, then one can get one's hands on certain useful relations amongst matrix coefficients of particular elements in the image. These relations will play a crucial role in Section 3.5.3 when we prove the main theorem of this thesis. They will be used to show that the cocycle we construct in the symmetric cube Selmer group is nontrivial.

3.1.1 Structure of G_2

We define G_2 to be the split simple group over $\mathbb{Q}$ of type G_2. Its Dynkin diagram therefore looks as in Figure 3.1.1.

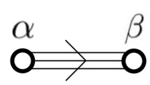


Figure 3.1.1: The Dynkin diagram of G_2

We are thus denoting the long simple root by α and the short simple root by β.

The root lattice looks as in Figure 3.1.2. There, the positive roots are labelled and the dominant chamber is shaded. Thus the positive roots are given as $\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta$.

Fixing a maximal torus $T \subset \mathrm{G}_2$ and a system of positive roots as above, we let P_α be the standard maximal parabolic subgroup of G_2 whose Levi contains α. We also let P_β be the other maximal parabolic subgroup of G_2; its Levi contains the root β. We let M_α and M_β be the standard Levi subgroups of P_α and P_β, respectively. Then M_α and M_β are both isomorphic to GL_2.

The smallest fundamental weight for G_2 is $\alpha+2\beta$, and the irreducible representation of G_2 of that highest weight is 7-dimensional. We call it the *standard representation* of G_2 and denote it by R_7. Its weights are shown in Figure 3.1.3.

We will be interested in this chapter in how R_7 restricts to the parabolic subgroup P_β. Let V_7 denote the space of R_7. For each weight γ of T in R_7, let $v_\gamma \in V_7$ be a fixed nonzero vector of that

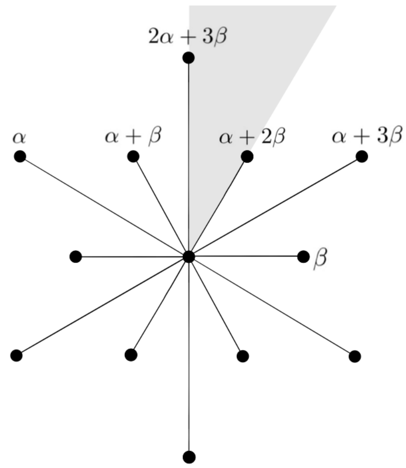


Figure 3.1.2: The root lattice of G_2

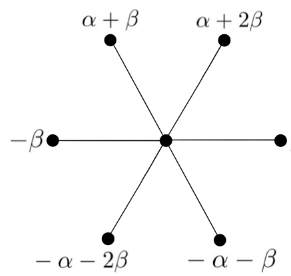


Figure 3.1.3: The weights of R_7

weight. Then we have a basis of V_7 given, in order, by

$$v_{\alpha+2\beta}, v_{\alpha+\beta}, v_\beta, v_0, v_{-\beta}, v_{\alpha-\beta}, v_{-\alpha-2\beta}.$$

In this basis, the Levi M_β, which we view as GL_2, is represented as

$$M_\beta \sim \begin{pmatrix} \mathrm{Std} & & \\ & \mathrm{Ad}^2 & \\ & & \mathrm{Std}^\vee \end{pmatrix}.$$

Here Std denotes the standard representation of GL_2, and $\mathrm{Ad}^2 = \mathrm{Sym}^2(\mathrm{Std}) \otimes \det^{-1}$ the 3-dimensional adjoint representation. The fact that the Levi M_β takes this shape can be seen from following root strings in the weight diagram for R_7 in the direction of β, as in Figure 3.1.4.

We also note that, in the basis above, the parabolic subgroup P_β is block upper triangular, as

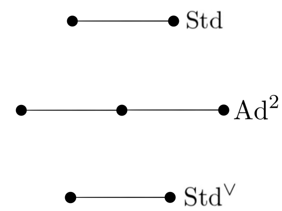


Figure 3.1.4: The Levi M_β under R_7

follows:

$$P_\beta \sim \begin{pmatrix} \mathrm{Std} & * & * \\ & \mathrm{Ad}^2 & * \\ & & \mathrm{Std}^\vee \end{pmatrix}.$$

The Weyl group W of G_2 is isomorphic to the dihedral group D_6 with 12 elements acting in the natural way on the root lattice. For γ a positive root of G_2, let w_γ be the reflection across the line perpendicular to γ. Then W is generated by w_α and w_β, and we have

$$W = \{1, w_\alpha, w_\beta, w_{\alpha\beta}, w_{\beta\alpha}, w_{\alpha\beta\alpha}, w_{\beta\alpha\beta}, w_{\alpha\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\alpha\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}, w_{-1}\}.$$

Here we amalgamate notation for products of w_α and w_β; so $w_{\alpha\beta} = w_\alpha w_\beta$, and so on. The final element w_{-1} is the longest element of W and it acts as -1 on the root lattice. It is equal to both $w_{\alpha\beta\alpha\beta\alpha\beta}$ and $w_{\beta\alpha\beta\alpha\beta\alpha}$.

Finally, we note that the group G_2 is self dual. We will often conflate G_2 and its dual group, though we remark that passing to the dual switches the long and short roots. So $\alpha^\vee$ is the short simple root for the dual G_2 and $\beta^\vee$ is the long one. Thus, on the dual side, the weights for the standard representation are

$$\pm(2\alpha^\vee + \beta^\vee),\ \pm(\alpha^\vee + \beta^\vee),\ \pm\alpha^\vee,\ 0.$$

3.1.2 Alternating trilinear forms and G_2

Let k be an algebraically closed field, and say for simplicity that k is of characteristic zero. Let V be the space of the standard representation of GL_7 over k. It is a classical fact that the group G_2 over k is the stabilizer of any alternating trilinear form which is *generic*, meaning that the orbit of this form must be Zariski open in GL_7. Let $e_1, \dots, e_7$ be a basis for V, and $e_1^\vee, \dots, e_7^\vee$ the dual basis for $V^\vee$. Then a standard example of such a trilinear form is given by

$$e_1^\vee \wedge e_2^\vee \wedge e_3^\vee + e_4^\vee \wedge e_5^\vee \wedge e_6^\vee + e_1^\vee \wedge e_4^\vee \wedge e_7^\vee + e_2^\vee \wedge e_5^\vee \wedge e_7^\vee + e_3^\vee \wedge e_6^\vee \wedge e_7^\vee.$$

(See, for example, [CH88].)

Making the permutation (2635)(47) on the indices turns this form into

$$-e_1^\vee \wedge e_4^\vee \wedge e_7^\vee - e_1^\vee \wedge e_5^\vee \wedge e_6^\vee + e_2^\vee \wedge e_3^\vee \wedge e_7^\vee + e_2^\vee \wedge e_4^\vee \wedge e_5^\vee + e_3^\vee \wedge e_4^\vee \wedge e_5^\vee,$$

and then making the change of basis $e_4 \mapsto -e_4$ and $e_5 \mapsto -e_5$ makes this

$$e_1^\vee \wedge e_4^\vee \wedge e_7^\vee + e_1^\vee \wedge e_5^\vee \wedge e_6^\vee + e_2^\vee \wedge e_3^\vee \wedge e_7^\vee - e_2^\vee \wedge e_4^\vee \wedge e_5^\vee + e_3^\vee \wedge e_4^\vee \wedge e_5^\vee.$$

Finally, for any $a \in k^\times$, we can make the change of basis

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) \mapsto (e_1, e_2, ae_3, a^{-1}e_4, ae_5, a^{-1}e_6, a^{-1}e_7)$$

to bring this form to

$$e_1^\vee \wedge e_4^\vee \wedge e_7^\vee + e_1^\vee \wedge e_5^\vee \wedge e_6^\vee + e_2^\vee \wedge e_3^\vee \wedge e_7^\vee - e_2^\vee \wedge e_4^\vee \wedge e_5^\vee + ae_3^\vee \wedge e_4^\vee \wedge e_5^\vee.$$

We record this in the following lemma.

Lemma 3.1.2.1. *For any $a \in k^\times$, the alternating 3-form*

$$e_1^\vee \wedge e_4^\vee \wedge e_7^\vee + e_1^\vee \wedge e_5^\vee \wedge e_6^\vee + e_2^\vee \wedge e_3^\vee \wedge e_7^\vee - e_2^\vee \wedge e_4^\vee \wedge e_5^\vee + ae_3^\vee \wedge e_4^\vee \wedge e_5^\vee$$

is generic.

For $a \in k^\times$, let G_a be the subgroup of GL_7 preserving the form in the lemma. Then $G_a \cong G_2$, and G_a is conjugate in GL_7 to the image of G_2 under the standard representation R_7 discussed in the previous section.

Lemma 3.1.2.2. *The subgroup of G_a, given on k-algebras A by*

$$T_a(A) = \{\mathrm{diag}(t_1, t_2, t_1 t_2^{-1}, 1, t_1^{-1} t_2, t_2^{-1}, t_1^{-1}) \mid t_1, t_2 \in \mathbb{G}_m(A)\}$$

is a maximal torus in G_a.

Proof. The group T_a is clearly a torus of rank 2, and it is easy to check that it preserves the form of Lemma 3.1.2.1. Since G_a is of rank 2, the lemma follows. □

We remark that, despite the choice of notation, T_a does not actually depend on the element a. However, we insist on this notation in order to be consistent with our notation for other subgroups of G_a, and to distinguish this torus from the torus T that appeared in the previous section, which was only a subgroup of G_2 and not immediately of GL_7.

We now study the root system of G_a in the basis $e_1, \ldots, e_7$. Write

$$[t_1, t_2] = \mathrm{diag}(t_1, t_2, t_1 t_2^{-1}, 1, t_1^{-1} t_2, t_2^{-1}, t_1^{-1}) \in T_a.$$

Abusing notation slightly, write

$$\alpha([t_1, t_2]) = t_1^{-1} t_2^2, \qquad \beta([t_1, t_2]) = t_1 t_2^{-1}.$$

We define various 1-parameter subgroups of G_a by defining them on A-points for k-algebras A as follows. For $x \in A$, let

$$g(\alpha, x) = \begin{pmatrix} 1 & & & & & & \\ & 1 & x & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & a^{-1}x & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}, \qquad g(\beta, x) = \begin{pmatrix} 1 & x & & & & & \\ & 1 & & & & & \\ & & 1 & -2x & -x^2 & & \\ & & & 1 & x & & \\ & & & & 1 & & \\ & & & & & 1 & -x \\ & & & & & & 1 \end{pmatrix},$$

$$g(\alpha+\beta,x)=\begin{pmatrix}1 & & x & & & & \\ & 1 & & 2x & & a^{-1}x^2 & \\ & & 1 & & & & \\ & & & 1 & & a^{-1}x & \\ & & & & 1 & & a^{-1}x \\ & & & & & 1 & \\ & & & & & & 1\end{pmatrix},$$

$$g(\alpha+2\beta,x)=\begin{pmatrix}1 & & & 2x & & & a^{-1}x^2 \\ & 1 & & & -x & & \\ & & 1 & & & -a^{-1}x & \\ & & & 1 & & & a^{-1}x \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1\end{pmatrix},$$

$$g(\alpha+3\beta,x)=\begin{pmatrix}1 & & & & x & & \\ & 1 & & & & & \\ & & 1 & & & & a^{-1}x \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1\end{pmatrix}, \qquad g(2\alpha+3\beta,x)=\begin{pmatrix}1 & & & & & x & \\ & 1 & & & & & -x \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1\end{pmatrix}.$$

Then for $\gamma \in \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}$, one checks easily the relations given by

$$[t_1,t_2]g(\gamma,x)[t_1,t_2]^{-1} = g(\gamma, \gamma([t_1,t_2])x).$$

One also checks that $g(\gamma,\cdot) \subset G_a$ by checking that these elements preserve the given generic alternating 3-form, and it follows that

$$\{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}$$

forms a system of positive roots for T_a in G_a.

We do one sample calculation checking that $g(\beta,\cdot) \subset G_a$ now. Write

$$e^\vee_{ijk} = e^\vee_i \wedge e^\vee_j \wedge e^\vee_k,$$

for $1 \le i,j,k \le 7$. Then the trilinear form stabilized by G_a is

$$e^\vee_{147} + e^\vee_{156} + e^\vee_{237} - e^\vee_{246} + ae^\vee_{345}.$$

Using that a matrix acts on the basis $e_1^\vee, \dots, e_7^\vee$ by its transpose, we compute then that

$$\begin{aligned}
&g(\beta, x)(e_{147}^\vee + e_{156}^\vee + e_{237}^\vee - e_{246}^\vee + ae_{345}^\vee) \\
&\qquad =(e_{147}^\vee + xe_{247}^\vee + xe_{157}^\vee + x^2 e_{257}^\vee) + (e_{156}^\vee + xe_{256}^\vee - xe_{157}^\vee - x^2 e_{257}^\vee) \\
&\qquad\quad + (e_{237}^\vee - 2xe_{247}^\vee - x^2 e_{257}^\vee) - (e_{246}^\vee + xe_{256}^\vee - xe_{247}^\vee - x^2 e_{257}^\vee) \\
&\qquad\quad + a(e_{345}^\vee - 2xe_{445}^\vee - x^2 e_{545}^\vee + xe_{355}^\vee - 2x^2 e_{455}^\vee - x^3 e_{555}^\vee) \\
&\qquad =(e_{147}^\vee + e_{156}^\vee + e_{237}^\vee - e_{246}^\vee + ae_{345}^\vee) + (x - 2x + x)e_{247}^\vee + (x - x)e_{157}^\vee \\
&\qquad\quad + (x^2 - x^2 - x^2 + x^2)e_{257}^\vee + (x - x)e_{256}^\vee \\
&\qquad =e_{147}^\vee + e_{156}^\vee + e_{237}^\vee - e_{246}^\vee + ae_{345}^\vee,
\end{aligned}$$

as desired.

Now we denote by $P_{a,\beta}$ the parabolic subgroup of G_a containing T_a along with all the positive roots for T_a in G_a and $-\beta$. One checks easily that if we let

$$g(-\beta, x) = \begin{pmatrix} 1 & & & & & & \\ x & 1 & & & & & \\ & & 1 & & & & \\ & & -x & 1 & & & \\ & & -x^2 & 2x & 1 & & \\ & & & & & 1 & \\ & & & & & -x & 1 \end{pmatrix},$$

then $g(-\beta, x) \in G_a$ and

$$[t_1, t_2] g(-\beta, x) [t_1, t_2]^{-1} = g(-\beta, \beta([t_1, t_2])^{-1} x).$$

Therefore for γ any positive root or $\gamma = -\beta$, the root subgroups corresponding to γ are the one-parameter subgroups $g(\gamma, \cdot)$ given above.

Proposition 3.1.2.3. *Let P_{232} be the standard parabolic subgroup of* GL_7 *of the form*

$$P_{232} = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ & & * & * & * & * & * \\ & & * & * & * & * & * \\ & & * & * & * & * & * \\ & & & & & * & * \\ & & & & & * & * \end{pmatrix}.$$

Then $P_{232} \cap G_a = P_{a,\beta}$.

Proof. Clearly $T' \subset P_{232}$ and $g(\gamma, \cdot) \subset P_{232}$ for any positive root γ or $\gamma = -\beta$. Therefore $P_{a,\beta} \subset P_{232}$.

To show the opposite inclusion, we use the Bruhat decomposition. Let

$$\tilde{w}_\alpha = \begin{pmatrix} 1 & & & & & & \\ & & -1 & & & & \\ & 1 & & & & & \\ & & & 1 & & & \\ & & & & & -a^{-1} & \\ & & & & a & & \\ & & & & & & 1 \end{pmatrix}, \qquad \tilde{w}_\beta = \begin{pmatrix} & 1 & & & & & \\ 1 & & & & & & \\ & & & & 1 & & \\ & & & -1 & & & \\ & & 1 & & & & \\ & & & & & & 1 \\ & & & & & 1 & \end{pmatrix}.$$

Then one checks easily that $\tilde{w}_\alpha, \tilde{w}_\beta \in G_a$. Also, $\tilde{w}_\alpha, \tilde{w}_\beta$ normalize the torus T', and they normalize the standard diagonal maximal torus in GL_7, which we denote T_7, and thus these elements are representatives for the Weyl groups of both G_a and GL_7.

Like in the previous section, we use amalgamated notation and let, for example, $\tilde{w}_{\alpha\beta} = \tilde{w}_\alpha \tilde{w}_\beta$. Let $s_\alpha = (23)(56) \in S_7$ be the permutation corresponding to $\tilde{w}_\alpha$ when viewing the Weyl group of GL_7 as the symmetric group on 7 elements. Similarly define $s_\beta = (12)(35)(67) \in S_7$, as well as $s_{\alpha\beta}$, and so on. Then one checks

$$s_{\alpha\beta} = (125763), \qquad s_{\beta\alpha} = (367521), \qquad s_{\alpha\beta\alpha} = (31)(26)(57), \qquad s_{\beta\alpha\beta} = (15)(37),$$
$$s_{\alpha\beta\alpha\beta} = (156)(273), \qquad s_{\beta\alpha\beta\alpha} = (165)(237), \qquad s_{\alpha\beta\alpha\beta\alpha} = (16)(27)(35),$$
$$s_{\beta\alpha\beta\alpha\beta} = (17)(25)(36), \qquad s_{\alpha\beta\alpha\beta\alpha\beta} = (17)(26)(35),$$

and this defines a homomorphism from the Weyl group of G_a to the Weyl group S_7 of GL_7 which is visibly injective. The Weyl group W_{232} of the Levi of P_{232} is the subgroup of S_7 which acts separately on the sets $\{1,2\}$, $\{3,4,5\}$, $\{6,7\}$. One sees from the list given above that the only elements of the Weyl group of G_a which are in W_{232} are 1 and s_β.

Let $M_{a,\beta}$ be the Levi of $P_{a,\beta}$. Writing W_a for the Weyl group of G_a and $W_{a,\beta}$ for that of M_β, we thus get an injective map

$$W_{a,\beta}\backslash W_a \hookrightarrow W_{232}\backslash S_7.$$

Let us identify $W_{a,\beta}\backslash W_a$ with the set $W^{P_{a,\beta}}$ of minimal length representatives of this quotient, so

$$W_{a,\beta}\backslash W_a \cong W^{P_{a,\beta}} = \{1, s_\alpha, s_{\alpha\beta}, s_{\alpha\beta\alpha}, s_{\alpha\beta\alpha\beta}, s_{\alpha\beta\alpha\beta\alpha}\}.$$

Write $W^{P_{232}}$ for the set of minimal length representatives for the quotient $W_{232}\backslash S_7$. Then we have an inclusion

$$W^{P_{a,\beta}} \hookrightarrow W^{P_{232}}.$$

Now the Bruhat decomposition gives a decomposition into disjoint sets,

$$\mathrm{GL}_7 = \coprod_{s\in W^{P_{232}}} P_{232}sP_{232}.$$

Similarly,

$$G_a = \coprod_{s\in W^{P_{a,\beta}}} P_{a,\beta}sP_{a,\beta}.$$

Because $W^{P_{a,\beta}}$ injects into $W^{P_{232}}$, a subdecomposition of the first decomposition above is given by

$$\mathrm{GL}_7 \supset \coprod_{s\in W^{P_{a,\beta}}} P_{232}sP_{232}.$$

Since $P_{a,\beta} \subset P_{232}$, we have

$$P_{a,\beta}sP_{a,\beta} \subset P_{232}sP_{232}$$

for any $s \in W^{P_{a,\beta}}$. Since for $s \neq 1$, $P_{232}sP_{232}$ is disjoint from P_{232}, this proves that $P_{a,\beta}sP_{a,\beta} \cap P_{232} = \emptyset$. Therefore we must have $P_{a,\beta} = P_{232} \cap G_a$, as desired. $\square$

The following lemma will be a key step for us in checking that the cocycle we construct later on will lie in the correct Bloch–Kato Selmer group.

Lemma 3.1.2.4. *Let $h \in P_{a,\beta}$, and write h_{ij} for the (i,j)-entry of the matrix h. Then we have the relations*

$$2h_{22}h_{13} - 2h_{12}h_{23} + h_{21}h_{14} - h_{11}h_{24} = 0$$

and

$$h_{22}h_{14} - h_{12}h_{24} - 2h_{21}h_{15} + 2h_{11}h_{25} = 0.$$

Proof. First we check this for h in the unipotent radical of $P_{a,\beta}$, which we denote by $N_{a,\beta}$. Any element $h \in N_{a,\beta}$ can be written as

$$h = g(\alpha, x_1)g(\alpha+\beta, x_2)g(\alpha+2\beta, x_3)g(\alpha+3\beta, x_4)g(2\alpha+3\beta, x_5).$$

Then one can compute using the expressions for these 1-parameter subgroups given above that

$$h = \begin{pmatrix} 1 & & x_2 & 2x_3 & x_4 & * & * \\ & 1 & x_1 & 2x_2 & -x_3 & * & * \\ & & 1 & & & * & * \\ & & & 1 & & * & * \\ & & & & 1 & * & * \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$

where the asterisks are certain polynomial combinations of $x_1, \ldots, x_5$. The matrix entries of this element clearly satisfy the relations listen in the statement of the lemma.

Now let h be any element of $P_{a,\beta}$ satisfying the relations given in the lemma. We will check that the entries of the matrices

$$g(\beta, x)h, \qquad g(-\beta, x)h, \qquad [t_1, t_2]h$$

also satisfy the relations stated in the lemma. For the first of these, let $h' = g(\beta, x)h$ and write $h' = (h'_{ij})$. We have

$$\begin{pmatrix} h'_{11} & h'_{12} \\ h'_{21} & h'_{22} \end{pmatrix} = \begin{pmatrix} h_{11} + xh_{21} & h_{12} + xh_{22} \\ h_{21} & h_{22} \end{pmatrix}$$

and

$$\begin{pmatrix} h'_{13} & h'_{14} & h'_{15} \\ h'_{23} & h'_{24} & h'_{25} \end{pmatrix} = \begin{pmatrix} h_{13} + xh_{23} & h_{14} + xh_{24} & h_{15} + xh_{25} \\ h_{23} & h_{24} & h_{25} \end{pmatrix}.$$

Thus

$$
\begin{aligned}
&2h'_{22}h'_{13} - 2h'_{12}h'_{23} + h'_{21}h'_{14} - h'_{11}h'_{24} \\
&\qquad = 2h_{22}(h_{13} + xh_{23}) - 2(h_{12} + xh_{22})h_{23} + h_{21}(h_{14} + xh_{24}) - (h_{11} + xh_{21})h_{24} \\
&\qquad = 2h_{22}h_{13} - 2h_{12}h_{23} + h_{21}h_{14} - h_{11}h_{24} + 2xh_{22}h_{23} - 2xh_{22}h_{23} + xh_{21}h_{24} - xh_{21}h_{24} \\
&\qquad = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&h'_{22}h'_{14} - h'_{12}h'_{24} - 2h'_{21}h'_{15} + 2h'_{11}h'_{25} \\
&\qquad = h_{22}(h_{14} + xh_{24}) - (h_{12} + xh_{22})h_{24} - 2h_{21}(h_{15} + xh_{25}) + 2(h_{11} + xh_{21})h_{25} \\
&\qquad = h_{22}h_{14} - h_{12}h_{24} - 2h_{21}h_{15} + 2h_{11}h_{25} + xh_{22}h_{24} - xh_{22}h_{24} - 2xh_{21}h_{25} + 2xh_{21}h_{25} \\
&\qquad = 0,
\end{aligned}
$$

as desired.

Now instead let $h' = g(-\beta, x)h$. Then we have

$$
\begin{pmatrix} h'_{11} & h'_{12} \\ h'_{21} & h'_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} + xh_{11} & h_{22} + xh_{12} \end{pmatrix}
$$

and

$$
\begin{pmatrix} h'_{13} & h'_{14} & h'_{15} \\ h'_{23} & h'_{24} & h'_{25} \end{pmatrix} = \begin{pmatrix} h_{13} & h_{14} & h_{15} \\ h_{23} + xh_{13} & h_{24} + xh_{14} & h_{25} + xh_{15} \end{pmatrix}.
$$

Thus

$$
\begin{aligned}
&2h'_{22}h'_{13} - 2h'_{12}h'_{23} + h'_{21}h'_{14} - h'_{11}h'_{24} \\
&\qquad = 2(h_{22} + xh_{12})h_{13} - 2h_{12}(h_{23} + xh_{13}) + (h_{21} + xh_{11})h_{14} - h_{11}(h_{24} + xh_{14}) \\
&\qquad = 2h_{22}h_{13} - 2h_{12}h_{23} + h_{21}h_{14} - h_{11}h_{24} + 2xh_{12}h_{13} - 2xh_{12}h_{13} + xh_{11}h_{14} - xh_{11}h_{14} \\
&\qquad = 0,
\end{aligned}
$$

and

$$\begin{aligned}
&h'_{22}h'_{14} - h'_{12}h'_{24} - 2h'_{21}h'_{15} + 2h'_{11}h'_{25} \\
&\quad =(h_{22} + xh_{12})h_{14} - h_{12}(h_{24} + xh_{14}) - 2(h_{21} + xh_{11})h_{15} + 2h_{11}(h_{24} + xh_{14}) \\
&\quad =h_{22}h_{14} - h_{12}h_{24} - 2h_{21}h_{15} + 2h_{11}h_{24} + xh_{12}h_{14} + xh_{12}h_{14} - 2xh_{11}h_{15} + 2xh_{11}h_{14} \\
&\quad =0,
\end{aligned}$$

as desired.

Finally, let $h' = [t_1, t_2]h$. Then we have

$$\begin{pmatrix} h'_{11} & h'_{12} \\ h'_{21} & h'_{22} \end{pmatrix} = \begin{pmatrix} t_1h_{11} & t_1h_{12} \\ t_2h_{21} & t_2h_{22} \end{pmatrix}$$

and

$$\begin{pmatrix} h'_{13} & h'_{14} & h'_{15} \\ h'_{23} & h'_{24} & h'_{25} \end{pmatrix} = \begin{pmatrix} t_1h_{13} & t_1h_{14} & t_1h_{15} \\ t_2h_{23} & t_2h_{24} & t_2h_{25} \end{pmatrix}.$$

Thus

$$2h'_{22}h'_{13} - 2h'_{12}h'_{23} + h'_{21}h'_{14} - h'_{11}h'_{24} = t_1t_2(2h_{22}h_{13} - 2h_{12}h_{23} + h_{21}h_{14} - h_{11}h_{24}) = 0$$

and

$$h'_{22}h'_{14} - h'_{12}h'_{24} - 2h'_{21}h'_{15} + 2h'_{11}h'_{25} = t_1t_2(h_{22}h_{14} - h_{12}h_{24} - 2h_{21}h_{15} + 2h_{11}h_{25}) = 0,$$

as desired, once again.

Now let $M_{a,\beta}$ denote the Levi subgroup of $P_{a,\beta}$. It is generated by elements of the form $g(\beta, x)$, $g(-\beta, x)$ and $[t_1, t_2]$. By the Levi decomposition, $P_{a,\beta} = M_{a,\beta}N_{a,\beta}$. We already showed that the matrices in $N_{a,\beta}$ satisfy the conclusion of the lemma, and by what we just showed, so do matrices in $M_{a,\beta}N_{a,\beta}$. So we are done. $\square$

3.2 Setup

The construction of nontrivial elements in the symmetric cube Selmer group that we will make in this chapter relies on certain results of an automorphic nature that were obtained in the previous chapters of this thesis, especially Chapter 2. We begin by recalling the results which will be relevant here.

