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NONLINEAR WAVES

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Sapporo, July 10-21, 1995

Edited by R. Agemi, Y. Giga and T. Ozawa

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NONLINEAR WAVES

Proceedings of the Fourth MSJ International Research Institute

Sapporo

July 10-21, 1995

VOLUME I

Edited by

Rentaro AGEMI

Yoshikazu GIGA

Tohru OZAWA

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PREFACE

The Fourth MSJ International Research Institute of the Mathematical Society of Japan was held in Sapporo from July 10 to 21, 1995. This insitute featured 4 expository lectures, 18 invited lectures, and 57 contribured lectures. The programme covered a broad range of topics in mathematical analysis of nonlinear waves. The organizing committee consisted of R. Agemi, Y. Giga, J. Ginibre, S. Klainreman, K. Okamoto, and T. Ozawa.

This volume, together with the next, is a preliminary edition of the Proceedings of the Fourth MSJ International Research Institute on Nonlinear Waves. The authorized version will be published from GAKUTO International Series, Mathematical Sciences and Applications, GAKKŌTOSHO CO., LTD., Tokyo.

The organizing committee would like to thank Professor Hiroshi Fujita, Professor Nobuyuki Kenmochi and Professor Mikio Namiki for their valuable advices. The organizing committee would like thank also Miss Hisako Iwai and the staff of Sapporo International Communication Plaza Foundation for their help with the details of the meeting.

The organizing committee gratefully acknowledges financial support from Commemorative Association for the Japan World Exposition (1970); Hokkaido University; Inoue Foundation for Sciences; Japan Society of Promotion of Sciences; Ministry of Education, Science and Culture, Japan; Sapporo International Communication Plaza Foundation; THE SUHARA MEMORIAL FOUNDATION; Sumitomo Foundation; Yamada Science Foundation. Thanks are also due to all contributors for their submission of original papers and to the referees for their efficient work.

Sapporo, March 1996

Rentaro AGEMI

Yoshikazu GIGA

Tohru OZAWA

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International Research Institute on Nonlinear Waves
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 Congratulatory address by K.Hikichi, Dean of Faculty of Science, Hokkaido University
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 An Introduction to Nonlinear Schrödinger Equations
 10:40-12:00 W.Strauss(Brown)
 Expository Lectures on Nonlinear Waves
- Session A** 2:00-3:00 M.Flato(Bourgogne)
 Nonlinear Relativistic Wave Equations in General Dimensions
 3:10-4:10 Y.Tsutsumi(University of Tokyo)
 Nonlinear Klein-Gordon Equations and Hörmander's Conjectures
 4:30-5:30 A.Inoue(Tokyo Institute of Technology)
 On a Construction of the Fundamental Solution for the Free Weyl Equation by Hamiltonian Path-Integral Method
- Session B** 2:00-3:00 F.Asakura(Osaka Electro-Communication University)
 Global Solutions with a Single Transonic Shock Wave for Quasi-linear Hyperbolic Systems
 3:10-4:10 P.T.Kan(Indiana)
 Hyperbolic Systems of Conservation Laws with Umbilic Degeneracy and Two Dimensional Transonic Gas Systems
 4:30-5:30 Li Tatsien(Fudan)
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- July 11** 9:00-10:20 J.Ginibre(Orsay)
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- Session A** 2:00-2:50 V.Georgiev(Sophia)
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 3:00-3:50 W.Dan(University of Tsukuba)
 Low-Frequency Asymptotic Expansion for Second Order Elliptic Equation Systems in two Dimensions
 4:00-4:50 K.Tsutaya(Hokkaido University)
 Global Existence of Small Amplitude Solutions for the Klein-Gordon-Zakharov Equations
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- Session B** 2:00-2:50 K.Taniguchi(University of Osaka Prefecture)
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 3:00-3:50 L.Vega(Pais Vasco)
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- 3:40-4:20 T.Matsuyama(Hakodate National College of Technology)
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- 4:30-5:10 H.Uesaka(Nihon University)
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- 5:20-6:00 K.Furuya(Ochanomizu University)
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- Session B** 2:00-2:50 J.C.Saut(Orsay)
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- 3:00-3:50 Y.Shibata(University of Tsukuba)
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- 4:00-4:50 C.M.Dafermos(Brown)
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- July 13** 9:00-9:50 J.Hoppe(ETH)
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- 11:10-12:00 N.Hayashi(Science University of Tokyo)
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Nonexistence of Global Solutions to Semilinear Wave Equations

- 5:20-6:00 T.Motai(Tokyo University of Foreign Studies)
Asymptotics of Energy for Wave Equations with a Nonlinear
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- Session B 2:00-2:30 Zhang Jian(University of Tokyo)
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- 2:50-3:40 C.K.Lin(National Cheng Kung University)
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- 11:10-12:00 H.Matano(University of Tokyo)
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- Session B 2:00-2:40 M.Kuwamura(Wakayama University)
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- 2:50-3:30 K.M.Hui(Academia Sinica)
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Equation
- 3:40-4:20 C.Dohmen(University of Bonn)
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- 4:30-5:10 T.Tassa(UCLA)
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- 5:20-6:00 S.T.Kusimov(Ufa State Aviation Technical University)
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Radiation Determination: Modern Methods and Means
- July 15 Open Symposium for Citizens at Sapporo International Communication
Plaza Foundation - Frontiers of Mathematical Sciences -
M.Tsutsumi(Waseda University)

- Nonlinearity and Wave
A.Matsumura(Osaka University)
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C.Dafermos(Brown)
Exploration in Mathematics
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- Session A** 2:00-2:40 H.Kozono(Nagoya University)
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4:30-5:10 M.Mei(Kanazawa University)
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- Session B** 2:00-2:40 N.Koiso(Osaka University)
Vortex Filament Equation and Semilinear Schrödinger Equation
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- July 19** 9:00-10:20 T.Miyakawa(Kyushu University)
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Weak Solutions for the Evolution Problems of Harmonic Maps on Nondecreasing Domains
3:40-4:20 F.Nakajima(Iwate University)
Nonlinear Mathieu's Equations
4:30-5:10 Hi Jun Choe(POSTECH)
Removable Singularity for Stationary Navier-Stokes System
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Partial Regularity for a Minimum Problem with Free Boundary
- Session B** 2:00-2:50 A.S.Fokas(Loughborough)
A New Class of Physically Important Integrable Equations

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A Note on Uniqueness for the Classical Solution of the MHD Equations in R^3
- 4:30-5:10 H.Morimoto(Meiji University)
A Remark on the Existence of the Navier-Stokes Flow with Non-Vanishing Outflow Condition
- 5:20-6:00 S.Takahashi(Tokyo Denki University)
On Decay Rate Estimates for the Navier-Stokes Equations
- Session B** 2:00-2:30 K.Matsuuchi(University of Tsukuba)
Instability of Pipe-Poiseuille Flow and the Global Nature of Disturbances
- 2:40-3:10 Y.Mizuta(Hokkaido University)
Analysis on Large Free Surface Deformation of Magnetic Fluid
- 3:20-3:50 D.Summers(Kyoto University)
Nonlinear Waves and Chaos in the Pierce Beam-Plasma System
- 4:00-4:30 G.A.Barsegian(Academy of Sciences of Armenia)
The Theory of G -Lines and Its Applications in Physics
- 4:40-5:10 S.Makhmutov(Hokkaido University)
Estimates of Spherical Derivative
- July 21** 9:00-9:50 S.Klainerman(Princeton)
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- 10:10-11:00 M.Tsutsumi(Waseda)
Regional Blow-up of Solutions to the Initial-Boundary Value Problem
- 11:10-12:00 H.Okamoto(Kyoto University)
New Family of Exact Solutions of the Navier-Stokes Equations
- Closing address by Y. Giga, Organizer

Global Solutions with a Single Transonic Shock Wave for Quasilinear Hyperbolic Systems

FUMIOKI ASAKURA

Abstract: We shall study global solutions containing a single transonic shock wave for general quasilinear hyperbolic system $U_t + F_x = G$. The presence of G brings about secondary waves and the amount of such wave is finite along characteristics whose speeds are away from zero. We shall show that global in time solutions exist provided $T.V.U_0$ and $\|G\|_1$: the L^1 -norm in the space variable are sufficiently small and the total amount of secondary wave along the transonic characteristic is uniformly bounded.

1. Introduction

We study the Cauchy problem for a general quasilinear hyperbolic system of the form:

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial x}F(U) = G(x, U), \quad (x, t) \in R \times R_+, \quad (1)$$

$$U(x, 0) = U_0(x), \quad x \in R. \quad (2)$$

Here U is a vector function which takes on values in an open set $\Omega \subset R^n$; F is a smooth map from Ω to R^n and G from $R \times \Omega$ to R^n . We assume that the system (1) is strictly hyperbolic, which says that the Jacobian matrix $F'(U)$ has n real distinct eigenvalues:

$$\lambda_1(U) < \lambda_2(U) < \cdots < \lambda_n(U), \quad U \in \Omega.$$

We also assume that each characteristic field is genuinely nonlinear:

$$R_j \cdot \text{grad } \lambda_j \neq 0 \quad \text{for } U \in \Omega, 1 \leq j \leq n$$

where $R_j(U)$ denote the right eigenvector of $F'(U)$ corresponding to $\lambda_j(U)$. Since solutions to these quasi-linear hyperbolic systems become singular in general after a finite time, we shall discuss weak solutions.

When the characteristic speeds are away from zero, a solution exists provided $T.V.U_0$ and the L^1 -norm of the quantity:

$$G(x) = \max\{|G(x, U)| + |G'_U(x, U)|; U \text{ in a small neighborhood of } U_0(x)\} \quad (3)$$

are sufficiently small and the solution converges, as $t \rightarrow \infty$, to the superposition of shock waves, rarefaction waves, and steady state solutions whose strengths and speeds depend only on the data at infinity (Liu [7]). If one of the characteristic speed can be zero (the flow is *transonic*) new phenomena occur; in the transonic 1-D flow along the contracting duct, a standing shock wave is dynamically unstable, and stable along the expanding duct (Liu [8]).

In this paper we shall discuss an intrinsic approach to the above phenomena. We assume that the p -th characteristic speed alone can be zero: there exists $\delta > 0$ such that

$$|\lambda_j(U)| \geq \delta, \quad j \neq p, \quad U \in \Omega \quad \text{and} \quad \mathcal{N}_p = \{U \in \Omega; \lambda_p(U) = 0\} \neq \phi. \quad (4)$$

Solutions to (1) are called steady state solutions if they do not depend on t :

$$\frac{d}{dx}F(U) = G(x, U), \quad x \in \mathbb{R}. \quad (5)$$

We say that $\bar{U}(x)$ defined by

$$\bar{U}(x) = \begin{cases} \bar{U}_1(x), & x < 0 \\ \bar{U}_2(x), & x > 0 \end{cases} \quad (6)$$

is a p -standing shock wave, if: (1) $\bar{U}_1(x)$ and $\bar{U}_2(x)$ are steady state solutions, (2) The Rankine-Hugoniot condition and the Lax entropy condition are satisfied along $x = 0$ with speed 0.

Let $\bar{U}(x)$ be a standing shock wave whose strength is denote by α_* . We shall study the global existence of solutions to (1) containing a single transonic shock wave, whose initial data are the perturbation of $\bar{U}(x)$ in the total variation norm:

$$U_0(x) = \begin{cases} \tilde{U}_1(x), & x < 0 \\ \tilde{U}_2(x), & x > 0, \end{cases} \quad (7)$$

$$\sum_{j=1,2} T.V. (\tilde{U}_j(x) - \bar{U}_j(x)) \ll \alpha_*. \quad (8)$$

In this case, the amount of secondary wave produced by j -th waves ($j \neq p$) is estimated in the same way as Liu [7] and those produced by p -th waves only remain. Let h denotes the mesh length, G the local maximum of $G(x)$, and σ the speed of the transonic shock wave. Our local interaction estimates indicate that the total amount of the secondary wave produced by p -th waves is

$$\sum \frac{|\sigma|Gh}{\alpha_*}, \quad (9)$$

where the summation runs along the transonic shock wave. Thus global solutions exist provided α_* , $\|G\|_1/\alpha_*^2$ are sufficiently small and the quantity (9) is uniformly bounded.

2. Construction of Approximate Solutions

Let h, k be mesh lengths satisfying the C-F-L condition. Let $\theta = \{\theta_n\}$ be an equidistributed sequence in $(0, 1)$ and m, n be integers such that $n \geq 0$. We set $A_{m,n} = (2(m + \theta_n)h, nk)$, which will be sampling points.

Approximate solutions are constructed by the random choice scheme introduced in Liu [8]. Since the single strong p -shock wave is involved, we apply the front tracking method introduced by Chern [1] which is to trace the location of the single strong shock waves. Suppose that the approximate solution is constructed by front tracking for $0 \leq t < nk$. We denote by $x = x_{\mathcal{F}}(t)$ the front of the single strong p -shock wave and $m_{\mathcal{F}}(n) = \lfloor x_{\mathcal{F}}(nk)/2h \rfloor$; by abuse of notation $m_{\mathcal{F}} = m_{\mathcal{F}}(n)$ and $x_{\mathcal{F}} = x_{\mathcal{F}}(t)$.

First, we solve the steady state equation (5) in the interval $2(m_{\mathcal{F}} - 1)h < x < x_{\mathcal{F}}$ and $x_{\mathcal{F}} < x < 2(m_{\mathcal{F}} + 2)h$ with the initial value:

$$U(2(m_{\mathcal{F}} - 1 + \theta)h) = U_h(A_{m_{\mathcal{F}}-1,n}), \quad U(2(m_{\mathcal{F}} + 1 + \theta)h) = U_h(A_{m_{\mathcal{F}}+1,n}) \quad (10)$$

respectively and denote these solutions by U^- and U^+ respectively. Next we solve the Riemann problem for the system of conservation laws:

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial x}F(U) = 0, \quad (x, t) \in R \times R_+, \quad (11)$$

$$U(x, nk) = \begin{cases} U^-(x_{\mathcal{F}}(nk)), & x < x_{\mathcal{F}}(nk) \\ U^+(x_{\mathcal{F}}(nk)), & x > x_{\mathcal{F}}(nk). \end{cases} \quad (12)$$

The solution $U(x, t)$ contains a relatively strong p -shock wave which separates the $p - 1$ -constant region U_{p-1} and the p -constant region U_p . We solve again the steady state equation (5) in the interval $2(m_{\mathcal{F}} - 1)h < x < x_{\mathcal{F}}$ and $x_{\mathcal{F}} < x < 2(m_{\mathcal{F}} + 2)h$ with the initial value:

$$U(x_{\mathcal{F}}(nk)) = U_{p-1}, \quad U(x_{\mathcal{F}}(nk)) = U_p \quad (13)$$

respectively and denote these solutions by $U_{m_{\mathcal{F}}-1}$ and $U_{m_{\mathcal{F}}+1}$ respectively which will be the approximation in the front region at $t = nk$. For $nk \leq t < (n + 1)k$, we continue the front to be

$$x_{\mathcal{F}}(t) : \quad \sigma_p(t - nk) + x_{\mathcal{F}}(nk)$$

and define the approximate solution U_h by

$$U_h(x, t) = \begin{cases} U_{m_{\mathcal{F}}-1}(x), & (2m_{\mathcal{F}} - 1)h < x < x_{\mathcal{F}}(t) \\ U_{m_{\mathcal{F}}+1}(x), & x_{\mathcal{F}}(t) < x < (2m_{\mathcal{F}} + 3)h. \end{cases} \quad (14)$$

3. Interaction Estimates and Existence of Global Solutions

If the single strong shock wave does not enter the interaction diamond $\Delta_{m,n}$, the local interaction estimates are obtained in the same manner as Liu [7]. We denote by α the set of waves issuing from $(2mh, nk)$ and by $\beta = \beta^R$ the set of waves issuing from $(2(m-1)h, nk)$ and entering $\Delta_{m,n}$. We define

$$\widehat{Q}(\Delta_{m,n}) = Q_0(\beta^R, \alpha) + \frac{G_m h}{\alpha_*^2} |\beta^R| \quad (15)$$

where $Q_0(\beta^R, \alpha)$ is Glimm's quadratic term, and G_m is the maximum of $G(x)$ for $2(m-1) < x < (2m+1)h$ and U in a small neighborhood of $U_0(x)$. We have

$$\gamma_j = \beta_j + \alpha_j + O(1)\widehat{Q}(\Delta_{m,n}), \quad 1 \leq j \leq n. \quad (16)$$

Now assume that $m_{\mathcal{F}}(n) = m$ and the strong p -shock wave β_* enter the interaction diamond. We denote, as before, by α the set of waves issuing from $(2(m-1)h, nk)$. The waves in α entering the interaction diamond $\Delta_{m,n}$ are denoted by α^R , and γ^R the set of waves issuing from $(2(m-2)h, nk)$ and entering $\Delta_{m-1,n}$. We define the quantity $Q_1(\Delta_{m,n}) = Q_1(\alpha, \gamma; \beta_*)$ by

$$Q_1(\alpha, \gamma; \beta_*) = \beta_* \sum_{l \geq p} |\alpha_l^R| + \sum_{l \text{ satisfying } 1 \leq l < p \text{ or } l \geq p, \alpha_l^L \neq 0} \{|\alpha_l \gamma_m^R|\};$$

and set

$$\widehat{Q}(\Delta_{m,n}) = Q_1(\Delta_{m,n}) + \frac{G_m h}{\alpha_*^2} (|\alpha^R| + |\gamma^R|) \quad (17)$$

Let ϵ denote the set of waves issuing from $(2(m_{\mathcal{F}}-1), (n+1)k)$ and leaving $\Delta_{m,n}$. Then ϵ have the estimates:

Lemma 1 (cf. Chern [1]) *Assume that $|\lambda_p(V_{p-1}(x))|, |\lambda_p(V_p(x))| \geq c\alpha_*$ ($c > 0$), and $U_L(x), U_M(x)$ and $U_R(x)$ are close to a constant vector U_0 . Then it follows that*

$$\epsilon_j = \begin{cases} \alpha_j + \gamma_j^R + O(1)\widehat{Q}(\Delta_{m,n}) + O(1)\frac{\sigma G_m h}{\alpha_*}, & 1 \leq j < p \\ \alpha_j^L + \gamma_j^R + O(1)\widehat{Q}(\Delta_{m,n}) + O(1)\frac{\sigma G_m h}{\alpha_*}, & p \leq j \leq n, \end{cases} \quad (18)$$

$$\beta'_* = \beta_* + \alpha_p^R + O(1)\widehat{Q}(\Delta_{m,n}) + O(1)\frac{\sigma G_m h}{\alpha_*}. \quad (19)$$

Here $O(1)$ depends only on U_L, U_R and the system.

Let β_* denote also the magnitude of the single strong shock wave crossing J . Let $W(J)$ denote the collection of waves crossing J other than β_* and

$$L(J) = \sum_{\alpha \in W(J)} |\alpha|.$$

$L(J)$ together with β_* measures the total variation of $U_{h,\theta}(x, t)$ on J .

The global interaction estimates are to show that the total amount of interaction in the Glimm approximate solutions is uniformly bounded.

Theorem 1 *If α_* and $\|G\|_1/\alpha_*^2$ are sufficiently small and $L(O) \ll \alpha_*$, then it follows that*

$$\sum_{\Delta} \widehat{Q}(\Delta) \leq O(1) \left\{ L(O) + \alpha_* + \frac{\|G\|_1}{\alpha_*^2} \right\}^2 + O(1) \sum_{n \geq 0} \frac{|\sigma_n| G_{m(n)} h}{\alpha_*} \quad (20)$$

where the single strong shock wave enters $\Delta_{m(n),n}$ with speed σ_n . If moreover the sign of speed of the single strong shock front in the Glimm approximations never changes, then

$$\sum_{\Delta} \widehat{Q}(\Delta) \leq O(1) \left\{ L(O) + \alpha_* + \frac{\|G\|_1}{\alpha_*^2} \right\}^2. \quad (21)$$

This theorem is proved by introducing the potential function $Q(J)$ defined by

$$Q(J) = Q_0(J) + Q_p(J) + \sum_{j \neq p} Q_j(J) \quad (22)$$

where

$$\begin{aligned} Q_0(J) &= \sum \{ |\alpha\beta|; \alpha, \beta \in W(J) \text{ and approaching} \} + \beta_* \sum \{ |\alpha|; \alpha \in W_a(J) \}, \\ Q_p(J) &= \sum \left\{ \frac{|\alpha|}{\alpha_*^2} \sum_{m=m_{\mathcal{F}}}^{m_0} G_m h; \alpha \text{ is any } p\text{-wave in } W(J) \text{ entering in } \Delta_{m_0, n}, m_{\mathcal{F}} < m_0 \right\} \\ &\quad + \sum \left\{ \frac{|\alpha|}{\alpha_*^2} \sum_{m=m_0}^{m_{\mathcal{F}}} G_m h; \alpha \text{ is any } p\text{-wave in } W(J) \text{ entering in } \Delta_{m_0, n}, m_0 < m_{\mathcal{F}} \right\}, \\ Q_j(J) &= \begin{cases} \sum \left\{ \frac{|\alpha|}{\alpha_*^2} \sum_{m \leq m_0} G_m h; \alpha \text{ is any } j\text{-wave in } W(J) \text{ entering in } \Delta_{m_0, n} \right\}, & 1 \leq j < p, \\ \sum \left\{ \frac{|\alpha|}{\alpha_*^2} \sum_{m \geq m_0} G_m h; \alpha \text{ is any } j\text{-wave in } W(J) \text{ entering in } \Delta_{m_0, n} \right\}, & p < j \leq n. \end{cases} \end{aligned}$$

In case the sign of σ never changes, for example $\sigma \geq 0$, we define

$$Q_p^s(J) = \frac{1}{\alpha_*} G_{m_{\mathcal{F}}} (2(m_{\mathcal{F}} + 1)h - x_{\mathcal{F}}) + \frac{1}{\alpha_*} \sum_{m > m_{\mathcal{F}}} G_m h$$

where the single strong shock wave crossing J issues from $(x_{\mathcal{F}}, nk)$ and

$$\widetilde{Q}(J) = Q_p^s(J) + Q(J).$$

If we can prove that the right side of (20) is uniformly bounded, by repeating the argument in Glimm [2], Liu [6] and [7], we have global solutions within $L^\infty(R_+; BV(R)) \cap \text{Lip}(R_+; L^1_{\text{loc}}(R))$. Thus we can find global solutions in the following cases.

Case 1. *The sign of speed of the single transonic shock front in the Glimm approximations never changes.*

Case 2. *A priori L^∞ bounds are obtained.*

Case 3. *The initial speed of the transonic shock wave is large compared to $G(x)$:*

$$\sigma(U_1(x_*), U_1(x_*)) = O(1)\alpha_*^{\frac{3}{2}}, \quad G(x) = O(1)\alpha_*^3. \quad (23)$$

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INTEGRABLE NONLINEAR EVOLUTION EQUATIONS IN MULTIDIMENSIONS

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Abstract: Some nonlinear partial differential equations of evolution type are presented, which are integrable (in fact, C-integrable, as explained below) in $N + 1$ dimensions (N space variables and 1 time variable, with arbitrary N , including $N \geq 3$). Particular attention is focussed on an equation which has a certain character of “universality”. In the introduction, this notion of “universality” is reviewed in the (1+1)-dimensional context, including its relevance to provide a heuristic explanation of the remarkable fact, that certain nonlinear PDEs - such as the (1+1)-dimensional Nonlinear Schrödinger Equation - are both widely applicable *and* integrable.

1. Introduction

The purpose and scope of this paper is to provide a terse overview of some recent developments, one of whose main outcomes is the identification of a *universal C-integrable nonlinear partial differential equation in $N+1$ dimensions* [4]. To illustrate this result - including the notions of “universality” and “C-integrability” - we review in the next Section - in a (1+1)-dimensional context - our heuristic explanation of the remarkable fact, that certain nonlinear evolution PDEs are both widely applicable and integrable [7,8,2]. In the subsequent Section 3, the universal C-integrable nonlinear partial differential equation obtained by a rather straightforward extension of the same approach to an $(N+1)$ -dimensional context, is exhibited [4]. A final Section 4 displays some other C-integrable nonlinear evolution PDEs in $N + 1$ dimensions [5].

This paper is mainly intended as a guide to the relevant literature, which is referred to herein, and whose presentation is followed almost *verbatim* whenever appropriate.

2. Universal equations: why certain nonlinear PDEs are widely applicable and integrable.

The relevance of the notion of “universality” to identify “important” hence “interesting” Nonlinear Partial Differential Equations (NLPDEs) is based on the following reasoning [7,2].

Consider a *large* class of NLPDEs, and assume that, by applying some convenient *limiting process* to *all* these NLPDEs, there obtains a specific NLPDE. It is then justified

to identify this NLPDE as “*universal*”, inasmuch as it is *uniquely associated with a large class of NLPDEs*.

Assume then that the *limiting process* that generates in this manner *the universal NLPDE* is, in some sense, *exact* (perhaps asymptotically see below), and that therefore it *preserves integrability*, namely, *it yields again an integrable equation whenever it is applied to an integrable equation*. It is then justified to expect that *the universal equation be integrable*. Indeed, while the property to be integrable is certainly not a generic feature of NLPDEs (on the contrary, it is quite exceptional), it is instead likely that a *large class of NLPDEs* - precisely because it is large - contain at least *one* integrable specimen. But this is then enough, according to the reasoning detailed above, to imply that *the universal NLPDE*, obtained from *all* the NLPDEs of the large class by a *limiting procedure that preserves integrability*, be itself *integrable*.

If the *limiting process* that generates the *universal equation* has moreover the property of being “*phenomenologically reasonable and relevant*,” namely to correspond to phenomenological circumstances that are likely to have applicative relevance, it is also justified to expect that *the universal NLPDE be widely applicable*; since the *large class of NLPDEs* from which it has been obtained is itself likely to contain - again, precisely because it is large - *several equations having applicative relevance*.

This train of reasoning has been proffered as heuristic explanation of the remarkable fact that *certain NLPDEs are both widely applicable and integrable* [7,8,2]. In particular much attention has been focussed on the large class of autonomous NLPDEs whose *linear part is dispersive* and whose *nonlinear part is* (in some very weak sense) *analytic* [7,8,2]. In the *extreme “weak field” limit*, in which all nonlinear terms are neglected and only the linear part of the equation is retained, the equations of this class possess, as special solutions, those describing *a single dispersive wave*; and there are indeed several cases in which such a solution represents the main phenomenological feature of the phenomenon modeled by the equation under investigation. It is then generally of interest to investigate how the phenomenological description gets modified in the *less drastic version of the “weak field” limit*, in which the main nonlinear effects, however weak, are consistently taken into account. It is then known (already since long ago; see, for instance, the papers by T.Taniuti and his school [10]) that: (i) the main effect, relevant on a “slow” time scale and a “coarse-grained” space scale, is an *amplitude modulation* of the dispersive wave; (ii) this phenomenon is indeed ruled by certain universal NLPDEs, of which the so-called *Nonlinear Schrödinger (NLS) equation*,

$$i\psi_t + \psi_{xx} + s |\psi|^2 \psi = 0 \quad , \quad \psi \equiv \psi(x, t) \quad , \quad (2.1)$$

is the prototype (but there also are others); (iii) these universal NLPDEs are indeed

both *widely applicable and integrable* [7,8,2].

For the mathematical physicist or applied mathematician, an appealing outcome of this line of reasoning is *the identification of NLPDEs which deserve focused attention*. This is important, since a methodological difficulty that has bedeviled the study of NLPDEs has been the following dichotomy. On the one hand, any approach applicable to “all” NLPDEs, or at least to vast classes of such equations, could not hope to go beyond the investigation of general properties, such as existence and uniqueness, forsaking any ambition to acquire a more detailed understanding of the behaviour of the solutions of such NLPDEs. On the other hand, for some special equations, there do exist mathematical techniques that allow a much deeper understanding of the behaviour of the solutions; but such equations tended to be classified as “flukes”, unworthy of much attention (especially if their solution could be mastered without recourse to sophisticated mathematical techniques). It is therefore obvious that the possibility to identify *universal NLPDEs* which are likely to be both *widely applicable and integrable* provides appealing candidates for focused attention.

We have used so far the terms “*integrable*” and “*integrability*” in a rather loose manner. Indeed, a universally accepted definition of this notion is, in the context of NLPDEs, still lacking. But for our purposes it is sufficient to recall the notions of “*S-integrability*” and “*C-integrability*.” loosely speaking, a NLPDE is *S-integrable* if it can be solved via the “*Spectral transform technique*” or the “*inverse Scattering method*,” it is “*C-integrable*” if it can be solved (i.e., linearized) by an appropriate “*Change of variables*” (for a somewhat more precise definition along these lines, see the Addendum in [2]; and note that this more precise definition entails that *C-integrable NLPDEs are also S-integrable*, while of course the converse is not necessarily true).

There exist *techniques to manufacture classes of S-integrable and C-integrable equations*; in the latter case, an obvious method is to start from a *linear* PDE, and obtain from it a *nonlinear* PDE via an invertible *Change of variables* that introduces some nonlinearity. In this manner, however, one is more likely to produce “*flukes*” than “*goldfishes*” (i.e., interesting specimens ; in the terminology of V.E.Zakharov [11]).

In the light of the above discussion there does however emerge a rather clear *strategy to arrive at universal equations* which are likely to be important hence interesting (“*goldfishes*”): by taking as starting point a (perhaps artificially manufactured) integrable equation, and by then applying to it a limiting process of the kind mentioned above. In this manner one obtains *universal NLPDEs*, which are likely to be *both integrable and widely applicable* (goldfishes!).

Let us briefly review the findings arrived at by applying this approach in the context mentioned above, which as we have seen yields NLPDEs describing the *amplitude*

modulation of a single dispersive wave, in a regime of *weak nonlinearity*. As mentioned above, the prototypical NLPDE yielded by this approach is the NLS equation (2.1); a NLPDE that is indeed widely applicable, and whose S-integrability, uncovered two decades ago [12], has been instrumental in promoting the remarkable surge of research on integrable NLPDEs and related topics, that has characterized the recent development of theoretical and mathematical physics, as well as applied and pure mathematics, over the last twenty or so years.

An apparent *paradox* has however been noted in this context [6]. The NLS equation (2.1) is *S-integrable* (provided the coefficient s is real; in which case this parameter can of course be eliminated, except for its sign, by appropriate rescaling of the dependent variable ψ); but it is not *C-integrable*. Yet the class of NLPDEs from which the NLS equation is extracted via an asymptotically exact, hence integrability-preserving, limiting process, also contains C-integrable specimens [1]. How come, then, that NLS is not itself C-integrable?

The mechanism that bypasses this paradox operates as follows [6]. If one starts from a C-integrable equation and applies to it the limiting reduction technique that generates the NLS equation (2.1), one discovers that a “miracle” occurs: the coefficient s in (2.1) turns out to vanish, hence the equation obtained starting from a *C-integrable* NLPDE is the *linear* Schrödinger equation (which is, of course, trivially C-integrable, being itself linear).

This observation removes the apparent paradox [1]. But in fact, as pointed out by W.Eckhaus, one can then go further. An appropriate interpretation of the vanishing of the parameter s in (2.1) is that, due to a “miraculous” cancellation, the amplitude modulation due to the nonlinear effects is prevented from showing up on the slow and coarse-grained time and space scales that characterize the emergence of the NLS equation (2.1). It is then appropriate to repeat the analysis on slower and coarser-grained time and space scales; and the equation that typically then emerges, to characterize the amplitude modulation of a single dispersive wave on such a scale (when one starts from a C-integrable NLPDE) is the “Eckhaus equation” [7,6,2]

$$i\psi_t + \psi_{xx} + [|\psi|^4 + 2(|\psi|^2)_x]\psi = 0 \quad , \quad \psi \equiv \psi(x, t) \quad . \quad (2.2)$$

This NLPDE is indeed *C-integrable*, as demonstrated by the (invertible) *Change of dependent variable* that linearizes it [7,6,2]:

$$\varphi(x, t) = \psi(x, t) \exp\left[\int_{-\infty}^x dx' |\psi(x', t)|^2\right] \quad , \quad (2.3a)$$

$$\psi(x, t) = \varphi(x, t) \left[1 + 2 \int_{-\infty}^x dx' |\varphi(x', t)|^2 \right]^{-\frac{1}{2}} \quad , \quad (2.3b)$$

$$i\varphi_t + \varphi_{xx} = 0 \quad . \quad (2.4)$$

Note that, for simplicity, in writing the transformations (2.3) we have implicitly assumed that $\psi(x, t)$ and $\varphi(x, t)$ both vanish as $x \rightarrow -\infty$ fast enough to make the integrals in the r.h.s of (2.3) converge.

This derivation of the Eckhaus equation provides an illustration of the research strategy outlined above. This approach has been rather extensively explored in the context of NLPDEs in 1+1 [7,8,2] and 2+1 dimensions [9,2]; and recently, by extending it to the $(N+1)$ -dimensional context, a *universal C-integrable NLPDE in $N+1$ dimensions* has been obtained: the “ $(N+1)$ -dimensional Eckhaus equation”. [4]

3. Universal C-integrable equation in $N+1$ dimensions

In this Section we continue to follow almost *verbatim* the Introduction to [4], and we thereby obtain the $(N+1)$ -dimensional Eckhaus equation and justify our qualifying it as “*universal*.”

The starting point of our treatment is the following C-integrable NLPDE in $N+1$ dimensions [3],

$$u_{tt} - \Delta u + u + (2+p)u^p u_t - 2\mathbf{v} \cdot \nabla u + [u^{2p} - \mathbf{v}^2 - (\nabla \cdot \mathbf{v})]u = 0 \quad , \quad (3.1a)$$

$$\mathbf{v}_t = pu^{p-1} \nabla u = \nabla u^p \quad . \quad (3.1b)$$

Here the scalar field u , and the N -vector field \mathbf{v} , are functions of the N -vector space coordinate \mathbf{r} and of the time t :

$$u \equiv u(\mathbf{r}, t) \quad , \quad \mathbf{v} \equiv \mathbf{v}(\mathbf{r}, t) \quad ; \quad (3.1c)$$

while the first-order N -vector differential operator ∇ , and the second-order scalar differential operator $\Delta = \nabla^2$, are the standard gradient and Laplacian in N -dimensional space.

This NLPDE is C-integrable for any choice of the parameter p , as demonstrated by the following linearizing transformation [3]:

$$w(\mathbf{r}, t) = u(\mathbf{r}, t) \exp[F(\mathbf{r}, t)] \quad , \quad (3.2a)$$

$$F_t(\mathbf{r}, t) = [u(\mathbf{r}, t)]^p , \quad (3.2b)$$

$$\nabla F(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}, t) , \quad (3.2c)$$

$$w_{tt}(\mathbf{r}, t) - \Delta w(\mathbf{r}, t) + w(\mathbf{r}, t) = 0 ; \quad (3.2d)$$

which is however applicable, see (3.2c), only to the subclass of solutions of (3.1) restricted by the condition that the vector $\mathbf{v}(\mathbf{r}, t)$ be irrotational,

$$\nabla \times \mathbf{v} = 0 . \quad (1.6e)$$

Note however that, thanks to (3.1b), it is sufficient that this requirement (3.2e) hold at any one time t_0 , for it to hold for all time.

Let us now restrict consideration to the case $p = 2$, and rewrite for convenience (3.1) in the form

$$u_{tt} - \Delta u + u = \epsilon[-4u^2u_t + 2\mathbf{v} \cdot \nabla u + (\nabla \cdot \mathbf{v})u] + \epsilon^2[-u^5 + \mathbf{v}^2u], \quad (3.3a)$$

$$\mathbf{v}_t = 2u\nabla u , \quad (3.3b)$$

which corresponds to (3.1) via the rescaling $u \rightarrow \epsilon^{\frac{1}{2}}u$, $\mathbf{v} \rightarrow \epsilon\mathbf{v}$.

Hereafter ϵ plays the role of “small parameter.” For $\epsilon = 0$, (3.3a) becomes the (dispersive) Klein-Gordon equation, and it admits the (real) solution

$$u(\mathbf{r}, t) = A \exp[i(kx - \omega t)] + c.c. . \quad (3.4)$$

This represents a single dispersive wave; for notational simplicity (but without loss of generality) we have assumed that this wave travels along the x -axis; here and below we accordingly denote by x the “first” component of the N -dimensional space-vector \mathbf{r} , and by the $(N - 1)$ -dimensional space-vector \mathbf{y} the remaining components:

$$\mathbf{r} \equiv (x, \mathbf{y}) . \quad (3.5)$$

The “wave vector” k of the dispersive wave (3.4) is an arbitrary real parameter; while the corresponding “frequency” ω is related to it by the “dispersion relation”

$$\omega = (1 + k^2)^{\frac{1}{2}} . \quad (3.6a)$$

Let us recall that, to the dispersive wave (3.4), is associated the “group velocity” (in the x -direction)

$$c = d\omega/dk = k/\omega = k(1 + k^2)^{-\frac{1}{2}} \quad . \quad (3.6b)$$

The amplitude A in (3.4) is an arbitrary complex *constant*; and it is of course a trivial exercise to verify that this is consistent with the requirement that (3.4) with (3.6a) satisfy (3.3a), *provided ϵ vanishes*. The question we wish to focus upon is, how does the solution (3.4) get modified, if ϵ is indeed small but not exactly zero?

The answer to this question is, that the main effect is to induce a *modulation* of the amplitude of the dispersive wave (3.4):

$$u(x, \mathbf{y}, t) = a\psi(\xi, \boldsymbol{\eta}, \tau) \exp[i(kx - \omega t)] + c.c. + \dots \quad (3.7)$$

Here the dots indicate additional correction terms, which of course disappear as $\epsilon \rightarrow 0$. As for the modulated amplitude ψ , as explicitly indicated in (3.7) it is a function of the “coarse-grained and slow” space and time variables $\xi, \boldsymbol{\eta}$ and τ , which are related to the space and time variables x, \mathbf{y} and t by the formulas

$$\xi = \epsilon\alpha(x - ct) \quad , \quad (3.8a)$$

$$\boldsymbol{\eta} = \epsilon\beta\mathbf{y} \quad , \quad (3.8b)$$

$$\tau = \epsilon^2\gamma t \quad . \quad (3.8c)$$

Of course $\boldsymbol{\eta}$ is a $(N - 1)$ -dimensional space vector; while the constant c in the r.h.s. of (3.8a) is the group velocity (3.6b). As for the constants a, α, β, γ , they are merely introduced to perform “cosmetic” rescalings of the variables, so as to write the final result (see below) in neater form.

The main finding of [4] is the NLPDE satisfied by the amplitude ψ . It reads as follows:

$$i\psi_\tau + \Delta\psi + [|\psi|^4 + 2(|\psi|^2)_\xi]\psi + if\psi + 2\mathbf{g} \cdot \nabla_{\boldsymbol{\eta}}\psi + (\nabla_{\boldsymbol{\eta}} \cdot \mathbf{g})\psi + \mathbf{g}^2\psi = 0, \quad (3.9a)$$

$$\mathbf{g}_\xi = \nabla_{\boldsymbol{\eta}}(|\psi|^2) \quad , \quad (3.9b)$$

$$f_\xi + 2|\psi|^2 f = 2i\mathbf{g} \cdot (\psi^* \nabla_{\boldsymbol{\eta}}\psi - \psi \nabla_{\boldsymbol{\eta}}\psi^*) + i(\psi^* \nabla_{\boldsymbol{\eta}}^2\psi - \psi \nabla_{\boldsymbol{\eta}}^2\psi^*). \quad (3.9c)$$

Here of course the complex scalar field ψ , as well as the real auxiliary fields f and \mathbf{g} (f being a scalar, \mathbf{g} an $(N - 1)$ -dimensional vector) are functions of the coarse-grained and slow variables $\xi, \boldsymbol{\eta}$ and τ :

$$\psi \equiv \psi(\xi, \boldsymbol{\eta}, \tau) \quad , \quad f \equiv f(\xi, \boldsymbol{\eta}, \tau) \quad , \quad \mathbf{g} \equiv \mathbf{g}(\xi, \boldsymbol{\eta}, \tau) \quad . \quad (3.10a)$$

Note that in (3.9) $\nabla_{\boldsymbol{\eta}}$ is the first-order $(N - 1)$ -vector differential operator (gradient) acting on the $(N - 1)$ -vector $\boldsymbol{\eta}$, while Δ indicates the Laplace operator in N -dimensional space, so that

$$\Delta \equiv \partial^2 / \partial \xi^2 + \nabla_{\boldsymbol{\eta}}^2 \quad . \quad (3.10b)$$

Let us emphasize that, in contrast to (3.1) and (3.3), in (3.9) the space variables enter asymmetrically; the preferred direction ξ is of course identified by the direction of propagation of the dispersive wave, see (3.4) and (3.7).

As for the condition (3.2e), it yields the restriction

$$\nabla_{\boldsymbol{\eta}} \times \mathbf{g}(\xi, \boldsymbol{\eta}, \tau) = 0 \quad . \quad (3.11)$$

Note that (3.9b) implies that it is sufficient that this condition (3.11) hold for one value ξ_0 , for it to hold for all values of ξ .

The *universal* character of (3.9) originates from the manner it has been obtained here; in this respect the analogy, in the $(1 + 1)$ -dimensional context, with the Eckhaus equation (2.2) is telling. It is moreover clear that, if there is no dependence on the $(N - 1)$ -vector variable $\boldsymbol{\eta}$, one can set $\mathbf{g} = 0$ (consistently with (3.9b)) and $f = 0$ (consistently with (3.9c)); then (3.9a) reduces to the $(1 + 1)$ -dimensional Eckhaus equation (2.2).

The fact that the NLPDE (3.9) is obtained, via a procedure which is exact in the asymptotic limit $\epsilon \rightarrow 0$, from the C-integrable equation (3.3), entails that it must be itself C-integrable. This is indeed demonstrated by the following invertible transformation, that is, in fact, essentially identical to the transformation (2.3) that linearizes the Eckhaus equation (2.2):

$$\varphi(\xi, \boldsymbol{\eta}, \tau) = \psi(\xi, \boldsymbol{\eta}, \tau) \exp \left[\int_{-\infty}^{\xi} d\xi' |\psi(\xi', \boldsymbol{\eta}, \tau)|^2 \right] \quad , \quad (3.12a)$$

$$\psi(\xi, \boldsymbol{\eta}, \tau) = \varphi(\xi, \boldsymbol{\eta}, \tau) \left[1 + 2 \int_{-\infty}^{\xi} d\xi' |\varphi(\xi', \boldsymbol{\eta}, \tau)|^2 \right]^{-\frac{1}{2}} \quad . \quad (3.12b)$$

(For simplicity, we have again assumed here that $\psi(\xi, \boldsymbol{\eta}, \tau)$ and $\varphi(\xi, \boldsymbol{\eta}, \tau)$ both vanish as $\xi \rightarrow -\infty$, sufficiently fast to guarantee convergence of the integrals in the r.h.s.

of these formulas). It is indeed easily seen [4], using (3.9b) and (3.9c), that via this transformation, the NLPDE (3.9) gets transformed into the linear Schrödinger equation

$$i\varphi_\tau + \Delta\varphi = 0 \quad . \quad (3.14)$$

These results provide our justification for considering (3.9) a *universal* equation, deserving to be singled out as worthy of focussed attention [4] and further study, and appropriately called *(N+1)-dimensional Eckhaus equation*.

4. Other C-integrable equations in N+1 dimensions

In this final Section we display some other *C-integrable NLPDEs in N+1 dimensions* which have been recently identified. The reader interested in more information on these equations than that reported here (which is in fact limited to the mere display of them) is referred to the original papers [3,5].

Notation: \mathbf{r} is an N -dimensional vector, ∇ the corresponding N -vector gradient operator, and $\Delta = \nabla^2$ the standard N -dimensional Laplacian. Three C-integrable NLPDEs read as follows:

$$w_t + \Delta w = -2\mathbf{v} \cdot \nabla w - [h(w) + \mathbf{w}^2(\nabla \cdot \mathbf{v})]w, \quad (4.1)$$

$$i\psi_t + \Delta\psi = -2\mathbf{v} \cdot \nabla\psi - [ih(|\psi|) + \mathbf{v}^2 + (\nabla \cdot \mathbf{v})]\psi, \quad (4.2)$$

$$u_{tt} - \Delta u + u = 2\mathbf{v} \cdot \nabla u - [2h(u) + uh'(u) - \mathbf{v}^2 - (\nabla \cdot \mathbf{v})]u. \quad (4.3)$$

Here we assume $w \equiv w(\mathbf{r}, t)$ and $u(\mathbf{r}, t)$ to be real, and $\psi \equiv \psi(\mathbf{r}, t)$ to be complex. Note that, for $N = 3$, the left-hand-sides of these equations correspond to linear PDEs of major physical relevance: the diffusion equation, the Schrödinger equation and the Klein-Gordon equation (in the 3-dimensional world we inhabit). The right-hand sides of these equations display the nonlinear parts; they contain the (arbitrarily given) function h (of the arguments shown in each case), and the N -vector $\mathbf{v} \equiv \mathbf{v}(\mathbf{r}, t)$, which is an auxiliary dependent variable whose time evolution is characterized in all three cases by the evolution equation

$$\mathbf{v}_t = \nabla h, \quad (4.4)$$

with $h \equiv h(w)$ or $h \equiv h(|\psi|)$ or $h \equiv h(u)$, as the case may be. Of course in the case of (4.1) and (4.3), but not in the case of (4.2), h and \mathbf{v} are restricted to be *real*.

Each of these equations, (4.1), (4.2) and (4.3) (of course each supplemented by (4.4)), is *C-integrable*, provided the N -vector field $\mathbf{v}(\mathbf{r}, t)$ is irrotational:

$$\nabla \wedge \mathbf{v}(\mathbf{r}, t) = 0. \quad (4.5a)$$

Note that this condition is compatible with the time evolution (4.4), and it is therefore sufficient that it hold, say, at the “initial” time t_0 ,

$$\nabla \wedge \mathbf{v}(\mathbf{r}, t_0) = 0, \quad (4.5b)$$

to insure that it hold for all time, see (4.5a).

The property of *C-integrability* entails the possibility to *solve by quadratures the initial-value problem* for each of these equations, for any initial data which satisfy the constraint (4.5b), as well as to *manufacture a large class of completely explicit solutions* (satisfying the same constraint) [3].

Note incidentally that, for $h(u) = u^p$, (4.3,4) coincide with (3.1).

Two other classes of C-integrable (coupled) NLPDEs read as follows:

$$[\gamma \partial/\partial t + \Delta]f_j = 2 \sum_{k=1, k \neq j}^n [A^2 + (\nabla f_j) \cdot (\nabla f_k)]/(f_j - f_k), \quad j = 1, \dots, n, \quad (4.6)$$

$$[\partial^2/\partial^2 - \Delta + M^2]f_j = 2 \sum_{k=1}^n [A^2 + (\nabla f_j) \cdot (\nabla f_k) - f_{j,t}f_{k,t}]/(f_j - f_k), \quad j = 1, \dots, n. \quad (4.7)$$

Here n is an arbitrary integer, $n \geq 1$, the quantities A^2 , M^2 and γ are arbitrary constants (which clearly can be eliminated by rescalings), and the functions $f_j \equiv f_j(\mathbf{r}, t)$ are the n dependent variables. The *linearizing transformation* which subtends the *C-integrability* of these systems of n nonlinearly coupled evolution PDEs in $N+1$ dimensions is *the relation between the coefficients of a polynomial of degree n and its n zeros* [5].

Note that again, for $N = 3$ (and for the two choices $\gamma = 1$ or $\gamma = i$ of the arbitrary constant γ), the left-hand-sides of these equations correspond to *linear* PDEs of major physical relevance, and (4.7) is a *relativistically invariant nonlinear* partial differential equation in our 4-dimensional space-time.

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THE INITIAL VALUE PROBLEM FOR SEMILINEAR SCHRÖDINGER EQUATIONS

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Abstract. We present local and global existence theorems of the initial value problem for semilinear Schrödinger equations which do not allow the classical energy estimates. To avoid this difficulty, we make strong use of S. Doi's method for linear Schrödinger type equations.

This paper is concerned with the initial value problem for semilinear Schrödinger equations of the form

$$\partial_t u - i\Delta u = F(u, \nabla u) \quad \text{in } (0, \infty) \times \mathbb{R}^N, \quad (1)$$

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{R}^N, \quad (2)$$

where $u(t, x)$ is \mathbb{C} -valued, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ ($j = 1, \dots, N$), $\nabla = (\partial_1, \dots, \partial_N)$, $\Delta = \partial_1^2 + \dots + \partial_N^2$, $i = \sqrt{-1}$ and $N \in \mathbb{N}$ is the spatial dimension. We assume that the nonlinear term $F(u, q)$ satisfies

$$F(u, q) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^{2N}; \mathbb{C}), \quad F(u, q) = O(|u|^p + |q|^p) \quad \text{near } (u, q) = 0$$

with some integer $p \geq 2$. In the same way as complex analysis, we define $\partial/\partial u$, $\partial/\partial \bar{u}$, $\partial/\partial q_j$ and $\partial/\partial \bar{q}_j$ by

$$\begin{aligned} \frac{\partial}{\partial u} &= \frac{1}{2} \left(\frac{\partial}{\partial v_0} - i \frac{\partial}{\partial w_0} \right), & \frac{\partial}{\partial \bar{u}} &= \frac{1}{2} \left(\frac{\partial}{\partial v_0} + i \frac{\partial}{\partial w_0} \right), \\ \frac{\partial}{\partial q_j} &= \frac{1}{2} \left(\frac{\partial}{\partial v_j} - i \frac{\partial}{\partial w_j} \right), & \frac{\partial}{\partial \bar{q}_j} &= \frac{1}{2} \left(\frac{\partial}{\partial v_j} + i \frac{\partial}{\partial w_j} \right), \end{aligned}$$

$$v_0 = \operatorname{Re} u, \quad w_0 = \operatorname{Im} u, \quad v_j = \operatorname{Re} q_j, \quad w_j = \operatorname{Im} q_j, \quad j = 1, \dots, N.$$

We introduce the following Sobolev spaces

$$\begin{aligned} H^m &= \left\{ u \in \mathcal{S}'(\mathbb{R}^N) \mid \|u\|_{H^m} = \|(1 - \Delta)^{m/2} u\|_{L^2} < +\infty \right\}, \\ H^{m,n} &= \left\{ u \in \mathcal{S}'(\mathbb{R}^N) \mid \|u\|_{H^{m,n}} = \|(1 + |x|^2)^{n/2} (1 - \Delta)^{m/2} u\|_{L^2} < +\infty \right\} \end{aligned}$$

where $m, n \in \mathbb{N} \cup \{0\}$ and $\mathcal{S}'(\mathbb{R}^N)$ is the set of all tempered distributions on \mathbb{R}^N . $\xi \in \mathbb{R}^N$ is a dual variable of $x \in \mathbb{R}^N$ under the Fourier transformation. For the sake of convenience, we put $D = -i\nabla$.