The aforementioned construction will also depend on certain standard conjectures, and we will also state these conjectures in this section.

3.2.1 Summary of previous results

Let F be a cuspidal holomorphic eigenform of weight k and level 1. Then $k \geq 12$. We fix a root α_p of the Hecke polynomial of F at p and let $s_p = v_p(\alpha_p)$.

Let π_F be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated with F. Because of the assumption that F has level 1, we have $\epsilon(1/2, \pi_F, \mathrm{Sym}^3) = -1$. In particular, $L(1/2, \pi_F, \mathrm{Sym}^3) = 0$.

We view π_F as a representation of the Levi $M_\alpha(\mathbb{A})$ of $P_\alpha(\mathbb{A})$. We then consider the representation Π which we define to be the Langlands quotient of the unitary parabolic induction

$$\iota_{P_\alpha(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})}(\pi_F \otimes \delta_{P_\alpha(\mathbb{A})}^{1/10}).$$

Let Π_f be its finite component.

We assume the following conjecture.

Conjecture 3.2.1.1. *There is a unique summand Π' of $L^2_{\mathrm{disc}}(\mathrm{G}_2(\mathbb{Q})\backslash\mathrm{G}_2(\mathbb{A}))$ which is equivalent to Π at all but finitely many places. If Π'_f denotes its finite component, then actually $\Pi'_f \cong \Pi_f$. The representation Π' is cuspidal. The archimedean component Π'_∞ of Π' is discrete series with Harish-Chandra parameter $\frac{k-4}{2}(2\alpha+3\beta)+\rho$ (where the positive compact roots are given by $2\alpha+3\beta$ and β, and $\rho = 3\alpha+5\beta$ denotes half the sum of positive roots).*

A more general conjecture than this is stated in Chapter 1 as Conjecture 1.5.4.1, and it is explained there how this more general conjecture reduces to Arthur's conjectures and a computation of an archimedean Arthur packet à la Adams–Johnson, along with a statement about the expected behavior of CAP representations.

Let

$$\lambda_0 = \frac{k-4}{2}(2\alpha + 3\beta).$$

This conjecture implies that there is only one irreducible representation of $\mathrm{G}_2(\mathbb{A}_f)$ which is equivalent to Π_f at all but finitely many places and which appears in the cuspidal cohomology of G_2 with coefficients twisted by the highest weight λ_0 representation V_{λ_0} of $\mathrm{G}_2(\mathbb{C})$, and this this representation is exactly Π_f. This statement is Conjecture 2.3.1.2 of Chapter 2.

In order to state the result obtained in Chapter 2 which we will use here, we need to introduce a Hecke algebra R_p^{sph} and the weight space $\mathfrak{X}$. So first we let $\mathfrak{X}$ be the functor on algebraic extensions L of $\mathbb{Q}_p$ defined by

$$\mathfrak{X}(L) = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), L^\times).$$

Then $\mathfrak{X}$ is represented by a rigid analytic space over $\mathbb{Q}_p$ called *weight space*. Its group of $\overline{\mathbb{Q}}_p$-points contains the group $X^*(T)$ of algebraic characters of the maximal torus T in G_2.

Let K_f be a fixed maximal compact subgroup of $\mathrm{G}_2(\mathbb{A}_f)$ which is hyperspecial at all places, and let K_f^p be the product of its components at all places except for p. We let

$$R_p^{\mathrm{sph}} = \mathbb{Z}_p[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)] \otimes_{\mathbb{Z}_p} C_c^\infty(K_f^p \backslash \mathrm{G}_2(\mathbb{A}_f^p)/K_f^p, \mathbb{Z}_p).$$

The component $\mathbb{Z}_p[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]$ is supposed to be viewed as the algebra $\mathcal{U}_p$ introduced in Section 2.2.2 or, more generally, in Section 4.1 of [Urb11]. We write u_t for the group-like element of $\mathbb{Z}_p[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]$ corresponding to $t \in T(\mathbb{Q}_p)$. The component $C_c^\infty(K_f^p \backslash \mathrm{G}_2(\mathbb{A}_f^p)/K_f^p, \mathbb{Z}_p)$ is a space of compactly supported smooth functions, and it is a spherical Hecke algebra which acts on irreducible admissible representations of $\mathrm{G}_2(\mathbb{A}_f^p)$ over a coefficient field containing $\mathbb{Z}_p$ which are spherical at all places (except for p and ∞, of course).

The algebra R_p^{sph} acts, for instance, on *finite slope p-stabilizations* of everywhere unramified irreducible admissible representations of $\mathrm{G}_2(\mathbb{A}_f)$ (see Section 4.1.9 of [Urb11], or Section 2.2.2 or A.1.3, for this notion). By the results of Section 2.3.1, the representation Π_f has a certain finite slope p-stabilization, denoted $\sigma(\Pi)$, corresponding to the choice of root α_p of the Hecke polynomial of F at p; $\sigma(\Pi)$ is given as a space by Π_f^p with corresponding action by $C_c^\infty(K_f^p \backslash \mathrm{G}_2(\mathbb{A}_f^p)/K_f^p, \mathbb{Z}_p)$, and then $u_t \in \mathbb{Z}_p[T(\mathbb{Q}_p)/T(\mathbb{Z}_p)]$ acts by multiplication by $\chi_0(t)$ for some unramified character

$\chi_0 : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_p^\times$ satisfying

$$\chi_0(\mu^\vee(p)) = \alpha_p^{\langle 2\alpha+3\beta,\mu^\vee\rangle} p^{\langle \beta,\mu^\vee\rangle}$$

for any cocharacter $\mu^\vee$ of T. Thus

$$v_p(\chi_0(\mu^\vee(p))) = \langle s_p(2\alpha+3\beta)+\beta, \mu^\vee\rangle, \tag{3.2.1.1}$$

and $\sigma(\Pi)$ has *slope* $s_p(2\alpha+3\beta)+\beta$ in the sense of the aforementioned references.

Theorem 3.2.1.2. *Let the setting be as above. Then under Conjecture 3.2.1.1, there are*

- *an open affinoid subdomain* $\mathfrak{U} \subset \mathfrak{X}$*,*
- *a finite cover* $\mathbf{w} : \mathfrak{V} \to \mathfrak{U}$ *with* $\mathfrak{V}$ *reduced,*
- *a point* $y_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ *with* $\mathbf{w}(y_0) = \lambda_0$*,*
- *a Zariski dense subset* $\Sigma \subset \mathfrak{V}(\overline{\mathbb{Q}}_p)$ *with* $\lambda_y = \mathbf{w}(y)$ *regular algebraic for every* $y \in \Sigma$*,*
- *for each* $y \in \Sigma$*, a finite slope* p*-stabilization* σ_y *of an irreducible, cohomological, cuspidal automorphic representation* π_y *of* $\mathrm{G}_2(\mathbb{A})$ *of weight* $\mathbf{w}(y)$ *and full level* K_f*,*
- *a* $\mathbb{Z}_p$*-algebra homomorphism* $\theta_{\mathfrak{V}} : R_p^{\mathrm{sph}} \to \mathcal{O}(\mathfrak{V})$*,*

satisfying the following properties:

- *The set* $\mathbf{w}(\Sigma)$ *contains every dominant regular algebraic weight outside of a proper Zariski closed subset of* $\mathfrak{X}$*;*
- *The specialization of* $\theta_{\mathfrak{V}}$ *at the point* y_0 *is the character of* R_p^{sph} *coming from* $\sigma(\Pi)$*;*
- *For each* $y \in \Sigma$*, the specialization* θ_y *of* $\theta_{\mathfrak{V}}$ *at* y *is the character of* R_p^{sph} *induced by the representation* σ_y*;*
- *For all* $y \in \Sigma$*, the component* $\pi_{y,p}$ *of* π_y *at* p *can be written as a normalized parabolic induction*

$$\iota_{P_\alpha(\mathbb{Q}_p)}^{\mathrm{G}_2(\mathbb{Q}_p)}(\chi)$$

where χ is the unramified character of $T(\mathbb{Q}_p)$ such that

$$\chi(\mu^\vee(p)) = \theta_y(u_{\mu^\vee(p)})p^{-\langle\lambda+\rho,\mu^\vee\rangle}. \tag{3.2.1.2}$$

Here, $\rho = 3\alpha + 5\beta$ is half the sum of positive roots.

Proof. Except for reducedness of $\mathfrak{V}$ as well as the very last point in the statement of the theorem, this is part of Theorem 2.3.2.3. If $\mathfrak{V}$ is not reduced, we simply pass to $\mathcal{O}(\mathfrak{V})^{\mathrm{red}}$, which has no effect on $\mathfrak{V}(\overline{\mathbb{Q}}_p)$. The last point just follows from a short computation using Proposition 2.2.3.2 showing that, after undoing certain normalizations, the unramified character from which $\pi_{y,p}$ is induced satisfies the formula stated in the theorem. □

3.2.2 Galois representations into G_2

In this section we state a conjecture which is a version of the global Langlands correspondence for cohomological automorphic representations π of $\mathrm{G}_2(\mathbb{A})$ of weight λ, with λ algebraic and dominant. If λ is regular and π is cuspidal, such representations π are discrete series with infinitesimal character given by $\lambda + \rho$.

Fix such a π. Recall (see Section 1.3, for example) that if ℓ is a finite prime at which π is unramified, then the local component π_ℓ of π at ℓ corresponds via the Satake isomorphism to a unique semisimple conjugacy class $s(\pi_\ell) \in \mathrm{G}_2(\overline{\mathbb{Q}}_p)$. We call $s(\pi_\ell)$ the *Satake parameter* of π_ℓ. As well, we get a character $\omega : \overline{\mathbb{Q}}_p[X^*(T^\vee)] \to \overline{\mathbb{Q}}_p$, and these have the property that if V is a finite dimensional representation of G_2 and χ_V is the character of V, viewed as an element of $\overline{\mathbb{Q}}_p[X^*(T^\vee)]$, then

$$\omega(\chi_V) = \mathrm{Tr}(s(\pi_\ell)|V).$$

In the following we will have occasion to consider the composition $\lambda \circ \chi_V$ for λ and χ_V as above. What is meant by this is the set of all integers $\langle\lambda, \mu\rangle$, counted with multiplicity, with μ a character of $T^\vee$ in the support of χ_V. So if μ occurs as a weight of $T^\vee$ in V with multiplicity n, then this contributes the number $\langle\lambda, \mu\rangle$ to the set $\chi_V \circ \lambda$, n times.

We now state the following conjecture.

Conjecture 3.2.2.1. *Let λ be a dominant algebraic weight of T and let π be a cohomological* *automorphic representation of* $\mathrm{G}_2(\mathbb{A})$ *with cohomological weight* λ*. Let* S *be the set of finite places at which* π *is ramified. Then there is a continuous representation*

$$\rho_\pi : G_{\mathbb{Q}} \to \mathrm{G}_2(\overline{\mathbb{Q}}_p)$$

such that:

(1) ρ_π *is unramified at all finite primes* $\ell \notin S \cup \{p\}$*;*

(2) If $\ell \notin S \cup \{p\}$*, then*

$$\rho_\pi(\mathrm{Frob}_\ell^{-1})^{\mathrm{ss}} \in s(\pi_\ell),$$

where $(\cdot)^{\mathrm{ss}}$ *denotes semisimplification and* $s(\pi_\ell)$ *is the Satake parameter of* π_ℓ*;*

(3) If π_p *is unramified, then for any faithful representation* $R : \mathrm{G}_2 \to \mathrm{GL}_n$*, the representation* $R \circ \rho|_{G_{\mathbb{Q}_p}}$ *is crystalline. Furthermore,* $R \circ \rho|_{G_{\mathbb{Q}_p}}$ *has Hodge-Tate weights given by*

$$\mathrm{HT}(R \circ \rho|_{G_{\mathbb{Q}_p}}) = \chi_R \circ (\lambda + \rho),$$

where χ_R *is the character of* R*,* $\chi_R \circ \lambda$ *is defined as above, and* $\rho = 3\alpha + 5\beta$ *is half the sum of positive roots.. Finally, the characteristic polynomial of the inverse* ϕ^{-1} *of the crystalline Frobenius* ϕ *on* $D_{\mathrm{crys}}(R \circ \rho|_{G_{\mathbb{Q}_p}})$ *is the same as the characteristic polynomial of* $R(s(\pi_p))$*.*

This conjecture is analogous to many known results and other conjectures on Galois representations attached to cohomological automorphic representations. The closest case to this which is known in the literature is in the work of Kret–Shin [KS20]. There they prove a similar result for G_2 and GSp_{2n} under some extra assumptions (including a Steinberg assumption that the representations in this chapter will not satisfy).

We believe that this conjecture is not too far out of reach. When π is cuspidal, if one could prove certain properties, including cuspidality, of the exceptional theta lift to PGSp_6 of such a π as in the conjecture (this theta lift is the one discovered by Ginzburg–Rallis–Soudry [GRS97]) then one could hope to construct the Galois representation into $\mathrm{GL}_7(\overline{\mathbb{Q}}_p)$. From there, one should be able to use the arguments in the paper [Che19] of Chenevier to show that the Galois representation obtained actually factors through $\mathrm{G}_2(\overline{\mathbb{Q}}_p)$.

3.3 Some p-adic Hodge theory

We now give various results whose purpose is to help establish the crystallinity of certain p-adic representations. Although not all of these results actually do this directly (Lemma 3.3.1.1 is a criterion for an extension to be de Rham, for instance) they are enough to combine with other parts of the theory to show crystallinity of the representations which we need to show are crystalline.

3.3.1 Two lemmas

We now give some lemmas; the first of these lemmas is a technical lemma which generalizes a result due to Skinner and Urban, see [Urb13a]. It is proved in much the same way.

Lemma 3.3.1.1. *Let V and W be de Rham representations of $G_{\mathbb{Q}_p}$. Let E be an extension,*

$$0 \to V \to E \to W \to 0.$$

Assume that all the Hodge–Tate weights of V are strictly negative. Writing $g : D_{\mathrm{dR}}(E) \to D_{\mathrm{dR}}(W)$ for the natural map, assume there is a subspace $D \subset D_{\mathrm{dR}}(E)$ such that

$$D_{\mathrm{dR}}(W) = g(D) \oplus \mathrm{Fil}^0(D_{\mathrm{dR}}(W)).$$

Then E is de Rham.

Proof. We first claim that $H^0(\mathbb{Q}, V \otimes B_{\mathrm{dR}}^+) = H^1(\mathbb{Q}, V \otimes B_{\mathrm{dR}}^+) = 0$. In fact, let $B = \bigoplus_{i=0}^{\infty} t^i B_{\mathrm{dR}}^+$ where $t \in B_{\mathrm{dR}}^+$ is the usual uniformizer. Then $B/tB \cong B_{\mathrm{HT}}^+$. Since V is de Rham with strictly negative Hodge–Tate weights, we have

$$H^0(\mathbb{Q}, V \otimes B_{\mathrm{HT}}^+) = H^1(\mathbb{Q}, V \otimes B_{\mathrm{HT}}^+) = 0$$

Then tensoring the exact sequence

$$0 \to B \xrightarrow{t} B \to B_{\mathrm{HT}}^+ \to 0$$

with V gives that multiplication by t on B induces an isomorphism $H^0(\mathbb{Q}, V \otimes B) \to H^0(\mathbb{Q}, V \otimes B)$

and an isomorphism $H^1(\mathbb{Q}, V \otimes B) \to H^1(\mathbb{Q}, V \otimes B)$. Thus, composing these isomorphisms with themselves enough times shows that these groups are zero. In particular, the summands $H^0(\mathbb{Q}, V \otimes B_{\mathrm{dR}}^+)$ and $H^1(\mathbb{Q}, V \otimes B_{\mathrm{dR}}^+)$ are zero, which proves the claim.

Now we consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] \arrow[d] & \mathrm{Fil}^0 D_{\mathrm{dR}}(E) \arrow[r] \arrow[d] & \mathrm{Fil}^0 D_{\mathrm{dR}}(W) \arrow[r] \arrow[d] & 0 \arrow[d] \\
D_{\mathrm{dR}}(V) \arrow[r] \arrow[d] & D_{\mathrm{dR}}(E) \arrow[r, "g"] \arrow[d, "f_E"] & D_{\mathrm{dR}}(W) \arrow[r, "\delta"] \arrow[d, "f_W"] & H^1(\mathbb{Q}, V \otimes D_{\mathrm{dR}}) \arrow[d] \\
(V \otimes \frac{B_{\mathrm{dR}}}{B_{dR}^+})^{G_{\mathbb{Q}_p}} \arrow[r] & (E \otimes \frac{B_{\mathrm{dR}}}{B_{dR}^+})^{G_{\mathbb{Q}_p}} \arrow[r, "g'"] & (W \otimes \frac{B_{\mathrm{dR}}}{B_{dR}^+})^{G_{\mathbb{Q}_p}} \arrow[r, "\delta'"] \arrow[d] & H^1(\mathbb{Q}, V \otimes \frac{B_{\mathrm{dR}}}{B_{dR}^+}), \\
 & & 0 &
\end{tikzcd}$$

which is commutative and has exact rows and columns. The fact that the groups in the top corners are zero follows from the claim above, and exactness in the third column is due to W being de Rham. Now by the hypothesis that $D_{\mathrm{dR}}(W) = g(D) \oplus \mathrm{Fil}^0(D_{\mathrm{dR}}(W))$, we see that $g(f_W(D)) = (W \otimes \frac{B_{\mathrm{dR}}}{B_{dR}^+})^{G_{\mathbb{Q}_p}}$, so that $g \circ f_W$ is surjective. Thus $f_E \circ g'$ is surjective, and so therefore is g'. Thus $\delta' = 0$, which in turn implies $\delta = 0$. Since V and W are de Rham, so therefore is E. $\square$

Next we state a lemma about the interpolation of crystalline periods.

Lemma 3.3.1.2 (Kisin)**.** *Let $\mathfrak{W}$ be a reduced affinoid rigid analytic space and $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathcal{O}(\mathfrak{W}))$ a continuous representation. Assume there is a Zariski dense subset $T \subset \mathfrak{W}(\overline{\mathbb{Q}}_p)$ such that for all $x \in T$, the specialization ρ_x of ρ at x is Hodge–Tate with Hodge–Tate weights $k_{1,x}, \ldots, k_{n,x}$, in increasing order. Assume furthermore that for any n, the subset T_n of $x \in T$ such that $k_{i+1,x} - k_{i,x} \geq n$ for all i, is Zariski dense. Finally, for n sufficiently large and for any $x \in \Sigma_n$, assume that ρ_x is crystalline and that the eigenvalues of the crystalline Frobenius for ρ_x are given by $\phi_i(x)p^{k_{i,x}}$ for some $\phi_i \in \mathcal{O}(\mathfrak{W})$. Then for any $x \in \Sigma$,*

$$D_{\mathrm{crys}}(\rho_x)^{\phi = \phi_1(x) k_{1,x}} \neq 0$$

More generally, for any $1 \leq k \leq n$, we have

$$D_{\text{crys}}(\wedge^k \rho_x)^{\phi=\prod_{i=1}^k \phi_i(x) k_{i,x}} \neq 0.$$

Proof. The last statement follows from the one before it, and that statement is Proposition 4.2.2 (i) in [SU06b] which, in turn, is derived from Corollary 5.15 in [Kis03]. $\square$

3.3.2 Extensions of ρ_F by $\rho_F(1)$

Let F be our modular eigenform of level 1 introduced in Section 3.2, and ρ_F its p-adic Galois representation. We will need a particular result which does not seem to be available in the literature, namely that any extension E of the form

$$0 \to \rho_F(1) \to E \to \rho_F \to 0$$

is semistable. The proof is inspired by an argument in the paper [PR94] of Perrin-Riou. We will find the dimension of the space of such extensions which are semistable using filtered (ϕ, N)-modules, and then show it is equal to dimension of the space of all extensions by interpreting this latter space as a Galois cohomology group.

First, for convenience, we will prove the following lemma.

Lemma 3.3.2.1. *Let α_p be, as before, a root of the Hecke polynomial of F at p. Then the numbers α_p^{-1}, $p^{-1}\alpha_p^{-1}$, $p^{-(k-1)}\alpha_p$, and $p^{-k}\alpha_p$ are all distinct.*

Proof. It suffices to prove that $\alpha_p \neq p^{k-1}\alpha_p^{-1}$ and $\alpha_p \neq p^k\alpha_p^{-1}$, or equivalently, that $\alpha_p^2 \neq p^{k-1}$ and $\alpha_p^2 \neq p^k$. Let a_p be the Fourier coefficient of F at p. If we did have $\alpha_p^2 = p^{(k-1)}$, then $a_p = \pm p^{(k-1)/2}$. But since F is level 1, a_p is an integer, so this is impossible. If we instead had $\alpha_p^2 = p^k$, then $a_p = \pm(p^{k/2} + p^{(k-2)/2})$. rewrite this as

$$a_p = \pm p^{(k-1)/2}(p^{1/2} + p^{-1/2}).$$

But since $(p^{1/2} + p^{-1/2}) > 2$, this would violates Deligne's theorem that the Ramanujan Conjecture holds for F. Thus the lemma is proved. $\square$

We consider the associated filtered (ϕ, N)-module of ρ_F, $D_{\mathrm{st}}(\rho_F)$. Since F is unramified at p, the nilpotent operator N on $D_{\mathrm{st}}(\rho_F)$ is zero. The Frobenius operator ϕ is invertible and acts with eigenvalues α_p^{-1} and $p^{-(k-1)}\alpha_p$. The filtration $\mathrm{Fil}^i D_{\mathrm{st}}(V)$ has two steps: We have $\mathrm{Fil}^i D_{\mathrm{st}}(\rho_F) = D_{\mathrm{st}}(\rho_F)$ for $i \leq 0$; for $1 \leq i \leq k$, we have $\mathrm{Fil}^i D_{\mathrm{st}}(\rho_F)$ is one-dimensional; and for $i > k$, we have $\mathrm{Fil}^i D_{\mathrm{st}}(\rho_F) = 0$.

Let $\mathrm{MF}(\phi, N)$ denote the category of filtered (ϕ, N) modules over $\overline{\mathbb{Q}}_p$. Recall that a filtered (ϕ, N)-module D is *admissible* if the Newton polygon and Hodge polygon of D meet at their endpoints, and for every filtered (ϕ, N)-submodule $D' \subset D$, the Newton polygon of D' lies above its Hodge polygon. It is a theorem of Colmez and Fontaine that the admissible filtered (ϕ, N)-modules are precisely those coming from semistable representations of $G_{\mathbb{Q}_p}$.

Lemma 3.3.2.2. *We have*

$$\dim_{\overline{\mathbb{Q}}_p} \mathrm{Ext}^1_{\mathrm{MF}(\phi,N)}(D_{\mathrm{st}}(\rho_F), D_{\mathrm{st}}(\rho_F(1))) = 5.$$

Proof. Let us write $D = D_{\mathrm{st}}(\rho_F)$ and $D[1] = D_{\mathrm{st}}(\rho_F(1))$. The underlying vector spaces of D and $D[1]$ can be considered the same, with the filtration of $D[1]$ equal to that of D but shifted up by 1, and the Frobenius $\phi[1]$ on $D[1]$ given in terms of the Frobenius ϕ on D by $\phi[1](v) = p^{-1}\phi(v)$ for any v in the underlying space of D or $D[1]$.

Consider an extension E in $\mathrm{Ext}^1_{\mathrm{MF}(\phi,N)}(D, D[1])$, so E sits in exact sequence

$$0 \to D[1] \to E \to D \to 0.$$

Let w_1 be any vector in $D[1]$ which generates $\mathrm{Fil}^2(D[1])$, and w_2 any other vector in $D[1]$ which spans $D[1]$ along with w_1. Then w_1, w_2 may be considered as vectors in E. Let w_3 be any vector in E mapping to the vector in D corresponding to w_1 in $D[1]$, and similarly let w_4 be any vector in E mapping to the vector in D corresponding to w_2 in $D[1]$. Then w_1, w_2, w_3, w_4 is a basis for E. If $p^{-1}A$ is the matrix of the Frobenius acting on $D[1]$ in the basis $\{w_1, w_2\}$, then by construction there is a matrix M such that, in the basis $\{w_1, w_2, w_3, w_4\}$, the Frobenius ϕ_E on E is given by the block upper triangular matrix

$$\begin{pmatrix} p^{-1}A & M \\ 0 & A \end{pmatrix}.$$

By Lemma 3.3.2.1, all the eigenvalues of the above matrix are distinct. Therefore there is a unique choice of w_3, w_4 as above such that

$$\phi_E = \begin{pmatrix} p^{-1}A & 0 \\ 0 & A \end{pmatrix}.$$

Then $\{w_1, w_2, w_3, w_4\}$ is the unique basis of E, up to scaling by a nonzero matrix in the center of $\mathrm{GL}_4(\overline{\mathbb{Q}}_p)$, such that $w_1 \in \mathrm{Fil}^2(D[1])$, such that w_3 maps to the vector corresponding to w_1 in D, and such that

$$\phi_E = \begin{pmatrix} p^{-1}A & 0 \\ 0 & A \end{pmatrix}.$$

Now we define a map

$$M_2(\overline{\mathbb{Q}}_p) \times \overline{\mathbb{Q}}_p \to \mathrm{Ext}^1_{\mathrm{MF}(\phi,N)}(D, D[1]),$$

where $M_2(\overline{\mathbb{Q}}_p)$ is the space of 2 by 2 matrices over $\overline{\mathbb{Q}}_p$. If $(B, c) \in M_2(\overline{\mathbb{Q}}_p) \times \overline{\mathbb{Q}}_p$, we define a filtered (ϕ, N) module $E(B, c)$ in $\mathrm{Ext}^1_{\mathrm{MF}(\phi,N)}(D, D[1])$ as follows. We declare $E(B, c)$ to be the linear span of a basis of four vectors, denoted v_1, v_2, v_3, v_4, with Frobenius $\phi_{B,c}$ defined in this basis by

$$\phi_{B,c} = \begin{pmatrix} p^{-1}A & 0 \\ 0 & A \end{pmatrix},$$

nilpotent endomorphism $N_{B,c}$ defined in this basis by

$$N_{B,c} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

and filtration defined by

$$\mathrm{Fil}^i(E(B,c)) = \begin{cases} E(B,c) & \text{if } i \le 0; \\ \overline{\mathbb{Q}}_p v_1 + \overline{\mathbb{Q}}_p v_2 + \overline{\mathbb{Q}}_p v_3 & \text{if } i = 1; \\ \overline{\mathbb{Q}}_p v_1 + \overline{\mathbb{Q}}_p (c v_2 + v_3) & \text{if } 2 \le i \le k-1; \\ \overline{\mathbb{Q}}_p v_1 & \text{if } i = k; \\ 0 & \text{if } i \ge k+1. \end{cases} \tag{3.3.2.1}$$

We must check four things: First, the assignment $(B, c) \mapsto E(B, c)$ is well defined (i.e., that $E(B, c)$

is a filtered (ϕ, N)-module which is an extension of D by $D[1]$); second, that this assignment is injective; third, that it is surjective; and fourth, that it is a $\overline{\mathbb{Q}}_p$-linear map. From this it follows that $(B, c) \mapsto E(B, c)$ is an isomorphism of $\overline{\mathbb{Q}}_p$ vector spaces, which will prove the lemma.

Well definedness. We will check that $E(B, c)$ is a filtered (ϕ, N)-module; then clearly the maps

$$a_1 w_1 + a_2 w_2 \mapsto a_1 v_1 + a_2 v_2, \quad D[1] \to E(B, c)$$

and

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \mapsto a_3 w_1 + a_4 w_2, \quad E(B, c) \to D$$

make $E(B, c)$ an extension of D by $D[1]$. Here we are viewing the vectors w_1, w_2 from above as a basis for both $D[1]$ and D, as we are viewing their underlying spaces as the same.

To check $E(B, c)$ is a filtered (ϕ, N)-module, we only need to check that $N_{B,c}\phi_{B,c} = p\phi_{B,c}N_{B,c}$. But,

$$\phi_{B,c}^{-1} N_{B,c} \phi_{B,c} = \begin{pmatrix} pA^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p^{-1}A & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & pB \\ 0 & 0 \end{pmatrix} = pN_{B,c},$$

as desired.

Injectivity. Let (B, c) and (B', c') be elements of $M_2(\overline{\mathbb{Q}}_p) \times \overline{\mathbb{Q}}_p$. Assume we have a commutative diagram of filtered (ϕ, N)-modules

$$\begin{tikzcd}
0 \arrow[r] & D[1] \arrow[r] \arrow[d, "="] & E(B, c) \arrow[r] \arrow[d, "\sim"] & D \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & D[1] \arrow[r] & E(B', c') \arrow[r] & D \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Call the middle vertical map ψ. Take v_1, v_2, v_3, v_4 to be a basis of $E(B, c)$ satisfying the properties listed for the basis w_1, w_2, w_3, w_4 at the beginning of the proof. Let v_1', v_2', v_3', v_4' be a similar basis for $E(B', c')$. Then by the diagram,

$$\psi(v_1) = v_1', \quad \psi(v_2) = v_2',$$

and there is a matrix $M_\psi \in M_2(\overline{\mathbb{Q}}_p)$, say

$$M_\psi = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

such that

$$\psi(v_3) = m_{11}v_1' + m_{12}v_2' + v_3',$$

and

$$\psi(v_4) = m_{21}v_1' + m_{22}v_2' + v_4'.$$

Because ψ must preserve Fil^1, this implies $m_{21} = m_{22} = 0$. Because it must also preserve Fil^2, we have $m_{12} = 0$. And because it must preserve Fil^{k-1}, we have

$$\psi(cv_2 + v_3) = cv_2' + m_{21}v_2' + v_3' \in \overline{\mathbb{Q}}_p(c'v_2' + v_3'),$$

which implies $m_{11} = c' - c$.

Now equivariance for the action of the Frobenius operator gives

$$\psi(Av_3) = p^{-1}Am_{11}v_2' + Av_3' \in \overline{\mathbb{Q}}_p v_3 + \overline{\mathbb{Q}}_p v_4.$$

Thus since A is invertible, we have $m_{11} = 0$ and hence $c = c'$. Therefore $\psi(v_i) = v_i'$ for $i = 1, 2, 3, 4$, and so N-equivariance gives also $B = B'$. Thus $(B, c) \mapsto E(B', c')$ is injective.

Surjectivity. We must show that any extension E fitting in an exact sequence

$$0 \to D[1] \to E \to D \to 0,$$

is of the form $E(B, c)$ for some $(B, c) \in M_2(\overline{\mathbb{Q}}_p) \times \overline{\mathbb{Q}}_p$. Let ϕ_E, N_E be the operators for this module E. Let v_1, v_2, v_3, v_4 to be a basis of E satisfying the properties listed for the basis w_1, w_2, w_3, w_4 at the beginning of the proof. Then since the nilpotent operators for $D[1]$ and D are zero (as V is crystalline) we must have that in this basis,

$$N_E = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

for some $B \in M_2(\mathbb{Q}_p)$. By compatibility of the filtration of E with those of $D[1]$ and D, it is not too hard to see that the filtration on E must satisfy

$$\mathrm{Fil}^i(E) = \begin{cases} E(B,c) & \text{if } i \leq 0; \\ \overline{\mathbb{Q}}_p v_1 + \overline{\mathbb{Q}}_p v_2 + \overline{\mathbb{Q}}_p v_3 & \text{if } i = 1; \\ \overline{\mathbb{Q}}_p v_1 & \text{if } i = k; \\ 0 & \text{if } i \geq k+1. \end{cases}$$

So we only need to explain why there is a $c \in \overline{\mathbb{Q}}_p$ such that

$$\mathrm{Fil}^i(E) = \overline{\mathbb{Q}}_p v_1 + \overline{\mathbb{Q}}_p(cv_1 + v_3) \text{ if } 2 \leq i \leq k-1.$$

Since

$$\mathrm{Fil}^1(E) = \overline{\mathbb{Q}}_p v_1 + \overline{\mathbb{Q}}_p v_2 + \overline{\mathbb{Q}}_p v_3,$$

we at least know that there are $c, d \in \mathbb{Q}_p$ such that

$$\mathrm{Fil}^i(E) = \overline{\mathbb{Q}}_p v_1 + \overline{\mathbb{Q}}_p(dv_1 + cv_2 + v_3) \text{ if } 2 \leq i \leq k-1.$$

But of course, this just equals

$$\overline{\mathbb{Q}}_p v_1 + \overline{\mathbb{Q}}_p(cv_2 + v_3),$$

after cancelling dv_1 with an element of the first summand. This proves surjectivity.

$\overline{\mathbb{Q}}_p$-*linearity.* This follows from unravelling the definitions of the $\overline{\mathbb{Q}}_p$-linear structure on $\mathrm{Ext}^1_{\mathrm{MF}(\phi,N)}(D, D[1])$. We omit the details. $\square$

We record a corollary to the proof of Lemma 3.3.2.2 that will be useful later.

Corollary 3.3.2.3. *Let E be an extension in $\mathrm{Ext}^1_{\mathrm{MF}(\phi,N)}(D_{\mathrm{st}}(\rho_F), D_{\mathrm{st}}(\rho_F(1)))$. Let $w_3, w_4 \in D_{\mathrm{st}}(\rho_F)$ be any basis and let A be the matrix of the crystalline Frobenius for $D_{\mathrm{st}}(V)$ in that basis. Then there is a basis v_1, v_2, v_3, v_4 of E and a matrix $B \in M_2(\overline{\mathbb{Q}}_p)$ such that v_3 maps to w_3 and v_4*

maps to w_4 under the map $E \to D_{\mathrm{st}}(\rho_F)$, such that the crystalline Frobenius ϕ_E for E has the form

$$\phi = \begin{pmatrix} p^{-1}A & 0 \\ 0 & A \end{pmatrix}$$

in this basis, and such that the monodromy operator N_E for E has the form

$$N_E = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

in this basis.

Now we compute the dimension of the corresponding group of extensions of Galois representations.

Lemma 3.3.2.4. *We have*

$$\dim_{\overline{\mathbb{Q}}_p} \mathrm{Ext}^1_{\mathrm{G}_{\mathbb{Q}_p}}(\rho_F, \rho_F(1)) = 5.$$

Proof. First we note

$$\mathrm{Ext}^1_{\mathrm{G}_{\mathbb{Q}_p}}(\rho_F, \rho_F(1)) \cong H^1(\mathbb{Q}_p, \rho_F^\vee \otimes \rho_F(1)).$$

The group on the right and side breaks up as

$$H^1(\mathbb{Q}_p, \mathrm{Ad}^2 \rho_F(1) \oplus \overline{\mathbb{Q}}_p(1)) \cong H^1(\mathbb{Q}_p, \mathrm{Ad}^2 \rho_F(1)) \oplus H^1(\mathbb{Q}_p, \overline{\mathbb{Q}}_p(1)).$$

The piece $H^1(\mathbb{Q}_p, \overline{\mathbb{Q}}_p(1))$ is 2-dimensional by a standard computation in Kummer theory, and the piece $H^1(\mathbb{Q}_p, \mathrm{Ad}^2 \rho_F(1))$ is 3 dimensional by a now classical computation using local Tate duality. □

Proposition 3.3.2.5. *Any extension E of Galois representations,*

$$0 \to \rho_F(1) \to E \to \rho_F \to 0$$

is semistable at p.

Proof. We first recall that any extension of admissible filtered (ϕ, N)-modules is again admissible. So, because of the equivalence of categories between semistable representations and admissible filtered (ϕ, N)-modules, we get an injection

$$\mathrm{Ext}^1_{\mathrm{MF}(\phi,N)}(D_{\mathrm{st}}(\rho_F), D_{\mathrm{st}}(\rho_F(1))) \hookrightarrow \mathrm{Ext}^1_{\mathrm{G}_{\mathbb{Q}_p}}(\rho_F, \rho_F(1)),$$

whose image is the group of semistable extensions of ρ_F by $\rho_F(1)$. By Lemmas 3.3.2.2 and 3.3.2.4, both the source and target have dimension 5, so this injection is an isomorphism. The proposition follows. $\square$

3.4 Pseudocharacters and Galois representations

The combination of Theorem 3.2.1.2 and Conjecture 3.2.2.1 provides us many crystalline Galois representations into $\mathrm{G}_2(\overline{\mathbb{Q}}_p)$, varying with weights $\lambda \in \mathfrak{X}$. Moreover, the eigenvalues of Frobenius elements Frob_ℓ with $\ell \neq p$, as well as the eigenvalues of the crystalline Frobenius, are varying analytically in this family.

We would like to construct a genuine family of Galois representation into, not just GL_7, but G_2. The now classical theory of pseudorepresentations would allow us to do the former, but not the latter. The tool which will allow us to do the latter was introduced by V. Lafforgue [Laf18]; this is the notion of pseudocharacter.

We will recall a bit of the theory of pseudocharacters, and then construct a G_2-pseudocharacter of $G_{\mathbb{Q}}$ which interpolates the aforementioned Galois representations.

3.4.1 Pseudocharacters

In this section we recall some of the theory of pseudocharacters. We will follow Böckle–Harris–Khare–Thorne [B$\ddot{+}$19] in our exposition. We begin with the definition.

Definition 3.4.1.1. Let G be a split reductive group over $\mathbb{Z}$, A a ring and Γ a group. Let G act on itself by conjugation, and let $\mathbb{Z}[G]$ be the ring of regular functions on G, on which G therefore acts as well. Let $\mathbb{Z}[G]^G$ be the subring of invariants for this action. Then a *G-pseudocharacter* of Γ over A is a collection Θ of ring maps

$$\Theta_n : \mathbb{Z}[G^n]^G \to \mathrm{Fun}(\Gamma^n, A),$$

where $\mathrm{Fun}(\Gamma, A)$ is the ring of A-valued functions on Γ, such that the maps Θ_n satisfy the following properties:

(1) Given any positive integers m, n, any function $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, any $f \in \mathbb{Z}[G^m]^G$, and any $\gamma_1, \dots, \gamma_n \in \Gamma$, we have

$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}),$$

where f^ζ is defined by

$$f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$$

for all $g_1, \dots, g_n \in \Gamma$;

(2) Given any positive integer n, any $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$, and any $f \in \mathbb{Z}[G^n]^G$, we have

$$\Theta_{n+1}(\hat{f})(\gamma_1, \dots, \gamma_{n+1}) = \Theta_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n\gamma_{n+1}),$$

where $\hat{f}$ is defined by

$$\hat{f}(g_1, \dots, g_{n+1}) = f(g_1, \dots, g_{n-1}, g_n g_{n+1}),$$

for all $g_1, \dots g_{n+1} \in \Gamma$.

If Γ and A have topologies, then we say Θ is *continuous* if for any $f \in \mathbb{Z}[G]^G$ and any n, the map $\Theta_n(f)$ is a continuous function $\Gamma^n \to A$.

As noted in [B$\ddot{+}$19], Lemma 4.3, a representation $\rho : \Gamma \to G(A)$ gives a G-pseudocharacter of Γ over A, which is denoted $\operatorname{Tr}\rho$ and is defined by

$$(\operatorname{Tr}\rho)_n(f) = f(\rho(\gamma_1), \dots, \rho(\gamma_n)).$$

The pseudocharacter $\operatorname{Tr}\rho$ only depends on ρ up to conjugation in $G(A)$. We also note that when $G = \mathrm{GL}_n$, this recovers the notion of pseudorepresentation of Taylor [Tay91]; the function $\Theta_1(\mathrm{Tr})$, where $\mathrm{Tr} \in \mathbb{Z}[\mathrm{GL}_n]^{\mathrm{GL}_n}$ denotes the usual trace, will be a pseudorepresentation in this case, and the rest of the pseudocharacter will be determined by this.

Also noted in [B+̈19], Lemma 4.4, is that we can change the ring. More precisely, if Θ is a G-pseudocharacter over a ring A and $\phi : A \to B$ is a ring homomorphism, then $\phi_*\Theta$, defined by $(\phi_*\Theta)_n(f) = \phi \circ \Theta_n(f)$ for $f \in \mathbb{Z}[G]^G$, is a G-pseudocharacter of Γ over B.

For such Θ, we can also change the group; if $\psi : \Gamma' \to \Gamma$ is a group homomorphism, then $\psi^*\Theta$, defined by $(\psi^*\Theta)_n(f) = \Theta_n(f) \circ \psi$ for $f \in \mathbb{Z}[G]^G$, is a G-pseudocharacter of Γ' over A.

The changes of groups and rings just described are also compatible with continuity as long as the maps ϕ and ψ as above are continuous.

We now state a fundamental result in the theory of pseudocharacters.

Theorem 3.4.1.2. *Let G be a split reductive group over $\mathbb{Z}$, let Γ be a group, and let k be an algebraically closed field. Let Θ be a G-pseudocharacter of Γ over k. Then there is a representation $\rho : \Gamma \to G(k)$ such that $\Theta = \operatorname{Tr} \rho$.*

Proof. This follows from Theorem 4.5 in [B+̈19]. $\square$

Actually, we remark something stronger can be said than this; the assignment $\rho \mapsto \operatorname{Tr} \rho$ is in fact a bijection if we restrict our attention to *completely reducible* ρ. See Definitions 3.3 and 3.5 in [B+̈19] for this notion.

3.4.2 Construction of a pseudocharacter

We now use the setup of Section 3.2 to construct a G_2-pseudocharacter Θ of the Galois group $G_{\mathbb{Q}}$ over an affinoid ring. We assume Conjectures 3.2.1.1 and 3.2.2.1 (and continue to do so throughout the rest of this chapter).

We retain the setting of Section 3.2.1. In particular, we have our modular form F of weight k, our automorphic representation π_F of $\mathrm{GL}_2(\mathbb{A})$, our Langlands quotient Π, its p-stabilization $\sigma(\Pi)$, and its cohomological weight $\lambda_0 = \frac{k-4}{2}(2\alpha+3\beta)$. There is a chosen root α_p of the Hecke polynomial of F at p whose slope is s_p.

We also have our global Hecke algebra R_p^{sph} which is spherical away from p and given by $\mathcal{U}_p$ at p. For ℓ a finite prime different from p, let

$$\mathcal{H}_{K_\ell}^{\mathrm{sph}} = C_c^\infty(K_\ell \backslash \mathrm{G}_2(\mathbb{Z}_\ell)/K_\ell)$$

be the spherical Hecke algebra over $\mathbb{Z}_p$ at ℓ. Then naturally $\mathcal{H}^{\mathrm{sph}}_{K_\ell}$ injects into R^{sph}_p.

Now since we are assuming Conjecture 3.2.1.1, we have the existence of the objects in Theorem 3.2.1.2. So we have our weight space $\mathfrak{X}$, the affinoid subdomain $\mathfrak{U} \subset \mathfrak{X}$, the finite cover $\mathbf{w} : \mathfrak{V} \to \mathfrak{U}$, the point y_0 in $\mathfrak{V}$ corresponding to $\sigma(\Pi)$, the subset $\Sigma \subset \mathfrak{V}(\overline{\mathbb{Q}}_p)$, the unramified cohomological automorphic representations π_y for $y \in \Sigma$, and the character $\theta_{\mathfrak{V}} : R^{\mathrm{sph}}_p \to \mathcal{O}(\mathfrak{V})$.

Furthermore, since we are assuming Conjecture 3.2.2.1, each of the automorphic representations π_y for $y \in \Sigma$ have corresponding Galois representations $\rho_y : G_{\mathbb{Q}} \to \mathrm{G}_2(\overline{\mathbb{Q}}_p)$, satisfying the properties listed in that conjecture. The automorphic representation Π also has a Galois representation attached to it as in Conjecture 3.2.2.1, but this is unconditional. By Proposition 1.5.2.3, it is given as follows. Let $\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the Galois representation attached to F by Deligne. View the target $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ of ρ_F as $M_\beta(\overline{\mathbb{Q}}_p)$ and let $j : M_\beta \to \mathrm{G}_2$ be the inclusion. Then Π has attached to it the Galois representation ρ_Π given by

$$\rho_\Pi = j \circ (\rho_F(-(k-2)/2)).$$

Here, $\rho_F(-(k-2)/2)$ denotes a Tate twist of ρ_F.

Let $\mathcal{O}(\mathfrak{V})^\circ$ be the subring of functions in $\mathcal{O}(\mathfrak{V})$ whose evaluations at every point have p-adic absolute value bounded above by 1. This ring defines a formal scheme whose rigid generic fiber is $\mathfrak{V}$. It follows from Noether normalization for $\mathcal{O}(\mathfrak{V})$ (plus a few details, which we omit) that $\mathcal{O}(\mathfrak{V})^\circ$ is finite over a power series ring over $\mathbb{Z}_p$ in two variables. The ring $\mathcal{O}(\mathfrak{V})^\circ$ is therefore profinite. We will use this in the proof of the following proposition.

Proposition 3.4.2.1. *With the setting as above, in particular assuming Conjectures 3.2.1.1 and 3.2.2.1, there is a continuous* G_2*-pseudocharacter* Θ° *of* $G_{\mathbb{Q}}$ *over* $\mathcal{O}(\mathfrak{V})^\circ$ *satisfying the following properties:*

Let Θ *be the pseudocharacter of* $G_{\mathbb{Q}}$ *obtained from* Θ° *by changing the ring from* $\mathcal{O}(\mathfrak{V})^\circ$ *to* $\mathcal{O}(\mathfrak{V})$*. Then*

(1) The pseudocharacter Θ *is continuous;*

(2) The pseudocharacters Θ° *and* Θ *are unramified at all finite primes* $\ell \neq p$*;*

(3) The pseudocharacter Θ_{y_0} *obtained from* Θ *by changing the ring via the point* $y_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ *is*

the G_2*-pseudocharacter of* $G_{\mathbb{Q}}$ *over* $\overline{\mathbb{Q}}_p$ *obtained from the Galois representation* ρ_Π*;*

(4) For any $y \in \Sigma$*, the pseudocharacter* Θ_y *obtained from* Θ *by changing the ring via* y *is the* G_2*-pseudocharacter of* $G_{\mathbb{Q}}$ *over* $\overline{\mathbb{Q}}_p$ *obtained from the Galois representation* ρ_y*.*

Proof. Let $\ell \neq p$ be a finite prime, and let $y \in \Sigma$. Then the automorphic representation π_y is unramified at ℓ; let $\pi_{y,\ell}$ be its local component at ℓ, and let $s_{y,\ell}$ be its Satake parameter, considered as an element of $T^\vee(\overline{\mathbb{Q}}_p)/W$, where W is the Weyl group of G_2. Let $\theta_{y,\ell} : \mathcal{H}^{\mathrm{sph}}_{K_\ell} \to \overline{\mathbb{Q}}_p$ be the character of the spherical Hecke algebra at ℓ obtained from $\pi_{y,\ell}$; it is the restriction of θ_y to $\mathcal{H}^{\mathrm{sph}}_{K_\ell} \subset R_p$.

Now recall that $s_{y,\ell}$ is obtained from θ_y via the following process. First we view $\mathcal{H}^{\mathrm{sph}}_{K_\ell} \cong \mathbb{Z}_p[X^*(T^\vee)]^W$ via the Satake isomorphism, and we lift the character θ_y to a character $\tilde{\theta}_y : \mathbb{Z}_p[X^*(T^\vee)] \to \overline{\mathbb{Q}}_p$. This defines a character $X^*(T^\vee) \to \overline{\mathbb{Q}}_p^\times$. Then there is a unique element $\tilde{s} \in T^\vee(\overline{\mathbb{Q}}_p)$ such that for any $\mu \in X^*(T^\vee)$, we have

$$\tilde{\theta}_y(\mu) = \mu(\tilde{s}).$$

Then we define $s_{y,\ell}$ to be the Weyl orbit of $\tilde{s}$. It is unique with the property that

$$\theta_{y,\ell}(\chi_V) = tr(s_{y,\ell}|V),$$

for any finite dimensional representation V of G_2, where χ_V is the character of V.

Let F be the fraction field of $\mathcal{O}(\mathfrak{Y})$. Then in exactly this same way, from the character $\theta_{\mathfrak{Y},\ell} : \mathcal{H}^{\mathrm{sph}}_{K_\ell} \to \mathcal{O}(\mathfrak{Y})$ obtained by restricting $\theta_{\mathfrak{Y}}$ to $\mathcal{H}^{\mathrm{sph}}_{K_\ell}$, we obtain an element $s_{\mathfrak{Y},\ell} \in T^\vee(F)/W$.

Now let A be obtained from $\mathcal{O}(\mathfrak{Y})$ by inverting finitely many elements $f_1, \ldots, f_r \in \mathcal{O}(\mathfrak{Y})$, such that $s_{\mathfrak{Y},\ell} \in T^\vee(A)/W$. Let $y \in \Sigma$ be a point which does not lie on any of the divisors of the f_i's. Let $\bar{s}_{\mathfrak{Y},\ell} \in T(\overline{\mathbb{Q}}_p)/W$ be the element obtained from $s_{\mathfrak{Y},\ell}$ by specializing at y. Then by construction,

$$\theta_{y,\ell}(\chi_V) = tr(\bar{s}_{\mathfrak{Y},\ell}|V),$$

for any V, from which it follows that

$$\bar{s}_{\mathfrak{Y},\ell} = s_{y,\ell}.$$

We will use this property in the construction of Θ°.

First we define Θ on Frobenius elements Frob_ℓ for $\ell \neq p$. Let $f \in \mathbb{Z}_p[\mathrm{G}_2^n]^{\mathrm{G}_2}$ and let $\ell_1, \ldots, \ell_n$

be primes (not necessarily distinct) different from p. Then we define

$$\Theta_n(f)(\mathrm{Frob}_{\ell_1}, \ldots, \mathrm{Frob}_{\ell_n}) = f(s_{\mathfrak{Y},\ell_1}, \ldots, s_{\mathfrak{Y},\ell_n}).$$

Before checking any properties of this assignment, we first claim that $\Theta_n(f) \in \mathcal{O}(\mathfrak{Y})^\circ$. We only know a priori that $\Theta_n(f) \in F$. So let $f_1, \ldots, f_r \in \mathcal{O}(\mathfrak{Y})$ now be elements such that $s_{\mathfrak{Y},\ell_1}, \ldots, s_{\mathfrak{Y},\ell_n} \in T(A)/W$ where A is obtained from $\mathcal{O}(\mathfrak{Y})$ by inverting the f_i's. Let y be a point in Σ which is not on the divisor of any f_i. Then by construction,

$$y(\Theta_n(f)(\mathrm{Frob}_{\ell_1}, \ldots, \mathrm{Frob}_{\ell_n})) = f(s_{y,\ell_1}, \ldots, s_{y,\ell_n}) = f(\rho_y(\mathrm{Frob}_{\ell_1}), \ldots, \rho_y(\mathrm{Frob}_{\ell_n})),$$

where we are using the defining property of the Galois representation ρ_y in the last equality.

Now by continuity of ρ_y and the compactness of $G_{\mathbb{Q}}$, a standard argument using the Baire category theorem implies that the representation ρ_y takes values in $\mathrm{G}_2(\mathcal{O}_E)$, at least up to conjugation, for some finite extension E of $\mathbb{Q}_p$. Therefore the element $f(\rho_y(\mathrm{Frob}_{\ell_1}), \ldots, \rho_y(\mathrm{Frob}_{\ell_n}))$ is integral. Since this is true for all such y, by density of Σ, the element $\Theta_n(f)(\mathrm{Frob}_{\ell_1}, \ldots, \mathrm{Frob}_{\ell_n})$ is actually in $\mathcal{O}(\mathfrak{Y})^\circ$, as claimed.

The rest of the proof is more or less standard. Let $g_1, \ldots, g_n$ be elements of $G_{\mathbb{Q}}$, and for each i let $\ell_{i,j}$, $j > 0$, be primes such that $\mathrm{Frob}_{\ell_{i,j}} \to g_i$ as $j \to \infty$. The ring $\mathcal{O}(\mathfrak{Y})^\circ$ is profinite, so we can define $\Theta_n(f)(g_1, \ldots, g_n)$ by defining it on the finite quotients of $\mathcal{O}(\mathfrak{Y})^\circ$ using the definition above for Frobenius elements and taking the limit as $j \to \infty$. By continuity, the resulting assignment $f \mapsto \Theta_n(f)$ is a pseudocharacter, as $f \mapsto y(\Theta_n(f))$ is one (in fact it is the pseudocharacter attached to ρ_y by construction). Then again by continuity, the pseudocharacter Θ must satisfy the properties claimed in the proposition. □

3.4.3 The pseudorepresentation T

We continue to assume Conjectures 3.2.1.1 and 3.2.2.1. Then in Proposition 3.4.2.1 we constructed a G_2-pseudocharacter Θ of $G_{\mathbb{Q}}$ over $\mathcal{O}(\mathfrak{Y})$, where $\mathfrak{Y}$ is as in Theorem 3.2.1.2. We now restrict Θ to a certain affinoid curve in $\mathfrak{Y}$ which we construct as follows.

Let $c = (c_1, c_2)$ be a pair of integers with $c_1, c_2 > 0$ and $c_1 \neq c_2$. Let $\mathfrak{U}$ be the affinoid subset of weight space $\mathfrak{X}$ over which $\mathfrak{V}$ lies, and $\mathbf{w} : \mathfrak{V} \to \mathfrak{U}$ the corresponding map, as in Conjecture 3.2.1.1. Define $\mathfrak{L} \subset \mathfrak{X}$ to be the Zariski closure of the set of weights $\lambda \in \mathbf{w}(\Sigma)$ of the form

$$\lambda_0 + n(c_1(2\alpha + 3\beta) + c_2(\alpha + 2\beta)), \qquad n \in \mathbb{Z}_{>0}.$$

We can and do choose c so that there are infinitely many such weights in $\mathfrak{L}(\overline{\mathbb{Q}}_p) \cap \mathbf{w}(\Sigma)$ and so that $\lambda_0 \in \mathfrak{L}(\overline{\mathbb{Q}}_p)$.

Then $\mathfrak{L}$ is a line in $\mathfrak{U}$. Let $\mathfrak{Z}'$ be the curve in $\mathfrak{V}$ cut out by it. Then the point $y_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ corresponding to $\sigma(\Pi)$ is in $\mathfrak{Z}'$, and we take $\mathfrak{Z}$ to be the irreducible component of $\mathfrak{Z}'$ containing y_0. We let $\Sigma_c = \Sigma \cap \mathfrak{Z}(\overline{\mathbb{Q}}_p)$.