When we try to solve (1)–(2), we have the difficulty of so-called loss of derivatives because the nonlinear term $F(u, \nabla u)$ contains ∇u . We remark here that $\nabla \bar{u}$ does not cause the loss of derivatives. Then the studies on the initial value problem (1)–(2) were mainly concerned with the case of

$$\operatorname{Im} \frac{\partial F}{\partial q_j}(u, q) \equiv 0, \quad j = 1, \dots, N \quad (3)$$

(see [1], [9], [14] and [17]). Recently, however, several researchers studied (1)–(2) without the restriction (3) (see [2], [3], [4], [5], [6], [10], [11], [12], [13], [16] [18]). Except for [13], these works are applications of the theory of linear Schrödinger-type equations (see S. Mizohata [15] and S. Doi [7], [8] for instance). In order to explain the basic idea of the linear theory, let us consider the following linear equation

$$\left(\partial_t - i\Delta + \sum_{j=1}^N b_j(t, x) \partial_j \right) u = 0 \quad \text{in } (0, T) \times \mathbb{R}^N,$$

where $b_j(t, x) \in C([0, T]; \mathcal{B}^\infty(\mathbb{R}^N))$ and $\mathcal{B}^\infty(\mathbb{R}^N)$ is the set of all C^∞ -functions on \mathbb{R}^N whose derivatives of any order are all bounded on \mathbb{R}^N . The basic idea is to find a transformation $u \mapsto Ku$ so that the commutator $[K, -i\Delta]K^{-1}$ eliminates the bad first order term $\sum \{\operatorname{Im} b_j(t, x)\} \partial_j$ in some sense.

[2], [10], [11], [12], [16] and [18] are applications of gauge transformation which is available for the case of $N = 1$ ([2], [10], [12]), radially symmetric case ([11]) and the case of

$$\partial_j \left(\operatorname{Im} \frac{\partial F}{\partial q_k}(u, \nabla u) \right) - \partial_k \left(\operatorname{Im} \frac{\partial F}{\partial q_j}(u, \nabla u) \right) = 0, \quad j, k = 1, \dots, N \quad (4)$$

for any $u \in C^1(\mathbb{R}^N)$ ([16], [18]). If $N = 1$, then the condition (4) is satisfied automatically.

For general nonlinear terms and general spatial dimension, first C. E. Kenig, G. Ponce and L. Vega ([13]) obtained the sharp smoothing estimate of $e^{it\Delta}$ and applied it to proving local existence theorem for (1)–(2) under the smallness condition on the initial data u_0 . Roughly speaking, they showed that $\partial_j(\partial_t - i\Delta)^{-1}$ ($j = 1, \dots, N$) were bounded operators and they constructed the inverse of Schrödinger-type operators, which corresponded to (1), provided that the solution was sufficiently small. Secondly H. Chihara ([3], [4], [5], [6]) made strong use of S. Doi's method ([7], [8]) to show local and global existence theorems for (1)–(2). Using S. Doi's method under an appropriate condition on $\operatorname{Im} b_j(t, x)$

($j = 1, \dots, N$), one can find a transformation $u \mapsto Ku$ so that $[K, -i\Delta]K^{-1}$ is a first order elliptic pseudo-differential operator which is stronger than $\sum\{\text{Im } b_j(t, x)\}\partial_j$. Combining [3], [4], [5] and [6], we have

Theorem 1 (Local existence) (i) *We assume $p \geq 3$. Let m_1 be a sufficiently large integer. Then for any $u_0 \in H^m$ ($m \geq m_1$), there exists a time $T_1 = T_1(\|u_0\|_{H^{m_1}}) > 0$ such that the initial value problem (1)–(2) possesses a unique solution $u \in C([0, T_1]; H^m)$.*

(ii) *We assume $p = 2$. Let m_2 be a sufficiently large integer. Then for any $u_0 \in H^m \cap H^{m-2,2}$ ($m \geq m_2$), there exists a time $T_2 = T_2(\|u_0\|_{H^{m_2}} + \|u_0\|_{H^{m_2-2,2}}) > 0$ such that the initial value problem (1)–(2) possesses a unique solution $u \in C([0, T_2]; H^m \cap H^{m-2,2})$.*

Theorem 2 (Global existence) (i) *We assume $N \geq 3$ and $p \geq 3$. Let m_3 be a sufficiently large integer. Then there exists a small constant $\delta_3 > 0$ such that for any*

$$u_0 \in \bigcap_{j=0}^2 H^{m-2j,j} \quad (m \geq m_3 + 2) \quad \text{satisfying} \quad \sum_{j=0}^2 \|u_0\|_{H^{m_3-2j,j}} \leq \delta_3,$$

the initial value problem (1)–(2) possesses a unique solution

$$u \in \bigcap_{j=0}^2 C([0, \infty); H^{m-2j,j}).$$

(ii) *We assume that $N = 2$, $p \geq 3$ and*

$$F_3(e^{i\theta}u, e^{i\theta}q) = e^{i\theta}F_3(u, q) \quad \text{for} \quad (u, q) \in \mathbb{C} \times \mathbb{C}^N, \quad \theta \in \mathbb{R} \quad (5)$$

where $F_3(u, q)$ is a homogeneous cubic part of $F(u, q)$ near $(u, q) = 0$. Let m_4 be a sufficiently large integer. Then there exists a small constant $\delta_4 > 0$ such that for any

$$u_0 \in \bigcap_{j=0}^1 H^{m-2j,j} \quad (m \geq m_4 + 2) \quad \text{satisfying} \quad \sum_{j=0}^1 \|u_0\|_{H^{m_4-2j,j}} \leq \delta_4,$$

the initial value problem (1)–(2) possesses a unique solution

$$u \in \bigcap_{j=0}^1 C([0, \infty); H^{m-2j,j}).$$

(iii) *We assume $N \geq 13$ and $p = 2$. Let m_5 be a sufficiently large integer. Then there exists a small constant $\delta_5 > 0$ such that for any*

$$u_0 \in \bigcap_{j=0}^2 H^{m-2j,j} \quad (m \geq m_5 + 2) \quad \text{satisfying} \quad \sum_{j=0}^2 \|u_0\|_{H^{m_5-2j,j}} \leq \delta_5,$$

the initial value problem (1)–(2) possesses a unique solution

$$u \in \bigcap_{j=0}^2 C([0, \infty); H^{m-2j,j}).$$

Remark 1 Since our analysis is based on the symbolic calculus of psuedo-differential operators, it is very troublesome to determine the minimum of m_1, m_2, m_3, m_4 and m_5 .

Remark 2 According to [2] or [10], if $N = 1$ and $p = 2$, then one can get local solutions to (1)–(2) in $H^m \cap H^{m-1,1}$. Therefore one seems to be able to improve the part (ii) of Theorem 1. If one would employ S. Mizohata's method ([15]) instead of S. Doi's method, one could probably get the local solutions to (1)–(2) in $H^m \cap H^{m-1,1}$ also for the case of $p = 2$ and $N \geq 1$.

We will explain the outline of the proofs. Basically Theorem 1 follows from the energy estimates. Theorem 2 is proved by the energy and the decay estimates.

Concerning the energy estimates, we treat the equation (1) as a 2×2 system for (u, \bar{u}) because the nonlinear term $F(u, \nabla u)$ contains not only ∇u but also $\nabla \bar{u}$. More precisely, first we diagonalize this system modulo bounded operators. In fact, the following symbol of 2×2 partial differential operator

$$\begin{bmatrix} |\xi|^2 & 0 \\ 0 & -|\xi|^2 \end{bmatrix} + \sum_{j=1}^N \begin{bmatrix} b_{11j}(t, x) & b_{12j}(t, x) \\ b_{21j}(t, x) & b_{22j}(t, x) \end{bmatrix} \xi_j,$$

$$b_{mnj}(t, x) \in C^1([0, T]; \mathcal{B}^\infty(\mathbb{R}^N)), \quad m, n = 1, 2, \quad j = 1, \dots, N,$$

can be easily diagonalized modulo bounded operators provided that $|\xi|$ is sufficiently large. Secondly we make use of the linear theory. Since the system becomes a couple of single Schrödinger type equations essentially, S. Doi's method obtains the energy inequality. We suppose here the Doi-type condition, that is to say that there exist functions

$$\phi_j(t, x) \in C([0, T]; \mathcal{B}^\infty(\mathbb{R})) \cap C^1([0, T]; L^1(\mathbb{R})) \quad (j = 1, \dots, N)$$

such that

$$|\operatorname{Im} b_{11j}(t, x)|, |\operatorname{Im} b_{22j}(t, x)| \leq \phi_j(t, x_j) \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^N, \quad j = 1, \dots, N.$$

Then we have a symbol of transformation

$$k(t, x, \xi) = \begin{bmatrix} e^{-p(t, x, \xi)} & 0 \\ 0 & e^{p(t, x, \xi)} \end{bmatrix},$$

$$p(t, x, \xi) = \sum_{j=1}^N \left(\int_0^{x_j} \phi_j(t, s) ds \right) \xi_j (1 + \xi_j^2)^{-1/2}.$$

It is easy to see that the operator $k(t, x, D)$ is an automorphism on $(L^2)^2$ in some sense. In the application to semilinear equations, we choose $\phi_j(t, s)$ ($j = 1, \dots, N$) satisfying

$$\sup_{\check{x}_j \in \mathbb{R}^{N-1}} \left| \frac{\partial F}{\partial q_j}(u(t, x), \nabla u(t, x)) \right| \leq \phi_j(t, x_j), \quad j = 1, \dots, N$$

where $\check{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N)$.

On the other hand, to get the decay estimates, we employ well-known methods. For the cubic nonlinearity, we combine the Gagliardo–Nirenberg inequalities and the operators $J_k = x_k + 2i(1+t)\partial_k$ ($k = 1, \dots, N$) (see [9] and [12] for instance). For the quadratic nonlinearity, we make use of the L^p – L^q estimates (see [14] and [17]).

Remark 3 Our analysis is very suited to cubic nonlinearity. In this case, we choose $\phi_j(t, s)$ ($j = 1, \dots, N$) as

$$\phi_j(t, s) = \begin{cases} M \int_{\mathbb{R}^{N-1}} |(1 - \Delta)^{[(N-1)/2]+2} u(t, x)|^2 d\tilde{x}_j & \text{(for local existence),} \\ M(1+t)^{-d} \int_{\mathbb{R}^{N-1}} |(1 - \Delta)^{[(N-1)/2]+1} Ju(t, x)|^2 d\tilde{x}_j & \text{(for global existence),} \end{cases}$$

where $M > 0$ is a constant, $[l]$ is the largest integer $\leq l$, $d = d(N) \geq 0$ is a decay-ratio determined by the spatial dimension N , and $J = (J_1, \dots, J_N)$. When the nonlinear term $F(u, q)$ satisfies the gauge invariance (5), our results are optimal. On the other hand, the condition of the part (iii) of Theorem 2 is stronger than that of [14]–[17], that is $N(p-1)^2/2p > 1$. The former follows from $N(p-1)^2/2p > 3$ with $p = 2$. In this case, for the integrability of $\phi_j(t, s)$ ($j = 1, \dots, N$) in s , we lose some time-decay. In other words, we need an extra time-decay. We should say that our method is not so good to deal with quadratic nonlinearity.

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LOW-FREQUENCY ASYMPTOTIC EXPANSION FOR SECOND ORDER ELLIPTIC SYSTEM IN A TWO DIMENSIONAL EXTERIOR DOMAIN

WAKAKO DAN

1. Introduction and main results

Let Ω be an unbounded domain in the 2-dimensional Euclidean space \mathbb{R}^2 having a boundary $\partial\Omega$ which is infinitely smooth and compact. Let $A(x, \partial)$ ($\partial = (\partial_1, \partial_2)$, $\partial_j = \partial/\partial x_j$, $x = (x_1, x_2) \in \mathbb{R}^2$) be an $m \times m$ matrix of differential operators of the form:

$$(1.1) \quad A(x, \partial) = A(\partial) + P(x, \partial)$$

$$A(\partial)\mathbf{u}(x) = \sum_{i,j=1}^2 A^{ij} \partial_i \partial_j \mathbf{u}(x), \quad P(x, \partial)\mathbf{u}(x) = \sum_{i,j=1}^2 \partial_i (P^{ij}(x) \partial_j \mathbf{u}(x)),$$

where \mathbf{u} is an m -dimensional column vector, A^{ij} and $P^{ij}(x)$ are $m \times m$ matrices. The elements of A^{ij} are all real constants and the elements of $P^{ij}(x)$ are real valued functions in $C_0^\infty(\mathbb{R}^2)$. Put $A^{ij}(x) = A^{ij} + P^{ij}(x)$ and $A(\xi) = \sum_{i,j=1}^2 A^{ij} \xi_i \xi_j$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. In the present paper, we consider the boundary value problem with spectral parameter k in the domain Ω :

$$(1.2) \quad (k^2 I + A(x, \partial))\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where I is the $m \times m$ unit matrix.

Now, we introduce the assumptions (A.1) – (A.4) below.

$$(A.1) \quad {}^t A^{ij}(x) = A^{ji}(x).$$

$$(A.2) \quad \exists C > 0 \text{ s.t. } \sum_{i,j=1}^2 (A^{ij}(\cdot) \partial_j \mathbf{u}, \partial_i \mathbf{u})_\Omega \geq C \|\nabla \mathbf{u}\|_\Omega^2,$$

for $\mathbf{u} \in L_{loc}^2(\Omega)$, $\nabla \mathbf{u} \in L^2(\Omega)$ and $\mathbf{u} = 0$ on $\partial\Omega$. Here, $\|\cdot\|_\Omega$ and $(\cdot, \cdot)_\Omega$ denote usual L^2 norm and inner product on Ω .

In view of (A.1) and (A.2), $A(x, \partial)$ is a strongly elliptic system. If we note that $P^{ij}(x)$ vanish for large $|x|$, we have $A(\xi)$ is a symmetric matrix and that

$$\exists \delta > 0 \quad \text{s.t.} \quad A(\xi) \geq \delta |\xi|^2 I \quad \text{for any } \xi \in \mathbb{R}^2.$$

Let $\lambda_1(\xi), \dots, \lambda_N(\xi)$ denote distinct characteristic roots of $A(\xi)$, then

$$(A.3) \quad \lambda_j(\xi), j = 1, \dots, N \text{ have constant multiplicity for all } \xi \in \mathbb{R}^2 \setminus \{0\}.$$

Put $\Sigma_j = \{\xi \in \mathbb{R}^2 \mid \lambda_j(\xi) = 1\}$, $j = 1, \dots, N$. The final assumption is that

$$(A.4) \quad \text{the Gaussian curvatures of the curves } \Sigma_j, j = 1, \dots, N \text{ do not vanish.}$$

Now, we shall give the notation. Let G be a domain in \mathbb{R}^2 . Put

$$\begin{aligned} S_R &= \{x \in \mathbb{R}^2 \mid |x| = R\}, \quad B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}, \quad \Omega_R = B_R \cap \Omega, \\ L_R^2(G) &= \{\mathbf{u} \in L^2(G) \mid \mathbf{u}(x) = 0 \text{ for } |x| > R\}, \\ \dot{H}^p(\Omega) &= \{\mathbf{u} \in H_{loc}^p(\Omega) \mid \partial_x^\alpha \mathbf{u} \in L^2(\Omega), 1 \leq |\alpha| \leq p\}, \\ S_R &= \{x \in \mathbb{R}^2 \mid |x| = R\}, \quad B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}, \quad \Omega_R = B_R \cap \Omega. \end{aligned}$$

Take a constant $a > 0$ so that $\partial\Omega \subset B_{a-1}$ and $A(x, \partial) = A(\partial)$ when $|x| > a - 1$. Let \mathbb{C} be the set of all complex numbers. Put

$$D = \{k \in \mathbb{C} \setminus \{0\} \mid -\pi/2 < \arg k < 3\pi/2\}, \quad D_\pm = \{k \in D \mid \Im k \gtrless 0\}.$$

When we assume that (A.1) – (A.4) are satisfied, Vainberg [14, 15, 17] proved that there exists an operator $R_k \in \mathbb{B}(L_a^2(\Omega), H_{loc}^2(\Omega))$ which depend meromorphically on the parameter $k \in D$ and the asymptotic expansion of the operator R_k as $|k| \rightarrow 0$ has the form

$$(1.3) \quad R_k = k^{-\alpha} \sum_{m=0}^{\infty} \sum_{n=0}^{m\ell} \left[\frac{k}{P(\log k)} \right]^m \log^n k P_{m,n},$$

where α is an integer, ℓ is a non-negative integer, P is a polynomial with constant coefficients, and $P_{m,n}: L_a^2(\Omega) \rightarrow H_{loc}^2(\Omega)$ are bounded operators independent of k . We can get expansion (1.3) from the corresponding expansion for the case of general elliptic problems. But the integers α and ℓ and the polynomial P was not known, even for equations of second order. Recently, Kleinmann and Vainberg [5] obtained the complete asymptotic expansion in the case that $A(x, \partial)$ coincides with the Laplace operator in some neighbourhood of infinity. We apply the idea of [5] to the system case.

We denote by \mathbf{u}_0 (m -dimensional column) and U_1 ($m \times m$ matrix) the solutions of the problems:

$$(1.4) \quad \begin{aligned} A(x, \partial)\mathbf{u}_0 &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u}_0 &= 0 \quad \text{on } \partial\Omega, \quad |\mathbf{u}_0| = O(1) \quad \text{as } |x| \rightarrow \infty, \end{aligned}$$

$$(1.5) \quad \begin{aligned} A(x, \partial)U_1 &= 0 \quad \text{in } \Omega, \\ U_1 &= 0 \quad \text{on } \partial\Omega, \quad |U_1 - E_0| = O(1) \quad \text{as } |x| \rightarrow \infty, \end{aligned}$$

where E_0 is a fundamental solution: $A(\partial)E_0 = \delta I$. Here, δ is Dirac's distribution in \mathbb{R}^2 . We can show that the uniqueness and the existence of solutions of (1.4) and (1.5).

Since we can show that the solution of (1.4) converges to some constant, we can define the constant matrix and vector as follows:

$$L = \lim_{|x| \rightarrow \infty} (U_1 - E_0), \quad \mathbf{b} = \lim_{|x| \rightarrow \infty} \mathbf{u}_0.$$

With this notation, we shall state our main results.

Theorem 1.1. *For the solution $\mathbf{u} = R_k \mathbf{f}$ of (1.2) with $\mathbf{f} \in L_a^2(\Omega)$ in the case that $B(x, \partial)$ is the identity, the following asymptotic expansion is valid when $k \in D$, $|k| \rightarrow 0$:*

$$(1.6) \quad R_k \mathbf{f} = \sum_{m=0}^N \sum_{n=-m}^{\infty} k^m \log^{-n} k G_{m,n} \mathbf{f} + \hat{\mathbf{u}}_N,$$

where $G_{m,n} \in \mathbb{B}(L_a^2(\Omega), H^2(\Omega_a))$ are independent of k and

$$\|\hat{\mathbf{u}}_N\|_{H^2(\Omega_a)} \leq C(a) |k \log k|^{N+1} \|\mathbf{f}\|_{L_a^2(\Omega)}.$$

The leading terms of the asymptotic expansion have the form

$$(1.7) \quad \mathbf{u} = \mathbf{u}_0(x) + U_1 \left(\left(\log k - \frac{i\pi}{2} \right) M + B - L \right)^{-1} \mathbf{b} + O(k \log k),$$

where

$$M = c_\pi \int_{S_1} A(\omega)^{-1} dS_1, \quad c_\pi = \left(\frac{1}{2\pi} \right)^2,$$

$$B = c_\pi \sum_{j=1}^N \int_{\Sigma_j} \log |\omega|^2 \frac{P_j(\omega)}{|\nabla \lambda_j(\omega)|} d\Sigma_j, \quad d\Sigma_j \text{ is the surface element of } \Sigma_j,$$

the matrices $P_j(\xi)$ are projections on the eigenspaces corresponding to the $\lambda_j(\xi)$.

Let Λ denotes the set of all poles of R_k in D .

Theorem 1.2. *Assume that (A.1)–(A.4) are valid. Then, $\Lambda \cap D_+ = \emptyset$.*

When $A(x, \partial) = A(\partial)$, we have $\Lambda \cap (\mathbb{R} \setminus \{0\}) = \emptyset$.

2. Sketch of the proof

As a fundamental solution: $A(\partial)E_0 = \delta I$, we adopt $E_0(x) = -\mathfrak{F}^{-1}[\text{p.v.} A(\xi)^{-1}]$. Then we have a representation of $E_0(x)$ such that

$$(2.1) \quad E_0(x) = \log |x| M + Q,$$

where M is a constant matrix. Furthermore, projecting on the eigenspaces corresponding to the $\lambda_j(\xi)$, we have that for fundamental solution: $(k^2 I + A(\partial))E_k = \delta I$,

$$(2.2) \quad E_k(x) = E_0(x) + \left(\log k - \frac{i\pi}{2} \right) M + B + F_k^1(x) + F_k^2(x),$$

where

$$(2.3) \quad \sup_{x \in \Omega} |F_k^1(x)| = O(k \log k) \quad \text{and} \quad \|F_k^2 * \mathbf{f}\|_{H^2(\mathbb{R}^2)} = O(k^2)$$

for $\mathbf{f} \in L_a^2(\Omega)$ as $k \rightarrow 0$.

Let us denote by η a particular function which is infinitely smooth, $\eta(x) = 0$ for $|x| < a - 1$ and $= 1$ for $|x| > a - 1/2$. For any smooth \mathbf{u} let us denote by \mathbf{g} the following function:

$$\mathbf{g}(\mathbf{u}) = A(\partial)(\eta \mathbf{u}) - \eta A(\partial)\mathbf{u}, \quad x \in \mathbb{R}^2.$$

Since $A(x, \partial) = A(\partial)$ for $|x| > a - 1$, for $\mathbf{u} = R_k \mathbf{f}$, $\eta \mathbf{u}$ satisfies the following in \mathbb{R}^2

$$(k^2 I + A(\partial))(\eta \mathbf{u}) = \mathbf{g}(\mathbf{u}) + \eta \mathbf{f},$$

where the right-hand side has compact support. Representation

$$(2.4) \quad \eta \mathbf{u} = E_k * (\mathbf{g}(\mathbf{u}) + \eta \mathbf{f})$$

follows immediately.

From (1.3),

$$(2.5) \quad \mathbf{u} = k^s \frac{Q(\log k)}{S(\log k)} + O(k^{s+1} \log^\gamma k), \quad \text{in } \Omega_a, \quad k \rightarrow 0,$$

where s is an integer, S is a polynomial with constant coefficients and Q is a polynomial, not identically zero, whose coefficients are functions of x .

Substituting (2.2) and (2.5) into both sides of the equality (2.4) and equating the leading terms which contain the multiplier k^s and using the uniqueness of the problems (1.4) and (1.5), we get the leading terms. If we have the leading terms, we can obtain the asymptotic expansion by the usual perturbation method.

3. An application

We shall discuss the rate of the local energy decay of solutions to the dynamical system:

$$(3.1) \quad \begin{cases} \partial_t^2 \mathbf{u}(t, x) - A(\partial)\mathbf{u}(t, x) = 0 & \text{in } \mathbb{R} \times \Omega, \\ \mathbf{u}(t, x) = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ \mathbf{u}(0, x) = \mathbf{f}^1(x), \quad \partial_t \mathbf{u}(0, x) = \mathbf{f}^2(x) & \text{in } \Omega, \end{cases}$$

where t denotes the time variable and $\partial_t = \partial/\partial t$. Put

$$\begin{aligned} \mathfrak{f} &= (\mathbf{f}^1, \mathbf{f}^2), \quad \mathfrak{L}\mathfrak{f} = (\mathbf{f}^2, A(\partial)\mathbf{f}^1), \\ (\mathfrak{f}, \mathfrak{g})_{\mathcal{H}} &= \sum_{i,j=1}^2 \int_{\Omega} A^{ij} \partial_j \mathbf{f}^1 \cdot \overline{\partial_i \mathbf{g}^1} dx + (\mathbf{f}^2, \mathbf{g}^2)_{\Omega}, \quad \|\mathfrak{f}\|_{\mathcal{H}}^2 = (\mathfrak{f}, \mathfrak{f})_{\mathcal{H}}, \\ \mathcal{H} &= \{\mathfrak{f} \mid \mathbf{f}^1 \in \dot{H}^1(\Omega), \mathbf{f}^2 \in L^2(\Omega), \mathbf{f}^1 = 0 \text{ on } \partial\Omega\}, \\ D(\mathfrak{L}) &= \{\mathfrak{f} \mid \mathbf{f}^1 \in \dot{H}^2(\Omega), \mathbf{f}^2 \in H^1(\Omega), \mathbf{f}^j = 0 \ (j = 1, 2) \text{ on } \partial\Omega\}. \end{aligned}$$

Let us adopt $(\mathfrak{f}, \mathfrak{g})_{\mathcal{H}}$ as the inner product of \mathcal{H} , then \mathcal{H} is a Hilbert space. Since \mathfrak{L} is skew self-adjoint in \mathcal{H} , from Stone's theorem it follows that \mathfrak{L} generates one parameter unitary group $\{U(t) \mid t \in \mathbb{R}\}$ (cf.[12]). Put $U(t)\mathfrak{f} = (\mathbf{u}(t, x), \partial_t \mathbf{u}(t, x))$. Then, $\mathbf{u}(t, x)$ is a unique solution to (3.1) with initial data $\mathbf{f}^1 \in \dot{H}^1(\Omega)$ and $\mathbf{f}^2 \in L^2(\Omega)$.

Definition. We shall say that Ω is non-trapping if there exists a $T > 0$ depending only on a and Ω such that all components of $U(t)\mathfrak{f}$, $\mathfrak{f} = (0, \mathbf{f}^2)$, belong to $C^\infty([T, \infty) \times \overline{\Omega}_a)$ for any $\mathbf{f}^2 \in L_a^2(\Omega)$.

Vainberg [16, 17] proves the following theorem.

Theorem 3.1. Assume that Ω is non-trapping and that (A.1)–(A.4) are valid. Then there exist positive constants α, β, C and T such that for integers s and j , $0 \leq s \leq 1$, $0 \leq j \leq 2$,

$$\sum_{|\gamma| \leq 2+s-j} \|\partial^\gamma R_k \mathbf{f}\|_{\Omega_a} \leq C |k|^{1-j} e^{T|\Im k|} \sum_{|\gamma| \leq s} \|\partial^\gamma \mathbf{f}\|_{\Omega}$$

for any $k \in \{k \in D \mid |\Im k| < \alpha \log |\Re k| - \beta\}$ and $\mathbf{f} \in L_a^2(\Omega)$.

Put $\mathcal{H}_\infty = \{\mathfrak{f} \mid \mathbf{f}^j \in C_0^\infty(\overline{\Omega}) \ j = 1, 2\}$. As is well known, we have

$$(3.2) \quad U(t)\mathfrak{f} = \frac{1}{2\pi} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{-ikt} (ikI + \mathfrak{L})^{-1} \mathfrak{f} dk$$

for any $\mathfrak{f} \in \mathcal{H}_\infty$ and $\sigma > 0$. Then, we have the local energy decay of solutions.

Theorem 3.2. Assume that Ω is non-trapping and that (A.1)–(A.4) are valid. Let $\mathfrak{f} \in D(\mathfrak{L})$ and $\text{supp } \mathfrak{f}^j \subset \Omega_a$, $j = 1, 2$. Put $U(t)\mathfrak{f} = (\mathbf{u}(t, x), \partial_t \mathbf{u}(t, x))$. Then, we have the local energy decay estimate:

$$(3.3) \quad \sum_{|\alpha| \leq 1} \|\partial^\alpha \mathbf{u}(t, \cdot)\|_{\Omega_a} + \|\partial_t \mathbf{u}(t, \cdot)\|_{\Omega_a} \leq C_1 |t|^{-1} \log^{-2} t \left[\sum_{|\alpha|=1} \|\partial^\alpha \mathbf{f}^1\|_{\Omega} + \|\mathbf{f}^2\|_{\Omega} \right]$$

with some positive constant C_1 depending only on a and Ω for $t \rightarrow \infty$.

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LANGMUIR SOLITARY WAVES IN A WEAKLY MAGNETIZED PLASMA

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Abstract : We consider a nonlocal nonlinear Schrödinger equation which extends the equation modelling the nonlinear evolution of Langmuir waves in a weak magnetic field under the assumption that the characteristic transverse length scales are sufficiently larger than longitudinal ones. We classify, according to the nonlinearity and the dimension, the existence and non existence of localized solitary waves. We also give optimal decay rates and symmetry properties of the solitary waves.

1. Introduction

We are interested here in some models for waves in a weakly magnetized plasma which lead to new interesting nonlinear Schrödinger equations. We refer to [5] and the bibliography therein for a detailed description of those models, which is only sketched below.

We consider first upper-hybrid waves in a weakly magnetized plasma (the electron plasma frequency w_{pe} is larger than the electron cyclotron frequency w_{ce} , $w_{pe} \gg w_{ce}$). In this case the dispersion law is

$$(1.1) \quad w(k) = w_{pe} \left[1 + \frac{3}{2} |k|^2 (r_{De})^2 + \frac{1}{2} \frac{w_{ce}^2}{w_{pe}^2} \frac{|k_{\perp}|^2}{|k|^2} \right]$$

where r_{De} is the Debye radius, k is the wave vector and k_{\perp} is the wave vector transverse to the direction x of the magnetic field. The equation for the complex envelope ψ of the high frequency waves writes in dimensionless variables

$$(1.2) \quad \Delta(i\psi_t + \Delta\psi) - \sigma\Delta_{\perp}\psi + \nabla \cdot (|\nabla\psi|^2 \nabla\psi) = 0.$$

Here Δ is the Laplacian $\partial_x^2 + \partial_y^2 + \partial_z^2$ and Δ_{\perp} the transverse Laplacian $\partial_y^2 + \partial_z^2$ and $\sigma = \frac{1}{2} w_{ce}^2 / w_{pe}^2$.

Equation (1.2) can be simplified if the wave condensate has characteristic longitudinal scales much smaller than the transverse ones, i.e. $|k_{\perp}| \ll |k|$, so that (1.1) becomes approximately

$$(1.3) \quad w(k) = w_{pe} \left[1 + \frac{3}{2} k_x^2 (r_{De})^2 + \frac{1}{2} \frac{w_{ce}^2}{w_{pe}^2} \frac{|k_{\perp}|^2}{k_x^2} \right].$$

Then (1.2) reduces formally to (in dimensionless variables) :

$$(1.4) \quad \frac{\partial^2}{\partial x^2} \left[i\psi_t + \frac{\partial^2}{\partial x^2} \psi \right] - \Delta_{\perp} \psi + \frac{\partial}{\partial x} \left[\left| \frac{\partial \psi}{\partial x} \right|^2 \frac{\partial \psi}{\partial x} \right] = 0.$$

Note that σ has been taken equal to one by a simple scaling transformation.

The aim of this paper is to study (1.4) for which no mathematical results seem to be known so far (on the other hand (1.2) with $\sigma = 0$ has been studied intensively by T. Colin [3], [4] in the three dimensional case).

More precisely we will consider (localized) solitary wave solutions for (1.4) in 2 or 3 spatial dimensions. We will in fact consider a slightly more general equation than (1.4), namely

$$(1.5) \quad \frac{\partial^2}{\partial x^2} \left(i\psi_t + \frac{\partial^2 \psi}{\partial x^2} \right) - \Delta_{\perp} \psi + \frac{\partial}{\partial x} \left(\left| \frac{\partial \psi}{\partial x} \right|^{\alpha} \frac{\partial \psi}{\partial x} \right) = 0,$$

$$\psi = \psi(\underline{x}, t), \quad \underline{x} \in \mathbb{R}^d, \quad d = 2, 3, \quad t \geq 0,$$

where $\alpha > 0$. The physical problem corresponds to $\alpha = 2$. We have denoted $\underline{x} = (x, x')$ where x' represents the transverse variables, namely $x' = y$ if $d = 2$, $x' = (y, z)$ if $d = 3$.

Setting $\varphi = \frac{\partial \psi}{\partial x}$, (1.5) reduces to

$$(1.6) \quad \begin{cases} \frac{\partial}{\partial x} (i\varphi_t + \varphi_{xx}) - \Delta_{\perp} \psi + \frac{\partial}{\partial x} [|\varphi|^{\alpha} \varphi] = 0, \\ \varphi = \frac{\partial \psi}{\partial x}. \end{cases}$$

We will completely classify the existence and non existence of localized solitary waves solutions of (1.6) and prove qualitative properties of the solitary waves : regularity, symmetries, decay. The results are parallel to those for the generalized KP equations studied in [1], [2]. The Cauchy problem for (1.6) will be studied in a subsequent paper.

2. Solitary waves

We shall denote for $d = 2, 3$, Y the closure of $\partial_x(C_0^{\infty}(\mathbb{R}^d))$ for the norm

$$\|\partial_x \psi\|_Y = \left(\|\nabla \psi\|_{L^2}^2 + \|\partial_x^2 \psi\|_{L^2}^2 \right)^{1/2},$$

where $\partial_x(C_0^{\infty}(\mathbb{R}^d))$ denotes the space of functions of the form $\partial_x \psi$ with $\psi \in C_0^{\infty}(\mathbb{R}^d)$ (i.e. the space of functions φ in $C_0^{\infty}(\mathbb{R}^d)$ such that $\int_{-\infty}^{\infty} \varphi(x, x') dx = 0$ for every $x' \in \mathbb{R}^{d-1}$).

By standard imbedding theorems, if $\varphi \in Y$ and $d = 3$, then $\varphi = \partial_x \psi$ where $\psi \in L^6(\mathbb{R}^3)$; if $d = 2$ and $u \in Y$ then $\varphi = \partial_x \psi$ where $\psi \in L_{\text{loc}}^q(\mathbb{R}^2)$, $\forall q < +\infty$. Note that for $d = 2$ the choice of $\psi \in L_{\text{loc}}^q$ such that $\varphi = \partial_x \psi$ is not unique, but two such ψ will differ by a function independent of x . Hence, only one of them (up to a constant) satisfies $v = \partial_y \psi \in L^2(\mathbb{R}^2)$. We assume in all what follows that when $\varphi \in Y$ and when we take $\psi \in L_{\text{loc}}^q$ with $\partial_x \psi = \varphi$, we also have $v = \partial_y \psi \in L^2$. We then denote $v = \partial_y \psi$ by $D_x^{-1} \varphi_y$.

Definition 2.1. A solitary wave of (1.6) is a solution of the type $e^{i\omega t} \Phi(x, x')$ where Φ is real valued and $\Phi \in Y$ and $\omega > 0$.

We are thus looking for “localized” solutions to the system

$$(2.1) \quad \begin{cases} -\omega\Phi_x + \Phi_{xxx} - \Delta_{\perp}\Psi + (|\Phi|^{\alpha}\Phi)_x = 0 \\ \Phi = \Psi_x, \end{cases}$$

which we write also

$$(2.2) \quad -\omega\Phi_{xx} + \Phi_{xxxx} - \Delta_{\perp}\Phi + (|\Phi|^{\alpha}\Phi)_{xx} = 0.$$

We will from now on assume that $w = 1$, since the scale change $\tilde{\Phi}(x, x') = \omega^{-1/\alpha}\Phi(x/\omega^{1/2}, x'/\omega)$ transforms the system (2.1) or (2.2) in Φ into the same in $\tilde{\Phi}$ but with $w = 1$.

At this stage it is worth noticing that (2.2) is (except for the absolute value in the nonlinear term) exactly the elliptic equation satisfied by the solitary wave solutions of the generalized Kadomtsev-Petviashvili equations (see [1] [2]). The next theorem concerns the non existence of solitary waves to (1.6).

Theorem 2.1

(i) Assume that $d = 2$. The equation (2.1) does not admit any nontrivial solitary wave Φ satisfying $\Phi = \Psi_x \in Y$, $\Phi \in H^1(\mathbb{R}^2) \cap L_{loc}^{\infty}(\mathbb{R}^2)$ and $\partial_y^2\Psi \in L_{loc}^2(\mathbb{R}^2)$, if $\alpha \geq 4$.

(ii) Assume that $d = 3$. The equation (2.1) does not admit any nontrivial solitary wave satisfying $\Phi = \Psi_x \in Y$, $\Phi \in H^1(\mathbb{R}^3) \cap L^{2(\alpha+1)}(\mathbb{R}^3) \cap L_{loc}^{\infty}(\mathbb{R}^3)$, $\partial_x^2\Phi$, $\partial_y^2\Psi$ and $\partial_z^2\Psi \in L_{loc}^{\infty}(\mathbb{R}^3)$, if $\alpha \geq \frac{4}{3}$.

Remark 2.1 : The assertion in (ii) rules out the existence of solitary waves in the physical case $\alpha = 2$, $d = 3$. This has been observed formally in [6].

Proof of Theorem 2.1 : It is very similar to the proof of Theorem 1.1 in [1] and based on Pohojaev type identities. The regularity assumptions of Theorem 2.1 are needed to justify them by a standard truncation argument which will be sketched here (see [1]).

In the two-dimensional case we first multiply (2.1)₁ by $x\Phi$ to get after several integrations by parts :

$$(2.3) \quad \int_{\mathbb{R}^2} \left[\frac{1}{2}\Phi^2 + \frac{3}{2}\Phi_x^2 - \frac{1}{2}\Psi_y^2 - \frac{\alpha+1}{\alpha+2}|\Phi|^{\alpha+2} \right] = 0.$$

Then we multiply (2.1)₁ by $y\Psi_y$ to get after several integration by parts

$$(2.4) \quad \int_{\mathbb{R}^2} \left[-\frac{1}{2}\Phi^2 - \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Psi_y^2 + \frac{1}{\alpha+2}|\Phi|^{\alpha+2} \right] = 0.$$

To prove a third identity we first remark that if $\Phi \in Y \cap L^{2(\alpha+1)}$ satisfies (2.1) in $\mathcal{D}'(\mathbb{R}^2)$, and if Y' is the dual space of Y , then Φ satisfies

$$-\Phi + \Phi_{xx} + |\Phi|^{\alpha}\Phi - D_x^{-1}\Psi_{yy} = 0 \text{ in } Y'$$

where $D_x^{-1}\Psi_{yy} \in Y'$ is defined by $\langle D_x^{-1}\Psi_{yy}, g \rangle_{Y'} = (\Psi_y, D_x^{-1}g_y)$ for any $g \in Y$. Here (\cdot, \cdot) denotes the scalar product in L^2 . Taking then the product of this last equation with $\Phi \in Y$, we obtain

$$(2.5) \quad \int_{\mathbb{R}^2} [-\Phi^2 - \phi_x^2 - \Psi_y^2 + |\Phi|^{\alpha+2}] = 0.$$

By a suitable combination of (2.3), (2.4), (2.5) we get easily

$$(2.6) \quad \int_{\mathbb{R}^2} \left[2\Phi^2 + \left(\frac{\alpha-4}{\alpha+2} \right) |\Phi|^{\alpha+2} \right] = 0,$$

which rules out the existence of solitary waves when $\alpha \geq 4$.

The case $d = 3$ is treated in a similar way. We sketch a formal proof which again can be made rigorous by a truncation argument. We multiply successively (2.1)₁ by $x\Phi$, $y\Psi_y$, $z\Psi_z$ and integrate to get

$$(2.7) \quad \int_{\mathbb{R}^3} \left[\frac{1}{2}\Phi^2 - \frac{1}{2}\Psi_y^2 - \frac{1}{2}\Psi_z^2 + \frac{3}{2}\Phi_x^2 - \frac{\alpha+1}{\alpha+2}|\Phi|^{\alpha+2} \right] = 0.$$

$$(2.8) \quad \int_{\mathbb{R}^3} \left[-\frac{1}{2}\Phi^2 + \frac{1}{2}\Psi_y^2 - \frac{1}{2}\Psi_z^2 - \frac{1}{2}\Phi_x^2 + \frac{1}{\alpha+2}|\Phi|^{\alpha+2} \right] = 0,$$

$$(2.9) \quad \int_{\mathbb{R}^3} \left[-\frac{1}{2}\Phi^2 - \frac{1}{2}\Psi_y^2 + \frac{1}{2}\Psi_z^2 - \frac{1}{2}\Phi_x^2 + \frac{1}{\alpha+2}|\Phi|^{\alpha+2} \right] = 0.$$

Integrating (2.1)₁ once in x and taking the duality product of the resulting equation with $\Phi \in Y$ as in dimension 2, one obtains

$$(2.10) \quad \int_{\mathbb{R}^3} [-\Phi^2 - \Phi_x^2 + |\Phi|^{\alpha+2} - \Psi_y^2 - \Psi_z^2] = 0.$$

We subtract (2.10) from two times (2.7) to get

$$(2.11) \quad \int_{\mathbb{R}^3} \left[2\Phi^2 + 4\Phi_x^2 - \left(\frac{3\alpha+4}{\alpha+2} \right) |\Phi|^{\alpha+2} \right] = 0.$$

Eliminating the $|\Phi|^{\alpha+2}$ term between (2.8) and (2.11) and using (by (2.8)-(2.9)) that

$$\int_{\mathbb{R}^3} [\Psi_y^2 - \Psi_z^2] = 0,$$

we obtain

$$(2.12) \quad \int_{\mathbb{R}^3} \left[\frac{3\alpha}{2}\Phi^2 + \frac{3\alpha-4}{2}\Phi_x^2 \right] = 0$$

which achieves to prove (ii). ▀

We proceed now to the existence of solitary waves.

Theorem 2.2.

(i) For $d = 2$ (resp. $d = 3$), the system (2.1) admits a solitary wave if $1 \leq \alpha < 4$ (resp. $1 \leq \alpha < \frac{4}{3}$).

(ii) Any solitary wave Φ is continuous and tends to zero at infinity. In the case $\alpha = 2$,

$$\Phi \in H^\infty(\mathbb{R}^2) = \bigcap_{k \geq 0} H^k(\mathbb{R}^2).$$

Proof : The proof of (i) is identical to the corresponding one for the solitary waves of the generalized KP equations (see [1] Theorems 3.1 and 3.2). It consists in considering the minimization problem

$$(2.13) \quad I_\lambda = \text{Inf} \{ \|\Phi\|_Y^2, \Phi \in Y, \text{ with } \int_{\mathbb{R}^d} |\Phi|^{\alpha+2} dx dx' = \lambda \},$$

where $x' = y$ if $d = 2$, $x' = (y, z)$ if $d = 3$ and $\lambda > 0$, and to use the concentration-compactness principle of P.L. Lions [7].

Once a minimizer $\Phi \in Y$ has been found, when $d = 3$ for example, there exists a Lagrange multiplier θ such that

$$(2.14) \quad -\Phi_{xx} + \Phi + D_x^{-1}v_y + D_x^{-1}w_z = \theta \Phi |\Phi|^\alpha \text{ in } Y'(\mathbb{R}^3)$$

where $D_x^{-1}v_y$ (resp. $D_x^{-1}w_z$) is the element of Y' (the dual of Y in the L^2 -duality) such that for any $\zeta \in Y$,

$$\begin{aligned} \langle D_x^{-1}v_y, \zeta \rangle_{Y', Y} &= (v, D_x^{-1}\zeta_y)_{L^2} \\ \text{(resp. } \langle D_x^{-1}w_z, \zeta \rangle_{Y', Y} &= (w, D_x^{-1}\zeta_z)_{L^2}). \end{aligned}$$

Moreover one easily checks (taking the $\langle Y', Y \rangle$ duality of (2.14) with Φ) that $\theta > 0$.

By taking the x -derivative of (2.14) in $\mathcal{D}'(\mathbb{R}^3)$, using the definition of $v = D_x^{-1}\Phi_y = \Psi_y$ (resp. $w = D_x^{-1}\Phi_z = \Psi_z$), and performing the scale change $\tilde{\Phi} = \theta^{1/\alpha}\Phi$, one sees that $(\tilde{\Phi}, \tilde{\Psi})$ satisfies (2.1) with $\omega = 1$ in $\mathcal{D}'(\mathbb{R}^3)$.

The proof of (ii) is the same as the corresponding one for the solitary waves of generalized KP equations (see [1 ; Theorem 4.1] and [2 ; Theorem 1.1]).

We shall conclude this chapter by stating qualitative properties of (1.6). To start with we prove that they decay to zero at infinity with an algebraic rate.

Theorem 2.3

(i) Any nontrivial solitary wave Φ of (1.6) in \mathbb{R}^2 satisfies

$$(2.15) \quad r^2 \Phi \in L^\infty(\mathbb{R}^2), \quad r^2 = x^2 + y^2.$$

(ii) Any nontrivial solitary wave Φ of (1.6) in \mathbb{R}^3 satisfies

$$(2.16) \quad r^\delta \Phi \in L^2(\mathbb{R}^3), \quad \forall \delta, \quad 0 \leq \delta < 3/2, \quad r = (x^2 + y^2 + z^2)^{1/2}.$$

Remark 2.2: The proof of Theorem 2.3 is valid for any Y solution of the elliptic equation (2.2). It is based on a careful analysis of the convolution equation equivalent to (2.2) (see [2]).

The reason why Φ cannot decay fast at infinity is obvious after writing (2.2) in Fourier variables, namely

$$(2.17) \quad \hat{\Phi} = -\frac{\xi_1^2}{|\xi|^2 + \xi_1^4} |\widehat{\Phi} |^\alpha \widehat{\Phi},$$

where ξ is the dual variable of x .

If Φ were exponentially decaying at infinity, $\hat{\Phi}$ and $|\widehat{\Phi} |^\alpha \widehat{\Phi}$ would be analytic by the Paley-Wiener theorem. But this is absurd since $\xi \mapsto \frac{\xi_1^2}{|\xi|^2 + \xi_1^4}$ is not analytic at the origin. Moreover it is even unlikely that Φ is integrable because if it were the case $\hat{\Phi}$ would be continuous, which is absurd in view of (2.17) except if $\int_{\mathbb{R}^d} |\Phi|^\alpha \Phi = 0$ (a fact that we don't know how to disprove).

We conclude this chapter by a symmetry property of the solitary waves. Unlike the previous ones it is valid only for the minimizers of (2.13) and not for any nontrivial solution of the elliptic problem (2.2).

A ground state is a solitary wave which minimizes the action

$$S(\Phi) = E(\Phi) + \frac{\omega}{2} \int_{\mathbb{R}^d} |\Phi|^2, \quad d = 2, 3$$

among all the nonzero solutions of (2.1), where E is the energy defined for $\Phi \in Y$ by

$$E(\Phi) = \frac{1}{2} \int_{\mathbb{R}^2} \left(\Phi_x^2 + (D_x^{-1} \Phi_y)^2 \right) - \frac{1}{(\alpha + 2)} \int_{\mathbb{R}^2} |\Phi|^{\alpha+2}$$

if $d = 2$, and

$$E(\Phi) = \frac{1}{2} \int_{\mathbb{R}^2} \left(\Phi_x^2 + (D_x^{-1} \Phi_y)^2 + (D_x^{-1} \Phi_z)^2 \right) - \frac{1}{(\alpha + 2)} \int_{\mathbb{R}^3} |\Phi|^{\alpha+2}$$

if $d = 3$.

Proceeding as in [2] for the solitary waves of the generalized KP equations, one can show that any solution constructed in Theorem 2.2 is a ground state and furthermore :

Theorem 2.4. *Let $x' = y \in \mathbb{R}$ if $d = 2$ and $x' = (y, z) \in \mathbb{R}^2$ if $d = 3$; then, up to a translation of the origin of coordinates in x' , any ground state Φ is radial in x' , that is Φ depends only on x and $|x'|$.*

3. Some remarks and open questions

1. Nothing is known concerning the uniqueness of the ground states of (2.1).
2. The existence of travelling waves of (1.6), that is solutions of the form $\varphi(x, x', t) = e^{i\omega t} \Phi(x - ct, x')$, $c \neq 0$ is open.
3. We conjecture that the solitary waves of (1.6) are unstable if $\alpha \geq 4/3$, $d = 2$ (resp. if $\alpha \geq 4/5$, $d = 3$).

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Structure of Positive Radial Solutions to Scalar Field Equations

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Abstract: We will prove classification results on positive radial solutions to nonhomogeneous scalar field equations in space dimension $N \geq 3$. The results will be given in terms of the data only.

1 Introduction

1.1 During the last two decades many authors considered questions of existence and uniqueness of nonnegative solutions to semilinear elliptic equations in \mathbf{R}^N that converge to zero if $|x| \rightarrow \infty$, the so-called ground states. This paper wants to contribute to this field by presenting a systematic approach to the classification problem for radial solutions to the scalar field equation

$$\Delta u - u + Q(|x|)u^p = 0 \quad \text{in } \mathbf{R}^N \quad (1.1)$$

with $N \geq 3$, $p > 1$ and nonnegative, continuous Q . This equation arises for instance in the study of standing wave solutions to Klein–Gordon– and nonlinear Schrödinger equations (see [1]).

The main results on this problem so far are as follows: The existence of positive solutions is ensured by the condition $Q(r) \leq C(1 + r^l)$ for some $l \in [0, (N - 1)(p - 1)/2]$ [3]. On the other hand there are no positive solutions if $Q(r)r^{-(N-1)(p-1)/2}$ is nondecreasing [8]. Uniqueness of ground states could be proved by Kwong [6] in case $Q \equiv 1$ and by Yanagida [9] for Q and rQ'/Q decreasing.