There is a natural map $\mathcal{O}(\mathfrak{V}) \to \mathcal{O}(\mathfrak{Z})$ corresponding to the inclusion of $\mathfrak{Z}$ into $\mathfrak{V}$. Thus we may change the target of our pseudocharacter Θ from $\mathcal{O}(\mathfrak{V})$ to $\mathcal{O}(\mathfrak{Z})$, as explained in Section 3.4.1. This gives us a continuous G_2-pseudocharacter of $G_{\mathbb{Q}}$ over $\mathcal{O}(\mathfrak{Z})$, which we denote $\Theta_{\mathfrak{Z}}$. It satisfies all the same properties at Θ listed in Proposition 3.4.2.1, except that in the point (3), we must replace Σ with Σ_c.

In the next section, we will be concerned with the Galois representation $\rho_{\mathfrak{Z}} : G_{\mathbb{Q}} \to \mathrm{G}_2(\overline{\mathrm{Frac}(\mathcal{O}(\mathfrak{Z}))})$ that we obtain from $\Theta_{\mathfrak{Z}}$ via Theorem 3.4.1.2. But right now we must study the corresponding 7-dimensional pseudorepresentation we obtain from Θ.

Recall from Section 3.1.1 that we have the 7-dimensional representation $R_7 : \mathrm{G}_2 \to \mathrm{GL}_7$. This representation induces a map of rings $\mathbb{Z}[\mathrm{GL}_7]^{\mathrm{GL}_7} \to \mathbb{Z}[\mathrm{G}_2]^{\mathrm{G}_2}$. Composing $\Theta_{\mathfrak{Z}}$ with this map gives a GL_7-pseudocharacter of $G_{\mathbb{Q}}$ over $\mathcal{O}(\mathfrak{Z})$, and evaluating this pseudocharacter on the function $\mathrm{Tr} \in \mathbb{Z}[\mathrm{GL}_7]^{\mathrm{GL}_7}$ gives a 7-dimensional pseudorepresentation of $G_{\mathbb{Q}}$ into $\mathcal{O}(\mathfrak{Z})$. We denote this pseudorepresentation by T in what follows. It is not difficult to check that T is the trace of the composition $R_7 \circ \rho_{\mathfrak{Z}}$.

The following proposition will be the first time in this chapter that we must use that F is not CM. This is to ensure that the adjoint representation $\mathrm{Ad}^2 \rho_F$ is irreducible. We note that in the proof of this proposition, we will use certain terminology (such as "de Rham" or "crystalline" and so on) in reference to certain pseudorepresentations; what is meant by this is that the corresponding semisimple representations satisfy the properties described by this terminology. We also say that a pseudorepresentation is *irreducible* if it is not the sum of two nonzero pseudorepresentations, and that it is *absolutely irreducible* if, after passing to any étale extension of the target, it is still irreducible.

Lemma 3.4.3.1. *The pseudorepresentation $T : G_{\mathbb{Q}} \to \mathcal{O}(\mathfrak{Z})$ constructed above is either absolutely irreducible, or, over $\mathcal{O}(\tilde{\mathfrak{Z}})$ for some finite cover $\tilde{\mathfrak{Z}}$ of $\mathfrak{Z}$, it is the sum of two pseudorepresentations, say $T = T_1 + T_2$. In this latter case, if T_{1,x_0} and T_{2,x_0} denote, respectively, the specializations of T_1 and T_2 at a point $x_0 \in \tilde{\mathfrak{Z}}(\overline{\mathbb{Q}}_p)$ above y_0, then (up to swapping T_1 and T_2) we have*

$$T_{1,x_0} = \mathrm{Tr}(\mathrm{Ad}^2 \rho_F), \qquad T_{2,x_0} = \mathrm{Tr}(\rho_F(-(k-2)/2)) + \mathrm{Tr}(\rho_F(-k/2)).$$

Proof. We will work over a sufficiently large extension $\tilde{\mathfrak{Z}}$ as in the statement of the lemma. We write $\widetilde{\Sigma}_c$ for the preimage of Σ_c in $\tilde{\mathfrak{Z}}(\overline{\mathbb{Q}}_p)$, and we write x_0 for a chosen preimage of y_0 in $\tilde{\mathfrak{Z}}(\overline{\mathbb{Q}}_p)$.

First we note that by the facts recalled in Section 3.1.2, we have,

$$T_{x_0} = \mathrm{Tr}(\rho_F(-(k-2)/2)) + \mathrm{Tr}(\mathrm{Ad}^2 \rho_F) + \mathrm{Tr}(\rho_F(-k/2)),$$

where T_{x_0} is the specialization of T at the point x_0. These three pieces are irreducible because F is level 1 and therefore not CM (see Lemma 1.5.2.5, for example).

Assume now that T is reducible. Then it is either the sum of three pseudorepresentations, or the sum of two. More precisely, there are four cases. In Case 1, $T = T_1 + T_2 + T_3$ with specializations at x_0 given by

$$T_{1,x_0} = \mathrm{Tr}(\mathrm{Ad}^2 \rho_F), \qquad T_{2,x_0} = \mathrm{Tr}(\rho_F(-(k-2)/2)), \qquad T_{3,x_0} = \mathrm{Tr}(\rho_F(-k/2)).$$

In Case 2, $T = T_1 + T_2$ with

$$T_{1,x_0} = \mathrm{Tr}(\mathrm{Ad}^2 \rho_F) + \mathrm{Tr}(\rho_F(-k/2)), \qquad T_{2,x_0} = \mathrm{Tr}(\rho_F(-(k-2)/2)).$$

In Case 3, $T = T_1 + T_2$ with

$$T_{1,x_0} = \mathrm{Tr}(\mathrm{Ad}^2 \rho_F) + \mathrm{Tr}(\rho_F(-(k-2)/2)), \qquad T_{2,x_0} = \mathrm{Tr}(\rho_F(-k/2)).$$

Finally, in Case 4, $T = T_1 + T_2$ with

$$T_{1,x_0} = \mathrm{Tr}(\mathrm{Ad}^2 \rho_F), \qquad T_{2,x_0} = \mathrm{Tr}(\rho_F(-(k-2)/2)) + \mathrm{Tr}(\rho_F(-k/2)).$$

We will show that we must be in Case 4 if T is not irreducible by ruling out the other three cases now.

Case 1. We first recall that the character χ_{R_7} of the standard representation R_7 is supported at the 7 coweights

$$\pm(2\alpha^\vee + \beta^\vee), \pm(\alpha^\vee + \beta^\vee), \pm\alpha^\vee, 0. \tag{3.4.3.1}$$

We also note that the Hodge–Tate weights of T_{x_0} are given by $\pm k, \pm k/2, \pm(k-2)/2, 0$, and are therefore distinct. Therefore, by looking at the Hodge–Tate–Sen weights of $\mathcal{L}_0$, we deduce the following. Since $\pm k, 0$ are the Hodge–Tate weights of T_{1,x_0}, and

$$\pm k = \langle \lambda_0 + \rho, \pm(2\alpha^\vee + \beta^\vee)\rangle, \qquad 0 = \langle \lambda_0 + \rho, 0\rangle,$$

we must have that for any $x \in \widetilde{\Sigma}_c$, the specialization $T_{1,x}$ has Hodge–Tate weights

$$\langle \lambda_x + \rho, \pm(2\alpha^\vee + \beta^\vee)\rangle, \qquad \langle \lambda_x + \rho, 0\rangle = 0,$$

where λ_x is the image of x in weight space. Similarly, for any $x \in \widetilde{\Sigma}_c$, the specialization $T_{2,x}$ has Hodge–Tate weights

$$\langle \lambda_x + \rho, \alpha^\vee\rangle, \qquad \langle \lambda_x + \rho, -(\alpha^\vee + \beta^\vee)\rangle,$$

and $T_{3,x}$ has Hodge–Tate weights

$$\langle \lambda_x + \rho, \alpha^\vee + \beta^\vee\rangle, \qquad \langle \lambda_x + \rho, -\alpha^\vee\rangle.$$

Moreover, we know that for any such x, each of $T_{1,x}$, $T_{2,x}$ and $T_{3,x}$ is crystalline; this is because their sum, which corresponds to a cohomological automorphic representation of weight λ_x, is crystalline by Conjecture 3.2.2.1. By (3.2.1.2), the eigenvalues of the crystalline Frobenius ϕ are given on $T_{1,x} + T_{2,x} + T_{3,x}$ by

$$(\theta_y(u_{\mu^\vee(p)})p^{-\langle \lambda+\rho,\mu^\vee\rangle})^{-1},$$

where $\mu^\vee$ are the coweights from (3.4.3.1), and y is the point below x in $\mathfrak{Z}(\overline{\mathbb{Q}}_p)$. The numbers $\theta_y(u_{\mu^\vee(p)})$ vary analytically in x.

Now by (3.2.1.1), the slopes of the crystalline Frobenius for $\rho^{\mathrm{ss}}_{x_0}$ are given by

$$-\langle s_p(2\alpha+3\beta)+\beta-\lambda_0-\rho,\mu^\vee\rangle,$$

for $\mu^\vee$ as in (3.4.3.1). Therefore, for $x \in \widetilde{\Sigma}_c$ sufficiently close to x_0, the slopes for $T_{1,x} + T_{2,x} + T_{3,x}$ are given by

$$-\langle s_p(2\alpha+3\beta)+\beta-\lambda_x-\rho,\mu^\vee\rangle.$$

By definition of $\mathfrak{Z}$, for any $x \in \widetilde{\Sigma}_c$ we have

$$\lambda_x = \lambda_0 + n_x(c_1(2\alpha+3\beta)+c_2(\alpha+2\beta))$$

for some integer $n_x > 0$. Thus for x sufficiently close to x_0, the slopes of the crystalline Frobenius for $T_{1,x} + T_{2,x} + T_{3,x}$ are given by

$$-\langle s_p(2\alpha+3\beta)+\beta-\lambda_x-\rho,\mu^\vee\rangle$$

for $\mu^\vee$ as in (3.4.3.1), and are therefore given by

$$\pm(2s_p-(k-1)-n_x(2c_1+c_2)), \quad \pm(s_p-((k-2)/2)-n_x(c_1+c_2)), \quad \pm(s_p-(k/2)-n_xc_1), \quad 0.$$

By above, the Hodge–Tate weights for $T_{1,x}$ are

$$\pm(k-1+n_x(2c_1+c_2)), \quad 0,$$

those for $T_{2,x}$ are

$$((k-2)/2)+n_xc_1, \quad -((k/2)+n_x(c_1+c_2)),$$

and those for $T_{3,x}$ are

$$(k/2) + n_x(c_1 + c_2), \quad -(((k-2)/2) + n_x c_1).$$

The Hodge polygon for $T_{3,x}$ therefore ends at $(2, n_x c_2 + 1)$. But, since $c_1 \neq c_2$, it is then impossible for the Newton polygon for $T_{3,x}$ to meet the Hodge polygon at the end points for n_x sufficiently large; the closest we can get is for $T_{3,x}$ to have slopes

$$-(s_p - ((k-2)/2) - n_x(c_1 + c_2)), \quad (s_p - (k/2) - n_x c_1),$$

in which case the Newton polygon ends at $(2, -1 + n_x c_2)$. This is a contradiction, so we have excluded Case 1.

Case 2. Arguing exactly as in Case 1 above, for $x \in \widetilde{\Sigma}_c$, we get crystalline pseudorepresentations $T_{1,x}$ and $T_{2,x}$, where $T_{2,x}$ has Hodge–Tate weights

$$((k-2)/2) + n_x c_1, \quad -((k/2) + n_x(c_1 + c_2)),$$

and the slopes of $T_{1,x} + T_{2,x}$ are given again by

$$\pm(2s_p - (k-1) - n_x(2c_1 + c_2)), \quad \pm(s_p - ((k-2)/2) - n_x(c_1 + c_2)), \quad \pm(s_p - (k/2) - n_x c_1), \quad 0.$$

Then the Hodge polygon for $T_{2,x}$ ends at $(2, -1 - n_x c_2)$. This is again a contradiction since no pair of slopes from the list above can sum to $-1 - n_x c_2$.

Case 3. This can be dealt with in a completely analogous way as Case 2. □

3.5 The lattice $\mathcal{L}$

We will now construct a lattice in the Galois representation attached to the pseudorepresentation T considered above. We will show that the specialization of this lattice at the point y_0 is an extension which factors through the short root parabolic subgroup of G_2, and we will use this extension to construct the desired cocycle in the symmetric cube Bloch–Kato Selmer group.

3.5.1 Construction of $\mathcal{L}$

We continue to assume throughout that Conjectures 3.2.1.1 and 3.2.2.1 hold.

Let us first take a moment to summarize the constructions made in Section 3.4, because they will be used here. First of all, at the beginning of Section 3.4.3, we constructed an affinoid curve $\mathfrak{Z}$. It is reduced and irreducible, and it is finite over a line $\mathfrak{L}$ in weight space $\mathfrak{X}$. The line $\mathfrak{L}$ contains a Zariski dense subset Σ_c of classical weights which become increasingly regular.

We also have the continuous G_2-pseudocharacter $\Theta_{\mathfrak{Z}}$ of $G_{\mathbb{Q}}$ over $\mathcal{O}(\mathfrak{Z})$ which, via the representation $R_7 : \mathrm{G}_2 \to \mathrm{GL}_7$, gives rise to a continuous pseudorepresentation $T : G_{\mathbb{Q}} \to \mathcal{O}(\mathfrak{Z})$. By Theorem 3.4.1.2, this gives us a representation $\rho_{\mathfrak{Z}} : G_{\mathbb{Q}} \to \mathrm{G}_2(\overline{\mathrm{Frac}(\mathcal{O}(\mathfrak{Z}))})$, and also the representation $R_7 \circ \rho_{\mathfrak{Z}}$ of which T is the trace. Standard arguments show that there is a finitely generated extension $\mathcal{R}$ of $\mathcal{O}(\mathfrak{Z})$ in $\overline{\mathrm{Frac}(\mathcal{O}(\mathfrak{Z}))}$ such that $R_7 \circ \rho_{\mathfrak{Z}}$ takes values in $\mathrm{GL}_7(\mathcal{R})$, and is continuous with this target. (See, for example, the proof of Lemma 6 in [Tay91].) By simply passing to this extension, we will assume that $R_7 \circ \rho_{\mathfrak{Z}}$ takes values in $\mathcal{O}(\mathfrak{Z})$. By passing to the normalization, we will also assume that $\mathcal{O}(\mathfrak{Z})$ is integrally closed, and therefore Dedekind. The continuity of T implies that $R_7 \circ \rho_{\mathfrak{Z}}$ preserves a projective $\mathcal{O}(\mathfrak{Z})$-module $\mathcal{P}$ of rank 7 in $\mathrm{Frac}(\mathcal{O}(\mathfrak{Z}))^7$.

Now let $\mathcal{A}$ be the completed local ring of the base change of $\mathcal{O}(\mathfrak{Z})$ to $\overline{\mathbb{Q}}_p$ at the maximal ideal corresponding to $y_0 \in \mathfrak{Z}(\overline{\mathbb{Q}}_p)$. Then $\mathcal{A}$ is a discrete valuation ring. Let $\mathfrak{m}$ be its maximal ideal. Let $\mathcal{P}_{\mathcal{A}} = \mathcal{P} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \mathcal{A}$. This is a Galois-stable free $\mathcal{A}$-submodule of $\mathrm{Frac}(\mathcal{A})^7$ of rank 7. The reduction of $\mathcal{P}_{\mathcal{A}}$ modulo $\mathfrak{m}$ is the same as the specialization of $\mathcal{P}$ at y_0.

We now construct another lattice $\mathcal{L}_{\mathfrak{Z}}$ in the space of $R_7 \circ \rho_{\mathfrak{Z}}$. We need to separate the construction into two cases based on Lemma 3.4.3.1, namely when the pseudorepresentation T is absolutely irreducible, and when it is not. But first, to start, we pick $g_0 \in G_{\mathbb{Q}}$ such that $\rho_F(-k/2)(g_0)$ has eigenvalues $\gamma_6, \gamma_7 \in \overline{\mathbb{Q}}_p$, and such that the numbers

$$\gamma_1 = \gamma_7^{-1}, \quad \gamma_2 = \gamma_6^{-1}, \quad \gamma_3 = \gamma_6\gamma_7^{-1}, \quad \gamma_4 = 1, \quad \gamma_5 = \gamma_7\gamma_6^{-1}, \quad \gamma_6, \quad \gamma_7 \tag{3.5.1.1}$$

are all distinct.

Construction of $\mathcal{L}$ when T is absolutely irreducible

Because T is assumed absolutely irreducible here, $R_7 \circ \rho_3$ is irreducible. Let v_7' be a vector in the specialization $\mathcal{P}_{y_0}$ of $\mathcal{P}$ at y_0 which is an eigenvector of the specialization of $\rho_{y_0}(g_0)$ of ρ_3 at y_0 with eigenvalue γ_7. This is possible since by construction

$$\rho_{y_0}^{\mathrm{ss}} \cong \rho_F(-(k-2)/2) \oplus \mathrm{Ad}^2 \rho_F \oplus \rho_F(-k/2),$$

and therefore the eigenvalues of $\rho_{y_0}(g_0)$ are given exactly by $\gamma_1, \ldots, \gamma_7$.

Take $\tilde{w}_7 \in \mathcal{P}$ mapping to v_7' under specialization at y_0. Consider the sublattice $\mathcal{L}_{\mathfrak{Z}}$ of $\mathcal{P}$ generated over $\mathcal{O}(\mathfrak{Z})[G_{\mathbb{Q}}]$ by $\tilde{w}_7$, and let $\mathcal{L} = \mathcal{L}_{\mathfrak{Z}} \otimes \mathcal{A} \subset \mathcal{P}_{\mathcal{A}}$; this lattice $\mathcal{L}$ is the same as the sublattice of $\mathcal{P}_{\mathcal{A}}$ generated over $\mathcal{A}[G_{\mathbb{Q}}]$ by $\tilde{v}_7$. Because T is irreducible, the lattices $\mathcal{L}_{\mathfrak{Z}}$ and $\mathcal{L}$ are of full rank 7. By passing to a finite normal extension of $\mathcal{O}(\mathfrak{Z})$ and localizing, we may assume $\mathcal{L}_{\mathfrak{Z}}$ is free.

Let $\overline{\mathcal{L}}$ be the reduction of $\mathcal{L}$ modulo $\mathfrak{m}$. This is the same as the specialization of $\mathcal{L}_{\mathfrak{Z}}$ at y_0. Then $\overline{\mathcal{L}}^{\mathrm{ss}} \cong \rho_{y_0}^{\mathrm{ss}}$, and so by construction, we have either

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & *_3 & *_2 \\ 0 & \mathrm{Ad}^2 \rho_F & *_1 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}, \tag{3.5.1.2}$$

with $*_1$ and $*_2$ nontrivial, or

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & 0 & *_2 \\ *_3 & \mathrm{Ad}^2 \rho_F & *_1 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}, \tag{3.5.1.3}$$

again with $*_1$ and $*_2$ nontrivial; this latter case is the same as

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \mathrm{Ad}^2 \rho_F & {}^t*_3 & *_1 \\ 0 & \rho_F(-(k-2)/2) & *_2 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}.$$

Construction of $\mathcal{L}$ when T is absolutely reducible

In this case, by passing to a finite normal extension of $\mathcal{O}(\mathfrak{Z})$ and localizing, we may assume by Lemma 3.4.3.1 that $T = T_1 + T_2$ where

$$T_{1,y_0} = \mathrm{Tr}(\mathrm{Ad}^2 \rho_F), \qquad T_{2,y_0} = \mathrm{Tr}(\rho_F(-(k-2)/2)) + \mathrm{Tr}(\rho_F(-k/2)).$$

We may further assume $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$ with $\mathcal{P}_1$ and $\mathcal{P}_2$ having respective Galois actions $\rho_{\mathfrak{Z},1}$ and $\rho_{\mathfrak{Z},2}$ whose respective specializations $\rho_{y_0,1}$ and $\rho_{y_0,2}$ at y_0 satisfy

$$\rho_{y_0,1} \cong \mathrm{Ad}^2 \rho_F, \qquad \rho_{y_0,2}^{\mathrm{ss}} \cong \rho_F(-(k-2)/2) \oplus \rho_F(-k/2).$$

Now let v_7' be a vector in $\mathcal{P}_{y_0,2}$ which is an eigenvector for $\rho_{y_0,2}(g_0)$ with eigenvalue γ_7. Let $\tilde{w}_7 \in \mathcal{P}_2$ mapping to v_7' under specialization at y_0. Consider the sublattice $\mathcal{L}_{\mathfrak{Z},2}$ of $\mathcal{P}_2$ generated over $\mathcal{O}(\mathfrak{Z})[G_{\mathbb{Q}}]$ by $\tilde{w}_7$. Let $\mathcal{L}_{\mathfrak{Z},1} = \mathcal{P}_1$, and $\mathcal{L}_{\mathfrak{Z}} = \mathcal{L}_{\mathfrak{Z},1} \oplus \mathcal{L}_{\mathfrak{Z},2}$.

We also let $\mathcal{L}_2 = \mathcal{L}_{\mathfrak{Z}} \otimes \mathcal{A}$, and $\mathcal{L}_1 = \mathcal{P}_1 \otimes \mathcal{A}$. Then letting $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, we have $\mathcal{L} \subset \mathcal{P}_{\mathcal{A}}$. By passing to a finite normal extension of $\mathcal{O}(\mathfrak{Z})$ and localizing, we may assume $\mathcal{L}_{\mathfrak{Z}}$ is free.

Let $\overline{\mathcal{L}}$ be the reduction of $\mathcal{L}$ modulo $\mathfrak{m}$. This time, we have by construction,

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & 0 & *_2 \\ 0 & \mathrm{Ad}^2 \rho_F & 0 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}, \tag{3.5.1.4}$$

with $*_2$ nontrivial.

Construction of a basis of $\mathcal{L}$

Let u_6 and u_7 be eigenvectors for $\rho_F(-k/2)(g_0)$ in the space of $\rho_F(-k/2)$, with respective eigenvalues γ_6 and γ_7. We view $\rho_F(-(k-2)/2)$ as $\rho_F(-k/2)^\vee$, and we let $\{u_1, u_2\}$ be the basis of $\rho_F(-(k-2)/2)$ dual to $\{u_7, u_6\}$. We also view $\mathrm{Ad}^2 \rho_F$ as a subspace of $\rho_F(-k/2) \otimes \rho_F(-k/2)^\vee$, and we let $\{u_3, u_4, u_5\}$ be the basis of $\mathrm{Ad}^2 \rho_F$ corresponding to

$$\{u_6 \otimes u_7^\vee, u_7 \otimes u_7^\vee - u_6 \otimes u_6^\vee, u_7 \otimes u_6^\vee\}$$

in $\rho_F(-k/2) \otimes \rho_F(-k/2)^\vee$. For $g \in G_{\mathbb{Q}}$, let us write

$$\rho_F(-k/2)(g) = \begin{pmatrix} g_{66} & g_{67} \\ g_{76} & g_{77} \end{pmatrix}$$

in the basis u_6, u_7 of $\rho_F(-k/2)$. Let us also write

$$d(g) = \det \rho_F(-k/2)(g) = g_{66}g_{77} - g_{67}g_{76}.$$

Then we have

$$\rho_F(-(k-2)/2)(g) = \frac{1}{d(g)} \begin{pmatrix} g_{66} & -g_{67} \\ -g_{76} & g_{77} \end{pmatrix}$$

in the basis u_1, u_2, and we have

$$(\mathrm{Ad}^2 \rho_F)(g) = \frac{1}{d(g)} \begin{pmatrix} g_{66}^2 & 2g_{66}g_{67} & -g_{67}^2 \\ g_{66}g_{76} & g_{66}g_{77} + g_{67}g_{76} & -g_{67}g_{77} \\ -g_{76}^2 & -2g_{76}g_{77} & g_{77}^2 \end{pmatrix}$$

in the basis u_3, u_4, u_5.

Let $\rho_{\overline{\mathcal{L}}}$ denote the action of $G_{\mathbb{Q}}$ on $\overline{\mathcal{L}}$, and $\rho_{\mathcal{L}}$ that on $\mathcal{L}$. Assume first $\overline{\mathcal{L}}$ has the form displayed in (3.5.1.2). Pick a basis $v_1, \ldots, v_7$ of $\overline{\mathcal{L}}$ satisfying the following properties: The vectors v_6, v_7 map respectively to u_6, u_7 in the quotient $\rho_F(-k/2)$; v_3, v_4, v_5 are vectors in the subrepresentation

$$\begin{pmatrix} \rho_F(-(k-2)/2) & *_3 \\ 0 & \mathrm{Ad}^2 \rho_F \end{pmatrix}$$

which map respectively to u_3, u_4, u_5 in the quotient $\mathrm{Ad}^2 \rho_F$; and the vectors v_1, v_2 are, respectively, the vectors u_1, u_2 in the subrepresentation $\rho_F(-(k-2)/2)$. Then in this basis, $\rho_{\overline{\mathcal{L}}}(g_0)$ has the form

$$\rho_{\overline{\mathcal{L}}}(g_0) = \begin{pmatrix} \gamma_7^{-1} & & * & * & * & * & * \\ & \gamma_6^{-1} & * & * & * & * & * \\ & & \gamma_6\gamma_7^{-1} & & & * & * \\ & & & 1 & & * & * \\ & & & & \gamma_7\gamma_6^{-1} & * & * \\ & & & & & \gamma_6 & \\ & & & & & & \gamma_7 \end{pmatrix}.$$

Then since all the entries on the diagonal of this matrix are distinct by assumption, it is possible to modify the basis $v_1, \ldots, v_7$ so that it still has the properties listed above, but all the asterisks are zero. That is, we can and will assume that actually,

$$\rho_{\overline{\mathcal{L}}}(g_0)v_i = \gamma_i v_i, \qquad i = 1, \ldots, 7, \tag{3.5.1.5}$$

where each γ_i is defined as in (3.5.1.1).

Similarly, if instead $\overline{\mathcal{L}}$ has the form displayed in (3.5.1.3), then we can pick a basis $v_1, \ldots, v_7$ of $\overline{\mathcal{L}}$ satisfying the following properties: The vectors v_6, v_7 map respectively to u_6, u_7 in the quotient $\rho_F(-k/2)$; v_3, v_4, v_5 are, respectively, the vectors u_3, u_4, u_5 in the subrepresentation $\mathrm{Ad}^2 \rho_F$; v_1, v_2 are vectors in the subrepresentation

$$\begin{pmatrix} \rho_F(-(k-2)/2) & 0 \\ *_3 & \mathrm{Ad}^2 \rho_F \end{pmatrix}$$

which map respectively to u_1, u_2 in the quotient $\rho_F(-(k-2)/2)$; and (3.5.1.5) holds.

Finally, if $\overline{\mathcal{L}}$ has the form displayed in (3.5.1.3), then T is reducible, and we let $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ be the reductions of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, modulo $\mathfrak{m}$. Then we pick a basis $v_1, \ldots, v_7$ of $\overline{\mathcal{L}}$ satisfying the following properties: The vectors v_6, v_7 are in $\overline{\mathcal{L}}_2$ and map respectively to u_6, u_7 in the quotient $\rho_F(-k/2)$; v_3, v_4, v_5 are, respectively, the vectors u_3, u_4, u_5 in the subrepresentation $\mathrm{Ad}^2 \rho_F$; v_1, v_2 are, respectively, the vectors u_1, u_2 in the subrepresentation $\rho_F(-(k-2)/2)$ of $\overline{\mathcal{L}}_2$; and (3.5.1.5) holds.

In any case, we have a basis $v_1, \ldots, v_7$ of $\overline{\mathcal{L}}$ which are eigenvectors for $\rho_{\overline{\mathcal{L}}}(g_0)$ with respective eigenvalues as in (3.5.1.5). By Hensel's lemma, we have eigenvectors $\tilde{v}_1, \ldots, \tilde{v}_7$ in $\mathcal{L}$ for $\rho_{\mathcal{L}}(g_0)$, with respective eigenvalues $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_7$, such that

$$\tilde{v}_i \equiv v_i \pmod{\mathfrak{m}},$$

and

$$\tilde{\gamma}_i \equiv \gamma_i \pmod{\mathfrak{m}}.$$

By Nakayama's lemma, $\tilde{v}_1, \ldots, \tilde{v}_7$ form a basis for $\mathcal{L}$.

The alternating trilinear form on $\overline{\mathcal{L}}$.

We know that $\rho_{\mathcal{A}}$ factors through $\mathrm{G}_2(\overline{\mathrm{Frac}(\mathcal{A})})$. By the remarks made in Section 3.1.2, $\rho_{\mathcal{A}}$ therefore preserves a generic alternating trilinear form $\langle \cdot, \cdot, \cdot \rangle^{\sim}$ on $\overline{\mathrm{Frac}(\mathcal{A})}^7$. Let $\tilde{v}_1^{\vee}, \ldots, \tilde{v}_7^{\vee}$ be the dual basis of $\tilde{v}_1, \ldots, \tilde{v}_7$ in the dual of the space $\overline{\mathrm{Frac}(\mathcal{A})}^7$, and write

$$\langle \cdot, \cdot, \cdot \rangle^{\sim} = \sum_{1 \leq i < j < k \leq 7} \tilde{a}_{ijk} \tilde{v}_i^{\vee} \wedge \tilde{v}_j^{\vee} \wedge \tilde{v}_k^{\vee},$$

for some $\tilde{a}_{ijk} \in \overline{\mathrm{Frac}(\mathcal{A})}$. Then for any triple (i, j, k) with $1 \leq i < j < k \leq 7$, we have

$$\tilde{a}_{ijk} = \langle v_i, v_j, v_k \rangle^{\sim} = \langle g_0 v_i, g_0 v_j, g_0 v_k \rangle^{\sim} = \tilde{\gamma}_i \tilde{\gamma}_j \tilde{\gamma}_k \tilde{a}_{ijk}.$$

Thus $\tilde{a}_{ijk} = 0$ for all such triples (i, j, k) except possibly

$$(i, j, k) = (1, 4, 7),\ (1, 5, 6),\ (2, 4, 6),\ (2, 3, 7),\ (3, 4, 5),$$

and $\tilde{a}_{ijk}$ is nonzero for at least one of these triples (i, j, k). Let $\mathcal{B}$ be the ring obtained from $\mathcal{A}$ by adjoining $\tilde{a}_{ijk}$ for these five triples (i, j, k). By scaling $\langle \cdot, \cdot, \cdot \rangle^{\sim}$, we may assume that $\tilde{a}_{ijk}$ is integral for these five triples (i, j, k), and that $\tilde{a}_{ijk} \in \mathcal{B}^{\times}$ for at least one such (i, j, k). If $\mathfrak{m}_{\mathcal{B}}$ is the maximal ideal of $\mathcal{B}$, and we write $a_{ijk} \in \overline{\mathbb{Q}}_p$ for the reduction of $\tilde{a}_{ijk}$ modulo $\mathfrak{m}_{\mathcal{B}}$, then this means

$$a_{ijk} \neq 0 \text{ for at least one } (i, j, k) = (1, 4, 7),\ (1, 5, 6),\ (2, 4, 6),\ (2, 3, 7),\ (3, 4, 5).$$

Furthermore, if T is reducible, then by replacing $\mathcal{L}_2$ with $c\mathcal{L}_2$ for a suitable $c \in \mathcal{A}$, we may assume the valuation of $\tilde{a}_{345}$ is small enough so that we instead have

$$a_{ijk} \neq 0 \text{ for at least one } (i, j, k) = (1, 4, 7),\ (1, 5, 6),\ (2, 4, 6),\ (2, 3, 7).$$

Finally, write $\langle\cdot,\cdot,\cdot\rangle$ for the reduction of $\langle\cdot,\cdot,\cdot\rangle^{\sim}$ modulo $\mathfrak{m}_{\mathcal{B}}$. Then $\langle\cdot,\cdot,\cdot\rangle$ is a nontrivial alternating trilinear form on $\overline{\mathcal{L}}$ preserved by the action of $G_{\mathbb{Q}}$, and

$$\langle\cdot,\cdot,\cdot\rangle = \sum_{1\le i<j<k\le 7} a_{ijk} v_i^{\vee}\wedge v_j^{\vee}\wedge v_k^{\vee}.$$

3.5.2 The shape of $\mathcal{L}$ modulo $\mathfrak{m}$

Let us begin by summarizing the construction of the previous section. We have a lattice $\mathcal{L}_{\mathcal{Z}}$ in our representation $R_7\circ\rho_3$ whose specialization $\overline{\mathcal{L}}$ at y_0 is of one of the following three forms: We either have

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & *_3 & *_2 \\ 0 & \mathrm{Ad}^2\rho_F & *_1 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}, \tag{3.5.2.1}$$

with $*_1$ and $*_2$ nontrivial, or

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & 0 & *_2 \\ *_3 & \mathrm{Ad}^2\rho_F & *_1 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}, \tag{3.5.2.2}$$

again with $*_1$ and $*_2$ nontrivial, or

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & 0 & *_2 \\ 0 & \mathrm{Ad}^2\rho_F & 0 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}, \tag{3.5.2.3}$$

with $*_2$ nontrivial. We will eventually rule out the possibility that $\overline{\mathcal{L}}$ has the shape displayed in (3.5.2.2) or (3.5.2.3).

To do this, we use the alternating trilinear form constructed in the previous section. Let $v_1,\ldots,v_7$ be the basis of $\overline{\mathcal{L}}$ constructed in the previous section. Then we saw that there are numbers a_{147}, a_{156}, a_{237}, a_{246}, and a_{345} in $\overline{\mathbb{Q}}_p$, at least one of which is nonzero, such that the trilinear form

$$\langle\cdot,\cdot,\cdot\rangle = a_{147}v_1^{\vee}\wedge v_4^{\vee}\wedge v_7^{\vee} + a_{156}v_1^{\vee}\wedge v_5^{\vee}\wedge v_6^{\vee} + a_{237}v_2^{\vee}\wedge v_3^{\vee}\wedge v_7^{\vee} + a_{246}v_2^{\vee}\wedge v_4^{\vee}\wedge v_6^{\vee} + a_{345}v_3^{\vee}\wedge v_4^{\vee}\wedge v_5^{\vee}$$

is preserved by the action of $G_{\mathbb{Q}}$ on $\overline{\mathcal{L}}$. Here $v_1^{\vee},\ldots,v_7^{\vee}$ is the dual basis to $v_1,\ldots,v_7$. Furthermore, if $\overline{\mathcal{L}}$ has the form (3.5.2.3), we have that one of a_{147}, a_{156}, a_{237}, a_{246} is nonzero.

For $g \in G_{\mathbb{Q}}$, let (g_{ij}) be the matrix of g in the basis $v_1, \ldots, v_7$. Then we have the following relations from the previous section:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \frac{1}{d(g)} \begin{pmatrix} g_{66} & -g_{67} \\ -g_{76} & g_{77} \end{pmatrix}, \tag{3.5.2.4}$$

and

$$\begin{pmatrix} g_{33} & g_{34} & g_{35} \\ g_{43} & g_{44} & g_{45} \\ g_{53} & g_{54} & g_{55} \end{pmatrix} = \frac{1}{d(g)} \begin{pmatrix} g_{66}^2 & 2g_{66}g_{67} & -g_{67}^2 \\ g_{66}g_{76} & g_{66}g_{77} + g_{67}g_{76} & -g_{67}g_{77} \\ -g_{76}^2 & -2g_{76}g_{77} & g_{77}^2 \end{pmatrix}, \tag{3.5.2.5}$$

for any $g \in G_{\mathbb{Q}}$.

We will now study the numbers a_{147}, a_{156}, a_{237}, a_{246}, and a_{345} using the matrix coefficients g_{ij} and the fact that $g \in G_{\mathbb{Q}}$ preserves the form $\langle \cdot, \cdot, \cdot \rangle$. Our goal is to prove that (3.5.2.1) holds, that $a_{147} = a_{156} = a_{237} = -a_{246} \neq 0$, and that $a_{345} \neq 0$. This will force the action of $g \in G_{\mathbb{Q}}$ on $\overline{\mathcal{L}}$ to factor through, not just G_2, but even its short root parabolic subgroup.

Lemma 3.5.2.1. *One of* $a_{147}, a_{156}, a_{246}, a_{237}$ *is nonzero.*

Proof. We assumed the conclusion of the lemma to be true when (3.5.2.3) holds. Thus we may assume either (3.5.2.1) or (3.5.2.2) holds. Then one of the matrix coefficients

$$g_{36}, g_{46}, g_{56}, g_{37}, g_{47}, g_{57}$$

is nonzero for some $g \in G_{\mathbb{Q}}$.

Now assume for sake of contradiction that all of $a_{147}, a_{156}, a_{246}, a_{237}$ are zero. Then $a_{345} \neq 0$.

We then compute

$$\begin{aligned}
0 = a_{156}(g^{-1})_{13} &= \langle g^{-1}v_3, v_5, v_6\rangle \\
&= \langle v_3, gv_5, gv_6\rangle \\
&= a_{345}(g_{45}g_{56} - g_{55}g_{46}) + a_{237}(g_{75}g_{26} - g_{25}g_{76}) \\
&= a_{345}(g_{45}g_{56} - g_{55}g_{46}), \\
0 = a_{246}(g^{-1})_{23} &= \langle g^{-1}v_3, v_4, v_6\rangle \\
&= \langle v_3, gv_4, gv_6\rangle \\
&= a_{345}(g_{44}g_{56} - g_{54}g_{46}) + a_{237}(g_{74}g_{26} - g_{24}g_{76}) \\
&= a_{345}(g_{44}g_{56} - g_{54}g_{46}).
\end{aligned}$$

We thus get

$$a_{345}\begin{pmatrix} g_{44} & g_{54} \\ g_{45} & g_{55} \end{pmatrix}\begin{pmatrix} g_{56} \\ -g_{46} \end{pmatrix} = 0$$

Since $a_{345} \neq 0$ and $d(g) \neq 0$, (3.5.2.5) gives

$$\begin{pmatrix} g_{66}g_{77} + g_{67}g_{76} & -g_{67}g_{77} \\ -2g_{76}g_{77} & g_{77}^2 \end{pmatrix}\begin{pmatrix} g_{56} \\ -g_{46} \end{pmatrix} = 0.$$

Now we have

$$\det\begin{pmatrix} g_{66}g_{77} + g_{67}g_{76} & -g_{67}g_{77} \\ -2g_{76}g_{77} & g_{77}^2 \end{pmatrix} = g_{77}^2(g_{66}g_{77} - g_{67}g_{76}) = d(g)g_{77}^2,$$

which is zero only when g_{77} is zero. Since F is not CM (it is level 1) and hence ρ_F has big image, $g_{77} = 0$ only for g in a measure zero subset of $G_{\mathbb{Q}}$. For such g, we can invert the matrix above and we find that

$$g_{56} = g_{46} = 0$$

for all g outside a subset of $G_{\mathbb{Q}}$ of measure zero. By continuity of $\rho_{\overline{\mathcal{L}}}$, $g_{56} = g_{46} = 0$ for all g.

Now we have to repeat this argument a couple more times with different matrix coefficients.

We have

$$
\begin{aligned}
0 = a_{147}(g^{-1})_{13} &= \langle g^{-1}v_3, v_4, v_7\rangle \\
&= \langle v_3, gv_4, gv_7\rangle \\
&= a_{345}(g_{44}g_{57} - g_{54}g_{47}) + a_{237}(g_{74}g_{27} - g_{24}g_{77}) \\
&= a_{345}(g_{44}g_{57} - g_{54}g_{47}), \\
0 = a_{237}(g^{-1})_{23} &= \langle g^{-1}v_3, v_3, v_7\rangle \\
&= \langle v_3, gv_3, gv_7\rangle \\
&= a_{345}(g_{43}g_{57} - g_{53}g_{47}) + a_{237}(g_{73}g_{27} - g_{23}g_{77}) \\
&= a_{345}(g_{43}g_{57} - g_{53}g_{47}).
\end{aligned}
$$

Thus,

$$a_{345}\begin{pmatrix} g_{43} & g_{53} \\ g_{44} & g_{54} \end{pmatrix}\begin{pmatrix} g_{57} \\ -g_{47} \end{pmatrix} = 0,$$

and so

$$\begin{pmatrix} 2g_{66}g_{67} & -g_{67}^2 \\ g_{66}g_{77} + g_{67}g_{76} & -g_{67}g_{77} \end{pmatrix}\begin{pmatrix} g_{57} \\ -g_{47} \end{pmatrix} = 0.$$

The determinant of the 2 by 2 matrix above is $-d(g)g_{66}^2$, and so the argument from above applies to show that $g_{57} = g_{47} = 0$ for all g.

Finally, we have

$$
\begin{aligned}
0 = a_{156}(g^{-1})_{15} &= \langle g^{-1}v_5, v_5, v_6\rangle \\
&= \langle v_5, gv_5, gv_6\rangle \\
&= a_{345}(g_{35}g_{46} - g_{45}g_{36}) + a_{156}(g_{65}g_{16} - g_{15}g_{66}) \\
&= a_{345}(g_{35}g_{46} - g_{45}g_{36}), \\
0 = a_{237}(g^{-1})_{15} &= \langle g^{-1}v_5, v_3, v_7\rangle \\
&= \langle v_5, gv_3, gv_7\rangle \\
&= a_{345}(g_{33}g_{47} - g_{43}g_{37}) + a_{156}(g_{63}g_{17} - g_{13}g_{67}) \\
&= a_{345}(g_{33}g_{47} - g_{43}g_{37}).
\end{aligned}
$$

Because we already know $g_{46} = g_{47} = 0$, this reduces to

$$g_{45}g_{36} = g_{43}g_{37} = 0.$$

by (3.5.2.5) this is the same as

$$-g_{67}g_{77}g_{36} = g_{66}g_{76}g_{37}.$$

By a similar argument as above, this forces $g_{36} = g_{37} = 0$.

This is a contradiction, since one of

$$g_{36}, g_{46}, g_{56}, g_{37}, g_{47}, g_{57}$$

is nonzero as we said above. Therefore one of $a_{147}, a_{156}, a_{246}, a_{237}$ must be nonzero, as desired. $\square$

Lemma 3.5.2.2. *We have*

$$a_{147} = a_{156} = a_{237} = -a_{246}.$$

Proof. We either have

$$g_{13} = g_{14} = g_{15} = g_{23} = g_{24} = g_{25} = 0,$$

or

$$g_{31} = g_{41} = g_{51} = g_{32} = g_{42} = g_{52} = 0.$$

In either case, we find

$$\begin{pmatrix} (g^{-1})_{11} & (g^{-1})_{12} \\ (g^{-1})_{21} & (g^{-1})_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}.$$

By (3.5.2.4), this equals

$$\begin{pmatrix} (g^{-1})_{11} & (g^{-1})_{12} \\ (g^{-1})_{21} & (g^{-1})_{22} \end{pmatrix} = \begin{pmatrix} g_{77} & g_{67} \\ g_{76} & g_{66} \end{pmatrix}.$$

Now we use this to compute

$$\begin{aligned} a_{156}g_{77} = a_{156}(g^{-1})_{11} &= \langle g^{-1}v_1, v_5, v_6\rangle \\ &= \langle v_1, gv_5, gv_6\rangle \\ &= a_{156}(g_{55}g_{66} - g_{65}g_{56}) + a_{147}(g_{45}g_{76} - g_{75}g_{46}). \end{aligned}$$

Since $g_{65} = g_{75} = 0$, this gives

$$a_{156}g_{77} = a_{156}g_{77}g_{55}g_{66} + a_{147}g_{45}g_{76}.$$

By (3.5.2.5), this is

$$a_{156}g_{77}d(g) = a_{156}g_{77}g_{77}^2g_{66} - a_{147}g_{77}g_{67}g_{76},$$

or, after rearranging,

$$a_{156}g_{77}g_{67}g_{76} = a_{147}g_{77}g_{67}g_{76}.$$

But by ρ_F having big image, $g_{77}g_{67}g_{76} \neq 0$ for some g. Thus $a_{156} = a_{147}$.

Next we compute

$$\begin{aligned} a_{237}g_{66} = a_{237}(g^{-1})_{22} &= \langle g^{-1}v_2, v_3, v_7\rangle \\ &= \langle v_2, gv_3, gv_7\rangle \\ &= a_{237}(g_{33}g_{77} - g_{73}g_{37}) + a_{246}(g_{43}g_{67} - g_{63}g_{47}) \\ &= a_{237}g_{33}g_{77} + a_{246}g_{43}g_{67} \\ &= d(g)^{-1}(a_{237}g_{66}^2g_{77} + a_{246}g_{66}g_{76}g_{67}). \end{aligned}$$

Thus, after rearranging, we find

$$-a_{237}g_{66}g_{76}g_{67} = a_{246}g_{66}g_{76}g_{67}.$$

We therefore have $a_{237} = -a_{246}$.

Finally, we compute

$$\begin{aligned}
a_{156}g_{67} = a_{156}(g^{-1})_{12} &= \langle g^{-1}v_2, v_5, v_6\rangle \\
&= \langle v_2, gv_5, gv_6\rangle \\
&= a_{237}(g_{35}g_{76} - g_{75}g_{36}) + a_{246}(g_{45}g_{66} - g_{65}g_{46}) \\
&= a_{237}g_{35}g_{76} + a_{246}g_{45}g_{66} \\
&= d(g)^{-1}(-a_{237}g_{67}^2 g_{76} - a_{246}g_{67}g_{77}g_{66}) \\
&= d(g)^{-1}a_{237}g_{67}(-g_{67}g_{76} + g_{77}g_{66}) \\
&= a_{237}g_{67}
\end{aligned}$$

where the second-to-last line follows because $a_{237} = -a_{246}$. Thus it follows that $a_{156} = a_{237}$, and this completes the proof. $\square$

By the two previous lemmas, we can (and will) assume

$$a_{147} = a_{156} = a_{237} = 1, \qquad a_{246} = -1.$$

Assume now that

$$g_{13} = g_{14} = g_{15} = g_{23} = g_{24} = g_{25} = 0.$$

Then $\mathrm{Ad}^2 \rho_F$ is a subrepresentation of $\overline{\mathcal{L}}$. The quotient $\overline{\mathcal{L}}/\mathrm{Ad}^2 \rho_F$ is the extension E of the form

$$\begin{pmatrix} \rho_F(-(k-2)/2) & *_2 \\ 0 & \rho_F(-k/2) \end{pmatrix}$$

with $*_2$ nontrivial. This extension E is therefore given by

$$E \sim \begin{pmatrix} g_{11} & g_{12} & g_{16} & g_{17} \\ g_{21} & g_{22} & g_{26} & g_{27} \\ 0 & 0 & g_{66} & g_{67} \\ 0 & 0 & g_{76} & g_{77} \end{pmatrix}.$$

Lemma 3.5.2.3. *Assume that*

$$g_{13} = g_{14} = g_{15} = g_{23} = g_{24} = g_{25} = 0$$

for all g, so that we have the extension E as above. Then the exterior square $\wedge^2 E$ has a subrepresentation a nontrivial extension of the form

$$\begin{pmatrix} \chi_{\mathrm{cyc}} & * \\ 0 & 1 \end{pmatrix}.$$

Proof. We first use the following fact: If

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

is an invertible matrix with n by n blocks, then

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

It follows from this that

$$\begin{pmatrix} (g^{-1})_{16} & (g^{-1})_{17} \\ (g^{-1})_{26} & (g^{-1})_{27} \end{pmatrix} = -\frac{1}{d(g)} \begin{pmatrix} g_{77} & g_{67} \\ g_{76} & g_{66} \end{pmatrix} \begin{pmatrix} g_{16} & g_{17} \\ g_{26} & g_{27} \end{pmatrix} \begin{pmatrix} g_{77} & -g_{67} \\ -g_{76} & g_{66} \end{pmatrix}.$$

Thus, computing the product, we find in particular,

$$\begin{pmatrix} (g^{-1})_{16} \\ (g^{-1})_{26} \end{pmatrix} = -\frac{1}{d(g)} \begin{pmatrix} g_{77}^2 g_{16} + g_{77}g_{67}g_{26} - g_{77}g_{76}g_{17} - g_{67}g_{76}g_{27} \\ g_{77}g_{76}g_{16} + g_{77}g_{66}g_{26} - g_{76}^2 g_{17} - g_{76}g_{66}g_{27} \end{pmatrix}. \tag{3.5.2.6}$$

Thus we compute

$$\begin{aligned} -\frac{1}{d(g)}(g_{77}^2 g_{16} + g_{77}g_{67}g_{26} - g_{77}g_{76}g_{17} - g_{67}g_{76}g_{27}) =& a_{156}(g^{-1})_{16} \\ =& \langle g^{-1}v_6, v_5, v_6 \rangle \\ =& \langle v_6, gv_5, gv_6 \rangle \\ =& a_{246}(g_{25}g_{46} - g_{45}g_{26}) \\ & + a_{156}(g_{15}g_{56} - g_{55}g_{16}) \\ =& (g_{45}g_{26} - g_{55}g_{16}) \\ =& -\frac{1}{d(g)}(g_{67}g_{77}g_{26} - g_{77}^2 g_{16}), \end{aligned}$$

where, in the last line, we used (3.5.2.5). Hence

$$g_{77}g_{76}g_{17} + g_{67}g_{76}g_{27} = 0$$

and it follows (again from ρ_F having big image) that

$$g_{77}g_{17} + g_{67}g_{27} = 0 \tag{3.5.2.7}$$

We also compute

$$\begin{aligned}
-\frac{1}{d(g)}(g_{77}^2 g_{16} + g_{77}g_{67}g_{26} - g_{77}g_{76}g_{17} - g_{67}g_{76}g_{27}) =& a_{147}(g^{-1})_{16} \\
=& \langle g^{-1}v_6, v_4, v_7\rangle \\
=& \langle v_6, gv_4, gv_7\rangle \\
=& a_{246}(g_{24}g_{47} - g_{44}g_{27}) \\
& + a_{156}(g_{14}g_{57} - g_{54}g_{17}) \\
=& \frac{1}{d(g)}(g_{44}g_{27} - g_{54}g_{17}) \\
=& -\frac{1}{d(g)}((g_{66}g_{77} + g_{76}g_{67})g_{27} + 2g_{77}g_{76}g_{17}).
\end{aligned}$$

But, of course, this equals $a_{156}(g^{-1})_{16}$, since $a_{147} = a_{156}$, and so we get from above,

$$(g_{66}g_{77} + g_{76}g_{67})g_{27} + 2g_{77}g_{76}g_{17} = g_{67}g_{77}g_{26} - g_{77}^2 g_{16}.$$

Rearranging gives

$$g_{76}(g_{77}g_{17} + g_{67}g_{27}) + g_{77}(g_{77}g_{16} + g_{76}g_{17} + g_{67}g_{26} + g_{66}g_{27}) = 0.$$

By (3.5.2.7), the first term in this sum is zero, so we get

$$g_{77}(g_{77}g_{16} + g_{76}g_{17} + g_{67}g_{26} + g_{66}g_{27}) = 0.$$

Hence

$$g_{77}g_{16} + g_{76}g_{17} + g_{67}g_{26} + g_{66}g_{27} = 0. \tag{3.5.2.8}$$

Next we compute

$$\begin{aligned}
-\frac{1}{d(g)}(g_{77}g_{76}g_{16} + g_{77}g_{66}g_{26} - g_{76}^2g_{17} - g_{76}g_{66}g_{27}) =& a_{237}(g^{-1})_{26} \\
=& \langle g^{-1}v_6, v_3, v_7\rangle \\
=& \langle v_6, gv_3, gv_7\rangle \\
=& a_{246}(g_{23}g_{47} - g_{43}g_{27}) \\
& + a_{156}(g_{13}g_{57} - g_{53}g_{17}) \\
=& \frac{1}{d(g)}(g_{43}g_{27} - g_{53}g_{17}) \\
=& \frac{1}{d(g)}(g_{66}g_{76}g_{27} + g_{76}^2g_{17}).
\end{aligned}$$

Therefore,

$$g_{77}g_{76}g_{16} + g_{77}g_{66}g_{26} = 0,$$

and so

$$g_{76}g_{16} + g_{66}g_{26} = 0. \tag{3.5.2.9}$$

With this preparation we can now proceed to prove the lemma. The extension E has basis v_1, v_2, v_6, v_7. Therefore, the exterior square $\wedge^2 E$ has the basis

$$v_2 \wedge v_1,\ (v_7 \wedge v_1 + v_2 \wedge v_6),\ v_6 \wedge v_1,\ (v_7 \wedge v_1 - v_2 \wedge v_6),\ v_7 \wedge v_2,\ v_7 \wedge v_6.$$

We now compute part of the matrix of g in this basis using (3.5.2.4). We have

$$\begin{aligned}
g(v_2 \wedge v_1) &= gv_2 \wedge gv_1 \\
&= \frac{1}{d(g)}(g_{77}v_2 - g_{67}v_1) \wedge (-g_{76}v_2 + g_{66}v_1) \\
&= \frac{1}{d(g)}(g_{77}g_{66} - g_{67}g_{76})v_2 \wedge v_1 \\
&= \frac{1}{d(g)}v_2 \wedge v_1.
\end{aligned}$$

We also have

$$\begin{aligned}
g(v_7 \wedge v_1 + v_2 \wedge v_6) =& gv_7 \wedge gv_1 + gv_2 \wedge gv_6 \\
=& \frac{1}{d(g)}(g_{77}v_7 + g_{67}v_6 + g_{27}v_2 + g_{17}v_1) \wedge (-g_{76}v_2 + g_{66}v_1) \\
&+ \frac{1}{d(g)}(g_{76}v_7 + g_{66}v_6 + g_{26}v_2 + g_{16}v_1) \wedge (-g_{77}v_2 + g_{67}v_1) \\
=& \frac{1}{d(g)}(g_{66}g_{27} + g_{76}g_{17} - g_{67}g_{26} - g_{77}g_{16})v_2 \wedge v_1 + \frac{1}{d(g)}(g_{67}g_{66} - g_{66}g_{67}) \\
&+ \frac{1}{d(g)}(g_{66}g_{77} - g_{67}g_{76})v_7 \wedge v_1 + \frac{1}{d(g)}(-g_{67}g_{76} + g_{66}g_{77})v_6 \wedge v_2 \\
&+ \frac{1}{d(g)}(-g_{77}g_{76} + g_{76}g_{77})v_7 \wedge v_2 \\
=& \frac{1}{d(g)}(g_{67}g_{26} + g_{77}g_{16})v_2 \wedge v_1 + (v_7 \wedge v_1 + v_2 \wedge v_6),
\end{aligned}$$

where, in the last line, we used (3.5.2.8).