As to the structure of the set of radial solutions, little is known. Using a method based

on Pohozaev type identities, Yanagida and Yotsutani [10], [11] succeeded in solving this problem, provided an auxiliary initial value problem (see (1.4) below) could be solved explicitly. So this approach remained restricted to very special nonhomogenities. Here we will show how to avoid the use of such explicitly given functions in applying their technique. We will state our results in terms of the data only.

1.2 Formulation of the Problem. Let $u = u(r)$ be a radially symmetric solution of the scalar field equation (1.1), i.e. let u satisfy

$$\begin{aligned} u'' + \frac{N-1}{r}u' - u + Q(r)u^p &= 0 & \text{in } \mathbf{R}^+ := (0, \infty) \\ u(0) = a, \quad u'(0) &= 0. \end{aligned} \quad (1.2)$$

Here Q is subject to the following conditions:

$$(Q) \quad \begin{cases} Q \geq 0, & r^{N-1}Q \in L^1(0, 1), \\ Q \in C^0(\mathbf{R}^+) & e^{-(p+1)r}r^{-(N-1)(p-1)/2}Q \in L^1(1, \infty). \end{cases} \quad (1.3)$$

Some results will require higher regularity of Q , which we will not state explicitly. As a rule all derivatives of Q that occur in the proofs should be continuous.

Let us introduce the auxiliary function ϕ , that solves the linear problem

$$\begin{aligned} \phi'' + \frac{N-1}{r}\phi' - \phi &= 0 & \text{in } \mathbf{R}^+ \\ \phi(0) = 1, \quad \phi'(0) &= 0. \end{aligned} \quad (1.4)$$

Note that ϕ is positive and $(\frac{r\phi'}{\phi}) \rightarrow r - \frac{N-1}{2} + o(1)$ if $r \rightarrow \infty$. Defining w by $u = w \cdot \phi$, we easily calculate that w solves

$$(P) \quad \begin{cases} (g(r)w')' + K(r)g(r)w^p = 0 & \text{in } \mathbf{R}^+, \\ w(0) = a, \quad w'(0) = 0, \end{cases} \quad (1.5)$$

where

$$K(r) := |\phi(r)|^{p-1}Q(r), \quad g(r) := r^{N-1}|\phi(r)|^2. \quad (1.6)$$

It is well known that problem (P) has a unique solution $w \in C^0([0, \infty)) \cap C^2((0, \infty))$, which is of one of the following types:

- (i) $w(\cdot, a)$ is a zero-hit solution, i.e. $w(\cdot, a)$ has a zero in \mathbf{R}^+ .
- (ii) $w(\cdot, a)$ is a slow-decay solution, i.e. $w(\cdot, a) > 0$ in \mathbf{R}^+ and $\lim_{r \rightarrow \infty} \left[\int_r^\infty \frac{1}{g} \right] w(r, a) = \infty$.
- (iii) $w(\cdot, a)$ is a fast-decay solution, i.e. $w(\cdot, a) > 0$ in \mathbf{R}^+ and $\lim_{r \rightarrow \infty} \left[\int_r^\infty \frac{1}{g} \right] w(r, a)$ exists and is finite.

Moreover, writing the problem in divergence form, we arrived at a formulation that is suitable for applying the technique of Yanagida and Yotsutani (see [10], [11]). Let us state their main classification result in case that g and K are given as in (1.6):

Theorem. [11] Let Q satisfy (1.3) and define

$$\begin{aligned} G(r) &:= \frac{2}{p+1}g(r)^2 \left[\int_r^\infty \frac{1}{g} \right] |\phi(r)|^{p-1}Q(r) - \int_0^r g(s)|\phi(s)|^{p-1}Q(s)ds \\ H(r) &:= \frac{2}{p+1}g(r)^2 \left[\int_r^\infty \frac{1}{g} \right]^{p+2} |\phi(r)|^{p-1}Q(r) - \int_r^\infty g(s) \left[\int_s^\infty \frac{1}{g} \right]^{p+1} |\phi(s)|^{p-1}Q(s)ds \end{aligned}$$

and

$$r_G := \inf \{ r \in \mathbf{R}^+ | G(r) < 0 \}, \quad r_H := \sup \{ r \in \mathbf{R}^+ | H(r) < 0 \}. \quad (1.7)$$

In case that $G(r) \geq 0$ define $r_G = \infty$ and in case that $H(r) \geq 0$ define $r_H = 0$.

Then the structure of the set of positive solutions to (P) is as follows:

- (a) If $r_G = \infty$ and $G \neq 0$, then the structure is of type Z, i.e. $w(\cdot, a)$ is a zero-hit solution for all $a > 0$.
- (b) If $r_G < \infty$ and $r_H = 0$, then the structure is of type S, i.e. $w(\cdot, a)$ is a slow-decay solution for all $a > 0$.
- (c) If $0 < r_H \leq r_G < \infty$, then the structure is of type M, i.e. there is an unique $a_0 > 0$, such that $w(\cdot, a)$ is a slow-decay solution for all $a < a_0$, a fast-decay solution for $a = a_0$ and a zero-hit solution for all $a > a_0$.
- (d) If $G \equiv 0$, then $w(\cdot, a)$ is a fast-decay solution for all $a > 0$.

In the sequel we will derive criteria on Q that imply the assumptions in this theorem.

2 Behaviour of G and H

The information on the structure of the set of positive solutions to (1.2) lies in the shape and the the relation of the positivity sets of the functions of G and H . We will try to gain some of this information via the knowledge of the limit behaviour of G and H near $r = 0$ and $r = \infty$ and the possible number of their extrema.

As a first step we calculate the derivatives of G and H .

2.1 Lemma.

$$G'(r) = \frac{2}{p+1}g(r)|\phi(r)|^{p-1}Q(r) \left\{ \Phi(r) - \frac{p+3}{2} \right\} = H'(r) \left[\int_r^\infty \frac{1}{g} \right]^{-(p+1)}, \quad (2.1)$$

where $\Phi(r)$ is given by

$$\Phi(r) := \frac{g(r)}{r} \left[\int_r^\infty \frac{1}{g} \right] \underbrace{\left(2(N-1) + (p+3) \left(\frac{r\phi'}{\phi} \right) + \left(\frac{rQ'}{Q} \right) \right)}_{=:s_1(r)}. \quad (2.2)$$

Note that the extrema of G and H are located at the same points in \mathbf{R}^+ .

2.2 Lemma on Φ . (i)

$$\int_r^\infty \frac{1}{g(t)} \underbrace{(2(N-2) + 4(\frac{t\phi'}{\phi}))}_{=: 2/(p+3)s_2(t)} dt, \Phi(r) = 2 \int_r^\infty \frac{1}{g(t)} dt s_1(r) \quad (2.3)$$

(ii) $\lim_{r \rightarrow 0} \Phi(r) = \frac{2(N-1) + (\frac{rQ'}{Q})(0)}{N-2}$.

(iii) $\lim_{r \rightarrow \infty} \Phi(r) = \frac{p+3+q_\phi}{2}$, where $q_\phi := \lim_{r \rightarrow \infty} (\frac{rQ'}{Q}) / (\frac{r\phi'}{\phi})$.

The proof mainly uses integration by parts and l'Hospital's rule.

2.3 A Technical Lemma. For large r we have

(i)

$$g(r) \int_r^\infty \frac{1}{g(t)} dt = \frac{1}{2} + o(\frac{1}{r}).$$

(ii) If $q_\phi = 0$, then

$$\frac{1}{g(r)|\phi(r)|^{p-1}Q(r)} \int_0^r g(t)|\phi(t)|^{p-1}Q(t) dt = \frac{1}{p+1} + \frac{\lambda_1 - \lim_{r \rightarrow \infty} (\frac{rQ'}{Q})}{(p+1)^2} \frac{1}{r} + o(\frac{1}{r}).$$

In deriving these identities we make use of the equation satisfied by ϕ , integration by parts and the asymptotic behaviour of $(\frac{r\phi'}{\phi})$ near infinity.

2.4 Lemma on G . (i) $G(0) = 0$, $\text{sign } G'(0) = \text{sign } \{\Phi(0) - \frac{p+3}{2}\}$.

(ii)

$$\lim_{r \rightarrow \infty} G(r) = \begin{cases} -\infty & \text{if } \lim_{r \rightarrow \infty} (\frac{rQ'}{Q}) < \lambda_1 \\ \infty & \text{if } \lim_{r \rightarrow \infty} (\frac{rQ'}{Q}) > \lambda_1. \end{cases}$$

The idea in proving (ii) is to insert the asymptotic expansions of Lemma 2.3 in the definition of G : If $q_\phi \neq 0$ the statement is clear. Otherwise the sign of the leading term in the expansion depends on the relation of $(\frac{rQ'}{Q})$ and λ_1 at infinity.

2.5 Lemma on H . (i)

$$\lim_{r \rightarrow 0} H(r) \begin{cases} \geq 0 & \text{if } \lim_{r \rightarrow 0} (\frac{rQ'}{Q}) < \lambda_0 \\ \leq 0 & \text{if } \lim_{r \rightarrow 0} (\frac{rQ'}{Q}) > \lambda_0 \end{cases}$$

(ii) $\lim_{r \rightarrow \infty} H(r) = 0$.

Let us remark that at this stage we could prove existence results for positive solutions similar to those of Ding & Ni [3]. However, we focus on structure results here.

3 Structure Results

3.1 Proposition. If $(\frac{rQ'}{Q}) \leq \lambda_0$ in $[r_0, \infty)$, then $\Phi(r) < \frac{p+3}{2}$ in $[r_0, \infty)$.

Proof: As $(\frac{r\phi'}{\phi})$ is monotone increasing and we have the above condition on Q , we can estimate for $r \in [r_0, \infty)$

$$\int_r^\infty \frac{1}{g(t)} s_2(t) dt > \left[\int_r^\infty \frac{1}{g(t)} dt \right] s_2(r) \geq \left[\int_r^\infty \frac{1}{g(t)} dt \right] s_1(r).$$

As $\Phi(r) \leq \frac{p+3}{2}$ near infinity from Lemma 2.4 and we have identity (2.3), the estimate follows immediately.

As an immediate consequence from Proposition 3.1 and (b) of the structure theorem in 1.2 we can state the following

3.2 Theorem. If $(\frac{rQ'}{Q}) \leq \lambda_0$ in \mathbf{R}^+ , then the structure of solutions to (1.2) is type S.

If we allow $(\frac{rQ'}{Q})$ to have larger values, extrema of G may occur. Then we use that the sign of ϕ' at successive points where $\Phi = \frac{p+3}{2}$ (or, equivalently, $G'(0) = 0$) must change. In other words, if $\Phi'|_{\Phi=\frac{p+3}{2}}$ does not change its sign twice, G can have at most one extremum. In this case we can obtain enough informations on G and H from the asymptotic behaviour to locate r_G and r_H .

In order to do so, we calculate

$$\Phi'(r)|_{\Phi(r)=\frac{p+3}{2}} = \frac{1}{rs_1(r)} \underbrace{\left\{ (\lambda_0 - (\frac{rQ'}{Q}))s_1(r) + \frac{p+3}{2}rs_1'(r) \right\}}_{=:\Psi(r)}.$$

In general the information we have on s_1 is not precise enough to prove something at this stage. So we play the same trick again, which we will show here for the case $(\frac{rQ'}{Q}) = \text{const}$.

3.3 Theorem. If $Q(r) = r^s$, then the structure of the set of positive solutions is

- (i) of type S, if $s \leq \lambda_0$;
- (ii) of type M, if $s \in (\lambda_0, \lambda_1)$;
- (iii) of type M or type Z, if $s = \lambda_1$;
- (iv) of type Z, if $s > \lambda_1$.

Proof: Due to 3.2 statement (i) is clear. Otherwise straightforward calculation show that rs_1'/s_1 can cross the level 1 at most once; to be precise: If $s > \lambda_1$, then $rs_1'/s_1 < 1$ and if $s \in (\lambda_0, \lambda_1)$, then there is a $r_0 > 0$, such that $rs_1'/s_1 < 1$ in $(0, r_0)$ and $rs_1'/s_1 > 1$ in (r_0, ∞) . As

$$\begin{aligned} r\Psi'(r)|_{\Psi(r)=0} &= \frac{2}{p+3} \left(\lambda_0 - (\frac{rQ'}{Q}) \right) s_1(r)^2 + (p+3)^2 r^2 \quad \text{and} \\ r \left(r\Psi'(r)|_{\Psi(r)=0} \right)' \Big|_{(r\Psi'(r)|_{\Psi(r)=0})=0} &= 2(p+3)^2 r^2 \left(1 - \frac{rs_1'(r)}{s_1(r)} \right), \end{aligned}$$

this in combination with the limit behaviour of $r\Psi'|_{\Psi=0}$ implies that this function is negative if $s > \lambda_1$ and that it has exactly one sign change (from minus to plus) in case $s \in (\lambda_0, \lambda_1)$. Tracing the way back to Ψ , Φ , observing carefully the limit behaviour of these functions, we see that G and H have at most one extremum. Thus from the limit behaviour of G and H we conclude (ii) and (iv). (iii) can be proved similarly by additionally taking care of Ψ'' in a point where Ψ and Ψ' vanish. \square

Besides the applicability to other nonhomogenities Q these arguments can be further developed to cover the critical cases left out in the lemmata 2.4 and 2.5 by calculating a more precise asymptotic expansion in 2.3. We intend to come back to this problem in a forthcoming paper.

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Asymptotic Integrability of Physical Equations

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1 Introduction

Many physical situations can be studied asymptotically using the so-called multi-scale expansion method (see for example [1]). This method yields a basic evolution equation which is formally valid at the leading asymptotic order, as well as a sequence of evolution equations which are formally valid at higher asymptotic orders. It turns out that for many important physical systems the basic evolution equation is an **integrable** equation (see [2] for a discussion of this remarkable fact). Each integrable evolution equation is a member of a hierarchy of infinitely many integrable equations. It is interesting that the evolution equation valid at the next asymptotic order, differs from the next member of the associated integrable hierarchy, only in the value of the numerical coefficients of the nonlinear terms. For example, idealized unidirectional water waves of small amplitude and large wave length satisfy [3] the equation

$$u_t + u_{xxx} + 6uu_x + \varepsilon(\alpha_1 u_{xxxxx} + \alpha_2 uu_{xxx} + \alpha_3 u_x u_{xx} + \alpha_4 u^2 u_x) + O(\varepsilon^2) = 0, \quad (1.1)$$

where $\alpha_1, \dots, \alpha_4$ are certain numbers. As $\varepsilon \rightarrow 0$, this equation becomes the Korteweg-deVries (KdV) equation,

$$v_t + v_{xxx} + 6vv_x = 0, \quad (1.2)$$

which is an integrable equation. Furthermore, the $O(\varepsilon)$ terms of equation (1.1) differ from the equation

$$v_t + v_{xxx} + 6vv_x + \varepsilon\alpha_1(v_{xxxxx} + 10vv_{xxx} + 20v_xv_{xx} + 30v^2v_x) = 0 \quad (1.3)$$

only in the numerical coefficients of the $O(\varepsilon)$ nonlinear terms. Equation (1.3) is the next number of the hierarchy of integrable equations associated with the KdV equation.

If the basic evolution equation is integrable, we say that the underline physical system is *asymptotically integrable* to $O(\varepsilon)$. It turns out that in certain cases it is possible to *formally extend the asymptotic integrability of the system to $O(\varepsilon^2)$* . For example, in the case of water waves, Kodama found [4] an explicit transformation which maps equation (1.1) to the integrable equation (1.3).

We have recently shown [5] that the *concept of the mastersymmetries* (introduced by the author and Fuchssteiner in the early 80's [6]) *provides an algorithmic approach to finding the transformations which map the physical equations to the integrable ones*.

Let $\tau(x, t)$ denote the mastersymmetry of the integrable equation

$$v_t + M(v) = 0, \quad (1.4)$$

where $M(v)$ denotes a smooth function depending on the variable v , and on its x derivatives v_x, v_{xx}, \dots . The defining property of the mastersymmetry is that

$$[\tau(x, t), M(v)]_L = M_1(v), \quad (1.5)$$

where $M_1(v)$ is the next commuting flow of the associated hierarchy of integrable equations; $[\cdot, \cdot]_L$ denotes the Lie commutator defined by $[A, B]_L = A'B - B'A$, and prime denotes Fréchet differentiation,

$$A'(u)B = \left. \frac{\partial}{\partial \varepsilon} A(u + \varepsilon B) \right|_{\varepsilon=0}, \text{ i.e. } A' = \frac{\partial A}{\partial u} + \frac{\partial A}{\partial u_x} \partial_x + \frac{\partial A}{\partial u_{xx}} \partial_x^2 + \dots; \partial_x = \frac{\partial}{\partial x}.$$

For example the mastersymmetry of the KdV is

$$\tau(x, t) = 8v^2 + 4v_{xx} + 2v_x \partial_x^{-1} v + x(v_{xxx} + 6vv_x). \quad (1.6)$$

Indeed, if

$$M(v) = v_{xxx} + 6vv_x, \quad (1.7)$$

then

$$[\tau(x, t), M(v)]_L = v_{xxxxx} + 10vv_{xxx} + 20v_xv_{xx} + 30v^2v_x, \quad (1.8)$$

which is the next commuting flow of the KdV hierarchy.

The importance of mastersymmetries follows from the following observation: Let v solve the equation (1.4). Let u be defined by

$$u = v + \varepsilon P(v). \quad (1.9)$$

Then u solves

$$u_t + M(u) + \varepsilon[P(u), M(u)]_L + O(\varepsilon^2) = 0. \quad (1.10)$$

If $P(u) = \tau(x, t)$, then the $O(\varepsilon)$ term of equation (1.10) becomes $M_1(u)$. If the $O(\varepsilon)$ term of equation (1.10) differs from M_1 only in numerical coefficients, it follows that the *transformation* $P(u)$ mapping an integrable equation (equation (1.4)) to a physical equation (equation (1.10)) can be constructed from the mastersymmetry of the integrable equation, by replacing the numerical coefficients in the mastersymmetry $\tau(x, t)$ with arbitrary constants.

Using the mastersymmetry of the KdV we have shown that equation (1.1) can be mapped to KdV itself (instead of equation (1.3)). Also we have shown that equation (1.1) can be mapped to other integrable equations. These equations are integrable generalizations of the KdV equation and of the Gardner equation (a linear combination of KdV and of the modified KdV equation).

2 Equations Related to KdV and to Nonlinear Schrödinger

Proposition 2.1 [5]. (i) Let v solve the KdV equation (1.2). Let u be defined by

$$u = v + \varepsilon \left(\lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial_x^{-1} v + \lambda_4 x (v_{xxx} + 6v v_x) \right), \quad (2.1)$$

where $\lambda_1 = 283/180$, $\lambda_2 = -37/60$, $\lambda_3 = 203/45$, $\lambda_4 = -19/30$, and ∂_x^{-1} denotes integration with respect to x . Then u solves equation (1.1) with $\alpha_1 = 19/10$, $\alpha_2 = 5/3$, $\alpha_3 = 23/6$, $\alpha_4 = -1/6$, which are the numbers appearing in the modeling of unidirectional idealized water waves.

(ii) Let v solve the integrable equation

$$v_t + v_{xxx} + 6v v_x + \nu \epsilon (v_{xxt} + 2v v_{xxx} + 4v_x v_{xx}) = 0. \quad (2.2)$$

Let u be defined by

$$u = v + \epsilon \left(\lambda_1 v^2 + \lambda_2 v_{xx} + \lambda_3 v_x \partial_x^{-1} v \right), \quad (2.3)$$

where $\lambda_1 = -173/180$, $\lambda_2 = -113/60$, $\lambda_3 = 89/45$, and let $\nu = -19/10$. Then u solves equation (1.1), where the values of $\alpha_1, \dots, \alpha_4$ are given above.

(iii) Let v solve the integrable equation obtained by adding the terms

$$3\rho\epsilon v^2 v_x + \rho\nu\epsilon^2 (v^2 v_{xxx} + 4v v_x v_{xx} + v_x^3) + \rho\nu^2\epsilon^3 (v_x^2 v_{xxx} + 2v_x v_{xx}^2), \quad (2.4)$$

to the lhs of equation (2.2). Let u be defined by

$$u = v + \epsilon (\lambda_1 v^2 + \lambda_2 v_x \partial_x^{-1} v), \quad (2.5)$$

where $\lambda_1 = 101/36$, $\lambda_2 = 89/45$, and let $\nu = -19/10$, $\rho = 113/15$. Then u solves equation (1.1) where the values of $\alpha_1, \dots, \alpha_4$ are given above.

Remark 2.1 Equation (2.2) was first derived in [7] using the bi-Hamiltonian approach (equation (2.2) is equation (26e) and (30) of [7] and equation (5.3) of [8]). The Lax pair of equation (2.2) and an interesting class of its solutions, called peakons, were given in [9]. The linearization of equation (2.2) using the inverse spectral method is given in [10]. Equation (2.2) has a rich mathematical structure only if $\nu = O(\epsilon^{-1})$; if $\nu = O(1)$ the spectral theory of equation (2.2) is very similar to that of the KdV equation [10].

The results of Proposition 2.1 can be generalized as follows. Let $v_t + M(v) = 0$ denote the KdV, or equation (2.2), or equation (2.4). Let v solve the equation $v_t + M(v) + \epsilon M_1(v) = 0$, where $M_1(v)$ denotes the first commuting flow of the KdV, or of equation (2.2), or of equation (2.4). Then it is possible to find a transformation of the form $u = v + \epsilon P(v)$, such that u solves (1.1). In the case that M is given by equation (1.7) and M_1 is given by (1.8), $P(v)$ is of the form of equation (2.3), and this is precisely Kodama's result. In the other two cases $P(v)$ involves $\lambda_1 v^2 + \lambda_2 v_{xx}$, and $\lambda_1 v^2$, respectively.

Proposition 2.1 and the above remark show that there exists a trade off between the complexity of the integrable equation and of the associated transformation. Indeed, KdV is the simplest equation, while equation (2.1) defines the most complicated transformation. The extreme opposite of the KdV equation is equation (2.4) together with its first commuting flow; in this case the associated transformation takes the simplest form $u = v + \epsilon \lambda v^2$.

Proposition 2.2 Let $v(x, t)$ solve the nonlinear Schrödinger equation

$$iv_t + v_{xx} + \rho|v|^2v = 0,$$

where v is a complex valued function and ρ is a constant. Let u be defined by

$$u = v + \epsilon \left[\lambda_1 v_x + \lambda_2 v \partial^{-1} |v|^2 + \lambda_3 x (v_{xx} + \rho |v|^2 v) \right]. \quad (2.6)$$

Then u solves the physically important equation [11], [12]

$$iu_t + u_{xx} + \rho |u|^2 u + \epsilon \left[\alpha_1 u_{xxx} + \alpha_2 (|u|^2 u)_x + \alpha_3 u (|u|^2)_x \right] + O(\epsilon^2) = 0. \quad (2.7)$$

3 Equations Related to Burgers and to Benjamin-Ono Equations

Proposition 3.1 [11]. Let $v(x, t)$ satisfy the Burgers equation

$$v_t = v_{xx} + 2vv_x. \quad (3.1)$$

Let $u(x, t)$ be defined by

$$u = v + \epsilon \left(\frac{1}{2} (\alpha_3 - 2\alpha_1) v^2 + \frac{\alpha_2 - 3\alpha_1}{2} v_x \partial_x^{-1} v + \frac{\alpha_1}{2} x (v_{xx} + 2vv_x) \right), \quad (3.2)$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants. Then u solves

$$u_t = u_{xx} + 2uu_x + \epsilon \left[\alpha_1 u_{xxx} + \alpha_2 uu_{xx} + \alpha_3 u_x^2 + \left(\alpha_3 + \frac{\alpha_2}{2} - \frac{3}{2} \alpha_1 \right) u^2 u_x \right] + O(\epsilon^2). \quad (3.3)$$

Remark 3.1 Unfortunately there exists a linear relation among the coefficients of the $O(\varepsilon)$ terms of equation (3.3). This is to be contrasted with the cases of the equations related to KdV, and to nonlinear Schrödinger, where the coefficients of the $O(\varepsilon)$ terms are arbitrary. Further discussion of this linear constraint among the coefficient of equation (3.3) can be found in [12].

Equation (3.3) (without a linear constraint among its coefficients) can be obtained from a certain asymptotic expansion of the acoustic waves,

$$\rho_t + \rho_{xx} + 2\rho\rho_x + \varepsilon \left[\frac{\rho_{xxx}}{4} - \frac{1}{\gamma+1}\rho\rho_{xx} + \frac{\gamma-1}{2(\gamma+1)}\rho_x^2 + \left(3\frac{\delta+\gamma}{(\gamma+1)^2} - 1 \right) \rho^2\rho_x \right] + O(\varepsilon^2),$$

where $\varepsilon\rho$ is the perturbation of the density,

$$\gamma = \frac{P_0''\rho_0}{2a_0^2}, \quad \delta = \frac{P_0'''\rho_0^2}{6a_0^2}, \quad \varepsilon = -\frac{\mu}{2\rho_0 a_0 l},$$

$P_0 = P(\rho_0)$ is the equilibrium pressure, a_0 is the speed of sound, l is a typical length, prime denotes differentiation, and μ is viscosity. The constraint satisfied by the coefficients of equation (3.3) corresponds to

$$8\delta + 6\gamma + 1 - 3\gamma^2 = 0.$$

Proposition 3.2. [12] Let $v(x, t)$ satisfy the Benjamin-Ono equation,

$$v_t = 2vv_x + Hv_{xx}, (Hf)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi, \quad (3.4)$$

where f denotes a principal value integral. Let $u(x, t)$ be defined by

$$u = v + \frac{\varepsilon}{2} \left(\lambda_1 v^2 + \lambda_2 H v_x + \lambda_3 x(2vv_x + H v_{xx}) \right). \quad (3.5)$$

Then, u satisfies equation

$$\begin{aligned} u_t &= 2uu_x + Hu_{xx} \\ &+ \varepsilon [\alpha u_{xxx} + \beta_1 (uHu_x)_x + \beta_2 uHu_{xx} + \beta_3 H(uu_x)_x + \gamma u^2 u_x] + O(\varepsilon^2), \end{aligned} \quad (3.6)$$

where the coefficients of equation (3.6) satisfy the numerical constraints

$$3\alpha + \beta_1 + \beta_2 + \beta_3 = 0, \quad (3.7)$$

and

$$\beta_1 + \beta_3 - \gamma = 0, \quad (3.8)$$

Remark 3.2 A particular case of equation (3.6) occurs in the modeling of long internal waves in a deep continuously stratified fluid, and was recently derived in [13]. In this case u denotes the horizontal velocity of the fluid and the coefficients α, \dots, γ can be expressed through the parameters of the fluid stratification. The particular case that the stratification profile can be approximated by a two-layer model with density ρ_1 in the upper (shallow) layer and ρ_2 in the lower (deep) layer, was studied in [14] and is described by equation (3.3) with

$$\alpha = \frac{27}{4} \left(\frac{4\delta^2}{9} - 1 \right), \quad \beta_1 = 6, \quad \beta_2 = \frac{3}{2}, \quad \beta_3 = \frac{27}{2}, \quad \gamma = -3.$$

4 Rigorous Considerations

It is possible to use the above formal results to study rigorously the initial value problem of the physical equations related to the integrable ones. A rigorous methodology for achieving this has been given in [15]. Let

$$u(x, 0) = u_0(x) + \varepsilon u_1(x). \quad (4.1)$$

Our methodology for solving the initial value problem for $u(x, t)$ with $u(x, 0)$ given by (4.1) involves the follow steps:

(i) Given u_0 and u_1 , define $\tilde{v}(x, 0, \varepsilon)$ and $v(x, 0, \varepsilon)$ by

$$\tilde{v}(x, 0, \varepsilon)u_0(x) + \varepsilon[u_1(x) - P(u_0(x))], \quad (4.2)$$

and by

$$u_0(x) + \varepsilon u_1(x) = v(x, 0, \varepsilon) + \varepsilon P(v(x, 0, \varepsilon)), \quad (4.3)$$

respectively.

(ii) Define $\tilde{v}(x, t, \varepsilon)$ and $v(x, t, \varepsilon)$ as the solutions of the integrable equation satisfying $\tilde{v}(x, 0, \varepsilon)$ and $v(x, 0, \varepsilon)$ respectively. Using the fact that $\tilde{v}(x, 0, \varepsilon)$ and $v(x, 0, \varepsilon)$ are close in the L_∞ norm, establish that

$$\sup_{x \in \mathbb{R}} |v(x, t, \varepsilon) - \tilde{v}(x, t, \varepsilon)| \leq C\varepsilon^2, \quad \text{for all } t, \quad (4.4)$$

where C is some time-independent constant.

(iii) Define $u(x, t, \varepsilon)$ and $\tilde{u}(x, t, \varepsilon)$ by

$$u(x, t, \varepsilon) = v(x, t, \varepsilon) + \varepsilon P(v(x, t, \varepsilon)), \quad (4.5)$$

and by

$$\tilde{u}(x, t, \varepsilon) = \tilde{v}(x, t, \varepsilon) + \varepsilon P(\tilde{v}(x, t, \varepsilon)), \quad (4.6)$$

respectively. Show that u and \tilde{u} are well defined, and furthermore obtain the large time behavior of $|u|_\infty$ and $|\tilde{u}|_\infty$. Using these results show that

$$|(u - \tilde{u})(\cdot, t, \varepsilon)|_\infty \leq C\varepsilon^2, \text{ for all } t > 0, \quad (4.7)$$

where C is time-independent constant.

The implementation of the above methodology to equations related to Burgers and to KdV is given in [15] and [16] respectively.

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A REMARK ON THE EXISTENCE OF THE NAVIER-STOKES FLOW WITH NON-VANISHING OUTFLOW CONDITION

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Abstract. The boundary value problem of the Navier-Stokes equations has been studied so far mainly under the vanishing outflow condition. We study this problem under non-vanishing outflow condition for a bounded domain D in \mathbf{R}^2 or \mathbf{R}^3 with multiply-connected smooth boundary. We consider the boundary value of the form $\mathbf{b} = \mu\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1$, where μ is a constant, $\boldsymbol{\beta}_0$ is the boundary value of gradient of a harmonic function and $\int_{\partial D} \boldsymbol{\beta}_i \cdot \mathbf{n} \, d\sigma = 0$ ($i = 0, 1$). We suppose also that the external force is a potential force. Then we can show the existence of solutions even for large μ with a discrete countable set of exceptional values, if $\boldsymbol{\beta}_1$ is sufficiently small.

1. Introduction

Let D be a bounded domain in \mathbf{R}^n , $n=2$ or 3 , with smooth boundary $\partial D = \cup_{i=1}^k \Gamma_i$ ($k \geq 2$), where Γ_i 's are connected components of the boundary and D is inside of Γ_k . Thus one might think that Γ_k is the outer wall of the vessel and Γ_i ($1 \leq i \leq k-1$) are the inside walls.

We consider the following stationary Navier-Stokes equations

$$(NS) \begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p & = \mathbf{f} & \text{in } D, \\ \operatorname{div} \mathbf{u} & = 0 & \text{in } D, \end{cases}$$

with the boundary condition

$$(BC) \mathbf{u} = \mathbf{b} \quad \text{on } \partial D.$$

The existence of solutions to (NS) (BC) is known in general context under the vanishing outflow condition

$$(H) \quad \int_{\Gamma_i} \mathbf{b} \cdot \mathbf{n} d\sigma = 0 \quad (1 \leq i \leq k),$$

where \mathbf{n} denotes the unit outward normal vector to the boundary ∂D (c.f. Leray [6], Ladyzhenskaya [5], Fujita [4]). The condition (H) is stringent than the following general outflow condition, which could be called here the non-vanishing outflow condition

$$(H)_0 \quad \int_{\partial D} \mathbf{b} \cdot \mathbf{n} d\sigma = 0$$

which is to be satisfied by the boundary value \mathbf{b} of any solenoidal vector \mathbf{u} . We are concerned with the existence proof of solution to (NS) (BC) where only $(H)_0$ is satisfied and (H) does not hold true. An affirmative result has been obtained by Amick [1] for 2-D case under a certain assumption of symmetry. Morimoto [7] [8], Morimoto-Ukai [9] showed examples for 2-D annular domain case.

In this paper, we shall show the existence of solutions to $(NS)(BC)$ and $(H)_0$ for a certain class of the boundary values for 2-D or 3-D general domain under consideration, namely without any hypothesis of symmetry. Specifically, we consider the boundary value problem

$$(NS)_\mu \quad \begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \\ \mathbf{u} = \mu \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 & \text{on } \partial D, \end{cases}$$

where μ is an arbitrary constant, $\boldsymbol{\beta}_0$ is the boundary value of gradient of a harmonic function and $\boldsymbol{\beta}_1$ satisfies $(H)_0$. Then we prove that the solution to $(NS)_\mu$ exists for every μ except for a discrete countable set which depends on $\boldsymbol{\beta}_0$ if $\boldsymbol{\beta}_1$ is sufficiently small (Theorem 1, Theorem 2).

A nontrivial example of such $\boldsymbol{\beta}_0$ is

$$\sum_{i=1}^{k-1} \nabla \left(\frac{q_i}{4\pi|x - a_i|} \right)$$

in 3-dimension case, where q_i 's are constants and a_i 's are points outside D , each a_i being enclosed by Γ_i .

2. Notations and results

We use the following function spaces. Let $L^2(D)$ be the set of all vector valued square integrable functions in D with the inner product (\cdot, \cdot) and the norm $\|\cdot\|$; $W_p^m(D)$ the Sobolev spaces; $H^m(D) = W_2^m(D)$; $C_{0,\sigma}^\infty(D)$ the set of all smooth solenoidal vector functions with compact support in D ; $H_\sigma = H_\sigma(D)$ the closure of $C_{0,\sigma}^\infty(D)$ in $L^2(D)$; $H_\sigma^1 = H^1(D) \cap H_\sigma(D)$; V the completion of $C_{0,\sigma}^\infty(D)$ in the Dirichlet norm $\|\nabla \cdot\|$. $\|\cdot\|_V$ stands for $\|\nabla \cdot\|$ while V' is the dual space of V , $\|\cdot\|_{V'}$ being the dual norm of $\|\cdot\|_V$.

By definition, $\mathbf{u} \in H_\sigma^1$ is called a weak solution to (NS) , (BC) if and only if

$$\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in V,$$

and $\mathbf{u} = \mathbf{b}$ on ∂D in the trace sense. If the weak solution \mathbf{u} is in $H^2(D)$, then it is called a strong solution.

Theorem 1. *Suppose that $\mathbf{f} \in V'$ is a potential force, that β_0 is the boundary value of gradient of a harmonic function $\varphi \in H^2(D)$, and that β_1 is in $H^{1/2}(\partial D)$ with*

$$\int_{\partial D} \beta_1 \cdot \mathbf{n} d\sigma = 0.$$

Then, there exists a discrete countable set $M \subset \mathbf{R}$ such that for each $\mu \in \mathbf{R} \setminus M$, there exists a weak solution to $(NS)_\mu$ if $\|\beta_1\|_{H^{1/2}(\partial D)} < C^$ for some positive constant $C^* = C^*(\nu, \mu, D, \beta_0)$.*

Remark 1. *For 2-D annular domain, a relevant result has been shown by Morimoto-Ukai [9] under a particular circumstance but for “large” β_1 .*

Remark 2. *If $|\mu|$, β_0 and β_1 are small, then the existence of the solution can be shown by means of a standard argument with Leray-Schauder’s principle (e.g. Ladyzhenskaya [5], Temam [10]).*

Theorem 2. *Suppose that $\mathbf{f} \in L^2(D)$ is a potential force, that β_0 is the boundary value of gradient of a harmonic function $\varphi \in H^3(D)$, and that β_1 is in $H^{3/2}(\partial D)$ with*

$$\int_{\partial D} \beta_1 \cdot \mathbf{n} d\sigma = 0.$$

Then, there exists a discrete countable set $M \subset \mathbf{R}$ such that for each $\mu \in \mathbf{R} \setminus M$, there exists a strong solution to $(NS)_\mu$ if $\|\beta_1\|_{H^{3/2}(\partial D)} < C_$ for some positive constant $C_* = C_*(\nu, \mu, D, \beta_0)$.*

3. Lemmas

By the Sobolev imbedding theorem and the Poincaré’s inequality, we have

Lemma 1. *There exist a positive constant c_1 which depends only on D , such that the inequality:*

$$\|\mathbf{w}\|_{L^4(D)} \leq c_1 \|\mathbf{w}\|_V, \quad \forall \mathbf{w} \in V$$

holds.

The next lemma is an easy consequence of integration by parts and Hölder’s inequality.

Lemma 2. *i) $((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w}) = -((\mathbf{u} \cdot \nabla) \mathbf{w}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in H_\sigma^1, \mathbf{w} \in V.$*
ii) $|((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w})| \leq \|\mathbf{u}\|_{L^4} \|\mathbf{v}\|_{L^4} \|\mathbf{w}\|_V, \quad \forall \mathbf{u}, \mathbf{v} \in H_\sigma^1, \mathbf{w} \in V.$

Lemma 3. *If $\mathbf{u}, \mathbf{v} \in H_\sigma^1$, then $(\mathbf{u} \cdot \nabla)\mathbf{v}$ belongs to V' and*

$$\|(\mathbf{u} \cdot \nabla)\mathbf{v}\|_{V'} \leq \|\mathbf{u}\|_{L^4}\|\mathbf{v}\|_{L^4}$$

holds.

Proof.

Let \mathbf{w} be an arbitrary element in V . According to Lemma 2,

$$|((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w})| = | - ((\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v}) | \leq \|\mathbf{u}\|_{L^4}\|\mathbf{v}\|_{L^4}\|\mathbf{w}\|_V.$$

Therefore, $(\mathbf{u} \cdot \nabla)\mathbf{v}$ belongs to V' and

$$\|(\mathbf{u} \cdot \nabla)\mathbf{v}\|_{V'} \leq \|\mathbf{u}\|_{L^4}\|\mathbf{v}\|_{L^4}.$$

Q.E.D.

Let $\varphi \in H_\sigma^1(D)$. Then, we define the operator $L(\varphi)$ through the identity:

$$v' \langle L(\varphi)\mathbf{w}, \mathbf{v} \rangle_{V'} \equiv ((\varphi \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\varphi, \mathbf{v}),$$

Lemma 4. *Let $\varphi \in H_\sigma^1(D)$. Then, the operator $L(\varphi)$ is a compact linear operator from $H_\sigma^1(D)$ to V' subject to:*

$$\|L(\varphi)\mathbf{w}\|_{V'} \leq 2\|\varphi\|_{L^4}\|\mathbf{w}\|_{L^4}, \quad \forall \mathbf{w} \in H_\sigma^1(D).$$

Proof.

Let \mathbf{w} be an arbitrary element in $H_\sigma^1(D)$. Then, according to Lemma 2, we can estimate as

$$\begin{aligned} & |((\varphi \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\varphi, \mathbf{v})| \\ &= | - ((\varphi \cdot \nabla)\mathbf{v}, \mathbf{w}) - ((\mathbf{w} \cdot \nabla)\mathbf{v}, \varphi) | \\ &\leq 2\|\varphi\|_{L^4}\|\mathbf{w}\|_{L^4}\|\nabla\mathbf{v}\|_{L^2} = 2\|\varphi\|_{L^4}\|\mathbf{w}\|_{L^4}\|\mathbf{v}\|_V. \end{aligned}$$

Therefore, $L(\varphi)\mathbf{w} \in V'$ and

$$\|L(\varphi)\mathbf{w}\|_{V'} \leq 2\|\varphi\|_{L^4}\|\mathbf{w}\|_{L^4} \leq 2c\|\varphi\|_{L^4}\|\mathbf{w}\|_{H^1}$$

holds true, c being a domain constant.

Let $\{\mathbf{w}_n\}$ be a bounded sequence in $H_\sigma^1(D)$. Since $H_\sigma^1(D)$ is compactly imbedded in $L^4(D)$, we can choose a subsequence $\{\mathbf{w}_{n'}\}$ which converges strongly in $L^4(D)$. Then by virtue of the above estimate, the sequence $\{L(\varphi)\mathbf{w}_{n'}\}$ converges strongly in V' . Q.E.D.

Let G be the Green operator of the Stokes equations, originally defined through Odqvist's Green function. Namely $\mathbf{v} = G\mathbf{F}$ gives the unique solution to the Stokes equations:

$$\begin{cases} -\Delta\mathbf{v} + \nabla p &= \mathbf{F} & \text{in } D, \\ \operatorname{div} \mathbf{v} &= 0 & \text{in } D, \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial D. \end{cases}$$

The operator G can be extended over to V' , for we have

Lemma 5. *The operator G is a bounded linear operator from V' to V and the estimate*

$$\|G\mathbf{F}\|_V \leq \|\mathbf{F}\|_{V'}$$

holds.

Proof.

We give a proof for the self-containedness. Let \mathbf{F} be an arbitrary element in V' . By Riesz' theorem, there exists a unique element $\mathbf{v} \in V$ such that

$$(\nabla \mathbf{v}, \nabla \mathbf{u}) =_{V'} \langle \mathbf{F}, \mathbf{u} \rangle_{V'}, \quad \forall \mathbf{u} \in V,$$

which we denote by $\mathbf{v} = G\mathbf{F}$. The estimate follows from this formula. Q.E.D.

Let P be the orthogonal projection from $L^2(D)$ to H_σ and $A = -P\Delta$ be the Stokes operator with domain $\mathcal{D}(A) = H^2(D) \cap V$. Then, $A\mathbf{v} = P\mathbf{F}$ if $\mathbf{F} \in L^2(D)$. We denote by $\|\cdot\|_{\mathcal{D}(A)}$ the graph norm of A , that is, $\|\mathbf{v}\|_{\mathcal{D}(A)} = \|A\mathbf{v}\|$. The next lemmas is well known (Cattabriga [3], Ladyzhenskaya [5]).

Lemma 6. *The operator $G : L^2(D) \rightarrow \mathcal{D}(A)$ is a bounded linear operator, namely we have the estimate*

$$\|G\mathbf{F}\|_{\mathcal{D}(A)} \leq \|\mathbf{F}\|_{L^2} \quad (\mathbf{F} \in L^2(D)).$$

From the identity:

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times \operatorname{rot} \mathbf{u},$$

we have the following lemma for the gradient of harmonic functions.

Lemma 7. *Let $\varphi \in H^2(D)$ be harmonic in D . Then, $\mathbf{e}_0 = \nabla \varphi$ satisfies the Stokes equations:*

$$(S) \begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{0} & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \end{cases}$$

as well as the Navier-Stokes equations (NS) with $\mathbf{f} = \mathbf{0}$.

Let $\beta_1 \in H^{1/2}(\partial D)$ be given. Consider the boundary value problem for the Stokes equations:

$$(S)_{\beta_1} \begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{0} & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \\ \mathbf{u} = \beta_1 & \text{on } \partial D. \end{cases}$$

It is well known that for any $\beta_1 \in H^{1/2}(\partial D)$ satisfying $(H)_0$, there exists a unique solution $\mathbf{b}_1 \in H^1(D)$ to $(S)_{\beta_1}$ such that the estimate

$$\|\mathbf{b}_1\|_{H^1(D)} \leq c \|\beta_1\|_{H^{1/2}(\partial D)}$$

holds. Moreover, if $\beta_1 \in H^{3/2}(\partial D)$, then the unique solution \mathbf{b}_1 belongs to $H^2(D)$ and is subject to the estimate

$$\|\mathbf{b}_1\|_{H^2(D)} \leq c\|\beta_1\|_{H^{3/2}(\partial D)}$$

(e.g. Cattabriga [3], Temam [10]).

Since $H^1(D)$ is continuously imbedded in $L^4(D)$, we have

Lemma 8. *There exists a constant c_2 which depends only on D , such that the inequality*

$$\|\mathbf{b}_1\|_{L^4(D)} \leq c_2\|\beta_1\|_{H^{1/2}(\partial D)}$$

holds.

Let $\mathbf{e}_0, \mathbf{b}_1$ be as above. We define the operators E_0, B_1 as

$$E_0 = L(\mathbf{e}_0), B_1 = L(\mathbf{b}_1).$$

Put $K = -GE_0 = -GL(\mathbf{e}_0)$. Using Lemma 4 and Lemma 5, we have

Lemma 9. *The operator $K : V \rightarrow V$ is a compact linear operator.*

If the data are smooth, then we have the following lemmas.

Lemma 10. *Let $\varphi \in H^2(D) \cap H_\sigma^1(D)$. Then, the operator $L(\varphi)$ is a compact linear operator from $H^2(D)$ to $L^2(D)$ subject to:*

$$\|L(\varphi)\mathbf{w}\|_{L^2(D)} \leq 2\|\varphi\|_{W_4^1(D)}\|\mathbf{w}\|_{W_4^1(D)}, \quad \forall \mathbf{w} \in H^2(D) \cap H_\sigma^1(D).$$

Similarly to Lemma 9, we can show

Lemma 11. *The operator K is a compact linear operator from $\mathcal{D}(A)$ to $\mathcal{D}(A)$.*

4. Proof of Theorems

Firstly, we prove Theorem 1. Without loss of generality, we can suppose $\mathbf{f} = \mathbf{0}$. Let \mathbf{e}_0 and \mathbf{b}_1 be chosen as above. Under our assumptions, \mathbf{e}_0 and \mathbf{b}_1 are in $H^1(D)$. In seeking the solution \mathbf{u} to the equation $(NS)_\mu$, we put $\mathbf{u} = \mathbf{w} + \mu\mathbf{e}_0 + \mathbf{b}_1$. Since $\mu\mathbf{e}_0$ satisfies (NS) and \mathbf{b}_1 satisfies (S) , we have the following formal equation for \mathbf{w}

$$(1) \quad \begin{cases} -\nu\Delta\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \mu\{(\mathbf{e}_0 \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{e}_0\} + \{(\mathbf{b}_1 \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{b}_1\} \\ \quad \quad \quad + \mu\{(\mathbf{b}_1 \cdot \nabla)\mathbf{e}_0 + (\mathbf{e}_0 \cdot \nabla)\mathbf{b}_1\} + (\mathbf{b}_1 \cdot \nabla)\mathbf{b}_1 + \nabla p = \mathbf{0} \text{ in } D, \\ \text{div } \mathbf{w} = 0 \text{ in } D, \\ \mathbf{w} = \mathbf{0} \text{ on } \partial D. \end{cases}$$

With the operator E_0, B_1 and G , we can rewrite the equations, obtaining

$$(2) \quad \mathbf{w} + \frac{\mu}{\nu} G E_0 \mathbf{w} = -\frac{1}{\nu} G(\mathbf{w} \cdot \nabla) \mathbf{w} - \frac{1}{\nu} G B_1 \mathbf{w} - \frac{\mu}{\nu} G B_1 \mathbf{e}_0 - \frac{1}{\nu} G(\mathbf{b}_1 \cdot \nabla) \mathbf{b}_1,$$

which is a well defined equation for $\mathbf{w} \in V$.

Since the operator $K = -G E_0$ is a compact linear operator on V (Lemma 9), the spectrum $\sigma(K)$ is a discrete countable set with a possible accumulating point $\{0\}$. Let $\frac{\mu}{\nu} \notin \sigma(K)$. Then the bounded inverse $(I - \frac{\mu}{\nu} K)^{-1} \in \mathcal{L}(V)$ exists. Applying this operator to (2), we obtain

$$(3) \quad \mathbf{w} = -\frac{1}{\nu} (I - \frac{\mu}{\nu} K)^{-1} \{G(\mathbf{w} \cdot \nabla) \mathbf{w} + G B_1 \mathbf{w} + \mu G B_1 \mathbf{e}_0 + G(\mathbf{b}_1 \cdot \nabla) \mathbf{b}_1\}.$$

Let us denote the right hand side of (3) by $\Phi \mathbf{w}$:

$$(4) \quad \Phi \mathbf{w} = -\frac{1}{\nu} (I - \frac{\mu}{\nu} K)^{-1} \{G(\mathbf{w} \cdot \nabla) \mathbf{w} + G B_1 \mathbf{w} + \mu G B_1 \mathbf{e}_0 + G(\mathbf{b}_1 \cdot \nabla) \mathbf{b}_1\}.$$

Thanks to the preceding lemmas, we obtain the following estimates:

$$(5) \quad \|G(\mathbf{w} \cdot \nabla) \mathbf{w}\|_V \leq \|(\mathbf{w} \cdot \nabla) \mathbf{w}\|_{V'} \leq \|\mathbf{w}\|_{L^4}^2 \leq c_1^2 \|\mathbf{w}\|_V^2,$$

$$(6) \quad \|G B_1 \mathbf{w}\|_V \leq \|B_1 \mathbf{w}\|_{V'} \leq 2 \|\mathbf{b}_1\|_{L^4} \|\mathbf{w}\|_{L^4} \leq 2c_1 c_2 \|\beta_1\|_{H^{1/2}(\partial D)} \|\mathbf{w}\|_V,$$

$$\|G B_1 \mathbf{e}_0\|_V \leq \|B_1 \mathbf{e}_0\|_{V'} \leq 2 \|\mathbf{b}_1\|_{L^4} \|\mathbf{e}_0\|_{L^4} \leq 2c_2 \|\beta_1\|_{H^{1/2}(\partial D)} \|\mathbf{e}_0\|_{L^4},$$

$$\|G(\mathbf{b}_1 \cdot \nabla) \mathbf{b}_1\|_V \leq \|(\mathbf{b}_1 \cdot \nabla) \mathbf{b}_1\|_{V'} \leq \|\mathbf{b}_1\|_{L^4}^2 \leq c_2^2 \|\beta_1\|_{H^{1/2}(\partial D)}^2,$$

where c_1, c_2 are the constants in Lemma 1 and Lemma 8.