This computes the first two columns of g in the basis chosen above; these two columns begin with

$$\begin{pmatrix} d(g)^{-1} & * \\ 0 & 1 \end{pmatrix},$$

and the rest of the entries are zero, showing that this extension is a subrepresentation of $\wedge^2 E$. Here the asterisk denotes

$$* = d(g)^{-1}(g_{67}g_{26} + g_{77}g_{16}).$$

Note that $d(g) = \det \rho_F(-k/2)(g) = \chi_{\mathrm{cyc}}^{-1}(g)$, so as long as $*$ is nontrivial, this is the desired extension.

So assume for sake of contradiction that $*$ is trivial. Then

$$g_{67}g_{26} + g_{77}g_{16} = 0.$$

Combining this with (3.5.2.9) gives

$$\begin{pmatrix} g_{77} & g_{67} \\ g_{76} & g_{66} \end{pmatrix} \begin{pmatrix} g_{16} \\ g_{26} \end{pmatrix} = 0.$$

Since the matrix

$$\begin{pmatrix} g_{77} & g_{67} \\ g_{76} & g_{66} \end{pmatrix}$$

is invertible, this implies $g_{16} = g_{26} = 0$. But then (3.5.2.8) becomes

$$g_{76}g_{17} + g_{66}g_{27} = 0.$$

Along with (3.5.2.7), this gives

$$\begin{pmatrix} g_{77} & g_{67} \\ g_{76} & g_{66} \end{pmatrix} \begin{pmatrix} g_{17} \\ g_{27} \end{pmatrix} = 0.$$

Like before, this implies $g_{17} = g_{27} = 0$ as well, which implies that E is a trivial extension. This is a contradiction, and the lemma is proved. □

We are now in a position to use p-adic Hodge theory to rule out the possibility that $\mathrm{Ad}^2 \rho_F$ is a subrepresentation of $\overline{\mathcal{L}}$.

Lemma 3.5.2.4. *One of*

$$g_{13}, g_{14}, g_{15}, g_{23}, g_{24}, g_{25}$$

is nonzero for some $g \in G_{\mathbb{Q}}$.

Proof. Assume for sake of contradiction that

$$g_{13} = g_{14} = g_{15} = g_{23} = g_{24} = g_{25} = 0.$$

Then $\mathrm{Ad}^2 \rho_F$ is a subrepresentation of $\overline{\mathcal{L}}$ with quotient given by the extension E from above. By Lemma 3.5.2.3, the exterior square $\wedge^2 E$ contains as a subrepresentation a nontrivial extension

$$\begin{pmatrix} \chi_{\mathrm{cyc}} & * \\ 0 & 1 \end{pmatrix}.$$

Let us call this extension E'. It is unramified at all primes except p because $\mathcal{L}$ is, and we will show now that E' has to also be crystalline at p. This will be a contradiction since E' will represent a nontrivial class in the Bloch–Kato Selmer group

$$H^1_f(\mathbb{Q}_p, \overline{\mathbb{Q}}_p(1)),$$

which itself is trivial.

So to get started, recall that we have the $\mathcal{O}(\mathfrak{Z})$-lattice $\mathcal{L}_{\mathfrak{Z}}$. The specialization of $\mathcal{L}_{\mathfrak{Z}}$ at y_0 is $\overline{\mathcal{L}}$. For $y \in \Sigma_c$, the specialization $\mathcal{L}_y$ of $\mathcal{L}_{\mathfrak{Z}}$ at y is crystalline at p with Hodge–Tate weights given by

$$-\langle \lambda + \rho, \mu^\vee \rangle,$$

and crystalline Frobenius eigenvalues given by

$$\theta_y(u_{\mu^\vee(p)})^{-1} p^{\langle \lambda+\rho, \mu^\vee \rangle},$$

for

$$\mu^\vee \in \{0, \pm\alpha^\vee, \pm(\alpha^\vee + \beta^\vee), \pm(2\alpha^\vee + \beta^\vee)\}.$$

Recall that the functions of y given by $\theta_y(u_{\mu^\vee(p)})$ are analytic on $\mathfrak{Z}$, and

$$\theta_{y_0}(u_{\mu^\vee(p)}) = \alpha_p^{\langle 2\alpha+3\beta, \mu^\vee \rangle} p^{\langle \beta, \mu^\vee \rangle}.$$

We will apply Kisin's lemma (Lemma 3.3.1.2) to various exterior powers of $\mathcal{L}_{\mathfrak{Z}}$. The hypotheses of that lemma are satisfied because the weights of points in Σ_c are increasingly regular. The Hodge–Tate weights $\langle \lambda_0 + \rho, \mu^\vee \rangle$ of $\overline{\mathcal{L}}$ are given in increasing order by

$$-(k-1),\ -\frac{k}{2}, -\frac{k-2}{2},\ 0,\ \frac{k-2}{2},\ \frac{k}{2},\ k-1,$$

corresponding respectively to the coweights

$$-(2\alpha^\vee + \beta^\vee),\ -(\alpha^\vee + \beta^\vee),\ -\alpha^\vee, 0, \alpha^\vee,\ \alpha^\vee + \beta^\vee,\ 2\alpha^\vee + \beta^\vee. \tag{3.5.2.10}$$

The corresponding values of $\theta_y(u_{\mu^\vee(p)})^{-1} p^{\langle \lambda_0+\rho, \mu^\vee \rangle}$ are, in order,

$$p^{-(k-1)}\alpha_p^2,\ p^{-(k-2)/2}\alpha_p,\ p^{-k/2}\alpha_p,\ 1,\ p^{k/2}\alpha_p^{-1},\ p^{(k-2)/2}\alpha_p^{-1},\ p^{k-1}\alpha_p^{-2}.$$

Now it follows from Proposition 3.3.2.5 that E is semistable. Write $D = D_{\mathrm{crys}}(\rho_F(-k/2))$, and let $u_3, u_4 \in D$ be eigenvectors for the crystalline Frobenius with respective eigenvalues $p^{k/2}\alpha_p^{-1}$ and $p^{-(k-2)/2}\alpha_p$. Since these numbers are distinct by Lemma **??**, u_3, u_4 are linearly independent. Then by Corollary 3.3.2.3, there is a basis w_1, w_2, w_3, w_4 of $D_{\mathrm{st}}(E)$ such that, in this basis, the crystalline Frobenius ϕ_E for $D_{\mathrm{st}}(E)$ is given by

$$\phi_E = \mathrm{diag}(p^{(k-2)/2}\alpha_p^{-1}, p^{-k/2}\alpha_p, p^{k/2}\alpha_p^{-1}, p^{-(k-2)/2}\alpha_p)$$

and the monodromy operator N_E for $D_{\mathrm{st}}(E)$ is given by

$$N_E = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

for some $B \in M_2(\overline{\mathbb{Q}}_p)$. Write

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

We claim first that $b_{12} = b_{22} = 0$. To show this, we apply Kisin's lemma to $\wedge^2\overline{\mathcal{L}}$, as was inspired by [Urb13b]. This shows that

$$D_{\mathrm{crys}}(\wedge^2\overline{\mathcal{L}})^{\phi = p^{-(k-1)}\alpha_p^2 \cdot p^{-(k-2)/2}\alpha_p} \neq 0.$$

Since $\mathrm{Ad}^2\,\rho_F \otimes E$ is the only subquotient of $\wedge^2\overline{\mathcal{L}}$ that can contribute an eigenvector for the crystalline Frobenius with this eigenvalue, it follows that

$$D_{\mathrm{crys}}(\mathrm{Ad}^2\,\rho_F \otimes E)^{\phi = p^{-(k-1)}\alpha_p^2 \cdot p^{-(k-2)/2}\alpha_p} \neq 0.$$

Now

$$D_{\mathrm{st}}(\mathrm{Ad}^2\,\rho_F \otimes E) = D_{\mathrm{crys}}(\mathrm{Ad}^2\,\rho_F) \otimes D_{\mathrm{st}}(E)$$

because $\mathrm{Ad}^2\,\rho_F$ is crystalline, and the monodromy operator on this space is given by $1 \otimes N_E$. It follows that

$$D_{\mathrm{st}}(\mathrm{Ad}^2\,\rho_F \otimes E)^{\phi = p^{-(k-1)}\alpha_p^2 \cdot p^{-(k-2)/2}\alpha_p} \neq 0.$$

This space is generated by vectors of the form $v' \otimes w_4$ where $v' \in D_{\mathrm{crys}}(\mathrm{Ad}^2\,\rho_F)$ is a crystalline Frobenius eigenvector with eigenvalue $p^{-(k-1)}\alpha_p^2$ (again we are using Lemma **??**). Therefore

$D_{\mathrm{crys}}(\mathrm{Ad}^2\,\rho_F \otimes E)$ must include a vector $v' \otimes w_4$ of this type, which is thus in the kernel of $1 \otimes N_E$. Hence $N_E v_4 = 0$, which implies $b_{12} = b_{22} = 0$.

Next we claim $b_{11} = 0$. To do this, we apply the same argument, but to the dual $(\wedge^2\overline{L})^\vee$ of $\wedge^2\overline{L}$ (or, equivalently, to $\wedge^5\overline{\mathcal{L}}$). The specializations $\mathcal{L}_y^\vee$ for $y \in \Sigma_c$ have the same Hodge–Tate weights and corresponding crystalline Frobenius eigenvalues as $\mathcal{L}_y$. So this gives

$$D_{\mathrm{crys}}((\wedge^2\overline{\mathcal{L}})^\vee)^{\phi = p^{-(k-1)}\alpha_p^2 \cdot p^{-(k-2)/2}\alpha_p} \neq 0.$$

It follows that

$$D_{\mathrm{crys}}(\mathrm{Ad}^2\,\rho_F \otimes E^\vee)^{\phi = p^{-(k-1)}\alpha_p^2 \cdot p^{-(k-2)/2}\alpha_p} \neq 0.$$

We also have

$$D_{\mathrm{st}}(\mathrm{Ad}^2\,\rho_F \otimes E) = D_{\mathrm{crys}}(\mathrm{Ad}^2\,\rho_F) \otimes D_{\mathrm{st}}(E)^\vee.$$

Let $w_1^\vee, w_2^\vee, w_3^\vee, w_4^\vee$ be the dual basis to w_1, w_2, w_3, w_4. Then the monodromy operator $N_{E^\vee}$ on $D_{\mathrm{st}}(E)^\vee$ is given in this basis by

$$N_{E^\vee} = \begin{pmatrix} 0 & 0 \\ -\,{}^tB & 0 \end{pmatrix}.$$

So by a similar argument as above, it follows that a vector $v' \otimes w_1^\vee$, with v' as above, is in the kernel of $N_{E^\vee}$. Thus the first column of the matrix representing $N_{E^\vee}$ is zero; in particular $b_{11} = 0$.

Now we come to the representation $\wedge^2 E$. We claim that

$$D_{\mathrm{st}}(\wedge^2 E)^{\phi=1} = D_{\mathrm{crys}}(\wedge^2 E)^{\phi=1}.$$

From this it will follow that the extension E' is crystalline, and we will be done.

To prove the claim, we note that $D_{\mathrm{st}}(\wedge^2 E)^{\phi=1}$ is the span of $v_1 \wedge v_4$ and $v_2 \wedge v_3$. But

$$N(v_1 \wedge v_4) = (N_E v_1) \wedge v_4 + v_1 \wedge (N_E v_4) = v_1 \wedge (b_{12} v_1 + b_{22} v_2) = 0,$$

and

$$N(v_2 \wedge v_3) = (N_E v_2) \wedge v_3 + v_2 \wedge (N_E v_4) = v_2 \wedge (b_{11} v_1 + b_{21} v_2) = b_{21} v_2 \wedge v_2 = 0,$$

so this proves the claim, and hence also the lemma. $\square$

It follows from the lemma that $\overline{\mathcal{L}}$ has the shape

$$\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & *_3 & *_2 \\ 0 & \mathrm{Ad}^2 \rho_F & *_1 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix},$$

as the other two possible shapes have just been ruled out.

We now prove a lemma that will show that $\overline{\mathcal{L}}$ factors through $\mathrm{G}_2(\overline{\mathbb{Q}}_p)$.

Lemma 3.5.2.5. *We have* $a_{345} \neq 0$*.*

Proof. Assume on the contrary that a_{345}. We will get a contradiction to Lemma 3.5.2.4. We compute

$$\begin{aligned} a_{156}(g^{-1})_{13} &= \langle g^{-1}v_3, v_5, v_6 \rangle \\ &= \langle v_3, gv_5, gv_6 \rangle \\ &= a_{345}(g_{45}g_{56} - g_{55}g_{46}) + a_{237}(g_{75}g_{26} - g_{25}g_{76}) \\ &= -a_{237}g_{76}g_{25}, \end{aligned}$$

hence

$$(g^{-1})_{13} = -g_{76}g_{25}.$$

We also compute

$$\begin{aligned} a_{237}(g^{-1})_{25} &= \langle g^{-1}v_5, v_3, v_7 \rangle \\ &= \langle v_5, gv_3, gv_7 \rangle \\ &= a_{345}(g_{33}g_{47} - g_{43}g_{37}) + a_{156}(g_{63}g_{17} - g_{13}g_{67}) \\ &= -a_{156}g_{67}g_{13}, \end{aligned}$$

and hence

$$(g^{-1})_{25} = -g_{67}g_{13}.$$

Therefore,

$$(g^{-1})_{13} = -g_{76}g_{25} = g_{76}(g^{-1})_{67}(g^{-1})_{13} = -d(g)^{-1}g_{67}g_{76}(g^{-1})_{13}.$$

Because the image of ρ_F is big, $d(g)^{-1}g_{67}g_{76} = 0$ only for a measure zero set of $g \in G_{\mathbb{Q}}$. Thus we must have $(g^{-1})_{13}$, and hence g_{13}, is identically zero, and by the equations above g_{25} is also identically zero.

We also compute

$$\begin{aligned}
a_{156}(g^{-1})_{15} &= \langle g^{-1}v_5, v_5, v_6\rangle \\
&= \langle v_5, gv_5, gv_6\rangle \\
&= a_{345}(g_{35}g_{46} - g_{45}g_{36}) + a_{156}(g_{65}g_{16} - g_{15}g_{66}) \\
&= -a_{156}g_{66}g_{15},
\end{aligned}$$

and hence

$$(g^{-1})_{15} = -g_{66}g_{15}.$$

Thus

$$(g^{-1})_{15} = g_{66}(g^{-1})_{66}(g^{-1})_{15} = d(g)^{-1}g_{66}g_{77}(g^{-1})_{15}.$$

Like above, this forces g_{15} to be identically zero.

Similarly, we compute

$$\begin{aligned}
a_{237}(g^{-1})_{23} &= \langle g^{-1}v_3, v_3, v_7\rangle \\
&= \langle v_3, gv_3, gv_7\rangle \\
&= a_{345}(g_{43}g_{57} - g_{53}g_{47}) + a_{237}(g_{73}g_{27} - g_{23}g_{77}) \\
&= -a_{237}g_{77}g_{23},
\end{aligned}$$

and hence

$$(g^{-1})_{23} = -g_{77}g_{23}.$$

Thus

$$(g^{-1})_{23} = g_{77}(g^{-1})_{77}(g^{-1})_{23} = d(g)^{-1}g_{77}g_{66}(g^{-1})_{23},$$

and once again, this forces g_{15} to be identically zero.

Finally, we have

$$
\begin{aligned}
a_{156}(g^{-1})_{14} =& \langle g^{-1}v_4, v_5, v_6\rangle \\
=& \langle v_4, gv_5, gv_6\rangle \\
=& a_{345}(g_{55}g_{36} - g_{35}g_{56}) + a_{147}(g_{75}g_{16} - g_{15}g_{76}) \\
& + a_{246}(g_{65}g_{26} - g_{25}g_{66}) \\
=& 0,
\end{aligned}
$$

where the vanishing at the end is because $g_{15} = g_{25} = 0$. Similarly, we have

$$
\begin{aligned}
a_{237}(g^{-1})_{24} =& \langle g^{-1}v_4, v_3, v_7\rangle \\
=& \langle v_4, gv_3, gv_7\rangle \\
=& a_{345}(g_{53}g_{37} - g_{33}g_{57}) + a_{147}(g_{73}g_{17} - g_{13}g_{77}) \\
& + a_{246}(g_{63}g_{27} - g_{23}g_{67}) \\
=& 0
\end{aligned}
$$

Thus g_{14} and g_{24} are identically zero. This is a contradiction and finishes the proof. □

3.5.3 The symmetric cube Selmer group

We now have the representation $\overline{\mathcal{L}}$, which is of the form

$$
\overline{\mathcal{L}} \sim \begin{pmatrix} \rho_F(-(k-2)/2) & *_3 & *_2 \\ 0 & \mathrm{Ad}^2\,\rho_F & *_1 \\ 0 & 0 & \rho_F(-k/2) \end{pmatrix}
$$

where $*_1$, $*_2$ and $*_3$ are all nontrivial. Writing again $v_1, \ldots, v_7$ for our chosen basis of $\overline{\mathcal{L}}$, we have that this representation preserves the alternating trilinear form

$$
\langle\cdot,\cdot,\cdot\rangle = v_1^\vee \wedge v_4^\vee \wedge v_7^\vee + v_1^\vee \wedge v_5^\vee \wedge v_6^\vee + v_2^\vee \wedge v_3^\vee \wedge v_7^\vee - v_2^\vee \wedge v_4^\vee \wedge v_6^\vee + a_{345} v_3^\vee \wedge v_4^\vee \wedge v_5^\vee,
$$

for some nonzero $a_{345} \in \overline{\mathbb{Q}}_p$. From here on we write $a = a_{345}$.

By Lemma 3.1.2.1, the representation $\overline{\mathcal{L}}$ factors through the G_2-subgroup $G_a(\overline{\mathbb{Q}}_p)$ of $\mathrm{GL}_7(\overline{\mathbb{Q}}_p)$.

Because of its shape, by Proposition 3.1.2.3, $\overline{\mathcal{L}}$ factors though the short root parabolic $P_{a,\beta}(\overline{\mathbb{Q}}_p)$.

For $g \in G_{\mathbb{Q}}$, let us continue to write

$$\begin{pmatrix} g_{66} & g_{67} \\ g_{76} & g_{77} \end{pmatrix}$$

for the bottom block of g in the basis $v_1, \ldots, v_7$, and

$$d(g) = g_{66}g_{77} - g_{67}g_{76}$$

for its determinant. Then $d(g) = \chi_{\mathrm{cyc}}^{-1}(g)$. Write g_{ij} as before for the other entries, and also write $g'_{ij} = d(g)g_{ij}$. Then we have

$$(g'_{ij}) = \begin{pmatrix} g_{66} & -g_{67} & g'_{13} & g'_{14} & g'_{15} & g'_{16} & g'_{17} \\ -g_{76} & g_{77} & g'_{23} & g'_{24} & g'_{25} & g'_{26} & g'_{27} \\ & & g_{66}^2 & 2g_{66}g_{67} & -g_{67}^2 & g'_{36} & g'_{37} \\ & & g_{66}g_{76} & g_{66}g_{77} + g_{67}g_{76} & -g_{67}g_{77} & g'_{46} & g'_{47} \\ & & -g_{76}^2 & -2g_{76}g_{77} & g_{77}^2 & g'_{56} & g'_{57} \\ & & & & & g'_{66} & g'_{67} \\ & & & & & g'_{76} & g'_{77} \end{pmatrix}.$$

Let us write E for the extension given by

$$E = \frac{1}{d(g)} \begin{pmatrix} g_{66} & -g_{67} & g'_{13} & g'_{14} & g'_{15} \\ -g_{76} & g_{77} & g'_{23} & g'_{24} & g'_{25} \\ & & g_{66}^2 & 2g_{66}g_{67} & -g_{67}^2 \\ & & g_{66}g_{76} & g_{66}g_{77} + g_{67}g_{76} & -g_{67}g_{77} \\ & & -g_{76}^2 & -2g_{76}g_{77} & g_{77}^2 \end{pmatrix}. \tag{3.5.3.1}$$

It is an extension

$$0 \to \rho_F(-(k-2)/2) \to E \to \mathrm{Ad}^2 \rho_F \to 0,$$

and it is a nontrivial extension by Lemma 3.5.2.4.

We want to show E is semistable at p. Most of our constructions in this chapter up until now, including that of E, depended on a choice of root α_p of the Hecke polynomial of F at p. However, certain choices of α_p may lead to problems when showing E is semistable. But it turns out that if one choice of α_p is problematic, we can switch α_p for the other root $p^{k-1}\alpha_p^{-1}$ and show that this other choice is no longer problematic. We make this precise in the following lemma.

Lemma 3.5.3.1. *Notation as above, there is a choice of α_p such that*

$$D_{\mathrm{crys}}(\mathrm{Sym}^2(\rho_F)(-1))^{\phi=p^{3-2k}\alpha_p^2} \cap \mathrm{Fil}^0 D_{\mathrm{crys}}(\mathrm{Sym}^2(\rho_F)(-1) = 0.$$

For such a choice of α_p, the extension E constructed above is semistable.

Proof. Let us begin with either choice of α_p. We apply Kisin's lemma (Lemma 3.3.1.2) to the lattice $\mathcal{L}_3$. As in the proof of Lemma 3.5.2.4, this shows that

$$D_{\mathrm{crys}}(\overline{\mathcal{L}})^{\phi=p^{-(k-1)}\alpha_p^2} \neq 0,$$

and hence that

$$D_{\mathrm{crys}}(E)^{\phi=p^{-(k-1)}\alpha_p^2} \neq 0.$$

Now consider the twist $E(k-2)$. This is an extension

$$0 \to \rho_F((k-2)/2) \to E(k-2) \to \mathrm{Ad}^2 \rho_F(k-2) \to 0.$$

We wish to apply Lemma 3.3.1.1 to $E(k-2)$. The representation $\rho_F((k-2)/2)$ has Hodge–Tate weights which are strictly negative, and $\mathrm{Fil}^0 \mathrm{Ad}^2 \rho_F(k-2)$ is 2-dimensional.

In fact, let us write

$$\mathrm{Ad}^2 \rho_F(k-2) = \mathrm{Sym}^2(\rho_F)(-1).$$

Let w_1, w_2 be a basis of $D_{\mathrm{crys}}(\rho_F)$ such that $\mathrm{Fil}^1(D_{\mathrm{crys}}(\rho_F))$ is generated by w_1. Write

$$w_{11} = w_1 \otimes w_1[-1], \qquad w_{12} = w_1 \otimes w_2[-1], \qquad w_{22} = w_2 \otimes w_2[-1]$$

for the corresponding basis of

$$D_{\mathrm{crys}}(\mathrm{Sym}^2(\rho_F)(-1)) = \mathrm{Sym}^2(D_{\mathrm{crys}}(\rho_F))[-1].$$

Then

$$\mathrm{Fil}^0 D_{\mathrm{crys}}(\mathrm{Sym}^2(\rho_F)(-1))$$

is generated by w_{11} and w_{12}.

Now we know from above that the image of $D_{\mathrm{crys}}(E(-1))^{\phi=p^{3-2k}\alpha_p^2}$ in

$$D_{\mathrm{crys}}(\mathrm{Sym}^2(\rho_F)(-1)) = D_{\mathrm{dR}}(\mathrm{Sym}^2(\rho_F)(-1))$$

is nontrivial. Call this image D'. It is equal to

$$D_{\mathrm{crys}}(\mathrm{Sym}^2(\rho_F)(-1))^{\phi=p^{3-2k}\alpha_p^2}.$$

Write $aw_1 + bw_2$, for some $a, b \in \overline{\mathbb{Q}}_p$, for a nonzero element in $D_{\mathrm{crys}}(\rho_F)^{\phi=p^{-(k-1)}\alpha_p}$. Then D' is spanned by

$$a^2 w_{11} + 2ab w_{12} + b^2 w_{22}.$$

If $D' \cap \mathrm{Fil}^0 D_{\mathrm{crys}}(\mathrm{Sym}^2(\rho_F)(-1))$ were nontrivial, then this would force $b = 0$, and so w_1 would be an eigenvector for the crystalline Frobenius for $D_{\mathrm{crys}}(\rho_F)$ with eigenvalue $p^{-(k-1)}\alpha_p$. If this is the case, then we make the same construction of this extension E but with the roots α_p and $p^{k-1}\alpha_p^{-1}$ switched. Then the above argument shows instead that the other eigenvector, call it w, for the crystalline Frobenius in $D_{\mathrm{crys}}(\rho_F)$ would have $w^2[-1] \in D'$. But writing $w = a'w_1 + b'w_2$ with $a', b' \in \overline{\mathbb{Q}}_p$, we necessarily have $b' \neq 0$ (for otherwise w would be a multiple of w_1, a contradiction to the fact that the Hecke polynomial at p has distinct roots for a level 1 form). Then in this case we now have

$$D' \cap \mathrm{Fil}^0 D_{\mathrm{crys}}(\mathrm{Sym}^2(\rho_F)(-1)) = 0.$$

Thus we can apply Lemma 3.3.1.1 to show that $E(k-2)$, and hence E, is de Rham. But a de Rham extension of crystalline representations is semistable, so we are done. $\square$

Lemma 3.5.3.2. *Let V_2 be a 2-dimensional vector space with basis e_1, e_2 and*

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

be a 2 by 2 matrix acting on V in this basis. Then h acts through the matrix

$$\begin{pmatrix} h_{11}^3 & 3h_{11}^2h_{21} & 3h_{11}h_{21}^2 & h_{21}^3 \\ h_{11}^2h_{21} & 2h_{11}h_{12}h_{21}+h_{11}^2h_{22} & 2h_{11}h_{22}h_{21}+h_{12}h_{21}^2 & h_{22}h_{21}^2 \\ h_{11}h_{12}^2 & 2h_{11}h_{22}h_{12}+h_{12}^2h_{21} & 2h_{11}h_{12}h_{21}+h_{11}h_{22}^2 & h_{22}^2h_{21} \\ h_{12}^3 & 3h_{22}h_{12}^2 & 3h_{22}^2h_{12} & h_{22}^3 \end{pmatrix}$$

on the basis

$$e_1^3,\ e_1^2e_2,\ e_1e_2^2,\ e_2^3$$

of $\mathrm{Sym}^3(V)$.

Proof. This is a straightforward computation. $\square$

We are now ready to prove the main theorem of this thesis. Recall that we are writing Ad^3 for the representation $\mathrm{Sym}^3(\mathrm{Std}) \otimes \det^{-1}$ of GL_2.

Theorem 3.5.3.3. *Let F be a cuspidal holomorphic eigenform of level 1 and weight k, with p-adic Galois representation ρ_F. Then under Conjectures 3.2.1.1 and 3.2.2.1, the Bloch–Kato Selmer group*

$$H_f^1(\mathbb{Q}, (\mathrm{Ad}^3\,\rho_F)^\vee(k/2))$$

is nontrivial.

Proof. We need to construct a nontrivial class in $H_f^1(\mathbb{Q}, (\mathrm{Ad}^3\,\rho_F)^\vee(k/2))$. This is the same as constructing a nontrivial extension E' of Galois representations over $\overline{\mathbb{Q}}_p$,

$$0 \to (\mathrm{Ad}^3\,\rho_F)^\vee(k/2) \to E' \to \overline{\mathbb{Q}}_p \to 0$$

which is unramified at all primes $\ell \neq p$ and which is crystalline at p. By duality, this is the same as constructing a nontrivial extension E'',

$$0 \to \overline{\mathbb{Q}}_p \to E'' \to (\mathrm{Ad}^3\,\rho_F)(-k/2) \to 0$$

which, again, is unramified at all primes $\ell \neq p$ and crystalline at p. We will construct E'' as a twist of a subrepresentation of $\wedge^2 E$, where E is the extension discussed at the beginning of this section, and then verify that it satisfies these ramification and crystallinity properties.

Recall that the action of $g \in G_{\mathbb{Q}}$ on E has the matrix expression (3.5.3.1) in the basis $v_1, \ldots, v_5$. For $i, j \in \{1,2,3,4,5\}$, write

$$v_{ij} = v_i \wedge v_j.$$

We compute the action of g on $\wedge^2 E$ in the basis

$$v_{12},\ v_{13},\ (v_{23} - v_{14}),\ (v_{24} + v_{15}),\ v_{25},\ (2v_{23} + v_{14}),\ (v_{24} - 2v_{15}),\ v_{34},\ v_{35},\ v_{45}.$$

More precisely, we compute the first five columns of the matrix of g in this basis.

For reference, here is the matrix of g in the basis $v_1, \ldots, v_5$ of E:

$$E = \frac{1}{d(g)} \begin{pmatrix} g_{66} & -g_{67} & g'_{13} & g'_{14} & g'_{15} \\ -g_{76} & g_{77} & g'_{23} & g'_{24} & g'_{25} \\ & & g_{66}^2 & 2g_{66}g_{67} & -g_{67}^2 \\ & & g_{66}g_{76} & g_{66}g_{77} + g_{67}g_{76} & -g_{67}g_{77} \\ & & -g_{76}^2 & -2g_{76}g_{77} & g_{77}^2 \end{pmatrix}.$$

For the first column of the matrix of g on $\wedge^2 E$, we compute

$$d(g)^2 g v_{12} = (g_{66}v_1 - g_{76}v_2) \wedge (-g_{67}v_1 + g_{77}v_2) = (g_{66}g_{77} - g_{67}g_{76})v_{12} = d(g)v_{12}.$$

For the second column, we have

$$\begin{aligned} d(g)^2 g v_{13} =& (g_{66}v_1 - g_{76}v_2) \wedge (g'_{13}v_1 + g'_{23}v_2 + g_{66}^2 v_3 + g_{66}g_{76}v_4 - g_{76}^2 v_5) = \\ =& (g_{76}g'_{13} + g_{66}g'_{23})v_{12} + g_{66}v^3 - g_{66}^2 g_{76}(v_{23} - v_{14}) - g_{66}g_{76}^2(v_{24} + v_{15}) + g_{76}^3 v_{25}. \end{aligned}$$

For the third, we have

$$\begin{aligned} & d(g)^2 g(v_{23} - v_{14}) \\ & \quad = (-g_{67}v_1 + g_{77}v_2) \wedge (g'_{13}v_1 + g'_{23}v_2 + g_{66}^2 v_3 + g_{66}g_{76}v_4 - g_{76}^2 v_5) \\ & \qquad - (g_{66}v_1 - g_{76}v_2) \wedge (g'_{14}v_1 + g'_{24}v_2 + 2g_{66}g_{67}v_3 + (g_{66}g_{77} + g_{67}g_{76})v_4 - g_{77}g_{76}v_5) \\ & \quad = (-g_{77}g'_{13} - g_{67}g'_{23} - g_{76}g'_{14} - g_{66}g'_{24})v_{12} - 3g_{66}^2 g_{67}v_{13} \\ & \qquad + (g_{66}^2 g_{67} + 2g_{66}g_{67}g_{76})(v_{23} - v_{14}) + (g_{67}g_{76}^2 + 2g_{66}g_{77}g_{76})(v_{23} - v_{14}) - 3g_{77}g_{76}^2 v_{25}. \end{aligned}$$

For the fourth, we have

$$
\begin{aligned}
&d(g)^2 g(v_{24}+v_{15})\\
&\quad=(-g_{67}v_1+g_{77}v_2)\wedge(g'_{14}v_1+g'_{24}v_2+2g_{66}g_{67}v_3+(g_{66}g_{77}+g_{67}g_{76})v_4-g_{77}g_{76}v_5)\\
&\qquad+(g_{66}v_1-g_{76}v_2)\wedge(g'_{15}v_1+g'_{25}v_2-g_{67}^2v_3-g_{77}g_{67}v_4+g_{77}^2v_5)\\
&\quad=(-g_{77}g'_{14}-g_{67}g'_{24}+g_{76}g'_{15}+g_{66}g'_{25})v_{12}-3g_{66}g_{67}^2v_{13}\\
&\qquad+(g_{67}^2g_{76}+2g_{66}g_{77}g_{67})(v_{23}-v_{14})+(g_{66}g_{77}^2+2g_{77}g_{67}g_{76})(v_{23}-v_{14})-3g_{77}g_{76}^2v_{25}.
\end{aligned}
$$

Finally, for the fifth column, we have

$$
\begin{aligned}
d(g)^2gv_{25}=&(-g_{67}v_1+g_{77}v_2)\wedge(g'_{15}v_1+g'_{25}v_2-g_{67}^2v_3-g_{77}g_{67}v_4+g_{77}^2v_5)=\\
&=(-g_{67}g'_{15}-g_{77}g'_{25})v_{12}+g_{67}v^3-g_{77}g_{67}^2(v_{23}-v_{14})-g_{77}^2g_{67}(v_{24}+v_{15})+g_{77}^3v_{25}.
\end{aligned}
$$

Thus the first five columns of the matrix of g acting on $\wedge^2 E$ begin with the matrix

$$
\frac{1}{d(g)^2}\begin{pmatrix}
d(g) & c_1(g) & c_2(g) & c_3(g) & c_4(g)\\
 & g_{66}^3 & -3g_{66}^2g_{67} & -3g_{66}g_{67}^2 & g_{67}^3\\
 & -g_{66}^2g_{76} & g_{66}^2g_{67}+2g_{66}g_{67}g_{76} & g_{67}^2g_{76}+2g_{66}g_{77}g_{67} & -g_{77}g_{67}^2\\
 & -g_{66}g_{76}^2 & g_{67}g_{76}^2+2g_{66}g_{77}g_{76} & g_{66}g_{77}^2+2g_{77}g_{67}g_{76} & -g_{77}^2g_{67}\\
 & g_{76}^3 & -3g_{77}g_{76}^2 & -3g_{77}g_{76}^2 & g_{77}^3
\end{pmatrix},
$$

and the rest of the entries of these first five columns are zero; here we write

$$
\begin{aligned}
c_1(g)&=g_{76}g'_{13}+g_{66}g'_{23},\\
c_2(g)&=-g_{77}g'_{13}-g_{67}g'_{23}-g_{76}g'_{14}-g_{66}g'_{24},\\
c_3(g)&=-g_{77}g'_{14}-g_{67}g'_{24}+g_{76}g'_{15}+g_{66}g'_{25},\\
c_4(g)&=-g_{77}g'_{15}-g_{67}g'_{25}.
\end{aligned}
$$

Now by Lemma 3.5.3.2, the matrix above, when conjugated by

$$\mathrm{diag}(1,1,1,-1,-1),$$

gives an extension of the symmetric cube of the representation

$$g \mapsto \begin{pmatrix} g_{66} & -g_{76} \\ -g_{67} & g_{77} \end{pmatrix}$$

by $d(g) = \chi_{\text{cyc}}^{-1}(g)$. This symmetric cube representation is the symmetric cube of

$$d(g)(\rho_F(-k/2))^{\vee} = d(g)\rho_F(-(k-2)/2) = \rho_F(-k/2),$$

and therefore equals

$$(\operatorname{Sym}^3 \rho_F)(-3k/2) = (\operatorname{Ad}^3 \rho_F)(-(k+2)/2).$$

Twisting by 1 thus gives an extension E'' of $\overline{\mathbb{Q}}_p$ by $(\operatorname{Ad}^3 \rho_F)(-k/2)$. It is unramified at all $\ell \neq p$ because the original lattice $\mathcal{L}_3$ was, and if we choose the right root of the Hecke polynomial of F at p (which we assume we have done) then it is semistable at p because E is, by Lemma 3.5.3.1. We just need to verify that E'' is a nontrivial extension, and that it is crystalline at p.

To see E'' is nontrivial, assume otherwise. Then

$$c_1(g) = c_2(g) = c_3(g) = c_4(g) = 0$$

for all $g \in G_{\mathbb{Q}}$. By Lemma 3.1.2.4, we also have the relations

$$2g_{77}g'_{13} + 2g_{67}g'_{23} - g_{76}g'_{14} - g_{66}g'_{24} = 0$$

and

$$g_{77}g'_{14} - g_{67}g'_{24} - 2g_{76}g'_{15} + 2g_{66}g'_{25} = 0.$$

Altogether, this gives the following linear system of relations:

$$\begin{pmatrix} g_{76} & g_{66} & & & & \\ g_{77} & g_{67} & g_{76} & g_{66} & & \\ 2g_{77} & 2g_{67} & -g_{76} & -g_{66} & & \\ & & g_{77} & g_{67} & -g_{76} & -g_{66} \\ & & g_{77} & g_{67} & 2g_{76} & 2g_{66} \\ & & & & g_{67} & g_{77} \end{pmatrix} \begin{pmatrix} g'_{13} \\ g'_{23} \\ g'_{14} \\ g'_{24} \\ g'_{15} \\ g'_{25} \end{pmatrix} = 0.$$

A quick row reduction brings the 6 by 6 matrix in this relation to

$$\begin{pmatrix} g_{76} & g_{66} & & & & \\ g_{77} & g_{67} & & & & \\ & & -3g_{76} & -3g_{66} & & \\ & & g_{77} & g_{67} & & \\ & & & & 3g_{76} & 3g_{66} \\ & & & & g_{77} & g_{67} \end{pmatrix}.$$

The determinant of the above matrix is $-9d(g)^3 \neq 0$, and so it is invertible, forcing

$$g'_{13} = g'_{23} = g'_{14} = g'_{24} = g'_{15} = g'_{25} = 0.$$

This contradicts Lemma 3.5.2.4, proving that the extension E'' is nontrivial.

It remains to show E'' is crystalline. We already know it is semistable. As it is an extension

$$0 \to \overline{\mathbb{Q}}_p \to E'' \to (\mathrm{Ad}^3\,\rho_F)(-k/2) \to 0,$$

its crystalline Frobenius eigenvalues are

$$1,\ p^{(3k-2)/2}\alpha_p^{-3},\ p^{k/2}\alpha_p^{-1},\ p^{-(k-2)/2}\alpha_p,\ p^{-(3k-4)/2}\alpha_p^3.$$

Let N be the monodromy operator for E''. Then the relation $N\phi = p\phi N$ shows that if N is nontrivial, then we must have that one of

$$p^{(3k-2)/2}\alpha_p^{-3},\ p^{k/2}\alpha_p^{-1},\ p^{-(k-2)/2}\alpha_p,\ p^{-(3k-4)/2}\alpha_p^3$$

equals p. But this would force either

$$\alpha_p^3 \in \{p^{(3k-4)/2}, p^{(3k-2)/2}\}$$

or

$$\alpha_p \in \{p^{(k-2)/2}, p^{k/2}\}.$$

In this first case, we would then have that

$$\alpha_p = \zeta p^k p^{-2/3} \text{ or } \zeta p^k p^{-1/3},$$

where ζ is a third root of unity. This is impossible, since α_p is a p-Weil number of weight $(k-1)/2$.

Now let a_p be the pth Fourier coefficient of F. Then in the second case, we must have

$$a_p = \alpha_p + p^{k-1}\alpha_p^{-1} = p^{(k-2)/2} + p^{k/2},$$

regardless of whether α_p is $p^{(k-2)/2}$ or $p^{k/2}$. But then

$$p^{(k-2)/2} + p^{k/2} = p^{(k-1)/2}(p^{1/2} + p^{-1/2}),$$

and since $p^{1/2}+p^{-1/2} > 2$ for any prime p, this would violate Deligne's theorem that the Ramanujan conjecture holds for F. Thus we must have $N = 0$ and E'' is crystalline, as desired. $\square$

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Appendix A: Background on eigenvarieties

The purpose of this appendix is to recall the theory set up in Urban's paper [Urb11] while also correcting an error; specifically we will correct Theorem 1.4.2 there. In this theorem, Urban interprets a formula from Franke's paper [Fra98] about $(\mathfrak{g}, K)$-cohomology in terms of the cohomology of locally symmetric spaces. The formula is almost correct, except for some considerations involving disconnectedness of maximal compact subgroups at infinity. The justification of the formula given in Urban's paper is also erroneous for another reason which involves convergence of Eisenstein series. We will correct both of these aspects of this formula below. Then we will explain how this correction affects the rest of the results in Urban's paper; it does so only in a minor way.

A.1.1 Franke's formula and the cuspidal character distribution

We start by setting the stage for the formula of Franke. Let G be a reductive group over $\mathbb{Q}$ with complexified Lie algebra $\mathfrak{g}$. Let $K_{\max}$ be a maximal compact subgroup of the group $G(\mathbb{A})$ of adelic points of G, and let $K_{\max}$ factor as $K_{\max} = K_{f,\max} K_{\infty,\max}$. Let A_G be the maximal split torus in the center of G. We fix a minimal parabolic $\mathbb{Q}$-subgroup $P_{\min}$ of G.

Then for any compact open subgroup $K_f \subset K_{f,\max}$, we can consider the locally symmetric space

$$X_G(K_f) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_G(\mathbb{R})^\circ K_f K_\infty^\circ,$$

where the symbol $(\cdot)^\circ$ denotes the connected component of the identity of the group which is decorates.

Fix a Cartan subgroup $\mathfrak{t}$ in $\mathfrak{g}$. Say $\mathfrak{t}$ is contained in the complexified Lie algebra of the Levi of $P_{\min}$. Given an ordering on the roots of $\mathfrak{t}$ in $\mathfrak{g}$ compatible with $P_{\min}$ and an integral weight λ of $\mathfrak{t}$, let V_λ be the irreducible representation of $G(\mathbb{C})$ of highest weight λ. Then V_λ^M naturally defines a local system on $X_G(K_f)$ for any compact open subgroup $K_f \subset K_{f,\max}$, and these local systems are compatible under changing K_f. For any i, we can then consider the cohomology space

$$H^i(X_G(K_f), V_\lambda)$$

as well as the space $H^i(X_G, V_\lambda)$, defined by

$$H^i(X_G, V_\lambda) = \varinjlim_{K_f} H^i(X_G(K_f), V_\lambda).$$

This latter space naturally carries an action of the Hecke algebra $C_c^\infty(G(\mathbb{A}_f))$, as well as an action of the group $\pi_0(K_{\infty,\max})$ of components of $K_{\infty,\max}$; it is the action of this group of components that was overlooked in Urban's paper. Indeed, up to a central twist, the cohomology space $H^i(X_G, V_\lambda)$ is the $(\mathfrak{g}_0, K_{\infty,\max}^\circ)$-cohomology of the space $\mathcal{A}_\lambda(G) \otimes V_\lambda$; here, $\mathfrak{g}_0$ is the complexified Lie algebra of G_0 where $G = G_0 A_G$ is the Langlands decomposition of G, and $\mathcal{A}_\lambda(G)$ is the space of automorphic forms on G which are killed by a power of the annihilator of V_λ in the center of the universal enveloping algebra of $\mathfrak{g}$, as is shown in [Fra98] and [FS98]. The $(\mathfrak{g}_0, K_{\infty,\max})$-cohomology of this same space therefore computes the $\pi_0(K_{\infty,\max})$-invariants of the space $H^i(X_M, V_\lambda^M)$, and similarly for any intermediate subgroup between $K_{\infty,\max}$ and $K_{\infty,\max}^\circ$ in place of $K_{\infty,\max}$.

Let us write

$$H^i_{\mathrm{cusp}}(X_G, V_\lambda)$$

for the cuspidal cohomology; it may be defined as the image of the $(\mathfrak{m}_0, K_{\infty,\max}^\circ)$-cohomology of $\mathcal{A}_\lambda^0(G) \otimes V_\lambda$, where $\mathcal{A}_\lambda^0(G)$ is the space of cusp forms in $\mathcal{A}_\lambda(G)$.

Now let P be a standard (with respect to $P_{\min}$) parabolic subgroup of G with Langlands decomposition $P = M_0 A_P N$, and let $M = M_0 A_P$. For $f \in C_c^\infty(G(\mathbb{A}_f))$, let $f_M \in C_c^\infty(M(\mathbb{A}_f))$ be defined by

$$f_M(m) = \int_{K_{f,\max}} \int_{N(\mathbb{A}_f)} f(kmnk^{-1})\, dn\, dk, \tag{A.1.1.1}$$

where the Haar measure of $K_{f,\max}$ is 1, and the Haar measure on $N(\mathbb{A}_f)$ is normalized with respect to the Iwasawa decomposition to satisfy

$$\int_{G(\mathbb{A}_f)} f(g)\, dg = \int_{K_{f,\max}} \int_{N(\mathbb{A}_f)} \int_{M(\mathbb{A}_f)} f(mnk)\, dm\, dn\, dk.$$

Let W be the Weyl group of $\mathfrak{t}$ in $\mathfrak{g}$, and let $\mathfrak{m}$ be the complexified Lie algebra of M. We define

$$W^P = \{w \in W \mid w^{-1}\gamma > 0 \text{ for all simple roots } \gamma > 0 \text{ in } \mathfrak{m}\},$$

and

$$W^P_{\text{Eis}} = \{w \in W^P \mid (w^{-1}\gamma)|_{\mathfrak{m}} \text{ is dominant for all simple roots } \gamma \text{ in } \mathfrak{m}\}. \tag{A.1.1.2}$$

We will write ρ for half the sum of the positive roots of $\mathfrak{t}$ in $\mathfrak{g}$, and ρ_P for half the sum of the positive roots of $\mathfrak{t}$ in the complexified Lie algebra of N. We also write, for any $w \in W$ and any weight λ,

$$w * \lambda = w(\lambda + \rho) - \rho.$$

We will also write $l(w)$ for the length of a Weyl group element w. Also, we write $V_\lambda^\vee$ for the dual of V_λ.

Then we have the following theorem, which corrects Theorem 1.4.2 in [Urb11].

Theorem A.1.1.1. *Let λ be a dominant regular weight of $\mathfrak{t}$. Let K_∞ be an open subgroup of $K_{\infty,\max}$. Then for any $f \in C_c^\infty(G(\mathbb{A}_f))$, we have*

$$\operatorname{Tr}(f|H^*(X_G, V_\lambda^\vee)^{\pi_0(K_\infty)}) = \sum_P \sum_{w \in W^P_{\text{Eis}}} (-1)^{l(w)+\dim(N)} \operatorname{Tr}(f_M|H^*_{\text{cusp}}(X_M, V^{M,\vee}_{w*\lambda+2\rho_P})^{\pi_0(K_\infty \cap P(\mathbb{R}))}),$$

where V_λ^M denotes the highest weight λ representation of M, the first sum is over all standard parabolic subgroups $P = MN$ of G, and the traces are computed as the alternating sum over the degree of cohomology of traces on each cohomology space;

$$\operatorname{Tr}(f|H^*(X_G, V_\lambda^\vee)^{\pi_0(K_\infty)}) = \sum_i (-1)^i \operatorname{Tr}(f|H^i(X_G, V_\lambda^\vee)^{\pi_0(K_\infty)}),$$

and similarly for the trace on the right hand side.

Proof. This follows from the equality of formulas (1) and (2) in Section 7.7 of Franke's paper [Fra98], as well as Poincaré duality as explained in section 1.4 of [Urb11]. Note that Franke's formula involved the $(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}))$-cohomology of the discrete spectrum. But the discrete spectrum splits into the cuspidal spectrum and the residual spectrum, and as is explained in the paper of Li–Schwermer [LS04], the residual spectrum does not contribute to cohomology because λ is regular. $\square$

We remark that Urban makes the claim in [Urb11] that the Eisenstein series contributing to the cohomology of $V_\lambda^\vee$ for regular λ are in the region of convergence; this does not seem to always be true, but it does not affect this result because of results in the paper of Li–Schwermer cited above.

Having to take invariants by $\pi_0(K_\infty \cap P(\mathbb{R}))$ in this theorem means that the definition cuspidal character distribution $I_0^\dagger(\cdot, \lambda)$ from Section 4.6 of [Urb11] must be modified, and we describe how to make this modification in the next section while recalling the relevant objects.

A.1.2 The cuspidal character distribution

We continue with the setting of the previous section, and in particular we will work with our reductive group G. We must now assume $G(\mathbb{R})$ has discrete series and that G is quasisplit. Let T be a maximal torus in G. We assume our fixed minimal parabolic $P_{\min}$ contains T. We write U for the unipotent radical of $P_{\min}$, so $P_{\min} = TU$.

We now introduce weight space. Let L be a finite extension of $\mathbb{Q}_p$. When G is split over $\mathbb{Q}$ or semisimple (as is the case for all groups considered in the main body of this Chapter 2) we write

$$\mathfrak{X}(L) = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), L^\times).$$

In general, we refer to Section 4.3.2 of [Urb11] for the definition that needs to be used.

The assignment $L \mapsto \mathfrak{X}(L)$ is represented by a rigid analytic space $\mathfrak{X}$ which we call weight space. We call $\lambda \in \mathfrak{X}(L)$ *arithmetic* if it can be factored as $\lambda = \lambda^{\mathrm{alg}}\epsilon$ for some finite order character ϵ of $T(\mathbb{Z}_p)$ and an algebraic weight λ^{alg} of T.

Next, let

$$T^- = \{t \in T(\mathbb{Q}_p) \mid tU(\mathbb{Z}_p)t^{-1} \subset U(\mathbb{Z}_p)\},$$

and

$$T^{--} = \left\{ t \in T^- \;\middle|\; \bigcap_{n \geq 0} tU(\mathbb{Z}_p)t^{-1} = \{1\} \right\}.$$

We will introduce the Hecke algebras denoted $\mathcal{U}_p$ and $\mathcal{H}_p$ and the ideal $\mathcal{H}'_p$ of $\mathcal{H}_p$.

Define the Hecke algebra $\mathcal{U}_p$ by

$$\mathcal{U}_p = \mathbb{Z}_p[T^-/T(\mathbb{Z}_p)].$$

We fix a model of G over $\mathbb{Z}_p$. Let $m > 0$ be an integer, and define

$$I_m = \{g \in G(\mathbb{Z}_p) \mid (g \bmod p^m) \in P_{\min}(\mathbb{Z}/p^m\mathbb{Z})\}.$$

Then I_m is the Iwahori subgroup of depth m in $G(\mathbb{Z}_p)$ corresponding to $P_{\min}$. The algebra $\mathcal{U}_p$ then may be identified with the subalgebra of $C_c^\infty(I_m \backslash G(\mathbb{Z}_p)/I_m, \mathbb{Z}_p)$ generated by the operators

$$u_t = \frac{1}{\mathrm{Vol}(I_m)} \mathrm{char}(I_m t I_m).$$

It is commutative.

Let

$$\mathcal{H}_p = \mathcal{U}_p \otimes_{\mathbb{Z}_p} C_c^\infty(G(\mathbb{A}_f^p), \mathbb{Q}_p),$$

and let $\mathcal{H}'_p$ be the ideal in $\mathcal{H}_p$ generated by $u_t \otimes f^p$ where $t \in T^{--}$ and $f^p \in C_c^\infty(G(\mathbb{A}_f^p), \mathbb{Q}_p)$. These Hecke algebras act, for example, on admissible representations of $G(\mathbb{A}_f)$ by convolution.

Let the maximal compact subgroup $K_{f,\max}$ of $G(\mathbb{A}_f)$ factorize as $G(\mathbb{Z}_p)K^p_{f,\max}$, where $K^p_{f,\max}$ is a maximal compact subgroup of $G(\mathbb{A}_f^p)$. Fix a finite extension L of $\mathbb{Q}_p$, and let $\lambda \in \mathfrak{X}(L)$ an L-valued weight of T. There is a space $\mathcal{D}_\lambda(L)$, defined in Section 3.2.6 of [Urb11], which defines compatible local systems on $X_G(I_m K_f^p)$ for compact open subgroups $K_f^p \subset K^p_{f,\max}$. As in Chapter 4 of [Urb11], the cohomology spaces of these local systems fit together to define a space

$$H^*(X_G, \mathcal{D}_\lambda(L)).$$

Here, we are using the notation X_G in place of Urban's notation $\widetilde{S}_G$. This cohomology space is a module for $\mathcal{H}_p$ which is admissible in the sense that each element of $\mathcal{H}_p$ defines an endomorphism of finite rank. Because of how this space can be defined as the cohomology of a local system, it also has an action of $\pi_0(K_{\infty,\max})$ which commutes with that of $\mathcal{H}_p$.

We then consider the finite slope subspace

$$H^*_{\mathrm{fs}}(X_G, \mathcal{D}_\lambda(L)),$$

as defined in Section 4.3 of [Urb11]. This is also a module for $\mathcal{H}_p$ and it carries a commuting action of $\pi_0(K^G_\infty)$. We will not need the precise definitions of these spaces here. Instead, in the next section, we will recall certain results of Urban which relate these spaces to classical cohomology spaces; we use only these results Chapter 2.

Now we define the overconvergent character distributions of Urban. For K_∞ an open subgroup of $K_{\infty,\max}$, f in $\mathcal{H}'_p$, and $\lambda \in \mathfrak{X}(L)$, let

$$I^\dagger_G(f, \lambda; K_\infty) = \mathrm{Tr}(f, H^*_{\mathrm{fs}}(X_G, \mathcal{D}_\lambda(L))^{\pi_0(K_\infty)}).$$

We then define a character distribution $I^\dagger_{G,0}(\cdot, \lambda; K_\infty)$ inductively on the rank of G, similarly to Section 4.6 of [Urb11]. The definition will make use of character distributions denoted $I^\dagger_{G,M}(\cdot, \lambda; K_\infty)$ and $I^\dagger_{G,M,w}(\cdot, \lambda; K_\infty)$, where M is a Levi of a standard parabolic P in G and $w \in W^P_{\mathrm{Eis}}$ (see (A.1.1.2) for this notation).

If the rank of G is 0, we define

$$I^\dagger_{G,0}(\cdot, \lambda; K_\infty) = I^\dagger_{G,G}(\cdot, \lambda; K_\infty) = I^\dagger_{G,G,1}(\cdot, \lambda; K_\infty) = I^\dagger_G(\cdot, \lambda; K_\infty).$$

Here the right hand side defines all the terms before it.

Next, assume we have made appropriate definitions for when the rank of G is strictly less than some $r > 0$. Let $f \in \mathcal{H}'_p$. Then when the rank of G equals r, we define for M a Levi of a standard parabolic P in G and $w \in W^P_{\mathrm{Eis}}$,

$$I^\dagger_{G,M,w}(f, \lambda; K_\infty) = I^\dagger_{M,0}(f^{\mathrm{reg}}_{M,w}, w * \lambda + 2\rho_P; K_\infty \cap P(\mathbb{R})),$$

where $f_{M,w}^{\mathrm{reg}}$ is the regularized constant term of f as defined in Urban's paper; we recall that if G is split, and if $f = u_t \otimes f^p \in \mathcal{H}'_p$ and $w \in W^P$, then by definition,

$$f_{M,w}^{\mathrm{reg}} = u_{w(t),M} \otimes f_M^p,$$

where f_M^p is a constant term, defined as in (A.1.1.1) (but without a factor at p). In general (when G is not split) the regularized constant term is defined as on p. 50 of [Urb11].