Let

$$c_0 = \max\{c_1^2, 2c_1 c_2, 2c_2, c_2^2\}, \quad \gamma_0 = \frac{c_0}{\nu} \|(I - \frac{\mu}{\nu} K)^{-1}\|_{\mathcal{L}(V)}, \quad \gamma_1 = \|\beta_1\|_{H^{1/2}(\partial D)}.$$

It is to be noted that γ_0 does not depend on β_1 . We obtain

$$\|\Phi \mathbf{w}\|_V \leq \gamma_0 \{\|\mathbf{w}\|_V^2 + \gamma_1 \|\mathbf{w}\|_V + \gamma_1 |\mu| \|\mathbf{e}_0\|_{L^4} + \gamma_1^2\}, \quad \forall \mathbf{w} \in V.$$

In order to apply the contraction mapping theorem, we first examine the quadratic equation in X :

$$(7) \quad \gamma_0 (X^2 + \gamma_1 X + \gamma_1 |\mu| \|\mathbf{e}_0\|_{L^4} + \gamma_1^2) = X.$$

We fix \mathbf{e}_0 and μ . If γ_1 is sufficiently small, then $\gamma_1 - \gamma_0^{-1} < 0$ and the discriminant

$$\Delta = (\gamma_1 - \gamma_0^{-1})^2 - 4\gamma_1 (|\mu| \|\mathbf{e}_0\|_{L^4} + \gamma_1)$$

is positive. Therefore the equation (7) has two positive roots. Let r_0 be the smaller one, that is,

$$(8) \quad r_0 = \frac{1}{2} \left\{ \frac{1}{\gamma_0} - \gamma_1 - \sqrt{\Delta} \right\}.$$

Now we introduce the following closed ball in V :

$$\mathcal{B}_0 = \{\mathbf{w} \in V \mid \|\mathbf{w}\|_V \leq r_0\}.$$

Lemma 12. *The non-linear operator $\Phi : V \rightarrow V$ maps \mathcal{B}_0 into \mathcal{B}_0 and is a contraction mapping on \mathcal{B}_0*

Proof.

Let \mathbf{w} be in \mathcal{B}_0 . Then

$$\|\Phi \mathbf{w}\|_V \leq \gamma_0 \{r_0^2 + \gamma_1 r_0 + \gamma_1 |\mu| \|e_0\|_{L^4} + \gamma_1^2\} = r_0.$$

Therefore $\Phi \mathbf{w}$ is in \mathcal{B}_0 .

Let \mathbf{w}_1 and \mathbf{w}_2 be in \mathcal{B}_0 . Then, we obtain

$$\begin{aligned} & \|\Phi \mathbf{w}_1 - \Phi \mathbf{w}_2\|_V \\ &= \left\| -\frac{1}{\nu} (I - \frac{\mu}{\nu} K)^{-1} G \{(\mathbf{w}_1 \cdot \nabla) \mathbf{w}_1 - (\mathbf{w}_2 \cdot \nabla) \mathbf{w}_2 + B_1(\mathbf{w}_1 - \mathbf{w}_2)\} \right\|_V \\ &\leq \frac{1}{\nu} \|(I - \frac{\mu}{\nu} K)^{-1}\|_{\mathcal{L}(V)} \left[\|G \{(\mathbf{w}_1 \cdot \nabla) \mathbf{w}_1 - (\mathbf{w}_2 \cdot \nabla) \mathbf{w}_2\}\|_V + \|G B_1(\mathbf{w}_1 - \mathbf{w}_2)\|_V \right]. \end{aligned}$$

Using (5), (6), it is easy to see that the estimates:

$$\begin{aligned} & \|G \{(\mathbf{w}_1 \cdot \nabla) \mathbf{w}_1 - (\mathbf{w}_2 \cdot \nabla) \mathbf{w}_2\}\|_V \\ &= \|G \{(\mathbf{w}_1 - \mathbf{w}_2) \cdot \nabla \mathbf{w}_1 + (\mathbf{w}_2 \cdot \nabla)(\mathbf{w}_1 - \mathbf{w}_2)\}\|_V \\ &\leq c_1^2 \|\mathbf{w}_1 - \mathbf{w}_2\|_V (\|\mathbf{w}_1\|_V + \|\mathbf{w}_2\|_V) \\ &\leq c_0 (\|\mathbf{w}_1\|_V + \|\mathbf{w}_2\|_V) \|\mathbf{w}_1 - \mathbf{w}_2\|_V \\ &\leq 2c_0 r_0 \|\mathbf{w}_1 - \mathbf{w}_2\|_V, \end{aligned}$$

and

$$\|G B_1(\mathbf{w}_1 - \mathbf{w}_2)\|_V \leq 2c_1 c_2 \|\beta_1\|_{H^{1/2}(\partial D)} \|\mathbf{w}_1 - \mathbf{w}_2\|_V \leq c_0 \gamma_1 \|\mathbf{w}_1 - \mathbf{w}_2\|_V$$

hold. Therefore,

$$\|\Phi \mathbf{w}_1 - \Phi \mathbf{w}_2\|_V \leq \gamma_0 (\gamma_1 + 2r_0) \|\mathbf{w}_1 - \mathbf{w}_2\|_V.$$

From (8), it follows that

$$\gamma_0 (\gamma_1 + 2r_0) = 1 - \gamma_0 \sqrt{\Delta} < 1.$$

Therefore $\Phi \mathbf{w}$ is a contraction mapping on \mathcal{B}_0 .

Q.E.D.

From what has been given, the proof of Theorem 1 is complete.

Using Lemma 6, Lemma 10 and Lemma 11, we can prove Theorem 2 similarly.

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STATIONARY CONFIGURATIONS OF A VORTEX FILAMENT EMBEDDED IN BACKGROUND FLOWS

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Abstract: We present a simple recipe for calculating the three-dimensional forms of a vortex filament in equilibrium with background flows of an inviscid incompressible fluid, based on the localized induction approximation. An analogy is drawn between stationary forms of a vortex filament in a steady flow and trajectories of a charged particle in a steady magnetic field, upon which our procedure relies. To illustrate some of advantages, we revisit the Kida class, invariant forms of a vortex filament moving through a still fluid. As a related topic, the motion of a heavy symmetrical body is discussed.

1. Introduction.

We consider a thin curved vortex tube (vortex filament) embedded in an inviscid incompressible fluid. In general, if the distribution of vorticity is specified at all points of the fluid, the velocity at each point is provided by the Biot-Savart law. The dynamical theorem of Kelvin and Helmholtz states that a vortex tube is convected with the fluid without change of strength (=circulation Γ). Therefore, in order to determine the self-induced velocity of a vortex filament, we have only to evaluate the Biot-Savart integral at points in the vicinity of the vortex tube with Γ taken as a constant.

The approach to retain the minimum of essence is the so called “localized induction approximation” [2, 5, 8]; the size σ of the vortex core is so thin that the self-induction at a point \mathbf{X} on the filament is dominated by the contribution from the neighboring segment of length $2L$. After some calculation, we obtain

$$\mathbf{X}_t = A\mathbf{X}_s \times \mathbf{X}_{ss} + \mathbf{V}(\mathbf{X}, t) \quad , \quad (1a)$$

$$A = \frac{\Gamma}{4\pi} \log \left(\frac{L}{\sigma} \right) \quad , \quad (1b)$$

where $\mathbf{X} = \mathbf{X}(s, t)$ denotes the filament curve as a function of the arclength s and the time t , and $\mathbf{V}(\mathbf{x}, t)$ denotes the externally imposed flow field. The subscript indicates the partial differentiation. For simplicity, A is assumed to be a constant.

It is worth noting that, in the absence of the external flow, (1) is a completely integrable equation, being reducible to a cubic nonlinear Schrödinger equation [3].

2. Stationary shapes of a vortex filament in a background flow.

Consider the shape of a filament which is in equilibrium with a steady flow $\mathbf{V}(\mathbf{x})$. This setting stipulates that each point on the filament be movable only along itself, namely, $\mathbf{X}_t = V_{\parallel} \mathbf{X}_s$ with V_{\parallel} being some function. Equation (1a) then reads

$$A \mathbf{X}_s \times \mathbf{X}_{ss} + \mathbf{V}(\mathbf{X}) = V_{\parallel} \mathbf{X}_s \quad . \quad (2)$$

Taking the vector product with $\mathbf{t} = \mathbf{X}_s$, (2) is transformed into

$$A \mathbf{X}_{ss} = \mathbf{X}_s \times \mathbf{V}(\mathbf{X}) \quad . \quad (3)$$

If we think of s as the time t , $\mathbf{V}(\mathbf{X})$ as the magnetic field, and A as m/q , then (3) is identifiable as the equation governing the motion of a charged particle, with mass m and charge q , in a magnetic field $\mathbf{V}(\mathbf{X})$. It follows that the static shape of a vortex filament in a steady external flow is equivalent to the trajectory of a charged particle moving subject to the Lorentz force.

By appealing to the Lagrangian formalism of classical mechanics, the fully nonlinear form becomes easily accessible. The Lagrangian \mathcal{L} is

$$\mathcal{L} = \frac{A}{2} \mathbf{X}_s^2 + \mathbf{X}_s \cdot \mathbf{A} \quad , \quad (4)$$

where \mathbf{A} is the vector potential associated with the external field such that $\mathbf{V} = \nabla \times \mathbf{A}$.

Equation (3) is rewritten in the Hamiltonian form with three degrees of freedom. According to Liouville-Arnol'd's theorem, two symmetries in space, other than the kinetic energy, are required for our system to be integrable by quadratures. Generically, if a vortex line is continued far enough, it exhibits chaotic behavior even in the static balance. This fact may be indicative of a complicated entanglement of vortex filaments in unsteady flows and still more so in turbulent flows.

3. A vortex filament without change of form.

A proper example that illustrates the benefit from the use of this analogy is a vortex filament traveling, without change of form, through a still fluid [4, 5]. Kida reasoned that such a motion is composed of three ingredients, namely, a translation with velocity V in a certain direction, say z -direction, a rotation about the same axis with angular velocity Ω , and a slipping motion along itself with speed c_0 . The resulting equation is

$$A \mathbf{X}_s \times \mathbf{X}_{ss} = -c_0 \mathbf{X}_s + \Omega \mathbf{e}_z \times \mathbf{X} + V \mathbf{e}_z \quad , \quad (5)$$

where \mathbf{e}_z is the unit vector in the z -direction, and c_0 , Ω , V are all constants. This equation is converted into the form of (3) as

$$A\ddot{\mathbf{X}} = \dot{\mathbf{X}} \times \mathbf{V}(\mathbf{X}) \quad , \quad (6a)$$

where

$$\mathbf{V} = -\Omega\mathbf{e}_z \times \mathbf{X} - V\mathbf{e}_z \quad , \quad (6b)$$

and a dot denotes the differentiation in s . The vector potential $\mathbf{A}(\mathbf{x})$ is provided, in cylindrical coordinates (r, ϕ, z) , by

$$\mathbf{A} = -\frac{V}{2}r\mathbf{e}_\phi + \frac{\Omega}{2}r^2\mathbf{e}_z \quad . \quad (7)$$

With this form, the Lagrangian \mathcal{L} for (6) is

$$\mathcal{L} = \frac{A}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - \left(\frac{V}{2}r^2\dot{\phi} - \frac{\Omega}{2}r^2\dot{z}\right) \quad . \quad (8)$$

By inspection, we immediately find that z and ϕ are both cyclic, and the first integrals are available at once:

$$P_z = \partial\mathcal{L}/\partial\dot{z} = A\dot{z} + \Omega r^2/2 = \text{const.} \quad , \quad (9a)$$

$$P_\phi = \partial\mathcal{L}/\partial\dot{\phi} = Ar^2\dot{\phi} - Vr^2/2 = \text{const.} \quad (9b)$$

These integrals, augmented by $|\dot{\mathbf{X}}|^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 = 1$, coincide with the set of equations handled by Kida. The advantage is that, by invoking the Lagrangian formalism, our treatment requires less ingenuity to gain the integrals.

Kida's equation (5) is interpreted as an equation describing the motion of Lagrange's top, a heavy rigid body, with axial symmetry, fixed at a stationary point in a gravity field.

Let $\mathbf{t}(t)$ and $\boldsymbol{\omega}(t)$ be, respectively, the unit vector along the axis of symmetry and the angular velocity of the body as functions of the time t . By definition,

$$\dot{\mathbf{t}} = \boldsymbol{\omega} \times \mathbf{t} \quad . \quad (10)$$

Multiplying (10) vectorially by \mathbf{t} , we have

$$\boldsymbol{\omega} = \mathbf{t} \times \dot{\mathbf{t}} + (\mathbf{t} \cdot \boldsymbol{\omega})\mathbf{t} \quad . \quad (11)$$

It should be born in mind that $\boldsymbol{\omega}$ is the angular velocity viewed from the inertial frame. Because of the rotational symmetry about \mathbf{t} , the angular momentum $\mathbf{m}(t)$ relative to the stationary point, viewed from the inertial frame, takes on the form:

$$\mathbf{m} = A(\mathbf{t} \times \dot{\mathbf{t}}) + C\omega_3\mathbf{t} \quad , \quad (12)$$

where A and C are the moments of inertia at the stationary point, and $\omega_3 = \mathbf{t} \cdot \boldsymbol{\omega}$. The rate of change of the angular momentum relative to some point is equal to the moment of the external force about that point, giving

$$\dot{\mathbf{m}} = \mathbf{t} \times (-mgl\mathbf{e}_z) \quad , \quad (13)$$

where m is the mass of the body, $-ge_z$ is the gravity acceleration with e_z being the unit vector in the z -direction taken vertically upwards, and l is the length of line segment connecting the stationary point to the center of mass. Identifying as $mgl = \Omega$ and integrating (13) in t with an appropriate choice of the integration constant, we regain Kida's equation (5) for the variable $X(t) = \int^t \mathbf{t} dt$.

We note that (13) is compatible with the conservation of the total energy. The total energy E is the sum of the kinetic and potential energies, and is written as

$$E = \frac{1}{2} \mathbf{m} \cdot \boldsymbol{\omega} + mgl e_z \cdot \mathbf{t} \quad . \quad (14)$$

On using (10), the time-derivative of E becomes

$$\begin{aligned} \dot{E} &= \dot{\mathbf{m}} \cdot \boldsymbol{\omega} + mgl e_z \cdot \dot{\mathbf{t}} \\ &= (\dot{\mathbf{m}} + mgl \mathbf{t} \times e_z) \cdot \boldsymbol{\omega} \quad . \end{aligned} \quad (15)$$

Thus (13) ensures the conservation of energy ($\dot{E} = 0$).

It deserves mention that (10) and (13), supplemented by (12), constitute the subset of the Euler-Poisson equations (the Lie-Poisson form) written in the inertial frame [7]. The remaining equation describes the temporal evolution of the inertia tensor.

The spinning top whose axis is permanently upright is called a sleeping top. It is likened to a straight-line vortex. The sleeping top loses its stability if the rotation speed becomes slower than a critical value: $\omega_3^2 < 4Amgl/C^2$. In the context of a vortex filament, the awakened top is nothing but the Hasimoto soliton [3].

Our analogy encompasses the motion of a spherical pendulum. By taking the vector product of (13) with \mathbf{t} , we obtain, with the aid of the subsidiary equation $\mathbf{t} \cdot \mathbf{t} = 1$,

$$A\ddot{\mathbf{t}} = -mgl[e_z - (\mathbf{t} \cdot e_z)\mathbf{t}] - A(\dot{\mathbf{t}} \cdot \dot{\mathbf{t}})\mathbf{t} - C\omega_3 \dot{\mathbf{t}} \times \mathbf{t} \quad . \quad (16)$$

We observe that (16) describes the motion of a particle with mass A and charge q constrained to the surface of a unit sphere and subjected to a magnetic as well as the gravity fields. Here \mathbf{t} stands for the particle position and the gravity acceleration is replaced by mgl/A . The magnetic field \mathbf{B} is that generated by a monopole sitting at $\mathbf{t} = \mathbf{o}$:

$$(q/A)\mathbf{B} = -(C\omega_3/A)\mathbf{t}/|\mathbf{t}|^3 \quad . \quad (17)$$

The first and second terms on the RHS of (16) signify that the particle is exerted by the gravity force, the normal component of which is being projected out. The third one is the centrifugal force. The last one is the Lorentz force due to the monopole field (17). Interestingly, the same analogy was discovered by Berry and Robbins [1].

The configuration space for Lagrange's top is all possible rotations, that is, the rotational group $SO(3)$ whose parameterization necessitates three variables. The standard way is the use of the Euler angles (θ, φ, ψ) . On the other hand, for the motion of a spherical pendulum, it suffices to specify the tip of the vector \mathbf{t} , that is, to specify θ and φ . It follows that reduction of freedom is achieved from $SO(3)$ to the unit sphere S^2 . The origin of the reduction is attributed to the rotational symmetry about the top axis \mathbf{t} . The

emergence of a magnetic monopole in the reduced system (16) is not accidental, but is well comprehensible within a general framework of the cotangent bundle reduction [6].

We regard $SO(3)$ as a principal S^1 -bundle over S^2 with the group $S^1 (\cong SO(2))$ acting freely from the right on $SO(3)$ by rotation about the top axis. The Hamiltonian \mathcal{H} is

$$\mathcal{H} = \frac{1}{2} \left[\frac{p_\theta^2}{A} + \frac{(p_\varphi - \cos \theta p_\psi)^2}{A \sin^2 \theta} + \frac{p_\psi^2}{C} \right] + mgl \cos \theta \quad , \quad (18)$$

and is invariant under the right action by S^1 [7]. Here $(p_\theta, p_\varphi, p_\psi)$ are components of the canonically conjugate momentum. The Lie algebra \mathcal{G} of S^1 is identified with the axial component ω_3 of the angular velocity, thus with the real number \mathbf{R} , and its dual \mathcal{G}^* is identified with the axial component $p_\psi (\cong \mathbf{R})$ of the angular momentum. The mechanical connection is coined from the formula $p_\psi/C : TSO(3) \rightarrow \mathcal{G}$ and its role is played by ω_3 :

$$\omega_3(\theta, \varphi, \psi, \dot{\theta}, \dot{\varphi}, \dot{\psi}) = \cos \theta \dot{\varphi} + \dot{\psi} \quad . \quad (19)$$

Given $\mu \in \mathcal{G}^*$, define the one-form $\alpha^\mu = \alpha_\theta d\theta + \alpha_\varphi d\varphi + \alpha_\psi d\psi$ on $SO(3)$ by

$$\alpha_\theta \dot{\theta} + \alpha_\varphi \dot{\varphi} + \alpha_\psi \dot{\psi} = \mu \omega_3 \quad . \quad (20)$$

The LHS is the pairing between $T_q^*SO(3)$ and $T_qSO(3)$ at $q \in SO(3)$, while the RHS is the pairing between \mathcal{G}^* and \mathcal{G} . It yields

$$(\alpha_\theta, \alpha_\varphi, \alpha_\psi) = (0, \mu \cos \theta, \mu) \quad . \quad (21)$$

We decompose the momentum on the level set of $p_\psi = \mu$ into two parts, based on (21), as

$$(p_\theta, p_\varphi, \mu) = (p_\theta, p_\varphi - \mu \cos \theta, 0) + (0, \mu \cos \theta, \mu) \quad . \quad (22)$$

This is a kind of horizontal-vertical decomposition, and the first part is orthogonal to the second one with respect to the metric derived from the kinetic energy. In the first part, $(\tilde{p}_\theta, \tilde{p}_\varphi) = (p_\theta, p_\varphi - \mu)$ may be regarded as the momentum on S^2 . The symplectic structure on T^*S^2 is induced from the canonical one on $T^*SO(3)$:

$$dp_\theta \wedge d\theta + dp_\varphi \wedge d\varphi + dp_\psi \wedge d\psi = d\tilde{p}_\theta \wedge d\theta + d\tilde{p}_\varphi \wedge d\varphi - \mu \sin \theta d\theta \wedge d\varphi \quad . \quad (23)$$

Since ω_3 is the connection one-form, the two-form $d\alpha^\mu$ on $SO(3)$ automatically drops to a two-form on S^2 , giving the last term called the *magnetic term*. This causes the Lorentz force due to the monopole field. The reduced Hamiltonian \mathcal{H}_μ on T^*S^2 is

$$\mathcal{H}_\mu = \frac{1}{2A} \left(\tilde{p}_\theta^2 + \frac{\tilde{p}_\varphi^2}{\sin^2 \theta} \right) + \left(mgl \cos \theta + \frac{\mu^2}{2C} \right) \quad . \quad (24)$$

Hamilton's equations for \mathcal{H}_μ endowed with (23) indeed give rise to (16)

As a reverse process, we are able to reconstruct the orbit on $SO(3)$ from that of the reduced system (16) on S^2 by quadrature. Here we inquire into the rotation angle of the plane perpendicular to the top axis or the value of ψ . To fix ideas, we assume a periodic

motion with period T of the top axis, though it usually executes a quasi-periodic motion. By virtue of the axial symmetry, the axial component $\omega_3 = \dot{\psi} + \dot{\varphi} \cos \theta (= \mu/C)$ of the angular velocity is constant. The rotation angle $\Delta\psi$ of the axis after one period is then, on using Stokes' theorem,

$$\begin{aligned} \Delta\psi &= \omega_3 T - \int_0^T \dot{\varphi} \cos \theta dt \\ &= \omega_3 T + \iint \sin \theta d\theta \wedge d\varphi - 2\pi k \quad . \end{aligned} \quad (25)$$

The second term stands for the solid angle swept out by the tip of axis. The number k is some integer originating from the singularity of polar coordinates (θ, φ) at $\theta = 0$ and π . In accordance with the general formula, the rotation angle is composed of a geometric and a dynamic part. The first term of (25) is the dynamic angle. The second term, corrected with $-2\pi k$, is referred to as the geometric angle, because it depends only on the geometry of the closed orbit and is independent of the speed with which the orbit is traversed.

As shown by Marsden [6], the geometric part included in (25) is, up to an integral multiple of 2π , the logarithm of the holonomy, that is, minus the integration of the mechanical connection ω_3 along the closed solution curve C_s on S^2 :

$$- \oint_{C_s} \omega_3 dt = - \frac{q}{\mu} \iint_D \mathbf{B} \cdot d\mathbf{S} \quad , \quad (26)$$

where the last two integrals are taken over the domain D on S^2 enclosed by the curve C_s . As expected, the geometric angle is naturally expressible as the magnetic flux.

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Mixed Problem for Wave Equations with Oblique Boundary Condition

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Dedicated to Professor Kenjiro Okubo on the Occasion of His 60th Birthday

abstract This paper is concerned with "well posed function spaces" for hyperbolic equations of the simplest type. In [4] Kōmura mentioned this subject without proof. We consider the mixed problem with oblique boundary condition which is not wellposed in L^2 -sense. First we introduce the space $Y = D(\Delta_\sigma) \times M$ in which the equation is well posed and where M is a measure space. Next we consider the infinitesimal generator $A = \begin{pmatrix} 0 & I \\ \Delta_\sigma & 0 \end{pmatrix}$ which generate a C_0 -semigroup on Y .

1. Introduction

This paper is concerned with "well posed function spaces" for hyperbolic equations of the simplest type. In [4] Kōmura mentioned this subject without proof. We consider the following mixed problem:

$$(1) \quad \begin{cases} \frac{\partial^2 u(t, \mathbf{x})}{\partial t^2} = \Delta u(t, \mathbf{x}) & \text{in } (0, \infty) \times \Omega, \\ \left(\frac{\partial}{\partial x_1} + B(t, \mathbf{x}) \right) u \Big|_{x_1=0} = 0 & \Omega = \{ \mathbf{x} = (x_1, \dots, x_n) | x_1 > 0 \} \subset \mathbf{R}^n \\ u(0, \mathbf{x}) = f(\mathbf{x}), & u_t(0, \mathbf{x}) = g(\mathbf{x}), \end{cases}$$

$$B(t, \mathbf{x})u(t, \mathbf{x}) \equiv b_0(t, \mathbf{x}) \frac{\partial u(t, \mathbf{x})}{\partial t} + \sum_{j=2}^n b_j(t, \mathbf{x}) \frac{\partial u(t, \mathbf{x})}{\partial x_j} + d(t, \mathbf{x})u(t, \mathbf{x}) = 0$$

where coefficients are C^∞ -functions defined in a neighborhood of $x_1 = 0$.

We think that the case of (1) is not well posed in L^2 -sense(see Remark). Then first we introduce the space $Y = D(\Delta_\sigma) \times M$ in which (1) is well posed and where M is a measure

space. Next we consider the infinitesimal generator $A = \begin{pmatrix} 0 & I \\ \Delta_\sigma & 0 \end{pmatrix}$ which generates a C_0 -semigroup on Y .

Remark.

Many authors studied about L^2 -well posedness for hyperbolic mixed problems of second order. (cf [3],[7],[8],[1],[9]) In general (2) is not well-posed in L^2 -sense. (cf. Miyatake [7],[8])

$$(2) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u \\ \left(-\frac{\partial}{\partial x} + b \frac{\partial}{\partial y} - c \frac{\partial}{\partial t} \right) u \Big|_{x=0} = 0 \\ u(0, x, y) = f(x, y), \quad u_t(0, x, y) = g(x, y). \end{cases}$$

In fact the following three cases;

$$(i) \quad c > 0, \quad b \neq 0,$$

$$(ii) \quad c = 1, \quad |b| < 1,$$

$$(iii) \quad c \leq 0, \quad |b| > -c,$$

(2) is not L^2 -well-posed. Therefore especially in the case that $c = 0$ it is not well-posed if $b \neq 0$. (Ikawa [3])

2. Functional spaces

In this section we shall explain our notations.

$(M, \|\cdot\|_M)$ denotes a measure space.

Let

$$\hat{M} = \{\hat{u}; u \in M\}, \quad \Xi = \{u; |\xi|^{-1} \hat{u} \in \hat{M}\}, \quad \xi = (\xi_1, \dots, \xi_n)$$

where $\hat{u}(\xi) = (\mathcal{F}u)(\xi)$; $\mathcal{F}u$ means the Fourier translation of u

Let

$$(X, \|\cdot\|_X); \quad X = {}^t(M \times \Xi)$$

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_X = \left\| (\hat{u}(\xi)^2 + |\xi|^{-2} \hat{v}(\xi)^2)^{1/2} \right\|_{\hat{M}}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in X$$

Δ_N denotes Laplacian with Neumann condition $\frac{\partial}{\partial x_1} u \Big|_{x_1=0} = 0$.

The domain of Δ_N is

$$D(\Delta_N) = \left\{ u \in D(\Delta); \frac{\partial}{\partial x_1} u \Big|_{x_1=0} = 0 \right\}$$

Δ_σ denotes Laplacian with boundary condition $\left(\frac{\partial}{\partial x_1} + B(t, \mathbf{x})\right)u \Big|_{x_1=0} = 0$

The domain of Δ_σ is

$$D(\Delta_\sigma) = \left\{ u \in D(\Delta); \left(\frac{\partial}{\partial x_1} + B(t, \mathbf{x})\right)u \Big|_{x_1=0} = 0 \right\}$$

Let

$$(Y, \|\cdot\|_\sigma); \quad Y = {}^t(D(\Delta_\sigma) \times \Xi)$$

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_\sigma = \left\| \begin{pmatrix} u - Su \\ v \end{pmatrix} \right\|_X, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in Y$$

where S satisfies the following assumptions:

Assumptions:

(A.1)

$$I - S : D(\Delta_\sigma) \longrightarrow D(\Delta_N)$$

(A.2)

$$\|\Delta S u\| \leq \text{Const}\|u\|$$

3. Existence of $(I - \lambda\Delta_\sigma)^{-1}$

To show the existence of $(I - \lambda\Delta_\sigma)^{-1}$, it is sufficient to show the following Theorem.

Theorem 1.

Let $R(I - \lambda\Delta_\sigma)$ be the range of $(I - \lambda\Delta_\sigma)$ then for some small $\lambda > 0$, we get

$$R(I - \lambda\Delta_\sigma) = \hat{M}^{-1}$$

Proof. It is sufficient to show that for any $f \in \hat{M}^{-1}$, there exist $u \in D(\Delta_\sigma)$ such that $(I - \lambda\Delta_\sigma)u = f$.

More precisely see [2].

4. Systems of evolution equations

Putting $u_t = v$, let us rewrite the problem (1) in the following form:

$$(3) \quad \begin{cases} \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta_\sigma & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \end{cases}$$

Set

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ \Delta_\sigma & 0 \end{pmatrix}$$

We shall formulate (3) as an abstract equation:

$$\begin{cases} \frac{d}{dt}U = AU & \text{in } Y, \\ U(0) = \begin{pmatrix} f \\ g \end{pmatrix}. \end{cases}$$

Theorem 2. *A generates a C_0 semigroup $\{U(t)\}$ on Y .*

proof. It is sufficient to show that there exist $\lambda > 0$ and $K' > 0$ such that;

$$1 - \lambda K' > 0$$

$$(4) \quad \|(I - \lambda A)U\|_\sigma \geq (1 - \lambda K')\|U\|_\sigma, \quad U \in Y.$$

See [2].

5. Application

As an applicatin model of abstract result obtained in the previous sections, let us consider the following mixed problem:

$$(5) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u & \text{in } (0, \infty) \times \Omega, \quad \Omega = \{(x, y) \in \mathbf{R}^2 | x > 0\} \\ \left. \left(\frac{\partial}{\partial x} + \sigma(y) \frac{\partial}{\partial y} \right) u \right|_{x=0} = 0 \\ u(0, x, y) = f(x, y), \quad u_t(0, x, y) = g(x, y). \end{cases}$$

Where σ in (5) satisfies following conditions.

- (i) There exists $r > 0$ such that $\sigma(y) = 0$ for any $|y| \geq r$.
- (ii) There exists $\rho > 0$ such that $|\sigma(y)| < \rho$ [ρ is sufficiently small].
- (iii) σ is sufficiently smooth.

Our plan is that;

Let σ be sufficiently small. Then we make S.

As is well known that (5) is not well posed in L^2 -sense(see Remark).

Notations

Let X be a Banach space with norm $\|\cdot\|_X$.

Let (X, \mathcal{M}, μ) be a finite measure space.

The symbol $L^\infty(X)$ stands for the space of all complex valued Bochner integrable functions defined on X that are esesentially bounded,i.e., such that

$$\|f\|_\infty = \text{esssup}\{f(x); x \in X\} = \inf\{\text{sup}\{|f(x)|; x \in X \setminus N\} | \mu(N) = 0\} < \infty$$

We denote by $L^\omega(X)$, the space of functions which are in $L^\infty(X)$ and which are tend to 0 as $x \rightarrow \infty$, i.e., such that for any $\varepsilon > 0$ there exists a compact set $K \subset X$ such that

$$\|f\|_{L^\infty(X \setminus K)} < \varepsilon$$

$C(X)$ denotes the function space of continuous functions defined on X with complex values.

$C_0(X) = C(X) \cap L^\infty(X)$ with norm $L^\infty(X)$. This space is also a Banach space. Let $M^1(X)$ be a dual space of $C_0(X)$. $M^1(X)$ is a Banach space with norm $\|\cdot\|_{M^1(X)}$, where

$$\|\mu\|_{M^1(X)} = \|\mu\|_{(C_0(X))'} = \sup\left\{\left|\int f(x)\mu(dx)\right| : \|f\|_{C_0(X)} \leq 1\right\}$$

Definition of S

Let

$$\mathcal{F}\mu(\xi) = \int e^{-ix\xi}\mu(dx) \quad \mu \in M = M^1(\Omega).$$

We shall define the operator S.

(I) Let w be a solution to following Dirichlet problem (6)

$$(6) \quad \begin{cases} \Delta w = 0 & \text{in } Q \setminus \{(0, y)\} \\ w = -\sigma(y)u_y(0, y) & x = 0 \\ w = 0 & x \neq 0, |y| = 2r \end{cases}$$

where $Q = \{(x, y) \in \mathbf{R}^2; |x| < c, |y| < 2r\}, c > 0$.

(II) Let $\beta \in C^\infty(\mathbf{R}^2)$ satisfy the followings.

$$(7) \quad \begin{cases} \beta(-x, y) = -\beta(x, y) \\ \beta(x, y) = 0 \\ \beta_x(0, y) = 1 \end{cases} \quad \begin{cases} |x| \geq c, & |y| > 2r \\ |y| \leq r. \end{cases}$$

(III) S is defined as $Su = \beta w$.

Remark.

From the definitions, we get

$$\left. \frac{\partial}{\partial x}(Su) \right|_{x=0} = \beta_x(0)w.$$

Then it follows that

$$(I - S)u \in D(\Delta_N) \quad \text{for any } u \in D(\Delta_\sigma)$$

and there exists some constant Const such that

$$\|\Delta(Su)\| = \|\Delta(\beta w)\| \leq \text{Const}(\|w\| + \|w_x\| + \|w_y\|).$$

Existence of w

Theorem 3. *There exists a solution w to (6).*

proof. We can take w as

$$\begin{cases} w(x, y) = v(x, y) & 0 \leq x < c, \quad -2r < y < 2r, \\ w(x, y) = v(-x, y) & -c < x < 0, \quad -2r < y < 2r, \\ w(x, y) = 0 & c \leq |x|, \quad 2r \leq |y|, \end{cases}$$

where v is a solution to (8).

$$(8) \quad \begin{cases} \Delta v = 0 & 0 < x < c \quad -2r < y < 2r \\ v(x, -2r) = v(x, 2r) = 0 & 0 < x < c \\ v(0, y) = -\sigma(y)u_y(0, y) & -2r \leq y \leq 2r \\ v(c, y) = 0 & -2r \leq y \leq 2r. \end{cases}$$

In fact the solution to (8) is that;

$$v(x, y) = \sum_{n=1}^{\infty} b_n \frac{\sinh n\pi \frac{c-x}{4r}}{\sinh \frac{n\pi c}{4r}} \cdot \sinh n\pi \frac{y-2r}{4r},$$

where

$$b_n = \frac{1}{2r} \int_{-2r}^{2r} f(y) \sin \left[n\pi \frac{y-2r}{4r} \right] dy,$$

$$f(y) = -\sigma(y)u_y(0, y).$$

Therefore, w is exists.

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WEIGHTED ESTIMATE FOR THE WAVE EQUATION

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Abstract

Following the approach of Strichartz, we use a complex interpolation method to derive $L^p - L^q$ estimate for the inhomogeneous linear wave equation. Suitable adaptation of the method of F. John enables us to obtain a refined $L^p - L^\infty$ estimate involving the weights $\tau_\pm = 1 + |t \pm |x||$. Together with the method of F. John we use also the Fourier representation of the kernel of the resolution operator for the linear wave equation and obtain $L^2 - L^2$ -weighted estimate. A complex interpolation gives $L^p - L^q$ -weighted estimate. An application of this estimate to the semilinear wave equation is also considered.

1. Introduction

A lot of works were devoted to the semilinear wave equation

$$\square u = F(u), \tag{1}$$

where $F(u) = O(|u|^\lambda)$ near $|u| = 0$ and $\lambda > 1$. Here and below \square denotes the d'Alembertian on \mathbf{R}^{n+1} .

When the initial data for the corresponding Cauchy problem are large, the solutions blows-up in finite time (see [18] for example). For this the focus of the works dealing with this general semilinear wave equation was concentrated on solutions with small initial data.

The existence of solutions with small initial data, for the case of space dimensions $n = 3$ was studied by F. John in the pioneer work [7], where he established that for $1 < \lambda < 1 + \sqrt{2}$ the solution of (1) blows-up in finite time, while for $\lambda > 1 + \sqrt{2}$ the solution exists globally in time. Therefore, the value $\lambda_0 = 1 + \sqrt{2}$ is critical for the semilinear wave equation (1).

To obtain the existence theorem in his work [7] F.John proved the following weighted L^∞ – estimate for the wave equation $\square u = F$ in \mathbf{R}^{3+1} with zero initial data

$$\|\tau_+^\alpha \tau_-^\beta u\|_{L^\infty} \leq C \|\tau_+^\gamma \tau_-^\delta F\|_{L^\infty}, \quad (2)$$

where $\tau_\pm = 1 + |t \pm |x||$ are the weights associated with the characteristic surfaces of the wave equation and the parameters $\alpha, \beta, \gamma, \delta$ satisfy the conditions

$$\alpha = 1 \quad , \quad \beta = \gamma - 2 < 1 \quad , \quad \delta > 1. \quad (3)$$

For general space dimensions it was W.Strauss who proposed in [21] the conjecture that the critical value for the nonlinearity is the positive root $\lambda_0(n)$ of the equation

$$(n - 1)\lambda^2 - (n + 1)\lambda - 2 = 0. \quad (4)$$

Here below we shall make a brief review of the results concerning this conjecture.

For $n = 2$ a proof of the conjecture was given by R.Glassey ([5], [6]). A blow-up result for arbitrary space dimensions when $1 < \lambda < \lambda_0(n)$ was established by T.Sideris [17].

The critical values $\lambda = \lambda_0(n)$ were studied by J.Schaeffer in [16] for $n = 2, 3$. A simplified proof was found by H.Takamura [28].

Another interesting effect is the influence of the decay rate of the initial data on the existence of global solutions. In this case the solution might blow-up in finite time when the initial data decay very slowly at infinity even in the supercritical case when $\lambda > \lambda_0(n)$. For the case $n = 3$ this effect was established by F.Asakura [3] for the supercritical case. The critical cases for $n = 2, 3$ were studied by K.Kubota [14] , K.Tsutaya [29], [30], [31], R.Agemi and H.Takamura [2]. For the case $n \geq 4$ and supercritical nonlinearity the blow-up result for slowly decaying initial data is due to H.Takamura [27].

On the other hand, the existence part of the conjecture of W.Strauss for $n > 3$ is much less elucidated. Recently, Y. Zhou [32] has found a complete answer for $n = 4$ by using suitable weighted Sobolev estimates and the method developed by S.Klainerman [8], [9], [10] for proving the existence of small amplitude solutions.

The existence of a global solution for the case $\lambda = (n + 3)/(n - 1)$ was established by W.Strauss [23] by the aid of the conformal methods and the classical Strichartz inequality.

Another partial answer was given by R.Agemi, K.Kubota, H. Takamura in [1] for a special class of integral nonlinearities in (1). The approach in this work follows the approach of F.John based on his estimate (2).

A complete proof of the conjecture of W.Strauss for spherically symmetric initial data was found by H.Kubo [13] (see also [11], [12]).

By using different estimates H.Lindblad and C.Sogge [15] obtained a similar result as well as the existence of solutions in the supercritical case, non spherically symmetric initial data and space dimensions $n \leq 8$.

Let us make a brief conclusions of the above review of results concerning the missing existence part in the conjecture of W.Strauss.

1.The methods based on the John estimate (2) enable one to control the L^∞ – norm of the solutions. They work very well when the Riemann function is nonnegative (i.e. for $n \leq 3$). A similar idea enables one to consider the case of spherically symmetric initial data.

2. The application of weighted Sobolev inequality in combination with the conformal energy estimate for the wave equation (as it was done in [32]) leads to a weaker restriction $n \leq 4$ (or may be $n \leq 7$ as it was mentioned in [32]) due to the singularity of the nonlinear function $F(u)$.

3. The application of the classical Strichartz inequality enables one to overcome the obstruction caused by the singularity of the nonlinear function, but leads only to local existence and uniqueness of the solution, when

$$1 < \lambda \leq \frac{n+3}{n-1}$$

(see [17]) or the global existence for $\lambda = (n+3)/(n-1)$ (see [23]).

The main purpose of this work is twofold.

In order to overcome the above difficulties and to prove the existence of a small amplitude solution for the general case of arbitrary space dimensions, non-spherically symmetric initial data and

$$\lambda_0(n) < \lambda < \frac{n+3}{n-1},$$

we shall combine the approaches of F.John and R.Strichartz so that a more refined L^p-L^q -estimate taking into account the influence of the weights τ_{\pm} shall be established. Therefore, this estimate will enable us to use the advantages of the both previous estimates due to F.John and R.Strichartz. Actually, we shall have a precise information about the decay rate of the solution with respect to τ_{\pm} weights and we shall be able to avoid the loss of derivatives typical for the Sobolev estimates.

On the other hand, an application of this estimate to the semilinear wave equation (1) gives the possibility to establish the conjecture of W.Strauss for any space dimensions $n \geq 2$ and non-spherically symmetric initial data.

2. Statement of the results

To state the weighted estimate we consider the Cauchy problem for the linear wave equation

$$\square u = F, \tag{5}$$

with zero initial data. For simplicity we shall assume that the supports of u and F lie in the light cone, that is

$$\text{supp} F(t, x) \subset \{|x| \leq t + R\}. \tag{6}$$

Our main weighted estimate has the following form

Theorem 1 *Suppose $1 < p, q < \infty$ satisfy*

$$\begin{aligned} \frac{1}{q} < \frac{1}{p}, \quad \frac{1}{q} + \frac{1}{p} \leq 1, \\ \frac{n-3}{2} < \frac{n}{q} - \frac{1}{p}, \end{aligned} \tag{7}$$

while the parameters $\alpha, \beta, \gamma, \delta$ satisfy

$$\begin{aligned} \alpha &< \frac{n-1}{2} - \frac{n}{q}, \\ \frac{n-1}{2p} - \frac{n+1}{2q} &< \beta = \gamma - \frac{n+1}{2} + \frac{n}{p} - \frac{1}{q} < \frac{n-1}{2} - \frac{n}{q}, \\ \delta &> 1 - \frac{1}{p}. \end{aligned} \quad (8)$$

Then the solution u satisfies the estimate

$$\|\tau_+^\alpha \tau_-^\beta u\|_{L^q(\mathbf{R}_+^{n+1})} \leq C \|\tau_+^\gamma \tau_-^\delta F\|_{L^p(\mathbf{R}_+^{n+1})}, \quad (9)$$

where $\tau_\pm = 1 + |t \pm |x||$ and $\mathbf{R}_+^{n+1} = \{(t, x) \in \mathbf{R}^{n+1} : t \geq 0\}$.

Remark 1. The assumptions (8) lead to the following estimates for γ, δ

$$\begin{aligned} \gamma &> \frac{n+1}{2} \left(1 - \frac{1}{q} - \frac{1}{p}\right) + \frac{1}{q}, \\ \delta &> 1 - \frac{1}{p}. \end{aligned}$$

It is clear that γ, δ are nonnegative. In general α and β in the above theorem can be positive or negative numbers. However, in the application to the semilinear wave equation we shall take $\alpha, \beta > 0$.

Remark 2 The assumptions (7) in the above theorem determine a triangle $\triangle ABC$ in the plane of $1/q, 1/p$ -coordinates with vertices

$$A \left(\frac{n-3}{2(n-1)}, \frac{n-3}{2(n-1)} \right), B \left(\frac{1}{2}, \frac{1}{2} \right), C \left(\frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)} \right).$$

The point A corresponds to the John estimate, while the point C corresponds to the Strichartz estimate.

To establish the conjecture of W. Strauss we consider the semilinear wave equation

$$\begin{aligned} \square u &= F(u), \\ u(0, x) &= \varepsilon f, \quad \partial_t u(0, x) = \varepsilon g, \end{aligned} \quad (10)$$

where f, g are compactly supported smooth functions such that

$$\text{supp } f \cup \text{supp } g \subseteq \{|x| \leq R\}, \quad (11)$$

while ε is a sufficiently small positive number. For the nonlinear function $F(u)$ we shall assume that $F(u) \in C^0$ near $u = 0$ and for some $\lambda > 1$ satisfies

$$\begin{aligned} |F(u)| &\leq C|u|^\lambda, \\ |F(u) - F(v)| &\leq C|u - v|(|u|^{\lambda-1} + |v|^{\lambda-1}) \end{aligned} \quad (12)$$

near $u, v = 0$.

The existence and the uniqueness of the local solution in $C([0, T]; L^q(\mathbf{R}^n))$ for $q = 2(n+1)/(n-1)$ and $1 < \lambda \leq (n+3)/(n-1)$ is established in [17] by using the Strichartz inequality. Therefore, it remains to examine the existence of global solution to (10) for

$$\lambda_0(n) < \lambda < \frac{n+3}{n-1}, \quad (13)$$

where $\lambda_0(n)$ is the positive root of (4) For this case we have the following

Theorem 2 *Suppose the assumptions (11), (12) and (13) are fulfilled with $\lambda_0(n)$ being the positive root of the equation*

$$(n-1)\lambda^2 - (n+1)\lambda - 2 = 0. \quad (14)$$

Then there exists $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$ the Cauchy problem (10) admits a global solution

$$u \in L_{\alpha, \beta}^q(\mathbf{R}_+^{n+1}).$$

Here $L_{\alpha, \beta}^q(\mathbf{R}_+^{n+1})$ denotes the Banach space of all measurable functions with finite norm

$$\|\tau_+^\alpha \tau_-^\beta u\|_{L^q(\mathbf{R}_+^{n+1})}.$$

We shall explain the main idea to establish the weighted estimate of Theorem 1.

The solution of the Cauchy problem (5) can be represented by the aid of a Fourier transform

$$u(t, x) = (2\pi)^{-n} \int_0^t \int_{\mathbf{R}^n} \exp(ix\xi) \frac{\sin((t-s)|\xi|)}{|\xi|} \hat{F}(s, \xi) d\xi ds, \quad (15)$$

where $\hat{F}(s, \xi) = \int \exp(-iy\xi) F(s, y) dy$ is the partial Fourier transform of F . It is clear that $u(t, x) = \int_0^t U(F)(t, s, x) ds$, where

$$U(F)(t, s, x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \exp(ix\xi) \frac{\sin((t-s)|\xi|)}{|\xi|} \hat{F}(s, \xi) d\xi. \quad (16)$$

The Fourier integral operator U can be imbedded into analytic family of operators U_z defined for $z \in \mathbf{C}$ and $F(s, y) \in C_0^\infty(\mathbf{R}^{n+1})$ as follows

$$U_z(F)(t, s, x) = c(n) \int_{\mathbf{R}^n} \exp(ix\xi) (t-s)^{\frac{n}{2}-z} |\xi|^{z-\frac{n}{2}} J_{\frac{n}{2}-z}((t-s)|\xi|) \hat{F}(s, \xi) d\xi, \quad (17)$$

where $J_\nu(s)$ is the Bessel function of order ν and $c(n) = \sqrt{\frac{2}{\pi}} (2\pi)^{-n}$. The above family was introduced by R.Strichartz [24], [25] in order to obtain $L^p - L^q$ estimate for the wave equation. Integrating over s we introduce the operator

$$W_z(F)(t, x) = \int_0^t U_z(F)(t, s, x) ds. \quad (18)$$

Since $J_{\frac{1}{2}}(s) = \sqrt{\frac{2}{\pi}} \frac{\sin(s)}{\sqrt{s}}$, we see that the solution u can be represented as

$$u(t, x) = W_{\frac{n-1}{2}}(F)(t, x). \quad (19)$$

Applying the formula (see [4])

$$\begin{aligned} \int_{\mathbf{R}^n} \exp(-iy\xi)(t-s)^{\frac{n}{2}-z} |\xi|^{z-\frac{n}{2}} J_{\frac{n}{2}-z}((t-s)|\xi|) d\xi &= \\ &= \frac{(2\pi)^{n/2} 2^z}{\Gamma(1-z)} ((t-s)^2 - |y|^2)_+^{-z} \end{aligned} \quad (20)$$

with $s_+^{-z} = s^{-z}$ for $s > 0$ and $s_+^{-z} = 0$ for $s \leq 0$, we get

$$U_z(F)(t, s, x) = \frac{(2\pi)^{n/2} 2^z}{\Gamma(1-z)} \int ((t-s)^2 - |x-y|^2)_+^{-z} F(s, y) dy. \quad (21)$$

For $\operatorname{Re} z < 1$ the integral in (21) is a classical one, while for $\operatorname{Re} z \geq 1$ it is necessary to consider (21) as the action of the distribution

$$K_z(t, s, x, y) = \frac{(2\pi)^{n/2} 2^z}{\Gamma(1-z)} ((t-s)^2 - |x-y|^2)_+^{-z} \quad (22)$$

on the test function $F(s, y)$.

The possibility to apply a complex interpolation for the strip $0 \leq \operatorname{Re} z \leq (n+1)/2$ relies on a combined use of (17) and (21). More precisely, the well-known Strichartz estimate uses the following L^∞ estimate on the line $\operatorname{Re} z = 0$

$$\|U_z(F)(t, s, \cdot)\|_{L^\infty} \leq C \|F(s, \cdot)\|_{L^1} \quad (23)$$

and this is a direct consequence of (21). Making the observation that the representation formula (21) keeps its classical sense for $\operatorname{Re} z < 1$, we can use this classical representation for the larger semiplane $\operatorname{Re} z < 1$ and we can prove a weighted L^∞ estimate for this semiplane. Our idea to derive such an estimate is to follow the approach of F. John and to obtain L^∞ -estimate with weights τ_\pm for $\operatorname{Re} z < 1$.

Our next step is to derive L^2 -weighted estimate on the line $\operatorname{Re} z = (n+1)/2$. For the purpose we shall use the representation (17). Then the kernel K_z can be represented by the oscillatory integral

$$K_z(t, s, x, y) = c(n) \int_{\mathbf{R}^n} \exp(i(x-y)\xi)(t-s)^{\frac{n}{2}-z} |\xi|^{z-\frac{n}{2}} J_{\frac{n}{2}-z}((t-s)|\xi|) d\xi. \quad (24)$$

In this case we split the space of variables (t, s, x, y) into two complementary domains Ω_1 and Ω_2 . For the first domain we can apply either the estimate

$$(t-s) - |x-y| \geq C(1 + |t - |x||)$$

showing that (21) is a classical function, either the estimate

$$1 + |s - |y|| \geq C(1 + |t - |x||)$$

leading directly to weighted L^2 estimate involving the weights τ_- . The second domain Ω_2 is determined by

$$\begin{aligned} |x| &\leq t-1 \quad , \quad s \geq \frac{t-|x|}{8}, \\ t-s &\geq \frac{t-|x|}{8} \quad , \quad 1+|s-|y|| \leq \varepsilon_1(t-|x|) \\ |t-s-|x-y|| &\leq \varepsilon_1(t-|x|), \end{aligned}$$

where ε_1 is a sufficiently small positive number. The main geometrical observation for this domain is that the angle between $y/|y|$ and $(x-y)/|x-y|$ is equivalent to

$$D(t, s, x) = \sqrt{\frac{(t-|x|)t}{(t-s)s}}.$$

Then we make a further partition in the ξ coordinates and in this way we have to consider two possibilities. If $\xi/|\xi|$ is not close to $(x-y)/|x-y|$, then we can use a stationary phase method for the oscillatory integral in (24). On the other hand, when $\xi/|\xi|$ is close to $(x-y)/|x-y|$ the angle between $\xi/|\xi|$ and $y/|y|$ is proportional to $D(t, s, x)$ according to our main geometrical observation. We can make change of variables

$$s \rightarrow \tau = s - |y|$$

in (24). Then the phase and the amplitude functions in the oscillatory integral (24) also will be changed. In this way the study of L^2 -weighted boundedness of the operator W_z can be reduced to the study of L^2 -boundedness of concrete Fourier integral operators, depending on the parameter $D(t, s, x)$. The crucial point now is the fact that the angle between $\xi/|\xi|$ and $y/|y|$ is proportional to $D(t, s, x)$ will assure nondegeneracy of the phase function. This will allow us to apply a suitable modification of a criteria due to E.Stein (see [19], [20]) for L^2 -boundedness of Fourier integral operators.