Then we set

$$I_{G,M}^{\dagger}(\cdot, \lambda; K_\infty) = \sum_{w \in W_{\mathrm{Eis}}^P} (-1)^{l(w)+\dim(N)} I_{G,M,w}^{\dagger}(\cdot, \lambda; K_\infty),$$

where N is the unipotent radical of P. Finally, we set

$$I_{G,0}^{\dagger}(\cdot, \lambda; K_\infty) = I_G^{\dagger}(\cdot, \lambda; K_\infty) - \sum_{M \neq G} I_{G,M}^{\dagger}(\cdot, \lambda; K_\infty),$$

where the sum is over all standard Levi subgroups of G other than G itself.

For $\lambda = \lambda^{\mathrm{alg}}\epsilon \in \mathfrak{X}(L)$ arithmetic, with ϵ of conductor p^m, and for $f = u_t \otimes f^p \in \mathcal{H}^p$, we also define the character distribution $I_G^{\mathrm{cl}}(\cdot, \lambda, K_\infty)$ by

$$I_G^{\mathrm{cl}}(f, \lambda; K_\infty) = |\lambda^{\mathrm{alg}}(t)|^{-1} \operatorname{Tr}(f | H^*(X_G, V_{\lambda^{\mathrm{alg}}}^{\vee}(L)(\epsilon))^{\pi_0(K_\infty)}).$$

We will not need to define $V_{\lambda^{\mathrm{alg}}}^{\vee}(L)(\epsilon)$ in general here, but for $\epsilon = 1$, the local system it defines is the same as that defined by $V_{\lambda^{\mathrm{alg}}}^{\vee}$ but with coefficients in L instead of $\mathbb{C}$. (We are viewing L as a subfield of $\mathbb{C}$ here.)

We take the opportunity here to note the normalization by $|\lambda^{\mathrm{alg}}(t)|^{-1}$ that appears in this definition. This is important, as it affects computations in Chapter 2. We also similarly define $I_{G,0}^{\mathrm{cl}}(\cdot, \lambda, K_\infty)$ by

$$I_{G,0}^{\mathrm{cl}}(f, \lambda; K_\infty) = |\lambda^{\mathrm{alg}}(t)|^{-1} \operatorname{Tr}(f | H_{\mathrm{cusp}}^*(X_G, V_{\lambda^{\mathrm{alg}}}^{\vee}(L)(\epsilon))^{\pi_0(K_\infty)}).$$

Then the results of Sections 4.6 and 4.7 in Urban's paper all have $\pi_0(K_\infty)$-invariant analogues, proved in exactly the same way as described in those sections. Note that, even if one just wants to know those results for $K_\infty = K^\circ_{\infty,\max}$, it is still necessary to prove them for all K_∞ in order to apply the induction arguments needed; even though $K^\circ_{\infty,\max}$ is connected, it can be the case that $K^\circ_{\infty,\max} \cap P(\mathbb{R})$ is disconnected for a given P. In fact this happens for all proper parabolic subgroups of G when $G = \mathrm{G}_2$.

As a consequence of the results of Urban, the character distributions $I^\dagger_{G,M,w}(\cdot, \lambda; K_\infty)$ are effective finite slope character distributions, and one can apply the theory developed in Chapter 5 of Urban's paper to construct eigenvarieties for them. We explain this in more detail in the coming sections.

A.1.3 Multiplicities

We now recall some results of Urban about the overconvergent character distributions. We retain the notation of the previous section; a good deal of notation was just introduced, so we warn the reader that it may be wise to review it.

Let $w \in W$, and let $f \in \mathcal{H}_p$ with $f = u_t \otimes f^p$, where $f^p \in C^\infty_c(G(\mathbb{A}^p_f), \mathbb{Q}_p)$ and $t \in T^-$. For $\lambda = \lambda^{\mathrm{alg}}\epsilon$ an arithmetic weight of $T(\mathbb{Z}_p)$, we define

$$f^{w,\lambda} = |(w * \lambda^{\mathrm{alg}} - \lambda^{\mathrm{alg}})(t)|^{-1} u_t \otimes f^p.$$

We extend the map $f \mapsto f^{w,\lambda}$ to a $\mathbb{Q}_p$-linear automorphism of $\mathcal{H}_p$ by linearity. Then we have the following theorem of Urban.

Theorem A.1.3.1. *Fix an open subgroup K_∞ of $K_{\infty,\max}$. Then for any $f \in \mathcal{H}^p$ and $\lambda = \lambda^{\mathrm{alg}}\epsilon$ arithmetic, we have*

$$I^{\mathrm{cl}}_G(f, \lambda; K_\infty) = \sum_{w \in W} (-1)^{l(w)} I^\dagger_G(f^{w,\lambda}, w * \lambda; K_\infty).$$

Proof. This is just Theorem 4.5.4 in [Urb11], except that we are keeping track of the $\pi_0(K_{\infty,\max})$-action as well. $\square$

Next we briefly discuss p-stabilizations. Let $m > 0$ be an integer, and write

$$I'_m = \{g \in G(\mathbb{Z}_p) \mid (g \bmod p^m) \in U(\mathbb{Z}/p^m\mathbb{Z})\}.$$

This is a subgroup of the Iwahori subgroup I_m. Here we recall that $P_{\min} = TU$ where U is the unipotent radical of $P_{\min}$. Then

$$I_m/I'_m \cong T(\mathbb{Z}/p^m\mathbb{Z}).$$

Let τ be a smooth admissible representation of $G(\mathbb{A}_f)$. Assume there is some finite order character ϵ of $T(\mathbb{Z}_p)$ factoring through $T(\mathbb{Z}/p^m\mathbb{Z})$ such that $\tau^{I'_m}$ contains a vector on which I_m acts via ϵ through the quotient I_m/I'_m. Then $\tau^{I'_m} \otimes \epsilon^{-1}$ has an I_m-fixed vector and therefore receives an action of $\mathcal{H}_p$. An irreducible subquotient of $\tau^{I'_m} \otimes \epsilon^{-1}$ for this action is called a p-stabilization of τ. If τ comes as the finite part of a cohomological automorphic representation appearing in the cohomology of $V^\vee_{\lambda^{\text{alg}}}$ for some algebraic weight λ, then we say such a p-stabilization is of weight λ where $\lambda = \lambda^{\text{alg}}\epsilon$.

Let σ be an irreducible admissible $\mathcal{H}_p$-module. Then $\mathcal{U}_p$ acts through scalars on σ because $\mathcal{U}_p$ is commutative. For $t \in T^-$, let a_t be the eigenvalue corresponding to u_t acting on σ. If $a_t \neq 0$ for some (equivalently, all) t, then we say σ is of finite slope. In this case we define a character $\mu_\sigma \in X^*(T) \otimes \mathbb{Q}$ by

$$v_p(\mu_\sigma(\lambda^\vee(t))) = v_p(a_t)$$

for all algebraic cocharacters $\lambda^\vee$ of T. The character μ_σ is called the slope of σ.

For σ again an irreducible admissible $\mathcal{H}_p$-module, we have that for any $f \in \mathcal{H}_p$, the trace of f on σ is well defined. So we write in this case

$$J_\sigma(f) = \mathrm{Tr}(f|\sigma).$$

Then we have the following theorem.

Theorem A.1.3.2. *Fix a finite extension L of $\mathbb{Q}_p$. Then for any irreducible admissible finite slope $\mathcal{H}_p$-module σ, any $\lambda \in \mathfrak{X}(L)$, and any open subgroup K_∞ of $K_{\infty,\max}$, there is an integer $m^i(\sigma, \lambda; K_\infty)$ such that for any $f \in \mathcal{H}'_p$, we have*

$$\mathrm{Tr}(f|H^*_{\mathrm{fs}}(X_G, \mathcal{D}_\lambda(L))^{\pi_0(K_\infty)}) = \sum_\sigma m^i(\sigma, \lambda; K_\infty) J_\sigma(f).$$

Proof. This is Proposition 4.3.5 in [Urb11], except that once again we must take into account the action of $\pi_0(K_{\infty,\max})$ on the objects involved. $\square$

For any σ, λ, and K_∞ as in the theorem, we can define

$$m^\dagger(\sigma,\lambda;K_\infty)=\sum_i(-1)^i m^i(\sigma,\lambda;K_\infty).$$

Then we have

$$I_G^\dagger(\cdot,\lambda;K_\infty)=\sum_\sigma m^\dagger(\sigma,\lambda;K_\infty)J_\sigma.$$

We call $m^\dagger(\sigma,\lambda;K_\infty)$ the overconvergent multiplicity of σ.

Similarly we can define multiplicities

$$m_0^\dagger(\sigma,\lambda;K_\infty),\quad m^{\mathrm{cl}}(\sigma,\lambda;K_\infty),\quad m_0^{\mathrm{cl}}(\sigma,\lambda;K_\infty),$$

by the requirements that

$$\begin{aligned}
I_{G,0}^\dagger(f,\lambda;K_\infty)&=\sum_\sigma m_0^\dagger(\sigma,\lambda;K_\infty)J_\sigma(f),\\
I_G^{\mathrm{cl}}(f,\lambda;K_\infty)&=\sum_\sigma m^{\mathrm{cl}}(\sigma,\lambda;K_\infty)J_\sigma(f),\\
I_{G,0}^{\mathrm{cl}}(f,\lambda;K_\infty)&=\sum_\sigma m_0^{\mathrm{cl}}(\sigma,\lambda;K_\infty)J_\sigma(f).
\end{aligned}$$

To see that the first of these multiplicities is well defined, we combine the above theorem with the relation (when G is split)

$$\mathrm{Tr}(f_{M,w}^{\mathrm{reg}}|\sigma_M)=\mathrm{Tr}(f|\,\mathrm{Ind}_{M(\mathbb{A}_f^p)}^{G(\mathbb{A}_f^p)}(\sigma_M)),$$

where M is a the Levi of a standard parabolic P, $w\in W^P$, $f=u_t\otimes f^p\in\mathcal{H}_p'$ with $t\in T^-$, and σ_M is a p-stabilization of a smooth admissible representation of $M(\mathbb{A}_f)$. This shows (by induction) that the distributions $I_{G,M,w}^\dagger(\cdot,\lambda;K_\infty)$ satisfy the conclusion of Theorem A.1.3.2 because $I_{M,0}^\dagger(\cdot,\lambda;K_\infty\cap P(\mathbb{R}))$ does. The other two multiplicities are well defined by the admissibility of the cohomology spaces used to define them.

We note that if G is not split, the same argument still works because the definition of $f_{M,w}^{\mathrm{reg}}$ in that case differs from ours by a character of $T(\mathbb{Q}_p)$.

We now give a few tools to compute overconvergent multiplicities in terms of classical ones. The first of these tools can be used to simplify computations involving Theorem A.1.3.1. Let $X^*(T)$ be the group of algebraic characters of T. Let us write $X^*(T)_{\mathbb{Q},+}$ for the set of rational characters whose projection onto $X^*(T/Z_G)$, where Z_G is the center of G, is in the $\mathbb{Q}_{\geq 0}$-span of the simple roots of T in $X^*(T) \otimes \mathbb{Q}$. Then we have the following theorem.

Theorem A.1.3.3. *Let $\lambda = \lambda^{\mathrm{alg}}\epsilon$ be an arithmetic weight and let K'_∞ be an open subgroup of $K_{\infty,\max}$. Let σ be an irreducible admissible $\mathcal{H}_p$-module with slope μ. Assume $\mu \notin X^*(T)_{\mathbb{Q},+}$. Then $m^\dagger(\sigma, \lambda; K'_\infty) = 0$.*

Proof. As explained in [Urb11], this just follows from the fact that $H^*_{\mathrm{fs}}(X_G, \mathcal{D}_\lambda(L))$ has an integral structure. □

If σ is an irreducible admissible $\mathcal{H}_p$-module with slope μ, and $\lambda = \lambda^{\mathrm{alg}}\epsilon$ is an arithmetic weight of T, then the twisted distribution $\sigma^{w,\lambda}$ defined via

$$f^{w,\lambda} \mapsto J_\sigma(f)$$

corresponds to a module of slope $\mu + w * \lambda^{\mathrm{alg}} - \lambda^{\mathrm{alg}}$. If for all $w \neq 1$, we have

$$(\mu + w * \lambda^{\mathrm{alg}} - \lambda^{\mathrm{alg}}) \notin X^*(T)_{\mathbb{Q},+},$$

then we say μ is noncritical with respect to λ. In this case, by Theorems A.1.3.3 and A.1.3.1, J_σ is a constituent of $I^{\mathrm{cl}}_G(f, \lambda; K_\infty)$ if and only if it is a constituent of $I^\dagger_G(f, \lambda; K_\infty)$.

Finally, we state one last result about multiplicities. Let L be a finite extension of $\mathbb{Q}_p$. We say a $\mathbb{Q}_p$-linear map $J : \mathcal{H}'_p \to L$ is a finite slope character distribution if for all finite slope $\mathcal{H}_p$-modules σ, there is an integer $m_J(\sigma)$ such that

$$J(f) = \sum_\sigma m_J(\sigma) J_\sigma(f),$$

for all $f \in \mathcal{H}'_p$, and such that for any f, there are only finitely many σ's for which $J_\sigma(f) \neq 0$ (so that the sum makes sense). Moreover, J is called effective if all of the integers $m_J(\sigma)$ are nonnegative.

Theorem A.1.3.4. *Let $q(G) = (1/2)\dim(G(\mathbb{R})/K_{\infty,\max})$. Then for any standard parabolic $P = MN$ in G, any $w \in W^P_{\mathrm{Eis}}$, any arithmetic weight λ of $T(\mathbb{Z}_p)$, and any open compact subgroup K_∞ of $K_{\infty,\max}$, the character distribution*

$$(-1)^{q(G)} I^{\dagger}_{G,M,w}(\cdot, \lambda; K_\infty)$$

is an effective finite slope character distribution.

Proof. This is Corollary 4.7.4 in [Urb11] except, as usual, we must keep track of the action of $\pi_0(K'_\infty)$. □

A.1.4 Eigenvarieties

We retain the setting of the previous section, and fix an open subgroup $K_\infty \subset K_{\infty,\max}$ and an open compact subgroup $K^p_f \subset K^p_{f,\max}$. Let S be the smallest set of primes (not including p or ∞) away from which K^p_f is hyperspecial. We work with the weight space $\mathfrak{X}$ in this section. Recall this is a rigid analytic space over $\mathbb{Q}_p$. For any rigid analytic space $\mathfrak{Z}$, let $\mathcal{O}(\mathfrak{Z})$ be the ring of global analytic functions on $\mathfrak{Z}$. So $\mathcal{O}(\mathfrak{X})$ is the ring of analytic functions on $\mathfrak{X}$.

Definition A.1.4.1. A $\mathbb{Q}_p$-linear map $J : \mathcal{H}'_p \to \mathcal{O}(\mathfrak{X})$ is called an $\mathfrak{X}$-family of effective finite slope character distributions if for every $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, the composition $J_\lambda = \lambda \circ J : \mathcal{H}'_p \to \overline{\mathbb{Q}}_p$ is a finite slope character distribution.

By construction, when multiplied by the appropriate sign as in Theorem A.1.3.4, the overconvergent character distributions $I^{\dagger}_{G,0}(\cdot, \lambda, K'_\infty)$ and $I^{\dagger}_{G,M,w}(\cdot, \lambda, K'_\infty)$ fit into $\mathfrak{X}$-families of effective finite slope character distributions, denoted respectively by $I^{\dagger}_{G,0}(\cdot; K'_\infty)$ and $I^{\dagger}_{G,M,w}(\cdot, K'_\infty)$.

Now let

$$R_{S,p} = \mathcal{U}_p \otimes_{\mathbb{Z}_p} C^\infty_c(K^{S\cup\{p\}}_{\max} \backslash G(\mathbb{A}^{S\cup\{p\}}_f)/K^{S\cup\{p\}}_{\max}, \mathbb{Z}_p).$$

Also let

$$\mathcal{H}_p(K^p_f) = \mathcal{U}_p \otimes_{\mathbb{Z}_p} C^\infty_c(K^p_f \backslash G(\mathbb{A}^{S\cup\{p\}}_f)/K^p_f, \mathbb{Q}_p).$$

Then $R_{S,p} \subset \mathcal{H}_p(K^p_f) \subset \mathcal{H}_p$. A p-stabilization of an irreducible smooth admissible representation τ of $G(\mathbb{A}_f)$, such that τ has a fixed vector by K^p_f, will induce a $\mathbb{Z}_p$-algebra homomorphism $R_{S,p} \to \overline{\mathbb{Q}}_p$.

Now let $\widehat{R}_{S,p}$ be the p-adic completion of $R_{S,p}$, and let

$$\tilde{R}_{S,p} = \widehat{R}_{S,p}[u_t,\ t \in T^-].$$

Then the characters of $\widehat{R}_{S,p}$ that extend to characters of $\tilde{R}_{S,p}$ are those of finite slope. The algebra $\tilde{R}_{S,p}$ is given the topology so that $\widehat{R}_{S,p}$ is open in $\tilde{R}_{S,p}$ with its p-adic topology.

For L a finite extension of $\mathbb{Q}_p$, define $\mathfrak{R}_{S,p}(L)$ to be the set of continuous $\mathbb{Z}_p$-algebra homomorphisms $\tilde{R}_{S,p} \to L$. Actually, this definition makes sense for any algebraic extension L of $\mathbb{Q}_p$, and so we can consider $\mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p)$. We give $\mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p)$ a topology by the metric

$$|\theta - \theta'| = \sup_{f \in \widehat{R}_{S,p}} |\theta(f) - \theta'(f)|, \qquad \theta, \theta' \in \mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p).$$

Given J an $\mathfrak{X}$-family of effective finite slope character distributions, in Section 5 of [Urb11], Urban constructs eigenvarieties $\mathfrak{E}_{J,K_f^p}$ as subsets of $\mathfrak{X}(\overline{\mathbb{Q}}_p) \times \mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p)$. We do not recall here the precise construction, but instead we will state the properties of them we will need. Note that points in $\mathfrak{X}(\overline{\mathbb{Q}}_p) \times \mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p)$ are just pairs (θ, λ) where θ is continuous $\overline{\mathbb{Q}}_p$-valued character of $\tilde{R}_{S,p}$ and λ is a weight in $\mathfrak{X}(\overline{\mathbb{Q}}_p)$. The eigenvarieties $\mathfrak{E}_{J,K_f^p}$ are ringed spaces whose underlying topological spaces are subspaces of $\mathfrak{X}(\overline{\mathbb{Q}}_p) \times \mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p)$. One can then give them the structure of a rigid analytic variety.

Note that $\mathfrak{E}_{J,K_f^p}$ has an obvious map to $\mathfrak{X}$ given on $\overline{\mathbb{Q}}_p$-points by the projection $\mathfrak{X}(\overline{\mathbb{Q}}_p) \times \mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p) \to \mathfrak{X}(\overline{\mathbb{Q}}_p)$. This is a map of rigid analytic spaces.

Now given θ a character of $\tilde{R}_{S,p}$ and $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, let $m_J(\theta, \lambda)$ be the integer such that

$$J_\lambda(f) = \sum_\theta m_J(\theta, \lambda)\theta(f).$$

Here the sum is over all such θ. Then $m_J(\theta, \lambda) \geq 0$ because J is effective. We have the following theorem.

Theorem A.1.4.2. *Let J be an $\mathfrak{X}$-family of effective finite slope character distributions. Then the eigenvariety $\mathfrak{E}_{J,K_f^p}$ is an equidimensional rigid analytic space over $\mathbb{Q}_p$ of dimension* $\dim(\mathfrak{X})$. *It is locally finite over $\mathfrak{X}$. Furthermore, $(\theta, \lambda) \in \mathfrak{X}(\overline{\mathbb{Q}}_p) \times \mathfrak{R}_{S,p}(\overline{\mathbb{Q}}_p)$ is a point of $\mathfrak{E}_{J,K_f^p}(\overline{\mathbb{Q}}_p)$ if and only if*

$m_J(\theta, \lambda) > 0$.

Proof. This is part of Theorem 5.3.7 in [Urb11]. The local finiteness follows from point (ii) of that theorem, as the spectral varieties $\mathfrak{Z}_J(f)$ there are locally finite over weight space. $\square$

This theorem is a result about analytic families of characters of $\tilde{R}_{S,p}$, and we would like to upgrade it to a theorem about character distributions $\mathcal{H}_p(K_f^p) \to \overline{\mathbb{Q}}_p$. We can do this at the expense of shrinking weight space. The resulting family will be a rigid analytic space which is finite over the part of the eigenvariety which sits above an open subdomain in $\mathfrak{X}$.

Let σ be an irreducible admissible finite slope representation of $\mathcal{H}_p(K_f^p)$, and define $m_J(\sigma, \lambda)$ to be the integer such that for any $f \in \mathcal{H}_p(K_f^p)$, we have

$$J_\lambda(f) = \sum_\sigma m_J(\sigma, \lambda) \operatorname{Tr}(f|\sigma).$$

Here the sum is over all such σ. Then we have the following theorem.

Theorem A.1.4.3. *Let J be an $\mathfrak{X}$-family of effective finite slope character distributions. Let $\lambda_0 \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ and let σ_0 be an irreducible admissible finite slope representation of $\mathcal{H}_p(K_f^p)$. Given any irreducible admissible finite slope representation σ of $\mathcal{H}_p(K_f^p)$, write θ_σ for its restriction to $R_{S,p}$.*

Assume $m_J(\sigma_0, \lambda_0) > 0$. Then there are

- *an open affinoid subdomain $\mathfrak{U} \subset \mathfrak{X}$,*
- *an open affinoid subdomain $\mathfrak{W} \subset \mathfrak{E}_{J,K_f^p}$ which is finite and generically flat over $\mathfrak{U}$,*
- *a finite flat covering $\mathfrak{V}$ of $\mathfrak{W}$,*
- *a point $x_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ above $(\theta_0, \lambda_0) \in \mathfrak{E}_{J,K_f^p}$,*
- *for all $x \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$, a nonempty finite set Π_x of irreducible admissible finite slope representations of $\mathcal{H}_p(K_f^p)$,*
- *for every $\sigma \in \Pi_x$, an integer $m_x(\sigma) > 0$,*
- *a nontrivial $\mathbb{Q}_p$-linear map $I_{\mathfrak{V}} : \mathcal{H}_p(K_f^p) \to \mathcal{O}(\mathfrak{V})$,*

satisfying the following properties:

- *For every* $x \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$*, if we write* λ_x *for the image of* x *in* $\mathfrak{X}(\overline{\mathbb{Q}}_p)$*, then we have that if* $\sigma \in \Pi_x$*, then* $m_J(\sigma, \lambda_x) > 0$*;*

- *For all* $x \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ *we have*
$$x \circ I_{\mathfrak{V}} = \sum_{\sigma \in \Pi_x} m_x(\sigma);$$

- *We have that* Π_x *consists of one representation* σ_x *and* $m_J(\sigma) = m_J(\lambda_x, \sigma_x)$ *is constant for all* x *in a Zariski dense subset of* $\mathfrak{V}(\overline{\mathbb{Q}}_p)$*;*

- *We have that* $\sigma_0 \in \Pi_{x_0}$*;*

- *If* $\theta_{\mathfrak{W}} : R_{S,p} \to \mathcal{O}(\mathfrak{W})$ *is the character corresponding to* $\mathfrak{W}$*, then for any* $f \in R_{S,p}$ *and* $f' \in \mathcal{H}_p(K_f^p)$*, we have*
$$I_{\mathfrak{V}}(ff') = \theta_{\mathfrak{W}}(f) I_{\mathfrak{W}}(f').$$

Proof. This is Proposition 5.3.10 in [Urb11]. □

Applied to the cuspidal overconvergent distribution $(-1)^{q(G)} I^{\dagger}_{G,0}(\cdot, \lambda; K_\infty)$, one can deduce the following.

Theorem A.1.4.4. *Let* $\lambda_0 \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ *be arithmetic and let* σ_0 *be an irreducible admissible finite slope representation of* $\mathcal{H}_p(K_f^p)$*. Assume* $m_0^{\dagger}(\sigma_0, \lambda_0) \neq 0$*. Then there are*

- *an open affinoid subdomain* $\mathfrak{U} \subset \mathfrak{X}$*,*

- *a finite cover* $\mathbf{w} : \mathfrak{V} \to \mathfrak{U}$*,*

- *a point* $y_0 \in \mathfrak{V}(\overline{\mathbb{Q}}_p)$ *with* $\mathbf{w}(y_0) = \lambda_0$*,*

- *a Zariski dense subset* $\Sigma \subset \mathfrak{V}(\overline{\mathbb{Q}}_p)$ *with* $\mathbf{w}(y)$ *arithmetic regular for every* $y \in \Sigma$*,*

- *for each* $y \in \Sigma$*, a nonempty finite set* Π_y *of finite slope p-stabilizations of irreducible, cohomological, cuspidal automorphic representations of* G *of weight* $\mathbf{w}(y)$ *and level away from* p *given by* K_f^p*,*

- *a* $\mathbb{Z}_p$*-algebra homomorphism* $\theta_{\mathfrak{V}} : R_{S,p} \to \mathcal{O}(\mathfrak{V})$*,*

- *a nontrivial $\mathbb{Q}_p$-linear map $I_{\mathfrak{Y}} : \mathcal{H}_p(K_f^p) \to \mathcal{O}(\mathfrak{Y})$,*

satisfying the following properties:

- *The specialization of $\theta_{\mathfrak{Y}}$ at the point y_0 is θ_0;*
- *The representation σ_0 is an irreducible component of the specialization of $I_{\mathfrak{Y}}$ at y_0;*
- *For each $y \in \Sigma$ and each $\sigma \in \Pi_y$, the specialization of $\theta_{\mathfrak{Y}}$ at y occurs in the representation of $R_{S,p}$ on $\sigma^{K_f^p}$;*
- *For each $y \in \Sigma$, the specialization I_y of $I_{\mathfrak{Y}}$ at y satisfies*

$$I_y(f) = \sum_{\sigma \in \Pi_y} m^{\mathrm{cl}}(\sigma, \mathbf{w}(y); K_\infty) \operatorname{Tr}(f|\sigma),$$

  for $f \in \mathcal{H}_p(K_f^p)$;

- *The set $\mathbf{w}(\Sigma)$ contains all sufficiently regular dominant algebraic weights in $\mathfrak{X}(\overline{\mathbb{Q}}_p)$;*
- *There is a Zariski closed subset of $\mathfrak{U}$ such that for $y \in \Sigma$ with $\mathbf{w}(y)$ not in this closed subset, Π_y only contains one representation.*

Proof. This is Theorem 5.4.4 in [Urb11], except that the last two points in the statement is refined; the second-to-last statement follows from the proof this theorem in [Urb11], stating that we can consider Σ to be the set of classical points with noncritical weights. The last point follows from the proof of Proposition 5.3.10 (b) in [Urb11]. $\square$