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AN INTRODUCTION TO NONLINEAR SCHRÖDINGER EQUATIONS

J. GINIBRE

Abstract : This introductory course is devoted to the mathematical study of the Cauchy problem and of the theory of scattering for a class of nonlinear Schrödinger (NLS) equations of the form

$$i \partial_t u = -(1/2)\Delta u + f(u) \quad . \quad (1)$$

Although the treatment is directed specifically to the NLS equation (1), it will present methods of general relevance to other similar equations such as the Korteweg-de Vries equation, some nonlinear wave equations and Klein Gordon equations, and others. The Cauchy problem will be treated primarily by contraction methods applied to the integral equation associated with (1). For that purpose, a review will be made of the space time integrability properties (or generalized Strichartz inequalities) associated with the free Schrödinger equation. A survey will then be given of the treatment of the Cauchy problem in \mathbb{R}^n for L^2 and H^1 data, in the spirit of the work of Kato. The Cauchy problem will then be considered in \mathbb{T}^n , and an introduction given to recent work by Bourgain on that problem, based on a modified version of the contraction method and extensive use of Fourier space estimates. Finally, the theory of scattering for the equation (1) will be briefly sketched, centered on the two basic questions of existence of the wave operators and of asymptotic completeness, and a review will be made of the current status of the first question, both in the short range case and in the limiting Coulomb-like case.

Introduction

The purpose of the present course is to provide an introduction to the theory of

the Cauchy problem and to the theory of scattering for a typical semilinear evolution equation, namely the so-called nonlinear Schrödinger (NLS) equation

$$i \partial_t u = -(1/2)\Delta u + f(u) \quad . \quad (0.1)$$

Here the unknown function u is a complex function defined in space time $\mathbb{R}^n \times \mathbb{R}$ or $\mathbb{T}^n \times \mathbb{R}$, Δ is the Laplacian in \mathbb{R}^n or \mathbb{T}^n , and $f(u)$ is a nonlinear interaction term, $f(u) \in \mathcal{C}^1(\mathbb{C}, \mathbb{C})$, a typical example of which is

$$f(u) = \lambda |u|^{p-1} u \quad (0.2)$$

with $\lambda \in \mathbb{R}$ and $1 < p < \infty$. Although the treatment will address specifically the equation (0.1), the basic notions and methods will have a wider range of applicability, and apply for instance to other evolution equations such as the nonlinear wave equation (NLW)

$$\square u + f(u) = 0 \quad , \quad (0.3)$$

the nonlinear Klein-Gordon equation (NLKG)

$$\square u + m^2 u + f(u) = 0 \quad . \quad (0.4)$$

where u and f are as before and $\square = \partial_t^2 - \Delta$, the generalized Korteweg-de Vries equation (GKdV)

$$\partial_t u + \partial_x^3 u = \partial_x f(u) \quad (0.5)$$

where u is now a real function defined in space time $\mathbb{R} \times \mathbb{R}$ or $\mathbb{T} \times \mathbb{R}$ and $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, and many others.

We shall concentrate on the following problems

(1) The Cauchy problem in \mathbb{R}^n

The problem consists in studying the existence, uniqueness and possibly additional properties of solutions of the equation (0.1) with prescribed initial data $u(t=0) = u_0$. It will be treated by the general and standard method whereby one splits it into two steps : one first proves the existence of local solutions (in time), namely solutions defined in a small time interval $[-T, T]$, by a contraction method, which in addition provides uniqueness and continuous dependence with respect to initial data. One then extends the local solutions to global ones by exploiting a priori estimates of the solutions, derived from exact or approximate conservation laws associated with invariance properties of the equation.

(2) The Cauchy problem in \mathbb{T}^n

The problem is formulated as before and split into the same two steps, the second of which is treated in very much the same way as in \mathbb{R}^n . The local resolution by contraction

however, is much more difficult than in the case of \mathbb{R}^n , but an important progress in that direction has been made recently by Bourgain [B1]. We shall give an introduction to this important work. The contraction method has to be suitably modified, and the crux of the matter then relies on hard estimates, performed in Fourier transformed variables.

(3) The asymptotic behaviour in time of the global solutions in \mathbb{R}^n , in the form of the theory of scattering.

We shall first present the basic concepts and problems of that theory. We shall then review the status of the first of those problems, namely the existence of the wave operators for the equation (0.1), first in the so-called short range case, where the situation is fairly well understood up to a certain point, and then in the limiting Coulomb case, where the ordinary wave operators fail to exist and have to be replaced by modified ones, and where only preliminary results exist.

The literature on the NLS equation is enormous, and no attempt will be made at any kind of completeness in the present notes where we shall mainly quote the references that are directly relevant to the material covered. A good bibliography up to '89 can be found in the monograph [C]. See also [CH] [K2]. The material covered in Part 1 and to a lesser extent, in the bulk of Part 3 is by now fairly standard. See for instance [C] [K2]. The present exposition largely follows [G1]. The material covered in Part 2 is more recent and probably has not reached a final stage. The present exposition is taken from [G2]. The end of Part 3 is taken from recent work of T. Ozawa and the author [O] [GO].

Because of space and time limitation, a number of questions that would naturally fit under the title of the present course have been completely omitted. Such is the case for instance of the use of compactness methods in the treatment of the Cauchy problem, of the study of smoothness of solutions beyond the level of H^1 -solutions (see below), of the study of solutions blowing up in finite time, of the study of stationary solutions, etc. Finally, nothing will be presented of the current trend in the subject, which consists in studying NLS equations with more general non linearities also depending on space derivatives of u . These topics however will be treated in other lectures at this Conference.

1. The Cauchy problem in \mathbb{R}^n

In this section we study the Cauchy problem for the equation (0.1) with initial data $u(t=0) = u_0$. We shall always assume that f satisfies the following assumption

(H1) $f \in C^1(\mathbb{C}, \mathbb{C})$, $f(0) = 0$, and for some p , $1 < p < \infty$, the following estimate holds for all $z \in \mathbb{C}$:

$$|f'(z)| \equiv \text{Max}(|\partial f / \partial \bar{z}|, |\partial f / \partial z|) \leq C(1 + |z|^{p-1}) \quad . \quad (1.1)$$

We define the free evolution group

$$U(t) = \exp\left(i\frac{t}{2}\Delta\right) \quad (1.2)$$

which solves the free Schrödinger equation $i\partial_t u = -(1/2)\Delta u$. This is a one parameter unitary group in the Sobolev space H^ρ for all $\rho \in \mathbb{R}$, where

$$H^\rho = \left\{ u : \|u; H^\rho\| \equiv \|(1-\Delta)^{\rho/2}u\|_2 < \infty \right\}$$

and $\|\cdot\|_r$ denotes the norm in $L^r \equiv L^r(\mathbb{R}^n)$. The Cauchy problem for (0.1) is formally equivalent to the integral equation

$$\begin{aligned} u(t) &= U(t) u_0 - i \int_0^t dt' U(t-t') f(u(t')) \\ &\equiv U(t) u_0 + (F(u))(t) \equiv (A(u))(t) \end{aligned} \quad (1.3)$$

where the second line defines F and A in an obvious way.

We now comment briefly on the sense in which we want the equations (0.1) and (1.3) to be satisfied and on their interrelation. We shall always want (0.1) to be satisfied in the weakest reasonable sense, so as to allow for solutions as general as possible, namely in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$. Under the assumption (H1), a sufficient condition for (0.1) to make sense term by term in \mathcal{D}' is that $u \in L^p_{loc}(\mathbb{R}^{n+1})$, and this will turn out to be weaker than all the assumptions that we shall need below to make meaningful statements on the Cauchy problem. As regards (1.3), we shall always need assumptions which imply that u and $f(u) \in \mathcal{C}(I, H^{-N})$ for some interval $I \subset \mathbb{R}$ and some N , so that $F(u)$ is well defined as a Bochner integral in H^{-N} and actually $F(u) \in \mathcal{C}^1(I, H^{-N})$, and all terms in (1.3) are well defined in $\mathcal{C}(I, H^{-N})$. The equivalence of (0.1) (with initial condition) and (1.3) then boils down to the identity

$$(i\partial_t + (1/2)\Delta) F(u) = f(u) \quad (1.4)$$

where all terms are well defined in $\mathcal{C}(I, H^{-N}) \cap \mathcal{C}^1(I, H^{-N-2}) \subset \mathcal{C}(I, H^{-N-2})$ and the formal computation leading to (1.4) is easily justified with the available regularity.

In order to study the local Cauchy problem, we first need some estimates on the free evolution.

1.a Properties of the free evolution

We denote by \mathcal{F} the Fourier transform in the x variable, by ξ the variable conjugate to x , and by \star_x and \star_t the convolution in x and t , with the subscript omitted when there is no risk of confusion. The free group $U(t)$ can be represented by

$$U(t) = \mathcal{F}^{-1} \exp\left(-i\frac{t}{2}\xi^2\right) \mathcal{F} = (2\pi it)^{-n/2} \exp\left(i\frac{x^2}{2t}\right) \star_x \quad (1.5)$$

so that $U(t)$ is bounded from L^1 to L^∞ for all $t \neq 0$, with

$$\| U(t) f \|_\infty \leq (2\pi|t|)^{-n/2} \| f \|_1 \quad (1.6)$$

Interpolating between (1.6) and unitarity in L^2 by the Riesz Thorin theorem, we obtain

$$\| U(t) f \|_r \leq (2\pi|t|)^{-\delta(r)} \| f \|_{\bar{r}} \quad (1.7)$$

for all r , $2 \leq r \leq \infty$, where (r, \bar{r}) denote pairs of dual (Hölder conjugate) exponents, $1/r + 1/\bar{r} = 1$, and $\delta(r) \equiv n/2 - n/r$.

Let $I \subset \mathbb{R}$ be an interval (possibly unbounded). We define the operators

$$(U \star f)(t) = \int_I dt' U(t-t') f(t') \quad (1.8)$$

$$(U \star_R f)(t) = \int_{I \cap \{t' \leq t\}} dt' U(t-t') f(t') \quad (1.9)$$

where f is defined in $\mathbb{R}^n \times I$ and suitably regular, the subscript R stands for retarded and reference to I is omitted for brevity. Using the Hardy-Littlewood-Sobolev inequality ([Hö I] p. 117) in time, we obtain immediately the following first family of estimates.

Lemma 1.1. *Let $1 < q_1, q_2 < \infty$ and $0 \leq \delta(r) = \frac{1}{q_1} + \frac{1}{q_2} < 1$. Then the following estimates hold*

$$\| U \star f; L^{q_1}(I, L^r) \| \leq C \| f; L^{\bar{q}_2}(I, L^{\bar{r}} \| \quad (1.10)$$

$$\| U \star_R f; L^{q_1}(I, L^r) \| \leq C \| f; L^{\bar{q}_2}(I, L^{\bar{r}} \| \quad (1.11)$$

with a constant C independent of I .

We shall obtain a second more useful family of estimates by using general duality arguments which are to a large extent independent of the equation at hand. Those arguments have a long history (see [S'] [GV3] [Y] [GV4] and references therein) and the estimates thereby obtained sometimes go under the name of generalized Strichartz inequalities. The basic fact is that if B is an operator and B^* its adjoint in a suitable context, then it is equivalent that B or B^* or B^*B be bounded. The relevant context is as follows. Let \mathcal{H} be a Hilbert space, X a Banach space, X^* the dual of X , let \mathcal{D} be a dense subspace of X (no topology is needed on \mathcal{D}), and let \mathcal{D}_a^* be the algebraic dual of \mathcal{D} , so that $X^* \subset \mathcal{D}_a^*$. Let B be a linear operator from \mathcal{D} to \mathcal{H} and B^* its adjoint, defined by

$$\langle B^*v, f \rangle_{\mathcal{D}} = \langle v, Bf \rangle$$

for all $f \in \mathcal{D}$ and all $v \in \mathcal{H}$, where $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ is the pairing between \mathcal{D}_a^* and \mathcal{D} and $\langle \cdot, \cdot \rangle$ is the scalar product in \mathcal{H} . Let $\| \cdot \|$ denote the norm in \mathcal{H} . We then have the following lemma, the proof of which is elementary (see [GV4]).

Lemma 1.2. *The following statements are equivalent :*

(1) *There exists b , $0 \leq b < \infty$ such that for all $f \in \mathcal{D}$*

$$\| Bf \| \leq b \| f; X \| \quad .$$

(2) *$\mathcal{R}(B^*) \subset X^*$ and there exists b , $0 \leq b < \infty$, such that for all $v \in \mathcal{H}$*

$$\| B^*v; X^* \| \leq b \| v \| \quad .$$

(3) *$\mathcal{R}(B^*B) \subset X^*$ and there exists b , $0 \leq b < \infty$, such that for all $f \in \mathcal{D}$*

$$\| B^*Bf; X^* \| \leq b^2 \| f; X \| \quad .$$

*The constant b is the same in all three parts, and under any of the conditions (1) (2) (3), B and B^*B extend to bounded operators from X to \mathcal{H} and to X^* respectively.*

The basic example of the previous situation is the following. Let $I \subset \mathbb{R}$ be an interval and $U(t)$ a unitary one parameter group in \mathcal{H} . Define

$$Bf = \int_I dt U(-t) f(t) \quad (1.12)$$

for any $f \in \mathcal{D}$, a space of suitably regular functions from I to \mathcal{H} . Then

$$(B^*v)(t) = U(t)v \quad (1.13)$$

and

$$((B^*B)f)(t) = \int_I dt' U(t-t') f(t') \quad , \quad (1.14)$$

where duality is defined through the scalar product in $L^2(I, \mathcal{H})$. Then the conditions of Lemma 1.2 are satisfied with $X = L^1(I, \mathcal{H})$, $X^* = L^\infty(I, \mathcal{H})$ and $b = 1$. In the applications, useful estimates arise from the finding of other spaces X satisfying one of (and therefore all) the conditions of that lemma for fixed \mathcal{H} , \mathcal{D} and for the previous B .

The following (obvious) corollary of Lemma 1.2 turns out to be extremely useful.

Corollary 1.1. *Let any of the conditions of Lemma 1.2 hold for the same \mathcal{H} , \mathcal{D} and B and two pairs (X_i, b_i) , $i = 1, 2$. Then for all choices of $i, j \in \{1, 2\}$ and all $f \in \mathcal{D}$*

$$\| B^*Bf; X_i^* \| \leq b_i b_j \| f; X_j \|$$

*and B^*B extends to a bounded operator from X_j to X_i^* .*

Briefly stated, the diagonal cases of the condition (3) of Lemma 1.2 imply the off diagonal cases.

The relevance of the basic example to the Cauchy problem for the evolution equation

$$i\partial_t u = L u + f$$

with L self-adjoint in \mathcal{H} , and with initial data $u(t=0) = u_0$ follows from the fact that the associated integral equation can be written as

$$u = B^* u_0 - i(B^* B)_R f \quad (1.15)$$

where B is defined by (1.12) with $U(t) = \exp(-itL)$ and the subscript R again means retardation (cf. (1.3) (1.9)), thereby suggesting to look for spaces satisfying the conditions of Lemma 1.2, to take $u_0 \in \mathcal{H}$ and look for solutions in X^* by ensuring that $f \in X$. This leads to the question of extending the estimates of Part (3) of Lemma 1.2 and of Corollary 1.1 from the non retarded operators $B^* B$ to the retarded ones $(B^* B)_R$. In many cases the diagonal estimates (from X to X^*) are proved at the same time for $B^* B$ and $(B^* B)_R$. However the retardation breaks the factorization and Corollary 1.1 in general does not apply. A similar result applies however in special cases that we now describe.

Definition 1.1. A Banach space X of distributions in space time is said to be *stable under time restriction* if the multiplication by the characteristic function of an interval I in time is a bounded operator in X , with norm uniformly bounded with respect to I .

Lemma 1.3. *Let \mathcal{H} , B , B^* defined by (1.12) (1.13) and X satisfy any of the conditions of Lemma 1.2 and let X be stable under time restriction. Then $(B^* B)_R$ is bounded from X to $L^\infty(I, \mathcal{H})$ and from $L^1(I, \mathcal{H})$ to X^* .*

The (easy) proof is omitted (see [GV4]). Further off diagonal cases of boundedness of the operator $(B^* B)_R$ then follow from diagonal cases and from Lemma 1.3 by interpolation.

It is now a straightforward matter to apply the previous duality arguments to the Schrödinger equation. We need the following definition

Definition 1.2. A pair of exponents (q, r) is said to be *admissible* if

$$0 \leq 2/q = \delta(r) (\equiv n/2 - n/r) < 1 \quad .$$

We then obtain [GV3] [Y] :

Lemma 1.4. *The following estimates hold :*

(1) *For any admissible pair (q, r)*

$$\| U(t)u; L^q(\mathbb{R}, L^r) \| \leq c_r \| u \|_2 \quad . \quad (1.16)$$

For any admissible pairs (q_i, r_i) , $i = 1, 2$, and any interval $I \subset \mathbb{R}$

$$\| U \star f; L^{q_1}(I, L^{r_1}) \| \leq c_{r_1} c_{r_2} \| f; L^{\bar{q}_2}(I, L^{\bar{r}_2}) \| \quad , \quad (1.17)$$

$$\| U \star_R f; L^{q_1}(I, L^{r_1}) \| \leq c_{r_1} c_{r_2} \| f; L^{\bar{q}_2}(I, L^{\bar{r}_2}) \| \quad . \quad (1.18)$$

Proof. Lemma 1.1 with $q_1 = q_2$ provides the diagonal cases of (1.17) (1.18). The diagonal case of (1.17) with $(q_1 = q_2 = q, r_1 = r_2 = r)$ is Part (3) of Lemma 1.2 and by that lemma implies (1.16) which is nothing but Part (2). Finally the diagonal cases of (1.18) imply the off diagonal cases by Lemma 1.3 and interpolation, as explained above. \square

Note that (1.10) (1.11) and (1.17) (1.18) form two different families of estimates, each one depending on two parameters. A remarkable feature of (1.17) (1.18) is that the exponents (q_i, r_i) in the LHS are completely decoupled from those in the RHS. In what follows, we shall make use almost exclusively of that second family, which allows for a satisfactory treatment of the Cauchy problem. It is nevertheless possible to obtain more general estimates by interpolation between the two families, thereby obtaining more refined results on that problem [K3] [K4].

1.b Choice of spaces for the Cauchy problem

The spaces where we shall try to solve the Cauchy problem are tailored to fit into the previous discussion (see esp. (1.15)) and to take advantage of the estimates of Lemma 1.4. Let $I \subset \mathbb{R}$ and $\rho \geq 0$. Corresponding to initial data $u_0 \in H^\rho$, we define

$$X^\rho(I) = \{u : u \in \mathcal{C}(I, H^\rho), \text{ and } u \in L^q(I, H_r^\rho) \text{ for all admissible } (q, r)\} \quad (1.19)$$

where

$$H_r^\rho = \left\{ u : \| u; H_r^\rho \| \equiv \| (1 - \Delta)^{\rho/2} u \|_r < \infty \right\} \quad .$$

The spaces $X^\rho(I)$ are defined by a family of seminorms parametrized by a semi-open interval and can be made into Fréchet spaces. However, it is technically more convenient to modify slightly the definition in order to obtain spaces that come out naturally as Banach spaces. Let

$$0 \leq 2/q_0 = \delta(r_0) \equiv \delta_0 < 1 \quad .$$

We define

$$\begin{aligned} X_{r_0}^\rho(I) &= \{u; u \in \mathcal{C}(I, H^\rho) \text{ and } u \in L^q(I, H_r^\rho) \text{ for } 0 \leq 2/q = \delta(r) \leq \delta_0\} \\ &\equiv (\mathcal{C} \cap L^\infty)(I, H^\rho) \cap L^{q_0}(I, H_{r_0}^\rho) \end{aligned} \quad (1.20)$$

which are Banach spaces with obvious norms. We also define local spaces

$$X_{(r_0)loc}^\rho(I) = \left\{ u : u \in X_{(r_0)}^\rho(J) \text{ for any } J \subset\subset I \right\}$$

in the standard way. In what follows, the subscript ρ will be omitted if $\rho = 0$. Lemma 1.4 Part (1) asserts that $u_0 \in H^\rho \Rightarrow U(\cdot) u_0 \in X^\rho(\mathbb{R})$, with accompanying norm estimates.

It will turn out that some values of ρ are better than others, at least for the global problem. To see this, we introduce the following assumption on f , which will not be used technically until Section (1.f).

(H2) *Gauge invariance* : There exists $V \in C^1(\mathbb{C}, \mathbb{R})$ with $V(0) = 0$, $V(z) = V(|z|)$ for all $z \in \mathbb{C}$, and such that $f(z) = \partial V / \partial \bar{z}$.

Equivalently, there exists $G \in C^1(\mathbb{R}^+, \mathbb{R})$ such that $f(z) = zG'(|z|^2)$, the relation with (H2) being that $V(z) = G(|z|^2)$.

Let $\langle \cdot, \cdot \rangle$ denote the scalar product in L^2 . Under the assumption (H2), taking the scalar products $2Im \langle u, (0.1) \rangle$ and $2Re \langle \partial_t u, (0.1) \rangle$, we obtain respectively

$$\partial_t \|u\|_2^2 = -Im \langle u, \Delta u \rangle + 2Im \langle u, f(u) \rangle = 0 \quad (1.21)$$

and

$$0 = -Re \langle \partial_t u, \Delta u \rangle + 2Re \langle \partial_t u, f(u) \rangle = \partial_t E(u) \quad (1.22)$$

where

$$E(u) = (1/2) \|\nabla u\|_2^2 + \int dx V(u) \quad (1.23)$$

so that formally the L^2 -norm $\|u\|_2$ and the energy $E(u)$ are conserved quantities for the evolution (0.1). Those conservation laws will play an essential role in the problem of existence of global solutions for $\rho = 0$ (L^2 solutions) and $\rho = 1$ (H^1 solutions), two values of ρ that will be therefore of special interest.

We now turn to some homogeneity arguments which will shed some light on the subsequent results. Under a change of function $u \rightarrow u_\lambda$ where $u_\lambda(x, t) = u(x/\lambda, t/\lambda^2)$, the leading norms in the definition of X^ρ , namely

$$\|\nabla|^\rho u; L^q(L^r)\|$$

pick up a factor λ^d where d is the dimension of that norm in x

$$d = n/r + 2/q - \rho = n/2 - \rho$$

independently of (q, r) for admissible (q, r) . If we want to show that the operator F (see (1.3)) is bounded in $X^\rho(I)$ for some bounded I and for f homogeneous in u of degree p , we shall need some estimate of the form

$$\|\nabla|^\rho F(u); L^q(I, L^r)\| \leq C \|\nabla|^\rho u; L^q(I, L^r)\|^p |I|^\theta \quad (1.24)$$

with $\theta \geq 0$, where the RHS has to be homogeneous in u of degree p , and the last factor can possibly come from an application of the Hölder inequality in time. Counting the dimension in x , taking into account the fact that F has dimension 2 because of the time integration, we obtain

$$2 + n/2 - \rho = p(n/2 - \rho) + 2\theta$$

so that

$$(p - 1)(n/2 - \rho) \leq 2 \quad . \quad (1.25)$$

That relation is important enough to call for a definition.

Definition 1.3. p is critical (resp. subcritical, resp. supercritical) at the level of H^ρ (or for short at the level ρ) iff $(p - 1)(n/2 - \rho) = 2$ (resp. < 2 , resp. > 2).

The L^2 and H^1 critical values are $p = 1 + 4/n$ and $p = 1 + 4/(n - 2)$ respectively and will play an important role in what follows. The lesson to be drawn from the previous argument (see esp. (1.25)) is that in order to treat an interaction with a power p by contraction (perturbation) methods, at least one norm is needed for which p is at most critical.

We now turn to the problem of uniqueness of solutions

1.c Uniqueness of solutions of the Cauchy problem

The uniqueness proof of solutions relies on the following basic argument (or variants thereof). If X and Y are two Banach spaces with $Y \subset X$, and if the operator A from Y to X is a contraction in the X norm on the bounded sets of Y , then the equation $u = Au$ has at most one solution in Y . This will be applied by choosing for X a space of the form $X_{r_0}(I)$ defined by (1.20). The basic uniqueness result can be stated as follows.

Proposition 1.1. *Let f satisfy (H1) and let $I \subset \mathbb{R}$ be an interval containing zero. Then the equation (0.1) with initial data $u(0) = u_0$ has at most one solution in the following situations :*

- (1) *If $p \leq 1 + 4/n$ and $u_0 \in L^2$, in the space $X_{p+1,loc}(I)$.*
- (2) *If $p \leq 1 + 4/(n - 2)$ and $u_0 \in H^1$, in the space*

$$C(I, L^2) \cap L_{loc}^\infty(I, L^{p+1})$$

- (3) *If $p \geq 1 + 4/n$, namely $(p - 1)(n/2 - \rho) = 2$ with $0 \leq \rho < n/2$ and $u_0 \in H^\rho$, in the space*

$$X_{r_0,loc}(I) \cap L_{loc}^k(I, L^s)$$

with

$$\rho < \delta(s) < \text{Min}(n/2, \rho + 1)$$

$$0 \leq 2/k \leq \delta(s) - \rho$$

$$\delta_0 \geq (n - 2\delta(s))(n - 2\rho)^{-1} \quad .$$

Before sketching the proof of Proposition 1.1, we make some comments on the various statements. Part (1) practically solves the local Cauchy problem in L^2 , because in that case $X = Y$. Part (2) is tailored to cover the case of H^1 solutions at minimal cost. It also implies the uniqueness of weak solutions, namely of solutions in $L^\infty(H^1)$ obtained by compactness, for H^1 -subcritical p . Finally Part (3) is a fairly general statement which comes out naturally from the estimates in the proof.

Sketch of proof. We first remark that uniqueness is a local problem in time as soon as the solutions under consideration have some continuity in time, which is required anyway to formulate the Cauchy problem. In fact let u_1 and u_2 be two solutions with $u_1(0) = u_2(0)$ and let $t_0 = \text{Inf} \{t : u_1(t) \neq u_2(t)\}$. Then $u_1(t_0) = u_2(t_0)$ and if t_0 is inside I , it is sufficient to prove that the local Cauchy problem with initial data $u_i(t_0)$ at time t_0 has at most one solution in the interval $[t_0, t_0 + T]$ for some small $T > 0$ to obtain a contradiction with the definition of t_0 .

The crux of the proof then consists in showing that the operator A (or equivalently F) defined by (1.3) is a contraction in the norm of $X_{r_0}(I)$ for $I = [0, T]$ and T sufficiently small on bounded sets of the spaces where uniqueness is expected to hold. For that purpose, we estimate by Lemma 1.4

$$\| F(u_1) - F(u_2); X_{r_0}(I) \| \leq C \| f(u_1) - f(u_2); L^{\bar{q}_1}(I, L^{\bar{r}_1}) \| \quad (1.26)$$

for admissible (q_1, r_1) . More precisely, under (H1), one separates f as $f = f_1 + f_2$ with $|f'_1| \leq 1$ and $|f'_2| \leq |u|^{p-1}$ and uses different (q_i, r_i) for f_1 and f_2 , thereby obtaining

$$\begin{aligned} \dots \leq C \{ & T \| u_1 - u_2; L^\infty(I, L^2) \| + \| u_1 - u_2; L^{q_2}(I, L^{r_2}) \| T^\theta \\ & \times \text{Max}_i \| u_i; L^k(I, L^s) \|^{p-1} \} \end{aligned} \quad (1.27)$$

for admissible (q_2, r_2) , by the Hölder inequality in space and time, with $\theta \geq 0$ and

$$\begin{cases} (p-1)(n/2 - \delta(s)) = \delta_1 + \delta_2 \end{cases} \quad (1.28)$$

$$\begin{cases} (p-1)2/k = 2(1-\theta) - (\delta_1 + \delta_2) \end{cases} \quad (1.29)$$

where $\delta_i = \delta(r_i)$, $i = 1, 2$ so that

$$(p-1)(n/2 - \delta(s) + 2/k) = 2(1-\theta) \leq 2 \quad . \quad (1.30)$$

Note in particular that by (1.30), the norm in $L^k(L^s)$ dimensionally belongs to a level of regularity at which p is at most critical, as was expected from the discussion in Section 1.b. The proof of the proposition then follows from the previous estimates by making suitable choices of the parameters r_1, r_2, s and r_0 . In particular,

Part (1) is proved by choosing $2/k = \delta(s) = \delta_1 = \delta_2 = \delta_0 (= \delta(p+1))$

Part (2) is proved by choosing $k = \infty$, $\delta(s) = \delta_1 = \delta_2 (= \delta(p+1))$, which is possible provided $\delta(p+1) < 1$, namely $(p-1)(n/2-1) < 2$, and which yields $\theta = 1 - \delta(p+1)$.

Part (3) is proved by a more general suitable choice. \square

1.d The local Cauchy problem in L^2

We first state the result in the L^2 -subcritical case [T3].

Proposition 1.2. *Let f satisfy (H1) with $p-1 < 4/n$. Then for any $u_0 \in L^2$, there exist $T_{\pm} > 0$ such that*

(1) *The equation (0.1) with initial data $u(0) = u_0$ has a unique solution*

$$u \in X_{p+1,loc}((-T_-, T_+)) \quad .$$

(2) *If $T_+ < \infty$ (resp. $T_- < \infty$), then $\|u(t)\|_2 \rightarrow \infty$ when $t \rightarrow T_+$ (resp. $t \rightarrow -T_-$).*

In addition

(3) *For $-T_- < T_1 < T_2 < T_+$, the map $u_0 \rightarrow u$ is continuous from a neighborhood of the original u_0 in L^2 to $X([T_1, T_2])$.*

Sketch of proof. One first proves that for $I = [-T, T]$ and T sufficiently small, the map $u \rightarrow A(u)$ is a contraction of a ball of $X_{p+1}(I)$ into itself. This provides the existence of a unique solution in that ball. The estimates needed for the proof are exactly the same as those in the proof of Proposition 1.2.

One then iterates the local resolution in order to extend the solution as far as possible, thereby obtaining a maximal solution which by uniqueness does not depend on the successive times of local resolution. Let $(-T_-, T_+)$ be the interval of existence of the maximal solution. Part (2) then follows from the fact (which itself follows from the estimates in the proof) that the time of local resolution can be estimated from below in terms of the norm of the initial data

$$T \geq M(\|u_0\|_2) \tag{1.31}$$

where $M(\cdot)$ is a strictly positive function which can always be taken non increasing, typically

$$M(R) \sim C R^{-N}.$$

for some (possibly large) N . The argument goes by contradiction. Suppose that Part (2) does not hold at T_+ . Then there exists $R > 0$ and an increasing sequence t_1, \dots, t_n, \dots tending to T_+ and such that $\|u(t_n)\|_2 \leq R < \infty$. One can then solve the local Cauchy problem with initial time t_n and initial data $u(t_n)$ in the interval $t_n + T$ where $T \geq M(R)$ is independent of n , so that for n large enough $t_n + T > T_+$, in contradiction with the definition of T_+ as the upper limit of the maximal interval of existence.

Part (3) follows from the contraction property in the interval of local resolution, and is then extended by suitable iteration. \square

We now comment briefly on the L^2 -critical case where $p = 1 + 4/n$. In that case, the contraction proof works without difficulty by using the same estimates, with however one difference, namely one can no longer estimate the time of local resolution in terms of the L^2 norm of the initial data alone as in (1.31), and consequently one can no longer prove Part (2) of the proposition. What remains thereof is the fact that, if $T_+ < \infty$, then

$$\| u; X_{p+1}([0, T_+)) \| = \infty$$

and the analogue at $-T_-$.

In those situations where the statements of Proposition 1.2 hold (with Part (2) possibly modified as just explained), we shall say that the Cauchy problem for the equation (0.1) with initial data in L^2 is *locally well-posed* in $X_{p+1}(\cdot)$.

We now turn to the second value of ρ of special interest, namely $\rho = 1$.

1.e The local Cauchy problem in H^1

The main result can be stated as follows [K1].

Proposition 1.3. *Let f satisfy (H1) with $p - 1 < 4/(n - 2)$ ($p < \infty$ if $n \leq 2$). Then the Cauchy problem for the equation (0.1) with initial data in H^1 is locally well posed in $X_{p+1}^1(\cdot)$.*

More precisely, the statement is that of Proposition 1.2 with L^2 replaced by H^1 and $X(\cdot)$ by $X^1(\cdot)$.

Sketch of proof. We recall that

$$X^1 = \{u : u \text{ and } \nabla u \in X.\} \quad .$$

The proof follows the same pattern as that of Proposition 1.2. The main technical step consists in proving that for suitably small T (actually depending only on $\|u_0; H^1\|$) the map A defined in (1.3) is a contraction of the norm of $X_{p+1}(I)$ in a suitable ball B_1 of $X_{p+1}^1(I)$, where $I = [-T, T]$. The contraction property follows from the estimates in the proof of Proposition 1.1 Part (2), and the only additional information that is needed is the fact that the ball B_1 is mapped into itself by A . For that purpose, one estimates in addition

$$\begin{aligned} & \| \nabla F(u); X_{r_0}(I) \| \leq C \| f'(u) \nabla u; L^{\bar{q}_1}(I, L^{\bar{r}_1}) \| \\ & \dots \leq C \{ T \| \nabla u; L^\infty(I, L^2) \| + \| \nabla u; L^{q_2}(I, L^{r_2}) \| T^\theta \\ & \quad \| u; L^k(I, L^s) \|^{p-1} \} \end{aligned} \quad (1.32)$$

in exactly the same way as in (1.27), and one then continues with exactly the same estimates as in the proof of Proposition 1.1.

The proof then proceeds by the same abstract arguments as in Proposition 1.2, except for the fact that an additional argument is required to prove the continuity of the solution as a function of the initial data from H^1 to $X_{p+1}^1(\cdot)$. \square

1.f The global Cauchy problem. A priori estimates

Extending the local solutions of Section 1.d and 1.e to global ones, namely to solutions defined for all times, relies on a priori estimates of solutions of the equation (0.1). Such estimates are exploited through an abstract argument which is independent of the equation. A variant thereof has already been given as a proof of Part (2) of Proposition 1.2. We now reformulate it in a slightly more constructive fashion. The basic assumptions are as follows.

(1) The Cauchy problem for the equation at hand with initial data in some space K (in practice : L^2 or H^1) is locally well-posed in some space $Y(\cdot)$ (in practice $X_{p+1}(\cdot)$ or $X_{p+1}^1(\cdot)$).

(2) The time of local resolution can be estimated from below in terms of the norm of the initial data in K :

$$T \geq M(\| u_0; K \|) \quad (1.33)$$

for some strictly positive (non increasing) function M .

(3) The solutions of the Cauchy problem with initial data $u_0 \in K$ at time t_0 are estimated a priori in the following sense :

For any $u_0 \in K$, for any $t_0 \in \mathbb{R}$, for any bounded interval $I \subset \mathbb{R}$ with $t_0 \in I$, there exists a constant $C(u_0, t_0, I)$ such that for any solution u defined in I (in a suitable sense, typically $u \in Y(I)$) and with $u(t_0) = u_0$, the following estimate holds for all $t \in I$:

$$\| u(t); K \| \leq C(u_0, t_0, I) \quad (1.34)$$

Without loss of generality, C can be assumed to be non decreasing as a function of I .

The estimate is “a priori” in the sense that “ $\exists C$ ” comes before “ $\forall u$ ”, namely C does not depend on u (and the particular way in which it is obtained). On the other hand C may (generally does) depend on u_0 and may (in many cases does) depend on I .

Under those assumptions, the Cauchy problem is globally well-posed, in the sense that the solutions extend to all $t \in \mathbb{R}$ (in other words $T_+ = T_- = +\infty$). In fact, one can iterate the local resolution and solve the Cauchy problem at time t_{k-1} ($1 \leq k < \infty$) with initial data $u(t_{k-1})$ up to time $t_k = t_{k-1} + T_k$ with a local resolution time $T_k \geq M(\| u(t_{k-1}); K \|)$ by (1.33). Now if the series $\sum T_k$ converges, then on the one hand T_k tends to zero, but on the other hand $T_k \geq M(C(u_0, t_0, I))$ where $I = [t_0, t_0 + \sum T_k]$

by (1.34), which is a contradiction. Therefore the series $\sum T_k$ diverges and u can be continued to all $t \geq t_0$.

In order to apply the abstract argument to the NLS equation, we are thus led to estimate the L^2 -norm of the solutions in the L^2 theory and the H^1 -norm in the H^1 theory. Under the assumption (H2) on f , the L^2 -norm estimate would follow immediately from the L^2 -norm conservation, while the H^1 -norm estimate can reasonably be expected to follow from the L^2 -norm and energy conservation (see (1.23)). Now the L^2 -norm is well defined for the local solutions of Proposition 1.2 since $X_{p+1}(\cdot) \subset \mathcal{C}(\cdot, L^2)$ while the energy is well defined for the local solutions of Proposition 1.3 since $X_{p+1}^1(\cdot) \subset \mathcal{C}(I, H^1)$ and

$$\int dx V(u) \leq C \left(\|u\|_2^2 + \|u\|_{p+1}^{p+1} \right) \quad (1.35)$$

is controlled by Sobolev inequalities for H^1 -subcritical $p < 1 + 4/(n-2)$, namely $p+1 < 2n/(n-2)$, so that the conservation laws make sense. On the other hand, the ‘‘proof’’ given in Section 1.b was completely formal and we now face an important issue : can one prove conservation laws at a level of smoothness of solutions where they make sense ? In order to appreciate the difficulty of the question, we first give an example where the answer is not known. Let $n \geq 3$ and let f be a single power (0.2) with $\lambda > 0$ and H^1 -supercritical $p > 1 + 4/(n-2)$. In that case, by compactness methods, it is easy to prove the existence of global solutions [GV2]

$$u \in (L^\infty \cap \mathcal{C}_w)(\mathbb{R}, H^1 \cap L^{p+1}) \quad (1.36)$$

for which obviously the energy is well defined and energy conservation therefore makes sense. One can even ensure that $E(u(t)) \leq E(u_0)$ for all $t \in \mathbb{R}$. It is not known however whether energy conservation holds in that case, nor is it known whether the solution is unique. Uniqueness would imply energy conservation by the previous inequality and the time reversal invariance of the equation, so that energy conservation might be a first step in the (presumably difficult) uniqueness question.

We now show that the expected conservation laws actually do hold under the additional assumption (H2) in the situations of Proposition 1.2 and 1.3. We concentrate on the case of the energy for H^1 solutions. Now for such solutions $u \in \mathcal{C}(H^1)$ so that $\Delta u \in \mathcal{C}(H^{-1})$ and $f(u) \in \mathcal{C}(L^{(p+1)/p}) \subset \mathcal{C}(H^{-1})$ so that $\partial_t u \in \mathcal{C}(H^{-1})$ and it does not make sense to take the scalar product in L^2 of the equation with $\partial_t u$. We therefore introduce a regularization. Let $\varphi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R}^+)$ with $\|\varphi_1\|_1 = 1$ and $\varphi_j(x) = j^n \varphi_1(jx)$, $j \in \mathbb{N}$. One can (but does not need to) take φ_1 radial decreasing for definiteness. When $j \rightarrow \infty$, φ_j tends to a delta function in the sense that the operator $\varphi_j \star_x$ tends to $\mathbb{1}$ strongly in L^r for $1 \leq r < \infty$ and in H^ρ for all $\rho \in \mathbb{R}$. Let u be an H^1 solution. Then $\varphi \star u \in \mathcal{C}^1(H^N)$ for all N and $\varphi \star u$ satisfies

$$i\partial_t(\varphi \star u) = -(1/2)\Delta(\varphi \star u) + \varphi \star f(u) \quad (1.37)$$

where $\varphi = \varphi_j$ and the subscript j is omitted for brevity. Now the formal computation of Section (1.b) makes perfectly good sense for $\varphi \star u$. We take $2Re < \partial_t(\varphi \star u), (1.37) >$ and obtain

$$\partial_t E(\varphi \star u) = 2Re < \varphi \star \partial_t u, f(\varphi \star u) - \varphi \star f(u) > \quad (1.38)$$

where the RHS fails to vanish because in (1.37) $\varphi \star$ is outside of f instead of being inside. We integrate (1.38) between t_1 and t_2 and substitute again (1.37) in the RHS, thereby obtaining

$$\begin{aligned} E(\varphi \star u(t_2)) - E(\varphi \star u(t_1)) &= -Im \int_{t_1}^{t_2} dt \\ &\{ < \varphi \star \nabla u, f'(\varphi \star u)(\varphi \star \nabla u) - \varphi \star (f'(u)\nabla u) > \\ &+ 2 < \varphi \star f(u), f(\varphi \star u) - \varphi \star f(u) > \} \quad (1.39) \end{aligned}$$

We next take the limit $j \rightarrow \infty$, namely $\varphi \star \rightarrow \mathbb{1}$ by applying the Lebesgue dominated convergence theorem to the time integral in the RHS. Since $u \in X_{p+1}^1$, one has $\nabla u \in L^q(L^{p+1})$ for compatible q and one verifies easily that $f(u) \in L^2(L^2)$ and $f'(u) \in L^\infty(L^{(p+1)/(p-1)})$. This suffices to ensure uniform boundedness with respect to φ of the integrand by an integrable function and pointwise convergence to zero almost everywhere in t , thereby completing the proof.

Note that the proof would not work if we knew only that $u \in \mathcal{C}(I, H^1)$. It is essential here that in X_{p+1}^1 , some superfluous smoothness in time can be traded against additional smoothness in space.

1.g The global Cauchy problem in L^2

Conservation of the L^2 -norm for L^2 solutions can be proved under the assumptions of Proposition 1.2 supplemented with (H2) by an argument similar to, but simpler than that of the previous section. Combining that result with the abstract globalization argument, we obtain easily

Proposition 1.4. *Let f satisfy (H1) (H2) with $p-1 < 4/n$. Then the Cauchy problem for the equation (0.1) with initial data in L^2 is globally well-posed in $X_{p+1,loc}(\mathbb{R})$, and the L^2 -norm is conserved.*

Note that the solution comes out only in $X_{p+1,loc}$ and not X_{p+1} , because there is no reason at this stage to expect a decay property at infinity such as $u \in L^q(\mathbb{R}, L^r)$ for $r > 2$.

We next comment on the critical case $p-1 = 4/n$. In that case, L^2 -norm conservation still holds, with the same proof. However this does *not* imply globalization because in that case the time of local resolution is *not* estimated in terms of the L^2 norm alone. The global L^2 problem remains open in the critical case (except of course for small data).

1.h The global Cauchy problem in H^1

Conservation of the L^2 -norm and of the energy for H^1 solutions can be proved under the assumptions of Proposition 1.3 supplemented with (H2), as explained in Section 1.f. It remains to ensure that those conservation laws control the H^1 -norm. That is obviously the case for positive V , but clearly V should not be too negative. A sufficient condition would be that the contribution of the negative part of V to the energy be estimated sublinearly in terms of $\| \nabla u \|_2^2$ for fixed $\| u \|_2$. In view of the Sobolev inequality

$$\| u \|_{2+4/n}^{2+4/n} \leq C \| u \|_2^{4/n} \| \nabla u \|_2^2 \quad (1.40)$$

this suggests that the L^2 -critical value $p = 1 + 4/n$ is also critical in that respect, and motivates the following assumption

(H3) Let $V_{\pm} = \text{Max}(\pm V, 0)$. Then

$$R^{-(2+4/n)} V_-(R) \rightarrow 0 \quad \text{when } R \rightarrow \infty \quad .$$

One can then prove with the use of (1.40)

Lemma 1.4. *Let f satisfy (H2) and (H3) and let $u \in H^1$. Then*

$$\| \nabla u \|_2^2 \leq 4E(u) + M(\| u \|_2)$$

for some non decreasing positive function M .

Combining the conservation laws, the previous estimate and the abstract globalization argument, we obtain easily

Proposition 1.5. *Let f satisfy (H1) (H2) and (H3) with $p - 1 < 4/(n - 2)$. Then the Cauchy problem for the equation (0.1) with initial data in H^1 is globally well-posed in $X_{p+1,loc}^1(\mathbb{R})$. Furthermore $\| u(t) \|_2$ and $E(u(t))$ are constant and the solutions belong to $L^\infty(\mathbb{R}, H^1)$.*

One may wonder what happens if the assumption (H3) is not satisfied. In the limiting case where

$$V_-(R) \leq a R^{2+4/n} + bR^2$$

it follows easily from (1.40) that the Cauchy problem is globally well-posed in H^1 for $\| u \|_2$ small.

If V does not satisfy other lower bounds than those following from (H1) (H2), one can still obtain by Sobolev inequalities an estimate of the form

$$y(t) \leq 2E(u) + C \| u \|_2^2 + C \| u \|_2^\beta y(t)^\alpha \equiv a + b y(t)^\alpha \quad (1.41)$$

with $\alpha = (p - 1)n/4$, possibly with $\alpha > 1$, for the quantity $y(t) = \|\nabla u(t)\|_2^2$. This implies that the Cauchy problem is globally well-posed in H^1 for small data in H^1 . In fact for $\alpha > 1$, if the curve $z = a + b y^\alpha$ intersects the line $z = y$ at (y_1, y_1) and (y_2, y_2) in the (y, z) plane, then (1.41) implies that $y(t)$ remains in the disconnected region $[0, y_1] \cup [y_2, \infty)$ so that $0 \leq y(t) \leq y_1$ for all times if $y(0) \leq y_1$ since $y(t)$ is continuous in t . More explicitly one obtains easily the a priori estimate

$$y(t) \leq a \alpha / (\alpha - 1)$$

which provides an a priori estimate of u in H^1 , under the condition

$$a b^{\alpha-1} < (\alpha - 1)^{\alpha-1} \alpha^{-\alpha}$$

which according to (1.41) is a condition of smallness of $\|u\|_2$ and $E(u)$, namely a condition of smallness of the initial data in H^1 .

Finally, a condition like (H3) is close to necessary for global existence of large solutions. In fact if $V(R) = -R^{p+1}$ with $p \geq 1 + 4/n$, one can show that all solutions with $E(u) < 0$ blow up in a finite time. Blowing up solutions have considerable physical interest and are the subject of intense investigation, both mathematical and numerical.

2. The Cauchy problem in \mathbb{T}^n

In this section we study the Cauchy problem for the NLS equation in \mathbb{T}^n . For convenience reasons, we normalize \mathbb{T}^n as $[0, 2\pi]^n$ (with suitable identification) and the equation as

$$i \partial_t u = -\Delta u + f(u) \tag{2.1}$$

instead of (0.1). We shall denote by \mathcal{F}_x and \mathcal{F}_t the partial Fourier transforms in x and t , and by τ the variable conjugate to t . The variable x now runs over \mathbb{T}^n , and the variable t over \mathbb{R} or sometimes \mathbb{T} . We use the notation $L^r \equiv L^r(\mathbb{T}^n)$, and in case of doubt we indicate the variable x or t by a subscript in the various function spaces. For instance $L_t^q(\mathbb{R}, L_x^r)$ denotes the space of L^q functions of t with values in L^r of x .

As mentioned already, the Cauchy problem can be split into the same two steps as in \mathbb{R}^n , namely the local problem and the globalization problem. Globalization proceeds via a priori estimates deduced from the conservation laws, which are the same in \mathbb{T}^n as in \mathbb{R}^n , and proceeds therefore in very much the same way as in Section 1.g and 1.h. We concentrate therefore on the local problem. More precisely, we shall provide an introduction to the recent work of Bourgain [B1], which represents an important progress in the treatment of that problem. The method combines two types of ingredients. The first one is a reorganization of the estimates through a suitable choice of the function spaces in such a way as to make trivial the linear estimates of the operators B^* and $(B^*B)_R$ that occur in the integral equation (see (1.15)) and to concentrate all the

difficulty on estimating the nonlinear interaction f . That part of the argument is not restricted to the NLS equation : it has been also applied to the KdV equation and various generalizations thereof. Nor is it restricted to the case of \mathbb{T}^n : it also applies to the case of \mathbb{R}^n . It has been applied in particular to the KdV equation in \mathbb{R} , thereby yielding significant improvements of previously known results [B2]. The second type of ingredients consists of specific inequalities for the Schrödinger equation in \mathbb{T}^n which are restrictions or variants of the Strichartz inequalities. Bourgain has conjectured and partly proved a general family of Strichartz type inequalities in that case. We now describe these two types of ingredients successively.

2.a General features of the method

The two essential features of the method are the following.

(1) One uses function space norms that are expressed in terms of the absolute values of the space time Fourier transforms of the functions under consideration and one performs the main estimates on those quantities. For that purpose, one uses primarily the L^2 based Sobolev spaces

$$H^{\rho,b} = \{u : \|u; H^{\rho,b}\| \equiv \| \langle \xi \rangle^\rho \langle \tau \rangle^b \widehat{u} \|_2 < \infty \} \quad (2.2)$$

where $\widehat{u} = \mathcal{F}u$ and $\langle \lambda \rangle = (1 + |\lambda|^2)^{1/2}$. One uses also more complicated spaces where $|\widehat{u}|$ belongs to various combinations of (possibly weighted) L^p spaces.

In order to use the Fourier transform in time in the local problem, namely with functions that are only defined in a bounded time interval, it is appropriate to truncate the integral equation (1.3) in time. Let $\psi_1 \in \mathcal{C}^\infty(\mathbb{R})$, $0 \leq \psi_1 \leq 1$, $\psi_1(t) = 0$ for $|t| \leq 1$, $\psi_1(t) = 0$ for $|t| \geq 2$ and define $\psi_T(t) = \psi_1(t/T)$ for $T > 0$. We replace the equation (1.3) by

$$u(t) = \psi_{T_0}(t) U(t)u_0 - i\psi_T(t) \int_0^t dt' U(t-t') f(u(t')) \quad (2.3)$$

with $T_0 \geq T$. Clearly any solution of (2.3) solves (1.3) in $[-T, T]$. We therefore concentrate on (2.3) and try to solve it globally for suitably small T . We take $T \leq 1$ from there on. T will be the time of local resolution of (1.3). Depending on the needs, one can take $T_0 = T$ or fix $T_0 = 1$. One can also use further truncations and replace $f(u)$ by $\psi_{T_1} f(\psi_{T_1} u)$ with $T_1 \geq T$. Since $U(t)$ is 2π periodic in t , one can also 2π -periodise ψ_T so as to work entirely in space time \mathbb{T}^{n+1} .

(2) In the standard method (see Section 1), one uses classical (Lebesgue, Sobolev, etc.) spaces H for the unknown function u , and one combines estimates of the linear operators B^* and $(B^*B)_R$ in spaces of that type with estimates of the nonlinear interaction f . An important point of Bourgain's method is to use classical spaces for the function $U(-t)u$, namely to use spaces X defined by

$$\|u; X\| = \|U(-t)u; H\| \quad . \quad (2.4)$$

(In the language of Quantum Mechanics, this consists in working in the so-called interaction representation). An immediate consequence is that the free evolution disappears from the linear estimates. In fact

$$\| \psi_{T_0} U(\cdot) u_0; X \| \leq C \| u_0; \mathcal{H} \| \Leftrightarrow \| \psi_{T_0} u_0; H \| \leq C \| u_0; \mathcal{H} \| \quad (2.5)$$

$$\| \psi_T(U \star_R f); X \| \leq C \| f; X' \| \Leftrightarrow \| Kf; H \| \leq C \| f; H' \| \quad (2.6)$$

where K is the operator defined by

$$(Kf)(t) = \psi_T(t) \int_0^t dt' f(t') \quad . \quad (2.7)$$

Of special interest are the spaces $X^{\rho,b}$ associated with the spaces $H^{\rho,b}$ defined by (2.2) :

$$X^{\rho,b} = \{u; \| u; X^{\rho,b} \| \equiv \| U(-t)u; H^{\rho,b} \| < \infty\} \quad . \quad (2.8)$$

The norm in $X^{\rho,b}$ can be written explicitly as

$$\| u; X^{\rho,b} \|^2 = \int d\xi d\tau \langle \xi \rangle^{2\rho} \langle \tau + \xi^2 \rangle^{2b} |\widehat{u}(\xi, \tau)|^2$$

and more generally, for an underlying linear equation $i\partial_t u = \omega(-i\nabla)u$ with

$$U(t) = \mathcal{F}_x^{-1} \exp[-it \omega(\xi)] \mathcal{F}_x \quad , \quad (2.9)$$

$$\| u; X^{\rho,b} \|^2 = \int d\xi d\tau \langle \xi \rangle^{2\rho} \langle \tau + \omega(\xi) \rangle^{2b} |\widehat{u}(\xi, \tau)|^2 \quad . \quad (2.10)$$

In that case, the estimate (2.5) of the free term becomes trivial :

Lemma 2.1

$$\| \psi_{T_0} U(\cdot) u_0; X^{\rho,b} \| = \| \psi_{T_0}; H^b \| \| u_0; H^\rho \| \leq C T_0^{1/2-b} \| u_0; H^\rho \| \quad .$$

(Note in particular that in order to solve (2.3) in $X^{\rho,b}$ with $b > 1/2$ and T small, it is better to take $T_0 = 1$ than $T_0 = T$, so as to work in a ball of fixed, i.e. T -independent, size).

A typical form of the estimate (2.6) is as follows.

Lemma 2.2 *Let $-1/2 < b' \leq 0 \leq b \leq b' + 1$ and $T \leq 1$. Then*

$$\| Kg; H_t^b \| \leq C T^{1-(b-b')} \| g; H_t^{b'} \| \quad , \quad (2.11)$$

$$\| \psi_T(U \star_R f); X^{\rho,b} \| \leq C T^{1-(b-b')} \| f; X^{\rho,b'} \| \quad (2.12)$$

for all $\rho \in \mathbb{R}$, and with the same constant C .

Proof. Intuitively, (2.11) expresses the fact that integrating over t in an interval $O(T)$ produces either a factor T at fixed regularity ($b = b'$) or a gain of one derivative in t uniformly in T ($b = b' + 1$), or any convex combination thereof.

The estimate (2.12) follows immediately by integration over ξ from (2.11) applied to $\widehat{g}_\xi(\tau) = \widehat{f}(\tau, \xi)$ for fixed ξ . The estimate (2.11) can be proved as follows. We write

$$\begin{aligned} \psi_T \int_0^t dt' f(t') &= \psi_T \int d\tau (i\tau)^{-1} (e^{it\tau} - 1) \widehat{f}(\tau) \\ &= \psi_T \sum_{k \geq 1} \frac{t^k}{k!} \int_{|\tau|T \leq 1} d\tau (i\tau)^{k-1} \widehat{f}(\tau) - \psi_T \int_{|\tau|T \geq 1} d\tau (i\tau)^{-1} \widehat{f}(\tau) \\ &\quad + \psi_T \int_{|\tau|T \geq 1} d\tau (i\tau)^{-1} e^{it\tau} \widehat{f}(\tau) = I + II + III \quad . \end{aligned} \quad (2.13)$$

We estimate the three terms successively in H^b .

$$\begin{aligned} \|I; H^b\| &\leq \sum_{k \geq 1} \frac{1}{k!} \|t^k \psi_T; H^b\| T^{1-k} \|f; H_t^{b'}\| \left\{ \int_{|\tau|T \leq 1} d\tau \langle \tau \rangle^{-2b'} \right\}^{\frac{1}{2}} \\ &\leq C T^{1-(b-b')} \|f; H^{b'}\| \quad \text{for all } b \geq 0, b' \leq 0 \quad . \end{aligned} \quad (2.14)$$

$$\begin{aligned} \|II; H^b\| &\leq \|\psi_T; H^b\| \|f; H^{b'}\| \left\{ \int_{|\tau|T \geq 1} d\tau |\tau|^{-2} \langle \tau \rangle^{-2b'} \right\}^{\frac{1}{2}} \\ &\leq C T^{1-(b-b')} \|f; H^{b'}\| \quad \text{for all } b \geq 0, b' > -\frac{1}{2} \quad . \end{aligned} \quad (2.15)$$

We estimate the integral J in III by

$$\begin{aligned} \|J; H^b\| &\leq \|f; H^{b'}\| \sup_{|\tau|T \geq 1} \tau^{-1} \langle \tau \rangle^{b-b'} \\ &\leq C T^{1-(b-b')} \|f; H^{b'}\| \quad \text{for all } b, b' \in \mathbb{R}, b - b' \leq 1 \quad , \end{aligned} \quad (2.16)$$

and similarly

$$\|J\|_2 \leq C T^{1+b'} \|f; H^{b'}\| \quad \text{for all } b' \geq -1 \quad , \quad (2.17)$$

so that

$$\begin{aligned} \|III; H^b\| &= \|\langle \tau \rangle^b (\widehat{\psi}_T \star \widehat{J})\|_2 \\ &\leq C \left(\|\tau\|^b \widehat{\psi}_T\|_1 \|J\|_2 + \|\widehat{\psi}_T\|_1 \|J; H^b\| \right) \\ &\leq C T^{1-(b-b')} \|f; H^{b'}\| \quad \text{for all } b \geq 0, b' \geq -1, b - b' \leq 1 \quad . \end{aligned} \quad (2.18)$$

The estimate (2.11) then follows from (2.13) (2.14) (2.15) (2.18). \square

Note that for $b - b' < 1$, the factor $T^{1-(b-b')}$ can be made small by taking T small and yields the small factor needed for contraction. For $b - b' = 1$, one has to extract such a small factor from the estimate of $f(\psi_T u)$ in terms of u .

We next show that Strichartz type inequalities of the type of (1.16) can be exploited in the previous framework. Those inequalities take the general form

$$\| U(\cdot)u; Y \| \leq C \| u \|_2 \quad (2.19)$$

for all $u \in L_x^2$, for some suitable space Y of functions of space time, thereby yielding estimates of solutions of the free equation in Y . We now derive therefrom estimates of general functions of space time.

Lemma 2.3. *Assume Y to be stable under multiplication by L_t^∞ , namely*

$$\| \psi f; Y \| \leq C \| \psi; L_t^\infty \| \| f; Y \| \quad \forall \psi \in L_t^\infty, \forall f \in Y \quad . \quad (2.20)$$

and assume that the estimate (2.19) holds for all $u \in L_x^2$. Then for any $b > 1/2$, the following estimate holds for all $f \in X^{0,b}$:

$$\| f; Y \| \leq C b^{1/2} (2b - 1)^{-1/2} \| f; X^{0,b} \| \quad . \quad (2.21)$$

Proof. We write

$$f = U(t) \int d\tau e^{it\tau} (\mathcal{F}_t U(-\cdot) f)(\tau)$$

and apply (2.19) and (2.20) for fixed τ with $\psi = e^{it\tau}$ and $u = (\mathcal{F}_t U(-\cdot) f)(\tau)$. We obtain

$$\| f; Y \| \leq C \int d\tau \| (\mathcal{F}_t U(-\cdot) f)(\tau); L_x^2 \| \quad (2.22)$$

and by the Schwarz inequality

$$\| f; Y \| \leq C \| f; X^{0,b} \| \left\{ \int d\tau \langle \tau \rangle^{-2b} \right\}^{1/2} \quad (2.23)$$

which yields (2.21). \square

Let now Y^θ , $0 \leq \theta \leq 1$ be a family of spaces interpolating between $Y^0 = L_{x \times t}^2$ and $Y^1 = Y$. From Lemma 2.3 we obtain by interpolation

Corollary 2.1. *Under the assumptions of Lemma 2.3 and with Y^θ as above, we obtain the following estimate for $b > \theta/2$*

$$\| f; Y^\theta \| \leq C \| f; X^{0,b} \| \quad . \quad (2.24)$$

The method for solving the local Cauchy problem will then be a contraction method for the equation (2.3) in function spaces of the previous type (see (2.4)), of which the spaces $X^{\rho,b}$ are the simplest examples. The linear estimates will be taken care of by Lemma 2.2 or variants thereof, and the crux of the matter will be to estimate the nonlinear interaction f in the previous norms, either directly or through the use of Strichartz type inequalities and of Lemma 2.3 and Corollary 2.1.

2.b Periodic Strichartz inequalities

The inequalities of interest arise as special answers to the following general problem :

Problem. Let $d \geq 1$, let $S \subset \mathbb{Z}^d$ and $q > 2$. Find the optimal constant $K_q(S)$ (or at least an estimate thereof) such that

$$\left\| \sum_{s \in S} u(s) \exp(i \langle x, s \rangle); L^q(\mathbb{T}^d) \right\| \leq K_q(S) \| u; \ell^2(S) \| \quad (2.25)$$

for all $u \in \ell^2(S)$.

The relation of that problem with the Schrödinger equation in $\mathbb{T}^n \times \mathbb{R}$ is as follows. Let $d = n + 1$ and $u \in \dot{L}^2(\mathbb{T}^n)$. Then

$$U(t)u \equiv f = \sum_m \hat{u}(m) \exp(imx - im^2t) \quad (2.26)$$

where we use the notation m instead of ξ to emphasize the fact that ξ takes values in \mathbb{Z}^n instead of \mathbb{R}^n . Let

$$S_d = \{(m, -m^2) : m \in \mathbb{Z}^n\} \subset \mathbb{Z}^d \quad . \quad (2.27)$$

The estimate (2.25) with $S = S_d$ becomes (up to normalization)

$$\| f; L^q(\mathbb{T}^{n+1}) \| \leq K_q(S_d) \| u \|_2 \quad (2.28)$$

namely a Strichartz type inequality (see (1.16)). A similar situation would arise for any equation $i\partial_t u = \omega(-i\nabla)u$ with $\omega : \mathbb{Z}^n \rightarrow \mathbb{Z}$ by taking

$$S = \{(m, -\omega(m)) : m \in \mathbb{Z}^n\} \quad .$$

Now with S_d an infinite set, it may happen that no such estimate as (2.28) holds. One considers therefore also truncated inequalities of the following type. Let χ_N be the characteristic function of the ball of radius N in \mathbb{Z}^n . and let

$$S_{d,N} = S_d \cap (\text{Supp } \chi_N \times \mathbb{Z}) \quad . \quad (2.29)$$

One then looks for estimates of the type

$$\| f; L^q(\mathbb{T}^{n+1}) \| \leq K_q(S_{d,N}) \| u \|_2 \quad (2.30)$$

for all $u \in L^2$ with $\chi_n \widehat{u} = \widehat{u}$, allowing for the possibility that $K_q(S_{d,N}) \rightarrow \infty$ as $N \rightarrow \infty$. In particular, if $K_q(S_{d,N}) \sim N^\beta$, then the estimate (2.30) is essentially equivalent to

$$\| f; L^q(\mathbb{T}^{n+1}) \| \leq C \| u; H_x^\beta \| \quad (2.31)$$

In the case of the Schrödinger equation, Bourgain has proposed and partly proved a conjecture on the values of $K_q(S_{d,N})$, motivated by a comparison of (2.31) with (1.16). In fact (1.16) admits a critical value $q = r = r_S = 2 + 4/n$. For $q \geq r_S$ it follows from a Sobolev inequality that

$$\| U(t)u; L^q(\mathbb{R}^{n+1}) \| \leq C \| U(t)u; L^q(\mathbb{R}, H_r^\beta) \| \leq C \| u; H^\beta \| \quad (2.32)$$

with $\beta \equiv \beta(q) = n/r - n/q = n/2 - (n+2)/q$. Comparing (2.31) and (2.32) suggests the first part of the following conjecture in the periodic case.

Conjecture 2.1. (2.30) holds with

$$K_q(S_{d,N}) \begin{cases} \leq C_q N^{\beta(q)} & \text{for } q > r_S \\ \ll N^\varepsilon & \text{for } q = r_S \\ \leq C_q & \text{for } q < r_S \end{cases}$$

where \ll means that for any $\varepsilon > 0$, there exists C_ε such that $\dots \leq C_\varepsilon \dots$. The second part of the conjecture is weaker than the corresponding result in \mathbb{R}^n by a small power N^ε , which is imposed by the existence of counter-examples. The third part of the conjecture is natural because of the embeddings of the L^q spaces on \mathbb{T}^{n+1} .

The conjecture 2.1 has been partially proved by the use of two methods. The first method applies to the case where $q = 2s$ is an even integer. By the Plancherel theorem, one can write

$$\| f; L^q(\mathbb{T}^{n+1}) \|^q = C \| \mathcal{F}(f^s); \ell^2(\mathbb{Z}^{n+1}) \|^2 \quad (2.33)$$

Let $(m, p) \in \mathbb{Z}^n \times \mathbb{Z}$ and $\widehat{u} = \chi_N \widehat{u}$. Then

$$\mathcal{F}(f^s)(m, p) = \sum' \widehat{u}(m_1) \cdots \widehat{u}(m_s) \quad (2.34)$$

where the sum runs over all $\{m_1, \dots, m_s\} \in (\mathbb{Z}^n)^s$ such that

$$\begin{cases} |m_i| \leq N, & 1 \leq i \leq s, \\ m_1 + \cdots + m_s = m, \\ m_1^2 + \cdots + m_s^2 = -p. \end{cases} \quad (2.35)$$

Let $r_{m,p}$ be the number of decompositions (2.35) (for fixed N). It follows from (2.34) by the Schwarz inequality that

$$\begin{aligned} \|\mathcal{F}(f^s); \ell^2(\mathbb{Z}^{n+1})\|^2 &= \sum_{m,p} \left| \sum' \widehat{u}(m_1) \cdots \widehat{u}(m_s) \right|^2 \\ &\leq \sum_{m,p} r_{m,p} \sum' |\widehat{u}(m_1) \cdots \widehat{u}(m_s)|^2 \\ &\leq \left(\sup_{m,p} r_{m,p} \right) \|u\|_2^{2s}, \end{aligned} \quad (2.36)$$

and the problem is reduced to estimating $r_{m,p}$. This can be done if n and s are not too large. For instance, for $q = 4$, i.e. $s = 2$, there are only two vectors m_1 and m_2 in \mathbb{Z}^n , and (2.35) determines both $m_1 + m_2$ and $|m_1 - m_2|^2 = -(2p + m^2)$. The number of solutions for (m_1, m_2) is then at most 2^n times (because of the signs of the components $(m_1 - m_2)_i$ of $m_1 - m_2$) the number of decompositions of the integer $-2p + m^2$ as a sum of n squares. In particular $r_{m,p} \leq 2$ uniformly in N if $n = 1$. The known results on that problem [Gr] essentially yield a proof of the conjecture 2.1 for $q = 4$ and $n \geq 1$. For $n = 1$ and $q = 6$, i.e. $s = 3$, the problem reduces to a quadratic diophantine equation, and the known results again yield a proof of the conjecture 2.1 with $K_6(S_{2,N}) \leq \exp[C \log N / (\log \log N)]$. The case $n = 1, q = 4$ had been obtained previously in [Z]. See also [KPV].

The second method used in the proof of the conjecture (2.1) is an extension to the periodic case of a general method of proof of the Strichartz inequalities in \mathbb{R}^n and consists in estimating the operator of convolution with the measure σ with Fourier transform

$$\mathcal{F}\sigma = \sum_{s \in S_{d,N}} \delta_s \quad .$$

The main result runs as follows ([B1], Proposition 3.82).

Proposition 2.1 *Let $n \geq 1$, $u \in L^2(\mathbb{T}^n)$ with $\|u\|_2 \leq 1$ and $\chi_N \widehat{u} = \widehat{u}$ and let $f = U(t)u$. Then for any $\lambda \geq N^{n/4}$, the following estimates hold*

$$\mu \{ (x, t) \in \mathbb{T}^{n+1} : |f(x, t)| \geq \lambda \} \leq C_q N^{q\beta(q)} \lambda^{-q} \quad (2.37)$$

for all $q > r_S$, and

$$\mu \{ (x, t) \in \mathbb{T}^{n+1} : |f(x, t)| \geq \lambda \} \ll N^\varepsilon \lambda^{-r_S} \quad (2.38)$$

where μ denotes the Lebesgue measure on \mathbb{T}^{n+1} .

The proof is difficult and we refer to the original article [B1]. If one splits

$$f = f_1 + f_2 = f \chi(|f| < N^{n/4}) + f \chi(|f| \geq N^{n/4})$$

then Proposition 2.1 is a weak type estimate on f_2 , namely $f_2 \in L_w^q(\mathbb{T}^{n+1})$ for all $q \geq r_S$, so that $f_2 \in L^q(\mathbb{T}^{n+1})$ for all $q > r_S$ by an elementary interpolation argument and with the expected estimate. On the other hand, for all $q \geq 2$, obviously

$$\| f_1; L^q(\mathbb{T}^{n+1}) \| \leq CN^{(1-2/q)n/4} \| f_1; L^2(\mathbb{T}^{n+1}) \|^{2/q} ,$$

so that the conjecture 2.1 follows from Proposition 2.1 provided $(1 - 2/q)n/4 \leq \beta(q)$, namely for $q \geq 2 + 8/n$.

Collecting the results obtained by both methods one finally obtains the following status for the conjecture 2.1.

Proposition 2.2 *The conjecture 2.1 is proved in the following cases :*

For $n = 1$, for $2 \leq q \leq 4$ and $q \geq 6$ ($= r_S$),

For $n = 2$, for $q \geq 4$ ($= r_S$),

For $n = 3$, for $q > 4$ ($> r_S = 10/3$),

For $n \geq 4$, for $q \geq 2 + 8/n$ ($> r_S = 2 + 4/n$).

Additional results follow from the embeddings of the L^q spaces on \mathbb{T}^{n+1} . In particular the conjecture 2.1 fails to be proved only by a factor N^ε for $n = 1$, $4 < q < 6$ and for $n = 2$, $q < 4$, whereas a finite power of N is missing for the other cases not covered by Proposition 2.2.

We conclude this section with an inequality for the case $n = 1$, which is closely related to Corollary 2.1. For $n = 1$ and $\omega(\xi) = \xi^k$ with k an integer ≥ 2 , the Strichartz inequality (2.19) can be shown to hold in \mathbb{R}^2 with $Y = L^q(\mathbb{R}^2)$ and $q = r_S = 6$ for $k = 2$ (more generally $q = r_S = 2(k + 1)$) [KPV]. Interpolating as in Corollary 2.1 to obtain $Y^\theta = L^4$, namely with $\theta = (k + 1)/2k = 3/4$, one obtains

$$\| f; L^4(\mathbb{R}^2) \| \leq C \| f; X^{0,b} \| \tag{2.39}$$

with $b > (k + 1)/4k = 3/8$. The argument does not apply to the periodic case, because the initial Strichartz inequality with $q = 6$ does not hold in that case. Nevertheless, one can derive the periodic analogue of (2.39) by direct estimation, and one recovers in addition the limiting case which was lost in the proof of Lemma 1.3 (see [B1] Proposition 2.6).

Lemma 2.4 *Let $n = 1$. Then the following estimate holds*

$$\| f; L^4(\mathbb{T}^2) \| \leq C \| f; X^{0,3/8} \| . \tag{2.40}$$

That estimate will be used in the next section.

2.c The local Cauchy Problem in \mathbb{T}^n

We mainly restrict our attention to a single power interaction (0.2). The results extend to more general interactions under suitable assumptions of smoothness of f and of power behaviour at infinity. We begin with a simple case where the results follow easily from the estimates of Section 2.b, namely with the case $n = 1$, $q = 4$ where the conjecture 2.1 holds with no power of N in the RHS.

Proposition 2.3. *Let $n = 1$ and $1 < p \leq 3$. Then the Cauchy problem for the equation (0.1) (0.2) with initial data in $L^2(\mathbb{T})$ is locally well posed in $L^4(\mathbb{T}^n \times \mathbb{R})$ or in $X^{0,b}$ for $3/8 \leq b < 5/8$, and globally well-posed in $L^\infty(\mathbb{R}, L^2) \cap L^4_{loc}(\mathbb{R}, L^4)$.*

Sketch of proof. As in Sections 1.d and 1.e, the crux of the proof consists in showing that the RHS of (2.3) defines a contraction in suitable balls of the spaces indicated, for sufficiently small T . We take ψ_T to be 2π periodic in t (see Section 2.a). Then

$$\begin{aligned} \|f(u); X^{0,-3/8}\| &\leq C \|f(u); L^{4/3}(\mathbb{T}^2)\| \leq C \|u; L^4(\mathbb{T}^2)\|^p \\ &\leq C \|u; X^{0,3/8}\|^p \end{aligned} \quad (2.41)$$

and a similar chain of inequalities holds for the difference $f(u_1) - f(u_2)$. The last inequality in (2.41) is (2.40), the first one is the dual thereof, and the middle one is obvious. On the other hand the operator K (see (2.7)) maps $X^{b'}$ to X^b boundedly for $-1/2 < b' \leq -3/8$, $3/8 \leq b < b' + 1$ ($< 5/8$). The result follows immediately. \square

We now turn to the more difficult cases where the basic estimate of Proposition 2.2 contains a positive power of N , which by (2.31) is equivalent to a loss of derivatives. The main result on the local Cauchy problem is the following :

Proposition 2.4. *Let $n \geq 1$, $\rho > 0$ and $2 \leq p - 1 < 4/(n - 2\rho)$. Assume that there exists q such that the conjecture 2.1 holds and that*

$$q > \text{Max}(p + 1, r_S) \quad , \quad (2.42)$$

$$(p + 1)\beta(q) < (p - 1)\rho \quad . \quad (2.43)$$

Then the Cauchy problem for the equation (0.1) (0.2) with initial data $u_0 \in H^\rho(\mathbb{T}^n)$ is locally well-posed in $X^{\rho,1/2}$.

We comment on the assumptions on p . The upper bound on p is the expected sub-criticality condition of p at the level of H^ρ (see Section 1.b). That condition of course occurs also in the treatment of the corresponding problem in \mathbb{R}^n [CW1]. The lower bound $p \geq 3$ is a smoothness condition on f and could probably be relaxed. Further restrictions on p arise from the need to use of Proposition 2.2 for some q satisfying (2.42)

(2.43). Note however that (2.42) (2.43) by themselves do not introduce any further restriction on p . In fact (2.43) reduces to $\rho > 0$ for $q = r_S$ because $\beta(r_S) = 0$, while for $q = p + 1$, (2.43) reduces to the subcriticality condition for p .

The proof of Proposition 2.4 is difficult and we refer to the original paper [B1]. A brief sketch can be found in [G2].

Further results on the Cauchy problem for the NLS equation in \mathbb{T}^n appear in [B3] [B4].

3. Asymptotic behaviour in time and scattering theory

We now come back to the case of the NLS equation (0.1) in \mathbb{R}^n in situations where the Cauchy problem is globally well-posed, typically in the space $X_{loc}^1(\mathbb{R})$ for initial data $u_0 \in H^1$ (see Section 1.h) and we address the question of describing and classifying the asymptotic behaviour in time of the global solutions. A possible method to attack this question consists in comparing the given dynamics with suitably chosen simpler asymptotic dynamics. That method applies to a wide variety of dynamical systems and in particular to systems defined by (linear or non linear) partial differential equations, and gives rise to the theory of scattering. We first describe the basic problems of scattering theory, on the specific example of the NLS equation (0.1).

3.a Generalities on scattering theory

In the case of a semilinear equation like (0.1), the first obvious candidate for an asymptotic dynamics is the free dynamics generated by the group $U(t)$. The comparison between the two dynamics gives rise to the following two questions.

(1) Let $v_+(t) = U(t) u_+$ be a solution of the free Schrödinger equation. Does there exist a solution u of the full equation (0.1) which behaves asymptotically as v when $t \rightarrow \infty$, typically in the sense that for Y a suitable Banach space

$$\| u(t) - v_+(t); Y \| \rightarrow 0 \quad \text{when } t \rightarrow +\infty \quad (3.1)$$

or rather

$$\| U(-t) u(t) - u_+; Y \| \rightarrow 0 \quad \text{when } t \rightarrow +\infty \quad (3.1')$$

which may be more appropriate if $U(\cdot)$ is not a bounded group in Y . This may occur in favourable cases for all $u_+ \in Y$, in less favourable cases only for u_+ in a dense subset of Y . If (3.1) or (3.1') holds, one can define the map $\Omega_+ : u_+ \rightarrow u(0)$. That map is called the wave operator for positive time, in the sense of the space Y . The problem of existence of u for given u_+ is referred to as the problem of existence of the wave operator. The same problem arises at $t \rightarrow -\infty$: for given u_- , one looks for a solution u of the full evolution equation which behaves as $U(t) u_-$ as $t \rightarrow -\infty$, and one defines

the map $\Omega_- : u_- \rightarrow u(0)$ as the wave operator for negative time. The functions (or vectors) u_{\pm} are called asymptotic states at $\pm\infty$. See Fig 1.

(2) Conversely, given a solution u of the full equation (0.1), does there exist asymptotic states u_+ and u_- such that $v_{\pm}(t) = U(t) u_{\pm}$ behaves asymptotically as $u(t)$ when $t \rightarrow \pm\infty$, typically in the sense that (3.1) or (3.1') and their analogues for negative time hold. If that is the case for any u with initial data $u(0)$ in Y for some $u_{\pm} \in Y$, one says that asymptotic completeness holds in Y .

Asymptotic completeness is a much harder problem than the existence of the wave operators, except in the case of small data where it follows as an immediate by-product of the methods used below to solve the latter problem. Asymptotic completeness for large data in the sense described above requires strong assumptions on the interaction term f , in particular some repulsivity condition, and proceeds through the derivation of a priori estimates for general solutions of the full equation. Here we shall only consider the problem of existence of the wave operators. We refer to the literature for the problem of asymptotic completeness for large data [GV1] [GV3] [HT] [LS] [T1] [T2].

It may (and in many cases does) happen that the free dynamics is inadequate or insufficient to describe the asymptotic behavior of the full dynamics. In particular we shall see below examples where the wave operators fail to exist. In that case one has to use more elaborate asymptotic dynamics. In such a case it may happen that the comparison dynamics $v(t)$ can be parametrized by asymptotic states u_{\pm} without having $v_{\pm}(0) = u_{\pm}$. Actually $v_{\pm}(0)$ need not even be defined, since $v_{\pm}(t)$ matters only through its asymptotic behaviour at $t \rightarrow \pm\infty$. See below for examples.

We now consider the problem of existence of the wave operators in more detail. We restrict our attention to positive times and drop the subscript $+$ on the given comparison dynamics $v(t)$. We do not assume for the moment that $v(t)$ is a solution of the free equation. We assume that the global Cauchy problem is well-posed for the full evolution equation, and we want to construct a solution thereof which behaves as $v(t)$ when $t \rightarrow +\infty$. Now let $t_0 > 0$ and let $u_{t_0}(t)$ be the solution of the Cauchy problem for the full equation with initial data $u(t_0) = v(t_0)$ at time t_0 . We may then expect that when $t_0 \rightarrow \infty$, u_{t_0} has a limit u in some sense, and that the limit satisfies the required property. This is translated into equations as follows. The Cauchy problem with initial time t_0 is formulated as the integral equation

$$u(t) = U(t - t_0) u(t_0) - i \int_{t_0}^t dt' U(t - t') f(u(t')) \quad . \quad (3.2)$$

On the other hand v satisfies the identity

$$v(t) = U(t - t_0) v(t_0) - i \int_{t_0}^t dt' U(t - t') (i \partial_t + (1/2)\Delta) v(t') \quad (3.3)$$

obtained by considering the first term in the RHS as a function of t_0 , say $h(t_0)$, and writing that

$$h(t) = h(t_0) + \int_{t_0}^t dt' h'(t') \quad .$$

Taking the difference of (3.2) and (3.3) with $u(t_0) = v(t_0)$ yields the equation

$$u(t) = v(t) + i \int_t^{t_0} dt' U(t-t') (f(u(t')) - (i \partial_t + (1/2)\Delta)v(t')) \quad (3.4)$$

the solution of which is u_{t_0} as defined above. If $v(t) = U(t) u_+$, (3.4) reduces to

$$u(t) = U(t) u_+ + i \int_t^{t_0} dt' U(t-t') f(u(t')) \quad . \quad (3.5)$$

We obtain an equation for the solution u we are looking for by formally taking the limit $t_0 \rightarrow +\infty$ in (3.4) or (3.5). Restricting our attention to the case where $v(t) = U(t) u_+$, we finally obtain the equation

$$u(t) = U(t) u_+ + i \int_t^{\infty} dt' U(t-t') f(u(t')) \quad . \quad (3.6)$$

The existence problem for the wave operator Ω_+ is thereby reformulated as the Cauchy problem with initial time at $+\infty$. We shall solve that problem in two steps :

(1) Solve (3.6) by a contraction method in an interval $[T, \infty)$. The contraction will require the presence of a small parameter. This will be ensured by taking the interval $[T, \infty)$ to be small, namely T to be large.

(2) Continue the solution thereby obtained to all times by using the known results on the Cauchy problem at finite times.

A second possibility to obtain a small contraction factor at the first step consists in taking the data (namely u_+) small. In that case one will be able to solve the problem globally in \mathbb{R} at the first step. As a consequence, the first step will yield the existence of global solutions and asymptotic completeness for small data as an immediate by-product.

In order to perform the first step above, we shall need at the very least that the integral in (3.6) converges at infinity in some sense, and for that purpose that $f(u)$ exhibits some time decay. This requires two conditions :

(1) The function u itself should have some time decay, namely one should try to solve (3.6) in a space of functions exhibiting some time decay in its definition. Clearly one should require that at least $v(t)$ satisfy that time decay, and for that purpose impose suitable conditions on the space of asymptotic states.

(2) The time decay of u should imply some time decay of $f(u)$, namely $f(u)$ should go to zero sufficiently fast as u goes to zero. If $f(u)$ behaves as a power p_1 at $u = 0$,

that condition appears in the form of lower bounds on p_1 . Clearly one expects those bounds to be all the stronger as the assumed time decay on u is the weaker.

In what follows we shall restrict our attention to situations of well-posedness of the H^1 Cauchy problem. Combining the requirements of that situation with the preceding remark, we shall impose on f the following condition, which is a reinforcement of (H1)

($\tilde{H}1$) $f \in C^1(\mathbb{C}, \mathbb{C})$, $f(0) = 0$ and for some p_1, p_2 with

$$1 < p_1 \leq p_2 < 1 + 4/(n - 2)$$

the following estimate holds for all $z \in \mathbb{C}$.

$$|f'(z)| \equiv \text{Max} (|\partial f/\partial z|, |\partial f/\partial \bar{z}|) \leq C (|z|^{p_1-1} + |z|^{p_2-1}) \quad .$$

We conclude that section by mentioning a property of the wave operators which is formally obvious from their definition, namely the intertwining property. Let $u(\cdot, u_0)$ be the solution of the Cauchy problem for the given equation (namely (0.1) in the present case) with initial data $u(0, u_0) = u_0$ at time zero. Then for all $s \in \mathbb{R}$

$$u(t + s, \Omega_{\pm} u_{\pm}) = u(t, \Omega_{\pm} U(s) u_{\pm}) \quad (3.7)$$

namely Ω_{\pm} intertwine the free evolution with the full evolution. The actual proof of that property will be an immediate by-product of the existence proof of the Ω_{\pm} .

We now proceed to implement the previous program with various choices of function spaces.

3.b Wave operators in H^1

The first natural choice of function spaces to solve (3.6) consists in taking the spaces $X_{r_0}^1(\cdot)$ defined in Section 1.b (see (1.19) (1.20)). By Lemma 1.4, solutions $U(t) u_+$ of the free equation belong to such spaces provided $u_+ \in H^1$. One can then solve the local Cauchy problem at infinity as follows

Proposition 3.1 *Let f satisfy ($\tilde{H}1$) with $p_1 \geq 1 + 4/n$. Then*

(1) *For any $u_+ \in H^1$, there exists $T = T(u_+)$ such that the equation (3.6) has a unique solution $u \in X_{p_2+1}^1(I)$ where $I = [T, \infty)$. Furthermore $u \in X^1(I)$ and u is a continuous function of $u_+ \in H^1$ with values in $X^1(I)$.*

(2) *The solution u admits u_+ as an asymptotic state in H^1 , namely*

$$\| U(-t) u(t) - u_+; H^1 \| \rightarrow 0 \quad \text{when } t \rightarrow \infty \quad . \quad (3.8)$$

Sketch of proof. Part (1) is proved by a contraction method whereby one proves that the RHS of (3.6) defines a contraction in the norm of $X_{p_2+1}(I)$ on the bounded sets

of $X_{p_2+1}^1(I)$. The estimates are exactly the same as in the proof of Proposition 1.3, which in turn were the same as in the proof of Proposition 1.1, with the proviso that one should now take $\theta = 0$ since the time interval is unbounded. One can choose again $r_1 = r_2 = s = p + 1$ for each $p = p_1, p_2$ and the lower bound on p_1 arises from (1.30) with $\theta = 0$ and from the fact that the best available decay for the L^s norm is obtained for compatible (k, s) .

Part (2) follows from the estimate

$$\begin{aligned} \| U(-t_2) u(t_2) - U(-t_1) u(t_1); H^1 \| &= \left\| \int_{t_1}^{t_2} dt' U(t_2 - t') f(u(t')); H^1 \right\| \\ &\leq \| U \star f; X_{p+1}^1([t_1, t_2]) \| \end{aligned} \quad (3.9)$$

for $T \leq t_1 \leq t_2$, followed by the same estimates as in the proof of Part (1). \square

As mentioned in Section 3.a, global existence and asymptotic completeness for small data follow immediately from the estimates in the previous proof.

Corollary 3.1 *Let f satisfy $(\tilde{H}1)$ with $p_1 \geq 1 + 4/n$. Then*

- (1) *There exists $R > 0$ such that for any $u_+ \in H^1$ with $\| u_+; H^1 \| \leq R$, the equation (3.6) and for any $t_0 \in \mathbb{R}$, the equation (3.5) have a unique solution in $X_{p_2+1}^1(\mathbb{R})$. The solution actually belongs to $X^1(\mathbb{R})$ and depends continuously on u_+ .*
- (2) *The wave operators Ω_{\pm} and their inverses are defined in a neighborhood of zero in H^1 and are local homeomorphisms. In particular asymptotic completeness holds in H^1 for small data.*

Before going to step (2) of the construction of the wave operators, we add the assumption (H2) and extend the conservation laws to infinite time.

Proposition 3.2 *Let f satisfy $(\tilde{H}1)$ and (H2) with $p_1 \geq 1 + 4/n$. Let $u_+ \in H^1$, $T \in \mathbb{R}$, $I = [T, \infty)$ and let $u \in X_{p_2+1}^1(I)$ be solution of the equation (3.6). Then*

$$\| u(t) \|_2 = \| u_+ \|_2 \quad \text{and} \quad E(u(t)) = (1/2) \| \nabla u_+ \|_2^2 \quad (3.10)$$

for all $t \in I$.

Proof. By Part (2) of Proposition 3.1, and the conservation laws at finite time, it suffices to show that $\int dt V(u(t)) \rightarrow 0$ when $t \rightarrow \infty$. Restricting our attention to the typical case of a single power $p_1 = p_2 = p$, we only have to show that $\| u(t) \|_{p+1} \rightarrow 0$. Now $u \in L^\infty(I, H^1) \cap C_L(I, H^{-1})$ and therefore u is uniformly Hölder continuous in L^{p+1} with exponent $(1 - \delta(p+1))/2$, which together with the fact that $u \in L^q(I, L^{p+1})$ for compatible $(q, p+1)$ implies the result. \square

Combining the previous results with the results of Sections 1.e and 1.h yields the existence of the wave operators in H^1 .

Proposition 3.3 *Let f satisfy $(\tilde{H}1)$ $(H2)$ and $(H3)$, with $p_1 \geq 1 + 4/n$. Then*

- (1) *For any $u_+ \in H^1$, the equation (3.6) has a unique solution $u \in X_{p_2+1,loc}^1(\mathbb{R}) \cap X_{p_2+1}^1(\mathbb{R}^+)$. Furthermore $u \in X_{loc}^1(\mathbb{R}) \cap X^1(\mathbb{R}^+)$ and u satisfies (3.8) and (3.10).*
- (2) *The wave operator Ω_+ is defined in H^1 , continuous and bounded.*

The same results hold for negative time.

We only remark that boundedness of Ω follows from (3.10) which implies that

$$\| \Omega_+ u_+ \|_2 = \| u_+ \|_2 \quad \text{and} \quad E(\Omega_+ u_+) = (1/2) \| \nabla u_+ \|_2^2 \quad . \quad (3.11)$$

Note also that under the assumptions made so far, the solutions u in part (1) of the proposition have no reason whatsoever to be dispersive at $-\infty$, namely to belong to $X^1(\mathbb{R}^-)$.

The main conclusion of this section is that with the decay available from the space $X^1(\mathbb{R})$, namely with asymptotic states in H^1 , the required lower bound on p_1 for the existence of the wave operators comes out as $p_1 \geq 1 + 4/n$. The question then arises how far down one can go by considering functions with a better time decay. Before going into that question, we first give a negative result [S1] which tells us that one cannot do better than $p > 1 + 2n$.

3.c Non existence of wave operators for $p \leq 1 + 2/n$

We restrict our attention to $n \geq 2$ since $n = 1$ is a special case, and we state the result in sufficient generality to cover not only the NLS equation (0.1) with a single power interaction but also other equations such as the Hartree equation with a potential $|x|^{-\gamma}$, $0 < \gamma \leq 1$. We need the dilation operator $D(t)$ defined for $t \in \mathbb{R}^+$ by

$$(D(t)f)(x) = t^{-n/2} f(x/t) \quad . \quad (3.12)$$

Proposition 3.4 *Let $n \geq 2$ and $0 \leq (p-1)n/2 = \delta(r) \leq 1$. Let f be a map from L^2 to $L^{\bar{r}}$ with $f(0) = 0$ such that*

- (1) *f is uniformly Lipschitz on bounded sets, namely*

$$\| f(u_1) - f(u_2) \|_{\bar{r}} \leq C(R) \| u_1 - u_2 \|_2 \quad \text{for } \| u_i \|_2 \leq R, \quad i = 1, 2 \quad .$$

- (2) *f is gauge covariant according to*

$$f(\omega u) = \omega f(u) \quad \text{for all } u \in L^2 \text{ and } \omega : \mathbb{R}^n \rightarrow \mathbb{C}, \quad |\omega| \equiv 1 \quad .$$

(3) f is homogeneous of degree p according to

$$f(D(t)u) = t^{-\delta(\tau)} D(t) f(u) \quad \text{for all } u \in L^2 \text{ and } t > 0$$

(4) $\text{Ker } f = 0$, namely $f(u) = 0 \Rightarrow u = 0$

Let $T \in \mathbb{R}$, $I = [T, \infty)$ and let $u \in \mathcal{C}(I, L^2)$ be a solution of (0.1) such that there exists $u_+ \in L^2$ such that

$$\| U(-t) u(t) - u_+ \|_2 \rightarrow 0 \quad \text{when } t \rightarrow \infty \quad . \quad (3.13)$$

Then $u = 0$ and $u_+ = 0$.

Sketch of proof. Let $\varphi \in L^2 \cap L^{\bar{r}}$ and $\text{Max}(T, 0) \leq t_1 \leq t_2$. Then

$$\langle \varphi, U(-t_2) u(t_2) - U(-t_1) u(t_1) \rangle = -i \int_{t_1}^{t_2} dt \langle U(t)\varphi, f(u(t)) \rangle \quad (3.14)$$

tends to zero by (3.13) when $t_1, t_2 \rightarrow \infty$.

Now when $t \rightarrow \infty$, one expects asymptotic behaviours of the type

$$\begin{aligned} U(t)\varphi &\sim (it)^{-n/2} \widehat{\varphi}(x/t) \\ u(t) &\sim U(t) u_+ \sim (it)^{-n/2} \widehat{u}_+(x/t) \end{aligned}$$

so that by assumption (3)

$$\langle U(t)\varphi, f(u(t)) \rangle \sim t^{-(p-1)n/2} \langle \widehat{\varphi}, f(\widehat{u}_+) \rangle \quad (3.15)$$

thereby making the integral in (3.14) divergent at infinity for $p \leq 1 + 2/n$, in contradiction with (3.13) (3.14), unless $f(\widehat{u}_+) = 0$ (since $\widehat{\varphi}$ is otherwise arbitrary) and therefore $u_+ = 0$ (by assumption (4)) and $u = 0$ by (3.13) and L^2 -norm conservation.

It turns out that the additional assumptions (1) (2) are sufficient to control the error in (3.15), namely to show that it can be written as $t^{-(p-1)n/2} o(1)$ when $t \rightarrow \infty$. \square

The negative result of Proposition 3.4 shows that for $p \leq 1 + 2/n$, one cannot expect the existence of asymptotic states in the sense of (3.1') even in the weakest sense considered so far, namely in L^2 . That result however is far from unexpected. Actually with $u \in L^2$, one expects dimensionnally that $u \sim |x|^{-n/2}$ at infinity, so that $|u|^{p-1} \sim |x|^{-\gamma}$ with $p-1 = 2\gamma/n$ and (0.1) with single power interaction (0.2) should be compared with a linear Schrödinger equation with potential $V(x) = |x|^{-\gamma}$. It is well known that the Coulomb-like case $\gamma = 1$ is the limiting case where the existence of the wave operators breaks down in linear scattering theory [RS]. We shall return to that question in Section 3.f. Before that however, we shall try to bridge or at least reduce the gap between the

lower bound at $p_1 = 1 + 4/n$ reached in Section 1.b and the Coulomb value $p = 1 + 2/n$ by using function spaces with better time decay than $X^1(\mathbb{R})$.

3.d Wave operators in $H^1 \cap \mathcal{F}(H^1)$

The time decay implied by the definition of $X(\mathbb{R})$ or $X^1(\mathbb{R})$, namely $u \in L^q(\mathbb{R}, L^r)$ for compatible (q, r) , is far from optimal for solutions of the free Schrödinger equation. In fact, the optimal time decay in L^r is obtained from (1.7) as

$$\| U(t) u_0 \|_r \leq C |t|^{-\delta(r)}$$

as soon as $u_0 \in L^{\bar{r}}$, and is seen to be optimal on the explicitly computable example where u_0 is Gaussian. That time decay is dimensionally twice better than that contained in $X(\mathbb{R})$. On the other hand, the spaces $L^{\bar{r}}(1 \leq \bar{r} < 2)$ are inconvenient as spaces of initial data because they are not preserved by the free evolution. In order to define convenient spaces, it is useful to introduce the (vector valued) operator

$$J(t) = x + it\nabla \quad . \quad (3.16)$$

That operator is the infinitesimal generator of Galilei transformations. In fact one checks easily that under the assumption (H2) on f , the NLS equation (0.1) is invariant under the Galilei transformation

$$u \rightarrow (G_v u)(x, t) = \exp [iv \cdot x - iv^2 t/2] u(x - vt, t)$$

for $v \in \mathbb{R}^n$, the infinitesimal form of which is

$${}_v(G_v u)|_{v=0} = (ix - t\nabla)u = i J(t)u \quad .$$

The operator $J(t)$ is unitarily equivalent both to x and to $it\nabla$. In fact it satisfies the commutation relations

$$J(t) = U(t) x U(-t) = U(t - t') J(t') U(t' - t) \quad (3.17)$$

for all $t \in \mathbb{R}$ and

$$J(t) = it M(t) \nabla M(-t) \quad (3.18)$$

for all $t \in \mathbb{R}$, $t \neq 0$, where

$$M(t) = \exp [ix^2/2t] \quad . \quad (3.19)$$

From the elementary Sobolev inequality

$$\| u \|_r \leq C_r \| u \|_2^{1-\delta(r)} \| \nabla u \|_2^{\delta(r)}$$

which holds for $0 \leq \delta(r) \leq 1$ ($\delta(r) \leq 1/2$ for $n = 1$, $\delta(r) < 1$ for $n = 2$) and from (3.18), it follows immediately that

$$\|u\|_r \leq C_r |t|^{-\delta(r)} \|u\|_2^{1-\delta(r)} \|J(t)u\|_2^{\delta(r)} \quad (3.20)$$

for the same values of r , and therefore that if in addition u depends on t in such a way that u and $J(t)u \in L^\infty(\mathbb{R}, L^2)$, u has optimal time decay in L^r -norm for the same values of r . This suggests to define the following spaces :

$$\Sigma = H^1 \cap \mathcal{F}(H^1) = \{u \in H^1 : xu \in L^2\} \quad (3.21)$$

and in analogy with (1.19) (1.20), for any interval $I \subset \mathbb{R}$

$$Y^1(I) = \{u : u \in \mathcal{C}(I, \Sigma) \text{ and } u, \nabla u \text{ and } J(t)u \in L^q(I, L^r) \text{ for all admissible } (q, r)\} \quad , \quad (3.22)$$

and for $0 \leq 2/q_0 = \delta(r_0) \equiv \delta_0 < 1$

$$Y_{r_0}^1(I) = \{u; u \in \mathcal{C}(I, \Sigma) \text{ and } u, \nabla u \text{ and } J(t)u \in L^q(I, L^r) \text{ for } 0 \leq 2/q = \delta(r) = \delta_0\} \quad . \quad (3.23)$$

The spaces $Y^1(I)$ can be made into Fréchet spaces and the spaces $Y_{r_0}^1(I)$ are Banach spaces with obvious norms. We also define the corresponding local spaces

$$Y_{(r_0)loc}^1(I) = \left\{ u : u \in Y_{(r_0)}^1(J) \text{ for any } J \subset\subset I \right\} \quad .$$

One checks easily that initial data in Σ generate solutions of the free Schrödinger equation in $Y^1(\mathbb{R})$, and in particular, by (3.20), with optimal time decay in L^r for $0 \leq \delta(r) \leq 1$.

Lemma 3.1. *Let $u_0 \in \Sigma$. Then $U(t) u_0 \in Y^1(\mathbb{R})$.*

Proof. For admissible (q, r)

$$\begin{aligned} \|J(t) U(t) u_0; L^q(\mathbb{R}, L^r)\| &= \|U(t) x u_0; L^q(\mathbb{R}, L^r)\| \\ &\leq C \|x u_0\|_2 \end{aligned} \quad (3.24)$$

by (3.17) and (1.16). □

Lemma 3.1 leads us to expect that Σ is a suitable space of asymptotic states to define the wave operators. As a preliminary step however we need to control the global Cauchy problem at finite times in spaces of the type $Y^1(\cdot)$. In doing so we have to face the technical difficulty that (contrary to ∇) the operator $J(t)$ is not a derivation

because it contains x . This is taken care of by the following lemma, which states that nevertheless $J(t)$ behaves as a derivation when applied to gauge invariant functions.

Lemma 3.2. *Let $f \in C^1(\mathbb{C}, \mathbb{C})$ satisfy (H2). Then*

$$J(t) f(u) = \partial_z f(u) J(t)u - \partial_{\bar{z}} f(u) \overline{J(t)u} \quad (3.25)$$

Proof. This follows easily from gauge invariance (H2) and from the commutation relation (3.18). \square

One can then prove

Proposition 3.5. *Let f satisfy ($\tilde{H}1$) (H2) (H3). Then the Cauchy problem for the equation (0.1) with initial data $u_0 \in \Sigma$ is globally well-posed in $Y_{loc}^1(\mathbb{R})$. The solutions satisfy L^2 -norm and energy conservation.*

Sketch of proof. The proof is an immediate extension of those of Propositions 1.3 and 1.5. The local Cauchy problem requires in principle one more estimate for the function $J(t)u$, but in practice and with the use of Lemma 3.2 that estimate is identical with that of ∇u in (1.32). The global problem requires an additional a priori estimate for $J(t)u$, but for $u \in X_{loc}^1(\mathbb{R})$, that quantity satisfies a linear inequality and is estimated by a variant of Gronwall's Lemma. \square

We can now begin the study of the wave operators in Σ [CW2] [GV1] [GOV] [S2]. The final result given below comes from [CW2], with the simplified proof from [GOV]. We follow exactly the same pattern as in the H^1 case. We first solve the local Cauchy problem at infinity in the form of the equation (3.6).

Proposition 3.6. *Let f satisfy ($\tilde{H}1$) and (H2) with $p_1 > 1 + 4/(n + 2)$ ($p_1 > 3$ if $n = 1$). Then*

(1) *For any $u_+ \in \Sigma$, there exists $T = T(u_+)$ such that the equation (3.6) has a unique solution $u \in Y_{p_2+1}^1(I)$, where $I = [T, \infty)$. Furthermore $u \in Y^1(I)$ and u is a continuous function of $u_+ \in \Sigma$ with values in $Y^1(I)$.*

(2) *The solution u admits u_+ as an asymptotic state in Σ , namely*

$$\| U(-t) u(t) - u_+; \Sigma \| \rightarrow 0 \quad \text{when } t \rightarrow \infty \quad (3.26)$$

Sketch of proof. The proof is an immediate extension of that of Proposition 3.1. Again one needs in principle an additional estimate for the function $J(t)u$, which by Lemma 3.2 is in practice identical with that of ∇u . The lower bound on p_1 however comes out different, actually weaker, and we now explain why. We need to control the norm of u in $L^k(L^s)$ which occurs in the estimate for ∇u (see (1.32) with $\theta = 0$) and

in a similar estimate for $J(t)u$. But now by (3.20) the available decay information on u is that for $0 \leq \delta(s) \leq 1$, $u \in L^k(L^s)$ for any k with $k \delta(s) > 1$ (instead of $k \delta(s) = 2$ for compatible (k, s)). We now take the optimal (k, s) , namely $\delta(s) = 1$ (for $n \geq 3$) and k close to 1, which by (1.30) with $\theta = 0$ gives $(p-1)(n/2+1) > 2$, namely the lower bound in Proposition 3.6. (For $n = 1$, one cannot go beyond $\delta(s) = 1/2$, which gives $p_1 > 3$). \square

Note at this point that in contrast with Proposition 3.1, the assumption (H2) is already needed at the stage of the local resolution, because one uses Lemma 3.2 to estimate $J(t)u$.

As in the case of the H^1 theory, global existence and asymptotic completeness for small data follow immediately from the estimates in the previous proof. We give an abbreviated statement as a reminder.

Corollary 3.2. *Let f satisfy $(\tilde{H}1)$ and (H2) with $p_1 > 1 + 4/(n+2)$ ($p_1 > 3$ if $n = 1$). Then the same statements as in Corollary 3.1 hold with H^1 replaced by Σ and X^1 by Y^1 .*

Combining the previous results, namely Propositions 3.3, 3.5 and 3.6, we finally obtain the existence of wave operators in Σ .

Proposition 3.7. *Let f satisfy $(\tilde{H}1)$ (H2) and (H3) with $p_1 > 1 + 4/(n+2)$ ($p_1 > 3$ if $n = 1$). Then*

- (1) *For any $u_+ \in \Sigma$, the equation (3.6) has a unique solution $u \in Y_{p_2+1,loc}^1(\mathbb{R}) \cap Y_{p_2+1}^1(\mathbb{R}^+)$. Furthermore $u \in Y_{loc}^1(\mathbb{R}) \cap Y^1(\mathbb{R}^+)$ and u satisfies (3.26) and (3.10)*
- (2) *The wave operator Ω_+ is defined in Σ , continuous and bounded.*

The same results hold for negative time.

The main conclusion at this stage is that with the time decay available from the space $Y^1(\mathbb{R})$, namely for asymptotic states in Σ , the required lower bound on p_1 for the existence of the wave operators comes out as $p_1 > 3$ for $n = 1$, $p_1 > 1 + 4/(n+2)$ for $n \geq 2$. In view of the negative result of Proposition 3.4, that result is optimal for $n = 2$ (actually also for $n = 1$), but not for $n \geq 3$ where there remains a gap between $1 + 2/n$ and $1 + 4/(n+2)$. We shall come back to that question in the next section.

We conclude this section with some remarks on the local resolution at infinity without the assumption (H2) for $u_+ \in \Sigma$ [GV1] [S2]. In that case one can solve the local Cauchy problem at infinity by contraction in a space of the type

$$Z_{r_0}(I) = \{u : u \in X_{r_0}^1(I) \text{ and } (1 + |t|)^{\delta(r)} \|u(t)\|_r \in L^\infty(I) \\ \text{for } 0 \leq \delta(r) \leq \delta_0 < 1\}$$

under the following condition on p_1

$$p_1 \delta(p_1 + 1) > 1$$

or equivalently

$$np_1^2 - (n+2)p_1 - 2 > 0 \quad (3.27)$$

namely for $p_1 > p_0(n)$ where $p_0(1) = (3 + \sqrt{17})/2$, $p_0(2) = 1 + \sqrt{2}$, $p_0(3) = 2$, $p_0(4) = (3 + \sqrt{17})/4$, etc. Under the same assumption, one also obtains global existence and asymptotic completeness for small data in Σ . On the other hand the assumption (H2) (or at least some substitute thereof) is needed to control the global Cauchy problem at finite times and therefore the existence of the wave operators for large data, so that this theory loses its advantage for that problem.

The magic values $p_0(n)$ also occur at other places : the condition $p > p_0(n)$ comes out naturally in the available proofs of asymptotic completeness for large data for repulsive interactions [HT] [T1]. They also occur, shifted by one unit in dimension, in the corresponding problems for nonlinear wave equations (see on that subject several other lectures in this Conference).

3.e Improvement of the lower bound on p_1

We have seen in the previous section that for $n \geq 3$ there remains a gap between the optimal value $1 + 2/n$ and the accessible value $1 + 4/(n+2)$ for the lower bound on p_1 . We shall now show that this gap can be reduced (and actually closed for $n = 3$) [GOV]. As explained in the proof of Proposition 3.6, the lower bound on p_1 comes from the condition (1.30) with $\theta = 0$ together with the control of u in $L^k(I, L^s)$. If $\|u\|_s$ has optimal time decay, then one can take any k with $k \delta(s) > 1$, thereby obtaining

$$p > 1 + 4/(n + 2\delta(s))$$

The lower bound $1 + 4/(n+2)$ was obtained by taking $\delta(s) = 1$, which was allowed by (3.20) and the definition of $Y^1(I)$, and the optimal result would be obtained if we could take $\delta(s) = n/2$ or at least $\delta(s)$ close to $n/2$, namely s infinite or at least very large. By the inequality

$$\|u\|_s \leq C|t|^{-\delta(s)} \| |J(t)|^{\delta(s)} u \|_2 \quad (3.28)$$

for $0 \leq \delta(s) < n/2$, which follows from Sobolev inequalities and from (3.18) and which generalizes the case $\delta(r) = 1$ of (3.20), the required time decay of u would hold under the condition $|J(t)|^\rho u \in L^\infty(I, L^2)$ for $\rho = n/2$ or at least ρ close to $n/2$. This suggests to try generalizing Σ and $Y^1(I)$ by defining

$$\Sigma^\rho = H^\rho \cap \mathcal{F}(H^\rho) \quad (3.29)$$

$$Y^\rho(I) = \{u : u \in \mathcal{C}(I, \Sigma^\rho) \text{ and } u, |\nabla|^\rho u \text{ and } |J(t)|^\rho u \in L^q(I, L^r) \text{ for admissible } (q, r)\} \quad (3.30)$$

with $1 \leq \rho \leq n/2$, and to try solving the local Cauchy problem at infinity in $Y^\rho(I)$ for asymptotic states in Σ^ρ . This however would require to estimate $|\nabla|^\rho f$ and $|J(t)|^\rho f$ in $L^q(L^r)$, and would thus require f to be at least \mathcal{C}^ρ . Now for f behaving as a power p_1 at $u = 0$, one must have $\rho \leq p_1$. The situation is then as follows.

For $n = 3$, $1 + 2/n = 5/3 > n/2 = 3/2$, and the previous scheme can work with $\rho = 3/2$ and $p_1 > 5/3$.

For $n \geq 4$, $1 + 2/n < n/2$. Combining the previous conditions, one finds

$$p_1 - 1 > \text{Max}(\rho - 1, 4/(n + 2\rho))$$

namely $p_1 > \rho_0(n)$ where $\rho_0(n)$ is the solution of the equation

$$(\rho - 1)(n + 2\rho) = 4 \quad . \quad (3.31)$$

One finds

$$\begin{aligned} 1 + 2/n < \rho_0(n) < 1 + \frac{4}{n+2} - \frac{1}{2} \left(\frac{4}{n+2} \right)^3 + \frac{1}{2} \left(\frac{4}{n+2} \right)^5 - \dots \\ < 1 + 4/(n+2) < 2 \leq n/2 \quad . \end{aligned}$$

One may therefore expect the previous scheme to work for $n = 3$, $\rho = 3/2$, $p_1 > 1 + 2/3$ and for $n \geq 4$, $p_1 > \rho = \rho_0(n)$. This is actually what happens, modulo some technical difficulties. The useful range for ρ and p_1 is

$$1 \leq \rho < \text{Min}(p_1, 2) \quad . \quad (3.32)$$

Restricting the attention to single power f for simplicity, one reinforces the smoothness assumption on f by replacing $(\tilde{H}1)$ by

$(\tilde{\tilde{H}}1)$ $f \in \mathcal{C}^1(\mathbb{C}, \mathbb{C})$, $f(0) = 0$, $f'(0) = 0$, and for some p with $1 < p < 1 + 4/(n - 2)$, the following estimate holds for all $z_1, z_2 \in \mathbb{C}$:

$$|f'(z_1) - f'(z_2)| \leq C \begin{cases} |z_1 - z_2|^{p-1} & \text{if } p \leq 2 \\ |z_1 - z_2| (\text{Max}_{i=1,2} |z_i|^{p-2}) & \text{if } p \geq 2 \quad . \end{cases}$$

One follows the same pattern as in Sections 3.b and 3.d. In order to cope with the fact that ρ is no longer an integer, it is convenient to modify the definition of the spaces Y^ρ and to define them in terms of Besov spaces of the space variable. One can then prove the expected results, namely

(1) The Cauchy problem for the equation (0.1) with initial data in Σ^ρ is globally well-posed in $Y_{loc}^\rho(\mathbb{R})$ for $1 \leq \rho < \text{Min}(p, 2)$.

(2) One can solve the local Cauchy problem in the form of the equation (3.6) in $Y^\rho([T, \infty))$ for asymptotic states $u_+ \in \Sigma^\rho$, provided $1 \leq \rho < \text{Min}(p, 2)$ and $p - 1 > 4/(n + 2\rho)$.

(3) This implies as before the existence of global solutions and asymptotic completeness for small data.

(4) Finally under the assumptions $(\tilde{H}1), (H2), (H3)$ and the previous conditions on ρ and p , the wave operators exist in Σ^ρ .

We refer to [GOV] for the details. The question of existence of the wave operators for $n \geq 4$ and $1 + 2/n < p_1 \leq \rho_0(n)$ remains open at this stage.

3.f Modified wave operators for $p_1 = 1 + 2/n$

We have seen in Section 3.c that the wave operators are not defined in any reasonable sense for $p_1 \leq 1 + 2/n$, in analogy with the case of the linear Schrödinger equation with long range potential $|x|^{-\gamma}$ with $\gamma \leq 1$. In the latter case however, it is well known that one can define modified wave operators which are suitable substitutes for the non existing ordinary ones, and it is natural to try to extend that theory to the nonlinear case. This is the purpose of this section. There exist only preliminary results in that case, and we restrict our attention to the case of space dimension $1 \leq n \leq 3$ and to the critical value $p = 1 + 2/n$ where such results exist. We also restrict our attention to single power interactions of the type (0.2). The treatment follows [O] [GO].

We come back to the general discussion of Section 3.a and try now to construct solutions of the full equation (0.1) that behave asymptotically as some asymptotic dynamics $v(t)$ which we no longer assume to be the free dynamics. We start again from the integral equation (3.4) and take the formal limit $t_0 \rightarrow \infty$ which we rewrite as follows

$$u(t) = v(t) + i \int_t^\infty dt' U(t-t') \{f(u(t')) - f(v(t')) - i((i\partial_t + (1/2)\Delta)v(t') - f(v(t')))\} \quad (3.33)$$

The trouble comes from the fact that $f(u)$ does not decay sufficiently fast in time for the integral in (3.6) to converge. However if v is a sufficiently good approximation to the asymptotic behaviour in time of solutions of (0.1), one may hope that $f(u) - f(v)$ has better time decay than $f(u)$, thereby making its contribution to the integral in (3.33) convergent. How good an approximation v really is should appear in the time decay at infinity of the function

$$\tilde{f}(t) = (i \partial_t + (1/2)\Delta) v(t) - f(v(t)) \quad (3.34)$$

which measures the failure of v to satisfy the equation (0.1). For given v , we shall now regard (3.33) as an equation for the difference $w = u - v$, namely

$$w(t) = w^{(0)}(t) + i \int_t^\infty dt' U(t-t') \{f(v(t') + w(t')) - f(v(t'))\} \quad (3.35)$$

where

$$w^{(0)}(t) = -i \int_t^\infty dt' U(t-t') \tilde{f}(t') \quad . \quad (3.36)$$

We are now faced with the following two problems :

- (1) Solve the equation (3.35) for w .
- (2) Choose the asymptotic dynamics v in order to obtain a good time decay of \tilde{f} .

We begin with the first problem and solve it as before in two steps : we first solve the equation (3.35) by a contraction method locally in a neighborhood of infinity in time, and we then extend the solution to all times by using the known results on the Cauchy problem at finite times. This will be possible under some general decay assumptions on v and \tilde{f} . We shall then construct v satisfying those assumptions.

In order to solve (3.35), we use the following spaces. Let $\theta > 0$, let (q, r) be an admissible pair and let $I = [T, \infty)$ with $T > 0$. We define

$$Z_{\theta, r}(I) = \{w : w \in \mathcal{C}(I, L^2) \cap L^q(I, L^r) \text{ and}$$

$$\|w; Z_{\theta, r}(I)\| \equiv \text{Sup}_{t \in I} t^\theta (\|w(t)\|_2 + \|w; L^q([t, \infty), L^r)\|) < \infty\} \quad . \quad (3.37)$$

Note that in (3.37) one can replace $\|w(t)\|_2$ by $\|w; L^\infty([t, \infty), L^2)\|$. If one does so, then the last two norms in (3.37) would be uniformly bounded in t if one had simply $w \in X_r(I)$ (see (1.20)). Here we expect w as a difference to have better time decay, in the sense that those two norms decay at $t^{-\theta}$ instead of simply being bounded. The existence result at infinity for (3.35) can now be stated as follows.

Proposition 3.8 *Let $f(u) = \lambda|u|^{2/n}$ and let $v \in \mathcal{C}([1, \infty), L^2) \cap L^\infty([1, \infty), L^\infty)$ satisfy*

$$\|v(t)\|_\infty \leq c_\infty t^{-n/2} \quad (3.38)$$

for some c_∞ sufficiently small and all $t \geq 1$, and

$$\|\tilde{f}(t)\|_2 \leq C t^{-(1+\theta_0)} \quad (3.39)$$

for some $\theta_0 > n/4$ and all $t \geq 1$. Then for $n/4 < \theta < \theta_0$ the equation (3.35) has a unique solution $w \in Z_{\theta, r}([1, \infty))$.

The proof proceeds by a contraction method in $Z_{\theta, r}([T, \infty))$ for T sufficiently large and continuation to all $t \geq 1$ by the results on the global L^2 theory (see Sections 1.d and 1.g).

We remark that the smallness condition on c_∞ can be taken independent of θ, r and that the solution w is independent of θ, r . Actually w belongs to $Z_{\theta, r}$ for all θ, r satisfying the assumptions. Furthermore, Proposition 3.8 is not restricted in dimension. The restriction to $n \leq 3$ comes later, from the fact that we cannot do better than $\theta_0 = 1$ in satisfying (3.39).

We now turn to the construction of the asymptotic dynamics $v(t)$ and explain the ideas on the example of the linear Schrödinger equation

$$i \partial_t u = -(1/2) \Delta u + V u \quad (3.40)$$

with time dependent potential $V = V(t, x)$ [Hö IV] [RS]. In that case, as a first choice of modified free evolution, one can try

$$v_1(t) = U(t) \exp[-i S(t, -i\nabla)] u_+ \quad (3.41)$$

for some real function $S(t, \xi)$ to be determined later, and a generic asymptotic state $u_+ \in L^2$. In order to prove the existence of modified wave operators by the method of Cook, one then has to prove that

$$(\partial_t S(t, -i\nabla) - V(t, x)) v_1(t) \in L^1(\mathbb{R}^+, L^2) \quad (3.42)$$

Asymptotically, one expects $-i\nabla$ to be equivalent to x/t , and this suggests to choose

$$\partial_t S(t, \xi) = V(t, t\xi) \quad (3.43)$$

the original choice made by Dollard in 1964. However the expected cancellation in (3.42) cannot be verified in a simple manner. A better choice for v can be made by exploiting the following decomposition of $U(t)$:

$$U(t) = i^{-n/2} M(t) D(t) \mathcal{F} M(t) \quad (3.44)$$

where $D(t)$ and $M(t)$ are the dilation operator (3.12) and the multiplication operator (3.19). That decomposition is a simple rewriting of (1.5). Note that the commutation relations (3.17) (3.18) imply

$$(x/t) U(t) M(-t) = U(t) M(-t) (-i\nabla) \quad (3.45)$$

We take as a second choice of modified free evolution

$$\begin{aligned} v_2(t) &= U(t) M(-t) \exp[-i S(t, -i\nabla)] u_+ \\ &= \exp[-i S(t, x/t)] U(t) M(-t) u_+ \\ &= \exp[i x^2/2t - i S(t, x/t)] (it)^{-n/2} \widehat{u}_+(x/t) \end{aligned} \quad (3.46)$$

where the second line follows from (3.45) and the third one from (3.44). Note in particular that $|v_2(t)|$ is computed explicitly as

$$|v_2(t)| = t^{-n/2} |\widehat{u}_+(x/t)| = |D(t) \mathcal{F}u_+| \quad (3.47)$$

and is independent of S . Cook's method now requires that

$$\begin{aligned} U(t) M(-t) \{ \partial_t S(t, -i\nabla) - V(t, -it\nabla) + x^2/2t^2 \} \\ \times \exp[-i S(t, -i\nabla)] u_+ \in L^1(\mathbb{R}^+, L^2) \end{aligned} \quad (3.48)$$

and the first two terms in the central bracket cancel exactly for the choice (3.43). Finally, a third possible choice consists in taking

$$v_3(t) = \exp[-i S(t, x/t)] U(t)u_+ \quad (3.49)$$

It has the virtue of differing from the free evolution by a phase only. The comparison between the various choices of v in L^2 is easy. In fact

$$\|v_2(t) - v_3(t)\|_2 = \|(M(t) - \mathbb{1})u_+\|_2 \rightarrow 0 \text{ when } t \rightarrow \infty$$

because the operator $M(t)$ tends strongly to $\mathbb{1}$ when $t \rightarrow \infty$, while

$$\|v_2(t) - v_1(t)\|_2 = \|(M(t) - \mathbb{1}) \exp[-i S(t, -i\nabla)]u_+\|_2$$

is shown to tend to zero as a by-product of the estimates that lead to (3.48).

We now turn to the NLS equation (0.1). In that case, with $f(u) = u g(|u|^2)$, the potential is replaced by $g(|u|^2)$, and since u is expected to be asymptotic to v , one is led to apply the previous method with the potential $V(t, x) = g(|v(t, x)|^2)$. One can make the same three choices (3.41) (3.46) (3.49) as before. The choice of v_2 has the advantage that $|v_2|$ and therefore V can be explicitly computed and is independent of the choice of S , thereby leading to an explicit expression rather than an implicit equation for S at a later stage.

The condition (3.42) (or (3.48)) required in Cook's method is now replaced by an estimate of the function \tilde{f} defined by (3.34) in order to ensure the assumption (3.39) of Proposition 3.8. With the most convenient choice $v = v_2$, one finds

$$\begin{aligned} \tilde{f}(t) = U(t) M(-t) \{ \partial_t S(t, -i\nabla) - g(|v_2(t, -it\nabla)|^2) + x^2/2t^2 \} \\ \times \exp[-i S(t, -i\nabla)]u_+ \end{aligned} \quad (3.50)$$

by the same computation as that leading to (3.48) in the linear case. This suggests again, in analogy with (3.43) to choose

$$\partial_t S(t, \xi) = g(|v_2(t, t\xi)|^2)$$

which for the present choice of f and in view of (3.47) yields

$$\partial_t S(t, \xi) = \lambda t^{-1} |\widehat{u}_+(\xi)|^{2/n} \quad (3.51)$$

which is solved by

$$S(t, \xi) = \lambda(\ell n t) |\widehat{u}_+(\xi)|^{2/n} \quad (3.52)$$

That choice turns out to be adequate for $n = 1, 2$, but not for $n = 3$ where it is not smooth enough in \widehat{u}_+ at $\widehat{u}_+ = 0$, and has to be replaced by

$$\tilde{S}(t, \xi) = \lambda(\ell n t) \left(t^{-4/3} + |\widehat{u}_+(\xi)|^2 \right)^{1/3} \quad (3.53)$$

We denote by \tilde{v}_i ($i = 1, 2, 3$) the modified free evolutions with S replaced by \tilde{S} .

With the previous choices of v and S , one can then perform the necessary estimates needed in Proposition 3.8.

Proposition 3.9. *For $n \leq 2$, let $u_+ \in L^2$ with $x^2 u_+ \in L^2$ and define S by (3.52). Then*

(1) v_2 defined by (3.46) satisfies (3.38) with $c_\infty = \|\widehat{u}_+\|_\infty$ and satisfies (3.39) for all $\theta_0 < 1$.

(2) For all $\theta < 1$ and all r with $0 \leq \delta(r) < 1$, $v_i - v_j \in Z_{\theta, r}([1, \infty))$ for all $i, j = 1, 2, 3$.

For $n = 3$, let $u_+ \in L^2$ with $x^2 u_+ \in L^2$ and $\widehat{u}_+ \in L^1$. Define S by (3.52) and \tilde{S} by (3.53). Then

(3) \tilde{v}_2 satisfies (3.38) with $c_\infty = \|\widehat{u}_+\|_\infty$ and satisfies (3.39) for all $\theta_0 < 7/9$.

(4) For all $\theta < 7/9$ and all r with $0 \leq \delta(r) < 1$, the differences $v_i - v_j$, $v_i - \tilde{v}_j$ and $\tilde{v}_i - \tilde{v}_j$ belong to $Z_{\theta, r}([1, \infty))$ for all $i, j = 1, 2, 3$.

We refer to the original papers for the proof of the estimates. Combining Propositions 3.8 and 3.9, we obtain the final result

Proposition 3.10. *Let $n \leq 3$, $f(u) = \lambda|u|^{2/n}u$. Let $u_+ \in L^2$ with $x^2 u_+ \in L^2$ and $\|\widehat{u}_+\|_\infty$ sufficiently small. If $n = 3$, assume in addition that $\widehat{u}_+ \in L^1$.*

Then the equation (0.1) has a unique solution $u \in X(\mathbb{R}^+) \cap X_{loc}(\mathbb{R})$ such that for any r with $0 \leq \delta(r) < 1$ and any $\theta < 1$ (resp. $\theta < 7/9$) for $n \leq 2$ (resp. for $n = 3$), $u - v \in Z_{\theta, r}([1, \infty))$ for $v = v_i$, $i = 1, 2, 3$ (and in addition, if $n = 3$ for $v = \tilde{v}_i$, $i = 1, 2, 3$).

Proof. The results for v_2 ($n \leq 2$) and \tilde{v}_2 ($n = 3$) follow from Propositions 3.8 and 3.9 Parts (1) and (3), and from the results on the global L^2 -problem. The results for other choices of v follow from the previous ones, from the uniqueness statements in Proposition 3.8 and from Proposition 3.9, Parts (2) and (4). It is fortunate in the case $n = 3$ that $7/9 > 3/4 = n/4$. . . \square

Under the assumptions of Proposition 3.10, we define the modified wave operator (for positive time) as the map $\Omega_+ : u_+ \rightarrow u(0)$. Whereas there was no doubt that the corresponding definition was adequate for the ordinary wave operators (see Section 3.a), this is less obvious in the present case, where the asymptotic v depends in a complicated (nonlinear) way on u_+ . The best justification is that Ω_+ as defined above again intertwines the free (unmodified !) evolution with the full evolution. In order to make a meaningful statement, we need a set of asymptotic states which is invariant under the free evolution. We define for $R > 0$

$$Y(R) = \{u_+ : u_+ \in L^2, x^2 u_+ \in L^2, \Delta u_+ \in L^2 \text{ and } \|\hat{u}_+\|_\infty < R\} \quad .$$

Let again $u(t, u_0)$ be the solution of (0.1) with $u(0, u_0) = u_0$. Then the intertwining property holds in the following sense

Proposition 3.11. *Let $n \leq 3$, let $f(u) = \lambda|u|^{2/n}u$ and let R satisfy the smallness condition of c_∞ in Proposition 3.8. Then for any $u_+ \in Y(R)$ and any $s, t \in \mathbb{R}$.*

$$u(t + s, \Omega_+ u_+) = u(t, \Omega_+ U(s)u_+) \quad .$$

The proof follows from the uniqueness statement of Proposition 3.10 and from estimates similar to, but significantly simpler than, those needed in the proof of Proposition 3.9.

We finally remark that the situation in Proposition 3.10 is less satisfactory than in the previous cases (Propositions 3.3 and 3.7) in several respects. In addition to the limitation $n \leq 3$ on the space dimension, the results of Proposition 3.10 are restricted to small data (in the L^∞ norm of \hat{u}_+) and the (L^2 -valued) wave operators are defined only on asymptotic states with additional smoothness and/or decay. Similar results hold for the (less singular) Hartree equation, with however no upper restriction on the dimension (see [GO] for details).

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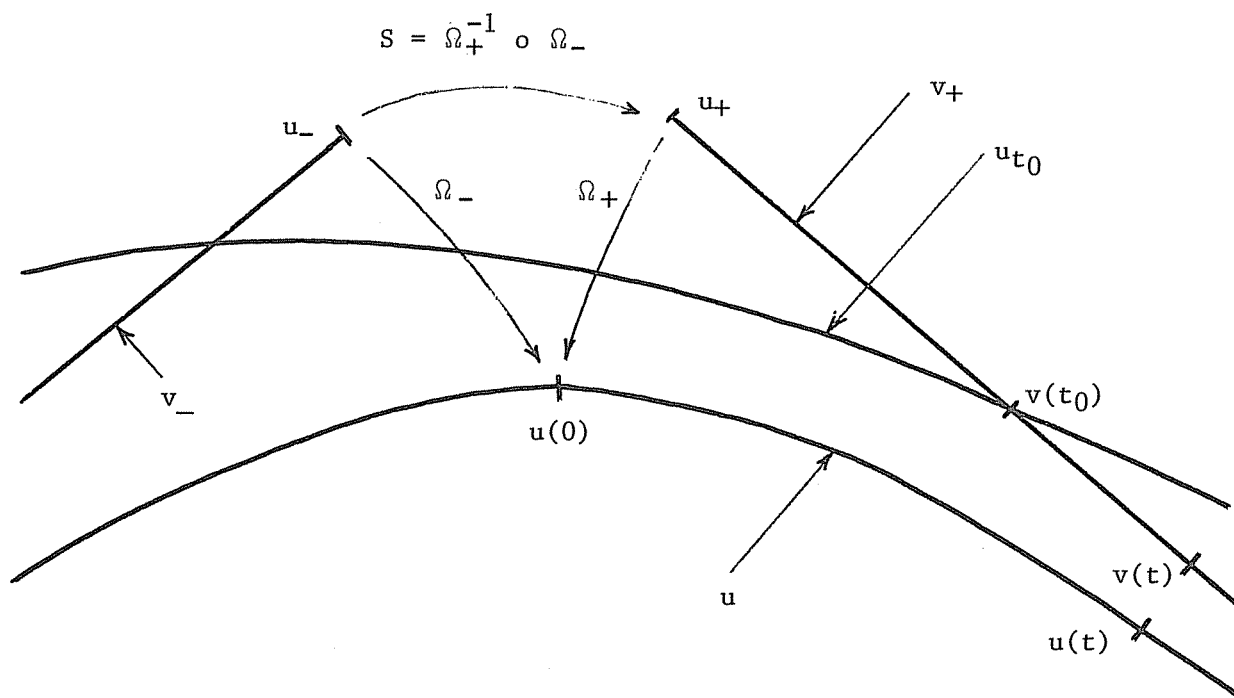


Fig. 1. Definition of the wave operators

GENERALIZED STRICHARTZ INEQUALITIES FOR THE WAVE EQUATION IN HOMOGENEOUS BESOV SPACES

J.Ginibre and G.Velo

Abstract: Strichartz inequalities for the wave equation in homogeneous Besov spaces are presented and the most significant steps needed for their proof are illustrated. The methods employed rely on the one side on specific estimates on the solutions of the wave equation and on the other side on abstract duality arguments of quite general character independent of the equation.

1. Introduction

In the first days of this conference, we heard many times the name of Strichartz in connection with some inequalities estimating space time integral norms of the solutions of the Cauchy problem for the wave equation

$$(\partial_t^2 - \Delta)u = f, \quad u(t=0) = u_0, \quad \partial_t u(t=0) = u_1 \quad (1.1)$$

in terms of similar norms of f and of suitable norms of u_0 and u_1 . The result of Strichartz [10] goes back to 1977 and, since then, many papers (among them [1],[3],[4],[7],[8],[9],[12]) have contributed to shape the inequalities to their actual general form. The objective of this lecture is to present a conceptually simple proof of the generalized Strichartz inequalities contained in a recent paper [5], to which we refer for details and for a more complete list of references. The proof is an expanded version of that written in [3] with, in addition, a treatment along the same lines of a limiting case derived in [7]. These estimates have proved themselves to be often indispensable in the study of the Cauchy problem and of scattering theory for the non linear wave equation. Particularly useful has been their formulation in Besov spaces for a natural treatment of the non linearities present in the equation.

In order to formulate the inequalities it is convenient to express the solution u of (1.1), that we consider in \mathbb{R}^{n+1} with $n \geq 2$, as a sum $u = v + w$, where v solves (1.1) with $f = 0$ and w solves (1.1) with $u_0 = u_1 = 0$. Using the notation $\omega = (-\Delta)^{1/2}$, $U(t) = \exp(i\omega t)$, $K(t) = \omega^{-1} \sin(\omega t)$ and $\dot{K}(t) = \cos(\omega t)$, v and w can be written as

$$v(t) = \dot{K}(t)u_0 + K(t)u_1 \quad (1.2)$$

and

$$w(t) = \int_0^t dt' K(t-t')f(t'). \quad (1.3)$$

For any operator $L(t)$ of the type $\omega^\lambda U(t)$, $\omega^\lambda K(t)$, $\omega^\lambda \dot{K}(t)$ with $\lambda \in \mathbb{R}$, we define the operators $L_R(t) = \chi_+(t)L(t)$ and $L_A(t) = \chi_-(t)L(t)$, with χ_\pm characteristic function of \mathbb{R}^\pm . This allows one to rewrite (1.3) for positive time as

$$w(t) = (K_R *_t \chi_+ f)(t), \quad (1.4)$$

where $*_t$ denotes the convolution in the time variable t . A similar expression with K_R replaced by K_A represents w for negative time. We restrict our attention from now on to positive time. The norm of the space $L^r \equiv L^r(\mathbb{R}^n)$, $1 \leq r \leq \infty$, is denoted by $\|\cdot\|_r$ and pair of Hölder conjugate exponents are denoted r, \bar{r} with $1/r + 1/\bar{r} = 1$, $1 \leq r \leq \infty$. The following multiples of the basic function $\alpha(r) \equiv (1/2 - 1/r)$, $\beta(r) = (n+1)\alpha(r)/2$, $\gamma(r) = (n-1)\alpha(r)$ and $\delta(r) = n\alpha(r)$ are of special interest. The expression $2\beta(r)$ represents the loss of derivatives in the pointwise in time estimate (2.21), $\gamma(r)$ is the exponent of the optimal decay in time of L^r solutions of the wave equation and $\delta(r)$ appears naturally in Hölder and Sobolev inequalities. The Fourier transform in \mathbb{R}^n is denoted by $\hat{\cdot}$. Convolution in $x \in \mathbb{R}^n$ is denoted by $*_x$, with subscript omitted when there is no risk of confusion. The initial data (u_0, u_1) for the problem (1.1) are taken from the space $\dot{H}_2^\mu \oplus \dot{H}_2^{\mu-1}$, with $\mu \in \mathbb{R}$, where the homogeneous Sobolev spaces \dot{H}_r^μ are defined by the expression

$$\dot{H}_r^\mu \equiv \dot{H}_r^\mu(\mathbb{R}^n) = \left\{ u : \|u; \dot{H}_r^\mu\| \equiv \|\omega^\mu u\|_r < \infty \right\} \quad (1.5)$$

for $1 \leq r \leq \infty$. Then the original Strichartz estimate takes the form

$$\|u; L^{r_S}(\mathbb{R}^{n+1})\| \leq C \left\{ \|(u_0, u_1); \dot{H}_2^{1/2} \oplus \dot{H}_2^{-1/2}\| + \|f; L^{\bar{r}_S}(\mathbb{R}^{n+1})\| \right\} \quad (1.6)$$

where $r_S = 2(n+1)/(n-1)$. The extended version of this inequality consists in replacing the space $L^{r_S}(\mathbb{R}^{n+1})$ (and $L^{\bar{r}_S}(\mathbb{R}^{n+1})$) by more general spaces such as $L^q(\mathbb{R}, \dot{H}_r^\rho)$ or L^q functions of the time variable with values in the Sobolev space \dot{H}_r^ρ or in Besov spaces of the space variables.

2. Statement and proof of the inequalities

In order to make this exposition as much self-contained as possible we will give the definition of homogeneous Besov spaces and we will recall their properties relevant to the subsequent discussion. The definition makes use of the Paley-Littlewood dyadic decomposition in which the dual space \mathbb{R}^n is partitioned into dyadic spherical shells. Let $\hat{\psi} \in C_0^\infty(\mathbb{R}^n)$ be a fixed function with $0 \leq \hat{\psi} \leq 1$, $\hat{\psi}(\xi) = 1$, for $|\xi| \leq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$. Then the functions $\hat{\varphi}_j(\xi) \equiv \hat{\varphi}_0(2^{-j}\xi)$, where $\hat{\varphi}_0(\xi) = \hat{\psi}(\xi) - \hat{\psi}(2\xi)$ and $j \in \mathbb{Z}$, have their support contained in $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and, for any $\xi \in \mathbb{R}^n \setminus \{0\}$, satisfy

$$\sum_{j \in \mathbb{Z}} \hat{\varphi}_j(\xi) = 1$$

with at most two non vanishing terms in the sum. These support properties make it convenient to define $\tilde{\varphi}_j \equiv \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ for all $j \in \mathbb{Z}$, so that

$$\varphi_j = \tilde{\varphi}_j * \varphi_j. \quad (2.1)$$

With any $u \in \mathcal{S}'(\mathbb{R}^n)$ we can associate the sequence of $C^\infty(\mathbb{R}^n)$ functions $\varphi_j * u$ to be considered as functions of the two variables j and x . For any $\rho \in \mathbb{R}$ and any r and s with $1 \leq r, s \leq \infty$ we define the homogeneous Besov spaces

$$\dot{B}_{r,s}^\rho \equiv \dot{B}_{r,s}^\rho(\mathbb{R}^n) = \left\{ u : \|u; \dot{B}_{r,s}^\rho\| \equiv \|2^{\rho j} \varphi_j * u; l_j^s(L_x^r)\| < \infty \right\} \quad (2.2)$$

where one takes first the L^r norm in the variable x and then the l^s norm in the variable j ([11] p.45 and p.238) and the homogeneous Triebel-Lizorkin spaces

$$\dot{F}_{r,s}^\rho \equiv \dot{F}_{r,s}^\rho(\mathbb{R}^n) = \left\{ u : \|u; \dot{F}_{r,s}^\rho\| \equiv \|2^{\rho j} \varphi_j * u; L_x^r(l_j^s)\| < \infty \right\} \quad (2.3)$$

with the L_x^r and l_j^s norms computed in the opposite order. The spaces $\dot{B}_{r,s}^\rho$ and $\dot{F}_{r,s}^\rho$ become Banach spaces once the quotient of $\mathcal{S}'(\mathbb{R}^n)$ by polynomials is taken. The Minkowski inequality implies

$$\begin{aligned} \dot{B}_{r,s}^\rho &\subset \dot{F}_{r,s}^\rho, & \text{for } \infty \geq r \geq s \geq 1 \\ \dot{B}_{r,s}^\rho &\supset \dot{F}_{r,s}^\rho & \text{for } 1 \leq r \leq s \leq \infty \end{aligned} \quad (2.4)$$

and the Hilbert space version of the Mihlin-Hormander multiplier theorem ([11] p.243) implies

$$\dot{H}_r^\rho = \dot{F}_{r,2}^\rho \quad (2.5)$$

for all $1 < r < \infty$. Comparison of (2.4) with (2.5) yields the inclusions $\dot{B}_{r,2}^\rho \subset \dot{H}_r^\rho$ for $2 \leq r < \infty$, $\dot{B}_{r,2}^\rho \supset \dot{H}_r^\rho$ for $1 < r \leq 2$. The Sobolev embeddings for the Besov spaces take the form

$$\dot{B}_{r_1,s}^{\rho_1} \supset \dot{B}_{r_2,s}^{\rho_2} \quad (2.6)$$

with $\rho_1, \rho_2 \in \mathbb{R}$, $1 \leq r_2 \leq r_1 \leq \infty$, $1 \leq s \leq \infty$ and $\rho_1 + \delta(r_1) = \rho_2 + \delta(r_2)$. The inclusion (2.6) is an elementary consequence of Young inequality applied to $\tilde{\varphi}_j * \varphi_j * u = \varphi_j * u$ and of the scaling properties in the variable j of the norms of $\tilde{\varphi}_j$ in L^r . We shall only need the spaces with $s = 2$ and, in that case, we shall omit that index and write $\dot{B}_r^\rho \equiv \dot{B}_{r,s}^\rho$. With the above notation and definition available the Strichartz inequalities can be stated in their generalized form.

Proposition. *Let $\rho_1, \rho_2, \mu \in \mathbb{R}$ and $2 \leq q_1, q_2, r_1, r_2 \leq \infty$ and let the following conditions be satisfied.*

$$0 \leq 2/q_i \leq \min(\gamma(r_i), 1), \quad \text{for } i = 1, 2 \quad (2.7)$$

$$(2/q_i, \gamma(r_i)) \neq (1, 1), \quad \text{for } i = 1, 2 \quad (2.8)$$

$$\rho_1 + \delta(r_1) - 1/q_1 = \mu \quad (2.9)$$

$$\rho_1 + \delta(r_1) - 1/q_1 = 1 - (\rho_2 + \delta(r_2) - 1/q_2) \quad (2.10)$$

(1) *Let $(u_0, u_1) \in \dot{H}^\mu \oplus \dot{H}^{\mu-1}$. Then v defined by (1.2) satisfies the estimates*

$$\|v; L^{q_1}(\mathbb{R}, \dot{B}_{r_1}^{\rho_1})\| + \|\partial_t v; L^{q_1}(\mathbb{R}, \dot{B}_{r_1}^{\rho_1-1})\| \leq C \|(u_0, u_1); \dot{H}_2^\mu \oplus \dot{H}_2^{\mu-1}\|. \quad (2.11)$$

(2) *For any interval $I \subset \mathbb{R}$, possibly unbounded, the following estimates hold*

$$\|K * f; L^{q_1}(I, \dot{B}_{r_1}^{\rho_1})\| \leq C \|f; L^{\bar{q}_2}(I, \dot{B}_{\bar{r}_2}^{-\rho_2})\|. \quad (2.12)$$

(3) *For any interval $I = [0, T)$, $0 < T \leq \infty$, the function $w = K_R * \chi_+ f$ defined by (1.4) satisfies the estimates*

$$\|w; L^{q_1}(I, \dot{B}_{r_1}^{\rho_1})\| + \|\partial_t w; L^{q_1}(I, \dot{B}_{r_1}^{\rho_1-1})\| \leq C \|f; L^{\bar{q}_2}(I, \dot{B}_{\bar{r}_2}^{-\rho_2})\|. \quad (2.13).$$

The constants C in (2.12) and (2.13) are independent of I .

The same results hold with \dot{B}_r^ρ replaced by \dot{H}_r^ρ everywhere, under the additional assumption that $r_i < \infty$ ($i = 1, 2$) whenever r_i occurs.

Before proceeding to the proof a more explicit description of the allowed region in the space of the parameters $(q_1, r_1; q_2, r_2)$ may be useful. This domain has a product structure which makes it possible to express the restrictions (2.7) and (2.8) in a natural

and simple way. For each pair of indices it is convenient to use the variables $(1/q, 1/r)$. For $n \geq 4$ the allowed region is represented by a quadrilateral $ABCD$ with $A = (0, 1/2)$, $B = (1/2, (n-3)/2(n-1))$, $C = (1/2, 0)$ and $D = (0, 0)$, corresponding to $(q = \infty, r = 2)$, $(q = 2, \gamma(r) = 1)$, $(q = 2, r = \infty)$ and $(q = \infty, r = \infty)$ respectively. See Fig.1. For $n = 3$ the quadrilateral reduces to the triangle ACD and for $n = 2$ it shrinks to the triangle AC_2D where $C_2 = (1/4, 0)$, corresponding to $(q = 4, r = \infty)$. See Fig.2. The boundary is allowed with the exception of the point B for $n \geq 3$. The points of the segment AB satisfy the equation $2/q = \gamma(r)$. The original result of Strichartz corresponds to the case $q = r \doteq r_S$ and $\beta(r) = 1/2$ and its representative point is at the intersection of the diagonal $1/q = 1/r$ with the segment AB for $n \geq 4$, AC for $n = 3$ and AC_2 for $n = 2$.

Proof of the Proposition. For fixed (q_i, r_i) , $i = 1, 2$, the parameters $\rho_1, \mu, -\rho_2$ are determined up to a common additive term, as can be seen from (2.9) and (2.10). Therefore, since the operator ω^λ maps isomorphically \dot{B}_r^{ρ} into $\dot{B}_r^{\rho-\lambda}$ for all $\lambda \in \mathbb{R}$, the inequalities (2.11), (2.12) and (2.13) for the general case will be a consequence of the same inequalities for the special case $\mu = 0$. On the other hand, the definitions of v and w (see (1.2), (1.3) and (1.4)) and of K, \dot{K} and U allow to reduce (2.11), (2.12) and (2.13) to similar estimates involving only U . The proposition will then result from the following inequalities:

$$\|U(\cdot)u; L^{q_1}(\mathbb{R}, \dot{B}_{r_1}^{\rho_1})\| \leq C\|u\|_2, \quad (2.11')$$

$$\|U * f; L^{q_1}(I, \dot{B}_{r_1}^{\rho_1})\| \leq C\|f; L^{\bar{q}_2}(I, \dot{B}_{r_2}^{-\rho_2})\| \quad (2.12')$$

for $I \subset \mathbb{R}$ and

$$\|U_R * f; L^{q_1}(I, \dot{B}_{r_1}^{\rho_1})\| \leq C\|f; L^{\bar{q}_2}(I, \dot{B}_{r_2}^{-\rho_2})\| \quad (2.13')$$

for $I = [0, T)$, $0 < T \leq \infty$ under the conditions (2.7) and (2.8) and

$$\rho_i + \delta(r_i) - 1/q_i = 0, \quad i = 1, 2. \quad (2.14)$$

From the Sobolev embeddings in Besov spaces (2.6) one immediately sees that, if (2.11'), (2.12') and (2.13') are satisfied for $P_1 = (1/q_1, 1/r_1)$, $P_2 = (1/q_2, 1/r_2)$, then they are satisfied for $P'_1 = (1/q'_1, 1/r'_1)$, $P'_2 = (1/q'_2, 1/r'_2)$ where $q'_1 = q_1$, $r'_1 \geq r_1$, $q'_2 = q_2$, $r'_2 \geq r_2$. Therefore, if $q_i > 2$, $i = 1, 2$, it will be sufficient to prove (2.11'), (2.12') and (2.13') for $P_i \in [A, B)$, $i = 1, 2$, namely for $2/q_i = \gamma(r_i)$, which, by (2.14), corresponds to $\rho_i = -\beta(r_i)$.

After those preliminary reductions we can start the proper proof. The general strategy consists first in writing pointwise estimates in time of norms involving space variables, then in integrating those estimates in the time variable to obtain the fully integrated inequalities

for special values of the parameters and, finally, in reaching the fully allowed region for the parameters by the use of abstract duality arguments.

At this initial stage of the proof the time variable is kept fixed, while only functions or distributions of the space variables \mathbb{R}^n are considered. Let f be such a function and let φ_j be the previously introduced dyadic decomposition in \mathbb{R}^n . From

$$\varphi_j * (U(t)f) = U(t)(\varphi_j * f) \quad (2.15)$$

and the unitarity of $U(t)$ in L^2 one gets trivially

$$\|\varphi_j * (U(t)f)\|_2 = \|\varphi_j * f\|_2, \quad (2.16)$$

while from

$$\varphi_j * (U(t)f) = (U(t)\varphi_j) * (\tilde{\varphi}_j * f) \quad (2.17)$$

and the Young inequality one obtains

$$\|\varphi_j * (U(t)f)\|_\infty \leq \|U(t)\varphi_j\|_\infty \|\tilde{\varphi}_j * f\|_1. \quad (2.18)$$

Now

$$(U(t)\varphi_j)(x) = \frac{2^{jn}}{(2\pi)^n} \int \exp\left(i2^j x \cdot \xi + i2^j t |\xi|\right) \hat{\varphi}_0(\xi) d\xi,$$

so that the L^∞ norm of $U(t)\varphi_j$ can be estimated for large positive j by the method of stationary phase ([6], Sec. 7.7) and for large negative j by taking the absolute value of the integrand. The computation leads to

$$\|\varphi_j * (U(t)f)\|_\infty \leq C 2^{jn} \min\left(1, (|t|2^j)^{-\frac{n-1}{2}}\right), \quad (2.19)$$

where the degeneracy of the phase is responsible for the decay exponent $(n-1)/2$. Interpolation between (2.16) and (2.18) and use of (2.19) yield the following fundamental pointwise estimate in time

$$\|\varphi_j * (U(t)f)\|_r \leq C \min\left(2^{2j\delta(r)}, |t|^{-\gamma(r)} 2^{2j\beta(r)}\right) \|\tilde{\varphi}_j * f\|_{\bar{r}} \quad (2.20)$$

for $2 \leq r \leq \infty$. The corresponding estimate involving Besov spaces

$$\|U(t)f; \dot{B}_r^{-\beta(r)}\| \leq C |t|^{-\gamma(r)} \|f; \dot{B}_{\bar{r}}^{\beta(r)}\| \quad (2.21)$$

is obtained by taking the l_j^2 norm of (2.20) after multiplication by the factor $2^{-j\beta(r)}$. Inequality (2.21) [1] [8], which is of interest by itself, will be used in the case $\gamma(r) < 1$.

We now proceed to integrate the pointwise estimates. For this purpose it is convenient to separate the two cases $q > 2$ and $q = 2$.

The case $q > 2$

Let f depend on time and rewrite (2.21) as

$$\|U_{(R)}(t - t')f; \dot{B}_r^{-\beta(r)}\| \leq C|t - t'|^{-\gamma(r)}\|f(t'); \dot{B}_r^{\beta(r)}\|, \quad (2.22)$$

where $U_{(R)}$ stands either for U or for U_R . Integrating over the variable t' , taking the L^q norm in the variable t with $2/q = \gamma(r) < 1$, and applying the Hardy-Littlewood-Sobolev inequality yield the integral estimate

$$\|U_{(R)} *_t f; L^q(I, \dot{B}_r^{-\beta(r)})\| \leq C\|f; L^{\bar{q}}(I, \dot{B}_r^{\beta(r)})\|. \quad (2.23)$$

Inequality (2.23) is a particular case of (2.12') and (2.13') in which $q_1 = q_2$, $r_1 = r_2$ and $2/q_i = \gamma(r_i)$. In this situation the domain and the range of the operator $U_{(R)} *_t$ are spaces in duality.

A simple way to eliminate this restriction for the unretarded estimates and to prove (2.11') is based on an abstract duality argument. That argument has a long history in the subject [3][4][9][10][12]. It has been applied to the present case in [3][5]. Since this argument can be found in another contribution to this volume by one of us [2], it will not be repeated in detail and free use of it will be made. Broadly speaking the duality argument states that if B is an operator from a dense subset \mathcal{D} of a Banach space X to a Hilbert space \mathcal{H} and B^* is its adjoint from \mathcal{H} to the algebraic dual \mathcal{D}_a of \mathcal{D} , then boundedness of B from X to \mathcal{H} (after extension to all of X) is equivalent to boundedness of B^* from \mathcal{H} to X^* , which is equivalent to boundedness of B^*B from X to X^* . In the situation of interest $\mathcal{H} = L^2 \equiv \dot{B}_2^0$, X can be any of the Banach spaces $X_r = L^{\bar{q}}(I, \dot{B}_r^{\beta(r)})$ with $0 \leq 2/q = \gamma(r) < 1$ (so that $X_r^* = L^q(I, \dot{B}_r^{-\beta(r)})$), and the relevant operators are

$$Bf = \int_I dt U(-t)f(t) \quad (2.24)$$

$$(B^*v)(t) = U(t)v \quad (2.25)$$

$$(B^*Bf)(t) = \int_I dt' U(t - t')f(t'). \quad (2.26)$$

Inequality (2.23) states that B^*B is bounded from X_r to X_r^* so that, by the duality argument, B is bounded from X_r to L^2 and B^* is bounded from L^2 to X_r^* . This proves (2.11') for $(1/q_1, 1/r_1) \in [A, B)$. Moreover boundedness of B from X_{r_1} to L^2 and of B^* from L^2 to $X_{r_2}^*$ yields boundedness of B^*B from X_{r_1} to $X_{r_2}^*$ which proves (2.12') for $(1/q_i, 1/r_i) \in [A, B)$, $i = 1, 2$.

In the study of the corresponding estimates for the retarded operator the following notation

$$((B^*B)_R f)(t) \equiv \int_I dt' U_R(t-t') f(t'). \quad (2.27)$$

will be useful. This operator does not enjoy the remarkable factorisation property of the unretarded one (see (2.27) versus (2.26)), so that its study requires additional arguments. The following chain of inequalities leads to boundedness of $(B^*B)_R$ from X_r to X_2^* :

$$\begin{aligned} \|(B^*B)_R f; X_2^*\| &= \sup_{t \in I} \left\| \int dt' U(-t') \chi_+(t-t') f(t'); \dot{B}_2^0 \right\| \\ &\leq C \sup_{t \in I} \|\chi_+(t-\cdot) f(\cdot); X_r\| \leq C \|f; X_r\| \end{aligned} \quad (2.28)$$

where we have used in the first equality the unitarity of $U(t)$, in the second inequality the boundedness of B and in the last one the fact that χ_+ is a uniformly bounded function of time. Interpolation of (2.23), which expresses boundedness of $(B^*B)_R$ from X_r to X_r^* , with (2.28) yields boundedness of $(B^*B)_R$ from X_{r_1} to $X_{r_2}^*$ with $r_2 \leq r_1$. Obviously the same result holds also for the advanced operator, which allows to remove restriction $r_2 \leq r_1$ the retarded operator being the adjoint of the advanced one. Thus (2.13') is proved for $(1/q_i, 1/r_i) \in [A, B)$ and, by the Sobolev inequalities, for all $q_i > 2$, $i = 1, 2$. This completes the proof in the case $q > 2$.

The case $q = 2$ for $n \geq 4$

Let f depend on time and rewrite (2.20) as

$$\|\varphi_j * (U_{(R)}(t-t') f(t'))\|_r \leq C \min(2^{2j\delta(r)}, |t-t'|^{-\gamma(r)} 2^{2j\beta(r)}) \|\tilde{\varphi}_j * f\|_{\bar{r}}. \quad (2.29)$$

This inequality will be used in the case $\gamma(r) > 1$. For fixed j , integrating over the variable t' , taking the L^2 norm in the variable t and applying the Young inequality yield the integral estimate

$$\|\varphi_j * (U_{(R)} * f); L^2(I, L^r)\| \leq C 2^{j(2\delta(r)-1)} \|\tilde{\varphi}_j * f; L^2(I, L^{\bar{r}})\|. \quad (2.30)$$

Taking the l_j^2 norm of (2.30) after multiplication by the factor $2^{-j(\delta(r)-\frac{1}{2})}$ yields

$$\|U_{(R)} * f; L^2(I, \dot{B}_r^{-(\delta(r)-\frac{1}{2})})\| \leq C \|f; L^2(I, \dot{B}_{\bar{r}}^{(\delta(r)-\frac{1}{2})})\| \quad (2.31)$$

which is a particular case of (2.12') and (2.13'), where $q_1 = q_2 = 2$ and $\gamma(r_1) = \gamma(r_2) > 1$. The general unretarded estimates (2.12') and (2.13') can now be obtained by exactly the same abstract duality argument as in the case $q > 2$. Similarly, the retarded estimates can be established in all situations except when $((1/q_1, 1/r_1), (1/q_2, 1/r_2)) \in (A, B) \times (B, C] \cup$

$(B, C] \times (A, B)$. In this case a more delicate argument involving a dyadic decomposition in time has to be used. \square

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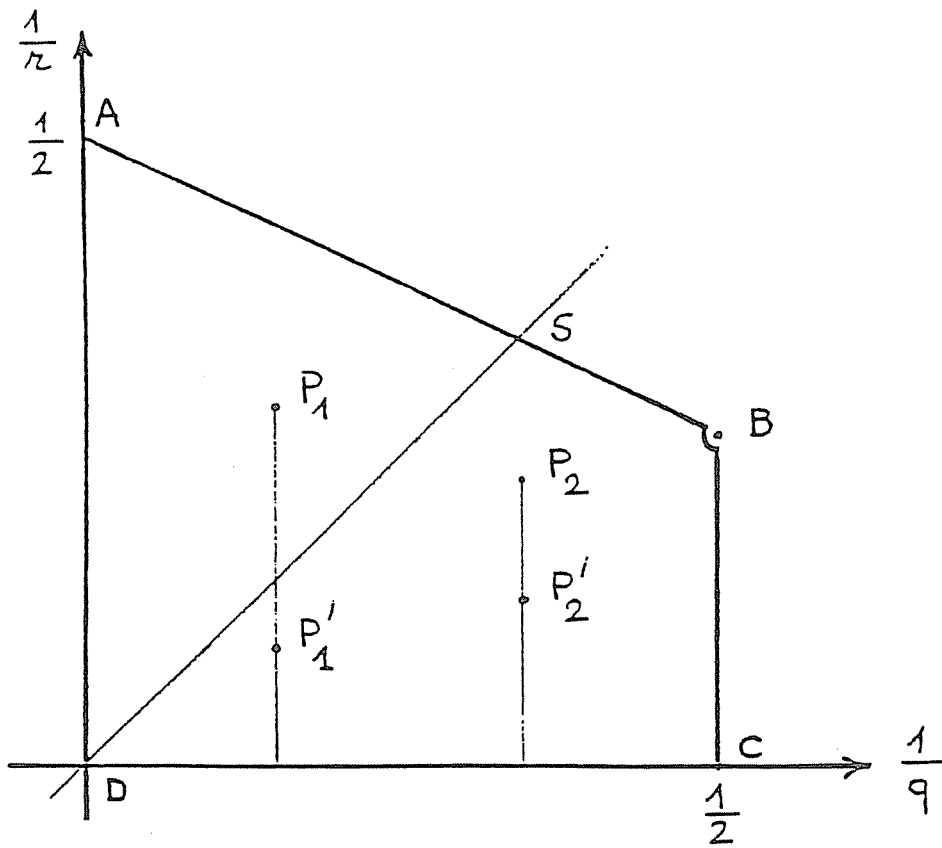


Fig. 1: The case $n \gg 4$

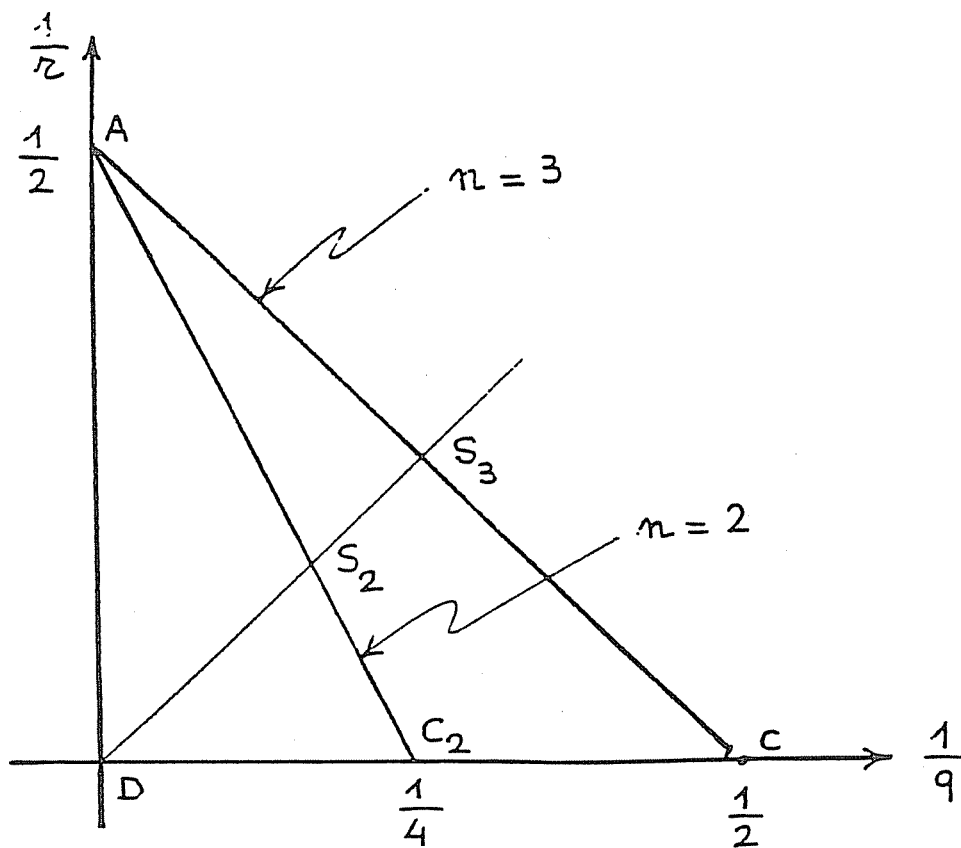


Fig. 2: The cases $n = 3$ and $n = 2$

ON THE INITIAL VALUE PROBLEM FOR THE DAVEY-STEWARTSON AND THE ISHIMORI SYSTEMS

NAKAO HAYASHI

Abstract. We consider the initial value problem for the Davey-Stewartson and the Ishimori systems. Our purpose is to investigate the minimal regularity assumptions necessary on the data which yield the local existence in time of small solutions to the systems.

1. Introduction We consider the Davey-Stewartson system

$$(DS) \quad \begin{cases} i\partial_t u + c_0 \partial_x^2 u + \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x \varphi, & (t, x, y) \in \mathbf{R}^3, \\ \partial_x^2 \varphi + c_3 \partial_y^2 \varphi = \partial_x |u|^2, \\ u(x, y, 0) = u_0(x, y), c_0, c_3 \in \mathbf{R}, \quad c_1, c_2 \in \mathbf{C} \end{cases}$$

and the Ishimori system

$$(I) \quad \begin{cases} i\partial_t u + c_4 \partial_x^2 u + \partial_y^2 u = F(u, \nabla u, \nabla \varphi), & (t, x, y) \in \mathbf{R}^3, \\ \partial_x^2 \varphi + c_8 \partial_y^2 \varphi = G(u, \nabla u), \\ u(x, y, 0) = u_0(x, y), \end{cases}$$

$$F = c_5 \frac{\bar{u}}{1 + |u|^2} ((\partial_x u)^2 + c_6 (\partial_y u)^2) + c_7 (\partial_x u \partial_y \varphi + \partial_y u \partial_x \varphi),$$

$$G = c_9 \frac{(\partial_x u)(\partial_y \bar{u}) - (\partial_x \bar{u})(\partial_y u)}{(1 + |u|^2)^2}, \quad c_4, c_8 \in \mathbf{R}, c_5, c_6, c_7, c_9 \in \mathbf{C}$$

The (D-S) system was first derived by Davey-Stewartson [8], Benney-Roskes [5] and Djordjevic-Redekopp [9] and model the evolution of weakly nonlinear water waves that

travel predominantly in one direction, but in which the wave amplitude is modulated slowly in horizontal directions. Independently Ablowitz and Haberman [2] and Cornille [7] obtained a particular form of (1.1) as an example of a completely integrable model which generalizes the one-dimensional Schrödinger equation. In [9] it was shown that the parameter c_3 can become negative when capillary effects are important.

When $(c_0, c_1, c_2, c_3) = (1, -1, 2, -1)$, $(-1, -2, 1, 1)$ or $(-1, 2, -1, 1)$ the system in (1.1) is referred to the inverse scattering literature as the DSI, DSII defocusing and DSII focusing respectively. In these cases several results concerning the existence of solitons or lump solutions have been established ([1],[3-4],[7],[11-13],[25]) by the inverse scattering techniques. As a matter of fact, cases where (1.1) is of inverse scattering type are exceptional. By Ghidaglia and Saut [14] the IVP (initial value problem) (1.1) was studied and classified as elliptic-elliptic (E-E), elliptic-hyperbolic (E-H), hyperbolic-elliptic (H-E) and hyperbolic-hyperbolic (H-H) according to the respective sign of $(c_0, c_3) : (+, +), (+, -), (-, +)$ and $(-, -)$. For the elliptic-elliptic and hyperbolic-elliptic cases, local and global properties of solutions were studied in the usual Sobolev spaces L^2, H^1, H^2 in [14].

We turn now to the Ishimori system (I). Ishimori [19] proposed the following system :

$$(II) \quad \begin{cases} \partial_t S = S \wedge (\partial_x^2 S + c_0 \partial_y^2 S) + c_1 (\partial_x \varphi \partial_y S + \partial_y \varphi \partial_x S) \\ \partial_x^2 \varphi + c_2 \partial_y^2 \varphi = c_3 S \cdot (\partial_x S \wedge \partial_y S) \\ S(x, y, 0) = S_0(x, y), \end{cases}$$

where $(c_0, c_1, c_2, c_3) = (-1, c_1, 1, 2)$ or $(1, c_1, -1, -2)$, $S(\cdot, t) : \mathbf{R}^2 \rightarrow \mathbf{R}^3$, $|S|^2 = 1$, $S \rightarrow (0, 0, 1)$ as $\sqrt{x^2 + y^2} \rightarrow \infty$ and \wedge denotes the wedge product in \mathbf{R}^3 . The IVP (II) is considered as a two-dimensional generalization of the Heisenberg equation in ferromagnetism. We put

$$S = (S_1, S_2, S_3) = \frac{1}{1 + |u|^2} (u + \bar{u}, -i(u - \bar{u}), 1 - |u|^2),$$

where $u : \mathbf{R}^2 \rightarrow \mathbf{C}$. Then it is clear that $S(\cdot, t) : \mathbf{R}^2 \rightarrow \mathbf{R}^3$, $|S|^2 = 1$, $u = (S_1 + iS_2)/(1 + S_3)$ and $S \rightarrow (0, 0, 1)$ as $\sqrt{x^2 + y^2} \rightarrow \infty$ if $u \rightarrow 0$ as $\sqrt{x^2 + y^2} \rightarrow \infty$. When $c_0 = 1, c_1 = 0$ (II) is reduced to the two dimensional Heisenberg equation. When $c_1 = 1$, (II) was studied formally by Konopelchenko and Matkarimov [21,22] by using the inverse scattering transform. By using the new variable u the Ishimori equations (II) can be written as

$$(I2) \quad \begin{cases} i\partial_t u + \partial_x^2 u + c_0 \partial_y^2 u \\ = 2 \frac{\bar{u}}{1+|u|^2} ((\partial_x u)^2 + c_0 (\partial_y u)^2) + ic_1 (\partial_x u \partial_y \varphi + \partial_y u \partial_x \varphi) \\ \partial_x^2 \varphi + c_2 \partial_y^2 \varphi = 2ic_3 \frac{(\partial_x u)(\partial_y \bar{u}) - (\partial_x \bar{u})(\partial_y u)}{(1+|u|^2)^2} \\ u(x, y, 0) = u_0(x, y) \end{cases}$$

which is a special case of (I).

Following the classification of the Davey-Stewartson systems used in [14], we classify (I) as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of $(c_4, c_8) : (+, +), (+, -), (-, +)$ and $(-, -)$.

In [24] Soyeur studied the case $(c_4, c_5, c_6, c_8, c_9) = (-1, 2, -1, 1, 4i)$ for (I) which corresponds to a hyperbolic-elliptic case and obtained local well-posedness results and a global existence of small solutions. His method is also useful to the elliptic-elliptic case of (I).

We now give a table which says works on local existence in time of solutions to (DS) and (I) systems

$y \backslash x$	I_1	I_2	I_3	I_4	I_5	I_6
(E-E) (DS)	[14]	[14]	[14]	[14]	[15]	[15]
(E-E) (I)	[24]	[24]	[24]	[24]	[15]	[15]
(E-H) (DS)	×	[6]	×	[23], [18], [6]	[15]	[15]
(E-H) (I)	×	×	×	[17]	×	[15]
(H-E) (DS)	[14]	[14]	[14]	[14]	[15]	[15]
(H-E) (I)	[24]	[24]	[24]	[24]	[15]	[15]
(H-H) (DS)	×	×	×	[23], [18]	[15]	[15]
(H-H) (I)	×	×	×	×	×	[15]

where I_1, \dots, I_6 imply the conditions on the data as follows :

$$I_1 : u_0 \in H^{m,0},$$

$$I_2 : u_0 \in H^{m,0}, \|u_0\|_{H^{m,0}} \text{ is sufficiently small,}$$

$$I_3 : u_0 \in H^{l,s}$$

$$I_4 : u_0 \in H^{l,s}, \|u_0\|_{H^{l,s}} \text{ is sufficiently small,}$$

$$I_5 : u_0 \text{ is in some analytic function space,}$$

$$I_6 : u_0 \text{ is small in some analytic function space,}$$

where

$$H^{l,s} = \{f \in L^2; \|(1+x^2+y^2)^{s/2}(1-\partial_x^2-\partial_y^2)^{l/2}f\| < \infty, l, s > 0\},$$

We next give a table which says works on global in time of solutions to (DS) and (I) systems.

$y \setminus x$	I_1	I_2	I_3	I_4	I_5	I_6
(E-E) (DS)	[14]	[14]	[14]	[14]	×	[15]
(E-E) (I)	×	×	×	[24]	×	[15, 16]
(E-H) (DS)	×	×	×	[6]	×	[15]
(E-H) (I)	×	×	×	×	×	[15, 16]
(H-E) (DS)	[14]	[14]	[14]	[14]	×	[15]
(H-E) (I)	×	×	×	[24]	×	[15, 16]
(H-H) (DS)	×	×	×	×	×	[15]
(H-H) (I)	×	×	×	×	×	[15, 16]

In this paper we concentrate our attention to the elliptic-hyperbolic (E-H) and hyperbolic-hyperbolic (H-H) cases. Without loss of generality we may take $(c_0, c_3) = (\pm 1, -1)$ in (DS), and $(c_4, c_8) = (\pm 1, -1)$ in (I). In these cases one has to assume that $\varphi(\cdot)$ satisfies the radiation condition.

$$(1.1) \quad \lim_{y \rightarrow \infty} \varphi(x, y, t) = \varphi_1(x, t), \quad \lim_{x \rightarrow \infty} \varphi(x, y, t) = \varphi_2(y, t),$$

where φ_1 and φ_2 are given functions. Under the radiation conditions (1.1) (DS) and (I) are written as after a rotation in the xy -plane and rescaling,

$$(1.2) \quad \begin{cases} i\partial_t u + Hu \\ = d_1|u|^2 u + d_2 u \int_y^\infty \partial_x |u|^2(x, y', t) dy' \\ + d_3 u \int_x^\infty \partial_y |u|^2(x', y, t) dx' + d_4 u \partial_x \varphi_1 + d_5 u \partial_y \varphi_2, \\ u(x, y, 0) = u_0(x, y), \end{cases}$$

$$(1.3) \quad \begin{cases} i\partial_t u + Hu = F_1(u) + F_2(u), \\ u(x, y, 0) = u_0(x, y), \end{cases}$$

where

$$\begin{aligned} F_1(u) &= \frac{\bar{u}}{1 + |u|^2} (d_6 (\partial_x u)^2 + d_7 (\partial_y u)^2 + d_8 (\partial_x u)(\partial_y u)) \\ F_2(u) &= d_9 (\partial_y u \partial_y \varphi_2 - \partial_x u \partial_x \varphi_1 \\ &\quad - \partial_y u \int_x^\infty G(u, \partial_x u, \partial_y u) dx' + \partial_x u \int_y^\infty G(u, \partial_x u, \partial_y u) dy'), \\ G(u, \partial_x u, \partial_y u) &= d_{10} \frac{(\partial_x u)(\partial_y \bar{u}) - (\partial_x \bar{u})(\partial_y u)}{(1 + |u|^2)^2}. \end{aligned}$$

where d_1, \dots, d_{10} are arbitrary constants, $H = \partial_x \partial_y$ when $c_0, c_4 = -1$ and $H = \partial_x^2 + \partial_y^2$ when $c_0, c_4 = 1$.

We state the results concerning (E-H) and (H-H) cases. Linares and Ponce [23] and Chihara [6] obtained the following results.

Theorem A [23]. *We assume that $u_0 \in H^{s,0} \cap H^{3,2} \equiv Y_s, s \geq 6, \varphi_1 = \varphi_2 \equiv 0$ and $\|u_0\|_{H^{6,0}} + \|u_0\|_{H^{3,2}}$ is sufficiently small. Then there exists a positive constant $T > 0$ and a unique solution u of (1.2) with $H = \partial_x \partial_y$ such that $u \in C([0, T]; Y_s)$.*

Theorem B [23]. *We assume that $u_0 \in H^{s,0} \cap H^{6,6} \equiv W_s, s \geq 12, \varphi_1 = \varphi_2 \equiv 0$ and $\|u_0\|_{H^{12,0}} + \|u_0\|_{H^{6,6}}$ is sufficiently small. Then there exists a positive constant $T > 0$ and a unique solution u of (1.2) with $H = \partial_x^2 + \partial_y^2$ such that $u \in C([0, T]; W_s)$.*

Theorem 1.1 [6]. *We assume that $u_0 \in H^{s,0}$, where s is a sufficiently large integer, $\varphi_1 = \varphi_2 \equiv 0$ and $\|u_0\|_{L^2} < 1/(2\sqrt{\max\{d_2, d_3\}}e)$. Then there exists a positive constant $T > 0$ and a unique solution u of (1.2) with $H = \partial_x^2 + \partial_y^2$ such that $u \in C_w([0, T]; H^{s,0}) \cap C([0, T]; H^{s-1,0})$.*

Theorem 1.2 [6]. *We assume that $u_0 \in \cap_{j=0}^5 H^{s-j,j}$, where s is a sufficiently large integer, $\varphi_1 = \varphi_2 \equiv 0$ and $\sum_{j=0}^5 \|u_0\|_{H^{s-3-j,j}}$ is sufficiently small. Then there exists a unique global solution u of (1.2) with $H = \partial_x^2 + \partial_y^2$ such that $u \in \cap_{j=0}^5 C_w([0, \infty); H^{s-j,j}) \cap C([0, \infty); H^{s-1-j,j})$.*

To state the results obtained in [15] we prepare notation and some function spaces. Let X be a Banach space with norm $\|\cdot\|_X$ and $B = (B_1, \dots, B_j)$ be a vector field of derivations. The generalized Sobolev space B^m is defined by

$$B^m = \{f \in L^2; \|f\|_{B^m} = \sum_{|\alpha| \leq m} \|B^\alpha f\|_{L^2} < \infty\},$$

where $B^\alpha = B_1^{\alpha_1} \dots B_j^{\alpha_j}$, $|\alpha| = \sum_{1 \leq k \leq j} \alpha_k$, $\alpha_k \in \mathbb{N} \cup \{0\}$.

Let $A > 0$. We define generalized analytic function space as follows :

$$G^A(B; X) = \{f \in X; \|f\|_{G^A(B; X)} = \sum_{\beta \in (\mathbb{N} \cup \{0\})^j} \frac{A^{|\beta|}}{\beta!} \|B^\beta f\|_X < \infty\}.$$

We introduce the first order differential operators $J_x = x + 2it\partial_x, J_y = y + 2it\partial_y, J_1 = y + it\partial_x, J_2 = x + it\partial_y, \Omega_{xy} = x\partial_y - y\partial_x$ and $\Omega_{12} = x\partial_x - y\partial_y$. By using these operators we define

$$\partial = (\partial_x, \partial_y), \quad R = (\partial, J_1, J_2), \quad \Gamma = (R, \Omega_{12})$$

$$\tilde{R} = (\partial, J_x, J_y) \quad \text{and} \quad \tilde{\Gamma} = (\tilde{R}, \Omega_{xy}).$$

These operators together with the identity form a Lie algebra.

Theorem 1 [15]. *We assume that*

$$u_0 \in G^A(\partial; H^{m,0}),$$

where $A > 0$ and $m \geq 3$. Then there exists a unique local solution u of (1.2) and a positive constant T such that

$$u(t, x) \in C([0, T]; G^{A_1}(\partial; H^{m,0})),$$

where $A_1 < A$.

Theorem 2 [15]. *We assume that*

$$u_0 \in G^A(\partial; H^{m,0}), \quad \|u_0\|_{G^A(\partial; H^{m,0})} < \frac{1}{2},$$

where $A > 0$ and $m \geq 4$. Then the same result as in Theorem 1 [15] holds for the IVP (1.3).

Theorem 3 [15]. *We assume that*

$$u_0 \in G^A(R(0); R^m(0)), \quad \|u_0\|_{G^A(R(0); R^m(0))} < \epsilon,$$

where $A > 0$, $m \geq 3$ and ϵ is a sufficiently small positive constant. Then there exists a unique global solution u of (1.2) with $H = \partial_x^2 + \partial_y^2$ such that

$$u(t, x) \in G^{A_1}(R(t); R^m(t)) \quad \text{for any } t \in \mathbf{R}^+,$$

where $A_1 < A$.

Theorem 3'[15]. *The result of Theorem 3 [15] holds true for the IVP (1.2) with $H = \partial_x \partial_y$ under the hypotheses of Theorem 3 with R replaced by \tilde{R} .*

Theorem 4 [15]. *The result of Theorem 3 [15] holds true with $m \geq 5$ for the IVP (1.3) with $H = \partial_x^2 + \partial_y^2$ (resp. (1.3) with $H = \partial_x \partial_y$), provided we replace in the hypotheses, R by Γ (resp. R by $\tilde{\Gamma}$).*

Remark 1. Theorem 3, 3' and 4 require the exponential decay on the data. In [16] that condition on the data was removed for the Ishimori system (1.3).

In order to state next results obtained in [17,18] we introduce the weighted Sobolev space $H_j^{m,s}$ defined by

$$H_j^{m,s} = \{f \in L^2(\mathbf{R}_j); \|f\|_{H_j^{m,s}} = \| \langle j \rangle^s \langle D_j \rangle^m f \|_{L_j^2} < \infty\}, \quad j = x, y,$$

where

$$\langle j \rangle = (1 + j^2)^{1/2}, \quad \langle D_j \rangle = (1 - \partial_j^2)^{1/2}, \quad D_j = -i\partial_j, \quad \text{and} \quad \|\cdot\|_{L_j^2}^2 = \int |\cdot|^2 dj.$$

Theorem 1 [18]. *We assume that $u_0 \in H^{\delta,0} \cap H^{0,\delta} \equiv Z_\delta, \delta > 1$, $\partial_x \varphi_1 \in H_x^{\delta,0}$, $\partial_y \varphi_2 \in H_y^{\delta,0}$ and $\|u_0\|_{H^{\delta,0}} + \|u_0\|_{H^{0,\delta}}$ is sufficiently small. Then there exists a positive constant $T > 0$ and a unique solution u of (1.2) such that $u \in C([0, T]; Z_\delta)$.*

Theorem 1 [17]. *We assume that $u_0 \in Z_4$, $\partial_x \varphi_1 \in H_x^{4,0} \cap H_x^{\delta,\delta}$, $\partial_y \varphi_2 \in H_y^{4,0} \cap H_y^{\delta,\delta}$, $\delta > 1$ and $\|u_0\|_{H^{4,0}} + \|u_0\|_{H^{0,4}} + \|\partial_x \varphi_1\|_{H_x^{\delta,\delta}} + \|\partial_y \varphi_2\|_{H_y^{\delta,\delta}}$ is sufficiently small for any t . Then there exists a positive constant T and a unique solution u of (1.3) with $H = \partial_x^2 + \partial_y^2$ such that $u \in C([-T, T]; Z_4)$.*

We now state the main result in this paper which is an improvement of [17, Theorem 1].

Theorem 1. *We assume that $u_0 \in Z_{1+\delta}$, $\partial_x \varphi_1 \in H_x^{1+\delta,0} \cap H_x^{\delta,\delta}$, $\partial_y \varphi_2 \in H_y^{1+\delta,0} \cap H_y^{\delta,\delta}$, $\delta > 1$ and $\|u_0\|_{H^{1+\delta,0}} + \|u_0\|_{H^{0,1+\delta}} + \|\partial_x \varphi_1\|_{H_x^{\delta,\delta}} + \|\partial_y \varphi_2\|_{H_y^{\delta,\delta}}$ is sufficiently small for any t . Then there exists a positive constant T and a unique solution u of (1.3) with $H = \partial_x^2 + \partial_y^2$ such that $u \in C([-T, T]; Z_{1+\delta})$.*

To prove Theorem 1 we use the following function space

$$\begin{aligned} X(T) &= \{f \in C([0, T]; L^2); \|f\|_{X(T)}^2 \\ &= \sup_{t \in [0, T]} \|f(t)\|_{X_1}^2 + \int_0^T \|f(t)\|_{X_2}^2 dt < \infty\}, \end{aligned}$$

where

$$\begin{aligned} \|f\|_{X_1} &= \| \langle D \rangle^{1+\delta} f \| + \| \langle \bar{x} \rangle^{1+\delta} f \|, \quad \|f\| = \|f\|_{L^2}, \\ \|f\|_{X_2} &= \| \langle x \rangle^{-\gamma} \langle D_x \rangle^{3/2+\delta} f \| + \| \langle y \rangle^{-\gamma} \langle D_y \rangle^{3/2+\delta} f \|. \end{aligned}$$

In the following three sections, we give the outline of the proof of Theorem 1.

§2 Linear Schrödinger Equations. In this section we consider the inhomogeneous Schrödinger equations

$$(2.1) \quad \begin{cases} i\partial_t u + \Delta u = f, & (t, x, y) \in \mathbf{R}^3, \\ u(x, y, 0) = u_0(x, y), \end{cases}$$

where $\Delta = \partial_x^2 + \partial_y^2$.

In the same way as in the proof of [18, Lemma 3.6 (3.20)] we have

Lemma 2.1. *Let u be the solution of (2.1) with $H = \partial_x^2 + \partial_y^2$. Then we have*

$$(2.1) \quad \begin{aligned} \|u\|_{X(T)}^2 &\leq C(\|u_0\|_{X_1}^2 + \int_0^t \|u(\tau)\|_{X_1}^2 + \|f(\tau)\|_{H^{0,\delta}} \|u(\tau)\|_{X_1} d\tau \\ &\quad + \sum_{j=x,y} \int_0^t |(\langle K_j \rangle \langle D_j \rangle^{1+\delta} f(\tau), \langle K_j \rangle \langle D_j \rangle^{1+\delta} u(\tau))| \\ &\quad + |(\langle D_j \rangle^{1+\delta} f(\tau), \langle D_j \rangle^{1+\delta} u(\tau))| d\tau \quad \text{for } t \in [0, T], \end{aligned}$$

where

$$K_j = \exp\left[\left(\int_0^j \langle \tau \rangle^{-\delta} d\tau\right) \frac{D_j}{\langle D_j \rangle}\right], \quad (f, g) = \int_{\mathbb{R}^2} f \bar{g} dx dy.$$

Remark 2. The operators K_j were used by Doi [10] first to prove the L^2 well posedness of linear Schrödinger equations.

Lemma 2.1 follows from the classical energy estimates and the smoothing property of solutions to (2.1) which is written as

$$\begin{aligned} &\sum_{j=x,y} (\| \langle D_j \rangle^m u(t) \|^2 + \int_0^t \| \langle j \rangle^{-\gamma} \langle D_j \rangle^{m+1/2} u(\tau) \|^2 d\tau) \\ &\leq C \sum_{j=x,y} (\| \langle D_j \rangle^m u_0 \|^2 + \int_0^t \| \langle j \rangle^{-\gamma} u(\tau) \|^2 d\tau \\ &\quad + \int_0^t |(\langle K_j \rangle \langle D_j \rangle^m f(\tau), \langle K_j \rangle \langle D_j \rangle^m u(\tau))| d\tau), \end{aligned}$$

(see [18, Lemma 3.2] for details).

§3 Nonlinear Estimates. In this section we present the estimates of nonlinear terms. We let

$$\begin{aligned} F_1(u) &= \frac{\bar{u}}{1 + |u|^2} (d_6 (\partial_x u)^2 + d_7 (\partial_y u)^2 + d_8 (\partial_x u)(\partial_y u)), \\ F_2(u) &= d_9 (\partial_y u \partial_y \varphi_2 - \partial_x u \partial_x \varphi_1 \\ &\quad - \partial_y u \int_x^\infty G(u, \partial_x u, \partial_y u) dx' + \partial_x u \int_y^\infty G(u, \partial_x u, \partial_y u) dy'), \\ G(u, \partial_x u, \partial_y u) &= d_{10} \frac{(\partial_x u)(\partial_y \bar{u}) - (\partial_x \bar{u})(\partial_y u)}{(1 + |u|^2)^2}. \end{aligned}$$

In the same way as in the proofs of [18, Lemma 2.3-2.5, Lemma 2.10, Lemmas 2.13, 2.14] we have by making use of the commutator estimates in the fractional order Sobolev space obtained in [20, Appendix]

Lemma 3.1. *We have for $\delta > 1$*

$$\begin{aligned} & \sum_{j=x,y} \int_0^T |(K_j < D_j >^{1+\delta} F_l(v(\tau)), K_j < D_j >^{1+\delta} u(\tau))| \\ & + |(< D_j >^{1+\delta} F_l(v(\tau)), < D_j >^{1+\delta} u(\tau))| d\tau \\ & \leq C(\|v\|_{X(T)}^2 + A) \|v\|_{X(T)} \|u\|_{X(T)}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=x,y} \int_0^T |(K_j < D_j >^{1+\delta} (F_l(v_1(\tau)) - F_l(v_2(\tau))), \\ & \quad K_j < D_j >^{1+\delta} (u_1(\tau) - u_2(\tau)))| \\ & + |(< D_j >^{1+\delta} (F_l(v_1(\tau)) - F_l(v_2(\tau))), < D_j >^{1+\delta} (u_1(\tau) - u_2(\tau)))| d\tau \\ & \leq C(\|v_1\|_{X(T)}^2 + \|v_2\|_{X(T)}^2 + A) \|v_1 - v_2\|_{X(T)} \|u_1 - u_2\|_{X(T)}, \end{aligned}$$

where

$$A = \sup_{t \in [-T, T]} \|\partial_x \varphi_1\|_{H_x^{\delta, \delta}} + \sup_{t \in [-T, T]} \|\partial_y \varphi_2\|_{H_y^{\delta, \delta}}.$$

§4 Proof of Theorem 1. We prove Theorem 1 by the classical contraction mapping principle. We define the operator Φ by $u = \Phi v$, where u is the solution of

$$(4.1) \quad \begin{cases} i\partial_t u + \partial_x^2 u + \partial_y^2 u = F_1(v) + F_2(v), \\ u(x, y, 0) = u_0(x, y), \end{cases}$$

where F_1 and F_2 are the same as those defined in (1.3), and

$$v \in X_{T, \rho} = \{f \in X(T); \|f\|_{X(T)} \leq \rho\}, \quad \rho \ll 1.$$

Applying Lemma 3.1 to Lemma 2.1 with $f = F_1(v) + F_2(v)$, we obtain by (4.1)

$$\|\Phi v\|_{X(T)} \leq \frac{1}{2} \|v\|_{X(T)}, \quad \text{and} \quad \|\Phi v_1 - \Phi v_2\|_{X(T)} \leq \frac{1}{2} \|v_1 - v_2\|_{X(T)}$$

under the conditions given in Theorem 1. This shows that Φ is a contraction mapping from $X_{T, \rho}$ into itself. Hence we have the result.

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NON-LOCAL CONSERVATION LAWS IN RELATIVISTIC THEORIES OF EXTENDED OBJECTS

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Abstract. The dynamics of an M -dimensional extended object whose $M+1$ dimensional world volume in $M+2$ dimensional space-time has vanishing mean curvature is formulated in term of geometrical variables (the first and second fundamental form of the time-dependent surface Σ_M). It is shown that the non-linear equations of motion for $\Sigma_M(t)$ can be viewed as consistency conditions of associated linear systems that give rise to the existence of non-local conserved quantities. For $M=2$, an explicit solution in terms of elliptic functions is exhibited, which is neither rotationally nor axially symmetric, and also yields 3-fold-periodic spacelike maximal hypersurfaces in $\mathbb{R}^{1,3}$.

1. Introduction

Consider the motion of an M -dimensional extended object $\Sigma_M(t)$ in \mathbb{R}^{M+1} . Any such motion gives rise to a $(M+1)$ -dimensional manifold \mathbb{M} in $(M+2)$ -dimensional space-time $\mathbb{R}^{1,M+1}$, whose boundaries (if Σ_M is compact) are Σ_M (initial time t_i) and Σ_M (final time t_f). Relativistically invariant dynamics for Σ_M can be formulated by subjecting \mathbb{M} to a variational principle, like the extremization of the volume-functional (generalizing [6]). The volume of \mathbb{M} may be given by introducing coordinates $(\varphi^\alpha)_{\alpha=0,\dots,M}$ on \mathbb{M} , describing $\mathbb{M} \subset \mathbb{R}^{1,M+1}$ by the $M+2$ coordinate-functions $x^\mu(\varphi^0, \dots, \varphi^M)$, calculating the metric $G_{\alpha\beta}$ induced by the flat Minkowski-metric $(\eta_{\mu\nu})_{\mu\nu=0,\dots,M+1} = \text{diag}(1, -1, \dots, -1)$ and integrating,

$$\begin{aligned}
 S &= \text{Vol}(\mathbb{M}) = \int d\varphi^{M+1} \sqrt{G} & (1) \\
 G &= (-)^M \det(G_{\alpha\beta}) \quad , \quad G_{\alpha\beta} = \frac{\partial x^\mu}{\partial \varphi^\alpha} \frac{\partial x^\nu}{\partial \varphi^\beta} \eta_{\mu\nu} \quad .
 \end{aligned}$$

Taking (1) as a starting point (with signature $(M) = (1, -1, \dots, -1)$) one may ask: what does extremality of S (considered as a functional of the x^μ) imply for $\Sigma_M(t)$, the shape of the extended object? Choosing $\varphi^0 = x^0 = t$, and the time dependence of the spatial parameters $\varphi = (\varphi^r)_{r=1, \dots, M}$ such that the motion of Σ_M (described by $\vec{x}(t, \varphi) = (x^1, \dots, x^{M+1})$) is always normal, i.e.

$$(G_{\alpha\beta}) = \begin{pmatrix} 1 - \dot{\vec{x}}^2 & 0 \dots 0 \\ 0 & \\ \vdots & -\partial_r \vec{x} \cdot \partial_s \vec{x} \\ 0 & \end{pmatrix} \quad (2)$$

the extremality condition(s)

$$\frac{1}{\sqrt{G}} \partial_\alpha \sqrt{G} G^{\alpha\beta} \partial_\beta x^\mu = 0 \quad (3)$$

$$\mu = 0, \dots, M+1$$

read:

$$\frac{\partial}{\partial t} \left(\sqrt{g} \sqrt{\frac{1}{1 - \dot{\vec{x}}^2}} \right) = 0 \quad (4)$$

$$\rho \cdot \ddot{\vec{x}} = \partial_r \frac{1}{\rho} g g^{rs} \partial_x \vec{x} \quad (5)$$

$$\rho = \rho(\varphi^1, \dots, \varphi^M) := \sqrt{\frac{g}{1 - \dot{\vec{x}}^2}}$$

where $\cdot = \frac{\partial}{\partial t}$, and g and g^{rs} are the determinant and inverse, respectively, of the (positive definite) metric $g_{rs} := \partial_r \vec{x} \partial_s \vec{x}$ on $\Sigma_M(t)$. The conservation law (4), “large area(densitie)s have to slow down, while small area(densitie)s speed up” (anticipating singularities as well as periodicity), encodes almost the complete dynamical information. To see this, one first notes that on a fixed compact surface $\Sigma_M(t = t_i)$ parameters $(\varphi^r)_{r=1, \dots, M}$ may be chosen such that the conserved (energy-)density is actually independent of φ , i.e.

$$\dot{\vec{x}}^2 + g/\lambda^{2M} = 1 \quad (6)$$

$$\lambda = \text{const.}$$

– as noted already in [2], (4) then ensures that (6) will hold for all t . Furthermore, as (5) and the orthogonality conditions (cp. (2))

$$\dot{\vec{x}} \partial_r \vec{x} = 0 \quad , \quad r = 1, \dots, M \quad (7)$$

are invariant under

$$\vec{x}(t, \lambda) \rightarrow \lambda \vec{x}\left(\frac{t}{\lambda}, \varphi\right) \quad (8)$$

(corresponding to $x^\mu \rightarrow \lambda x^\mu$ in (3)), one could put $\lambda = 1$ in (6), with the understanding, that each motion with $\lambda \neq 1$ can be obtained from a $\lambda = 1$ motion via (8). In any case, one can show that, since (6) and (7), i.e.

$$\dot{\vec{x}} = \pm \sqrt{1 - g/\lambda^{2M}} \vec{n} \quad , \quad (9)$$

\vec{n} = surface normal, holds (cp. [1]; for a Hamiltonian formulation of (9) see [3], and for general dependence of the normal velocity on \sqrt{g} , [4]), the equations of motion (5) are automatically satisfied – apart from points where $\dot{\vec{x}} = 0$. It is convenient to write (9) in the form

$$\dot{\vec{x}} = -\sin\theta \vec{n} \quad (10)$$

$$\theta = \theta(t, \varphi^1, \dots, \varphi^M) \in (-\pi/2, +\pi/2)$$

One should note that choosing the conserved energy density ρ to be constant on Σ (i.e. independent of φ) is a matter of convenience, not necessity; eq. (5) is a consequence of (4) and (7), resp. (10), for any ρ , and in the considerations that will follow one could equally think of $\sin^2\theta$ as being given by $1 - g/\rho^2(\varphi)$, rather than $1 - g/\lambda^{2M}$. Leaving the density ρ unspecified one would keep full Diff Σ invariance of the equations.

2. Formulation of the Dynamics of Σ_M in Terms of Geometrical Variables

The simple first-order form of the dynamics, (10) (resp. (9)), allow one to easily derive the basic equations,

$$\dot{g}_{rs} = -2\sin\theta h_{rs} \quad (11)$$

$$\dot{h}_{rs} = (\nabla_r \nabla_s - h_{ra} g^{ab} h_{sb}) \sin\theta \quad (12)$$

for the components of the metric tensor, and the second fundamental form

$$h_{rs} := -\partial_{rs}^2 \vec{x} \cdot \vec{n} \quad ; \quad (13)$$

(∇_r) are the covariant derivatives (with respect to φ^r) on Σ_t , i.e.

$$\nabla_a \nabla_b f = \partial_{ab}^2 f - \gamma_{ab}^c \partial_c f \quad (14)$$

$$\gamma_{ab}^c = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$$

for any function $f: \Sigma_t \rightarrow \mathbb{R}$. Note that the gauge-fixing (cp.(6)) has left one with a residual SDiff Σ_t -invariance, i.e. invariance of the equations under reparametrisations

$$\varphi^r \rightarrow \tilde{\varphi}^r(\varphi^1, \dots, \varphi^M) \quad , \quad J = \det \frac{\partial \tilde{\varphi}^r}{\partial \varphi^s} = 1 \quad (15)$$

and that θ (remember that $\cos^2\theta = g/\lambda^{2M}$) is an ‘observable’. Also note that (5), with $\rho = \lambda^M = \text{const}$, implies

$$\ddot{\vec{x}} \cdot \vec{n} = -\cos^2\theta \cdot H \quad , \quad H := g^{rs} h_{rs} \quad (16)$$

= mean curvature

(as well as $\ddot{\vec{x}} \partial_r \vec{x} = -\frac{1}{2} \partial_r (g/\lambda^{2M}) = \sin\theta \cos\theta \partial_r \theta$ – which is zero at the turning points, $\dot{\vec{x}}(\tilde{t}, \tilde{\varphi}) = 0$); taking the time-derivative of $\sin\theta := \vec{n} \cdot \dot{\vec{x}}$ one obtains

$$\dot{\theta} = \cos\theta H \quad (17)$$

(for $\theta \neq 0$, this could have been obtained directly from (11), $-2 \sin \theta \cos \theta \dot{\theta} \lambda^{2M} = \dot{g} = g g^{rs} \dot{g}_{rs} = -2g \sin \theta H$). Calculating $-\ddot{x}^\mu n_\mu$, n^μ being normal to \mathbb{M} in $\mathbb{R}^{1,M+1}$,

$$n^\mu = \begin{pmatrix} -\tan \theta \\ \frac{\vec{n}}{\cos \theta} \end{pmatrix} \quad (18)$$

one can check that $\frac{-\dot{\theta}}{\cos \theta}$ is indeed the curvature of any $\varphi = \text{const}$ curve (worldline) in \mathbb{M} (as it should, according to (17), to add up to zero, with the spatial principal curvatures). In any case, (11) and (12) imply

$$\ddot{g}_{rs} = \cot \theta \dot{g}_{rs} \dot{\theta} + \frac{1}{2} \dot{g}_{ra} g^{ab} \dot{g}_{bs} - 2 \sin \theta \nabla_r \nabla_s \sin \theta \quad (19)$$

(where $\dot{\theta}$, (cp) (17), could be replaced by $-\frac{1}{2} \cot \theta g^{ab} \dot{g}_{ab}$). Modulo the gauge-fixing, (19) is equivalent to the original minimal hypersurface equations, while the Weingarten map $T : T_b^a = g^{ac} h_{cb} = h_b^a$, whose eigenvalues are the principal curvatures κ_r , satisfies

$$\dot{T} = (T^2 + \nabla \cdot \nabla) \sin \theta \quad . \quad (20)$$

Note that the rate of change of the volume enclosed by Σ_M , respectively its total area, are given by

$$\dot{V} = - \int \sin \theta \cos \theta d^M \varphi, \quad \dot{A} = - \int \sin \theta \cos \theta H d^M \varphi \quad (21)$$

3. Zero Curvature Condition and Non-Local Conserved Quantities.

The fact that the dynamical equations (5) are automatically satisfied as a consequence of gauge-fixing conditions, (7), and a conservation law, (4), – which too can be stated as a condition on the metric of \mathbb{M} – may also be used in the following way: Consider hypersurfaces $\Sigma_{t_i}, \Sigma_{t_f}$ and motions in between such that for $t_i \leq t \leq t_f$ all points of the surface have non-vanishing velocity. The projection of \mathbb{M} onto \mathbb{R}^{M+1} will then be a euclidean domain $\mathbb{M}_E \subset \mathbb{R}^{M+1}$ (with Σ_{t_i} and Σ_{t_f} as boundary), parametrized by t and $(\varphi^r)_{r=1, \dots, M}$, and with the euclidean metric

$$(G_{ij}^E)_{ij=1, \dots, M+1} = \begin{pmatrix} g_{rs} = (\rho \cdot \cos \theta)^{\frac{2}{M}} \bar{g}_{rs} & 0 \\ & \vdots \\ 0 \dots 0 & \dot{\vec{x}}^2 = \sin^2 \theta \end{pmatrix} \quad . \quad (22)$$

Again one may choose $\rho(\varphi) = \lambda^M = \text{const}$, for simplicity. As (22) contains the entire information about \mathbb{M} , the minimal hypersurface equations should be equivalent to the flatness of \mathbb{M}_E , i.e. the vanishing of the curvature-tensor

$$\begin{aligned} R_{ijkl}^E &= \frac{1}{2} (\partial_{il}^2 G_{jk}^E + \partial_{jk}^2 G_{il}^E - \partial_{ik}^2 G_{jl}^E - \partial_{jl}^2 G_{ik}^E) \\ &+ (G^E)^{mn} (\Gamma_{m,il} \Gamma_{n,jk} - \Gamma_{m,ik} \Gamma_{n,jl}) \quad . \end{aligned} \quad (23)$$

One major advantage of this formulation is that the minimal hyper-surface equations (due to the definition of the curvature tensor, $(\nabla_i \nabla_l - \nabla_l \nabla_i)x^j \equiv -R_{ilk}^E x^k$) are therefore the compatibility conditions $([\partial_i + \Gamma_i, \partial_l + \Gamma_l]x^j = 0)$ of the linear system of equations

$$(\partial_i + \Gamma_i)\psi = 0 \quad i = 1 \dots M + 1 \quad , \quad (24)$$

with $(M+1) \times (M+1)$ matrices $(\Gamma_i)_k^j := \Gamma_{ik}^j$. Explicitely, one finds

$$\Gamma_c = \begin{pmatrix} \gamma_{cb}^a & \frac{1}{2}g^{ad}\dot{g}_{dc} \\ -\frac{\dot{g}_{ac}}{2\sin^2\theta} & \cot\theta\partial_c\theta \end{pmatrix}_{a,b,c=1 \dots M} \quad (25)$$

$$\Gamma_N = \begin{pmatrix} \frac{1}{2}g^{ad}\dot{g}_{db} & -\sin\theta\cos\theta\partial^b\theta \\ \cot\theta\partial_i\theta & \dot{\theta}\cot\theta \end{pmatrix} \quad (26)$$

Considering

$$\phi^{(r)}(\varphi^1 \dots \varphi^M, t) = \psi(\varphi^1, \dots, \varphi^r + \omega^r, \dots, \varphi^M, t) \psi^{-1}(\varphi^1, \dots, \varphi^r, \dots, \varphi^M, t) \quad r = 1 \dots M \quad (27)$$

(ψ the matrix of fundamental solutions of (24) and, for definiteness, taking Σ_M to be an M -torus, with $\varphi^r \in [0, \omega^r]$), satisfying

$$\partial_i \phi^{(r)} = [\phi^{(r)}, \Gamma_i] \quad , \quad (28)$$

non-local conserved charges

$$Q_{rm} = Tr(\phi^{(r)})^m \quad (29)$$

can be deduced from (24) – expressable in terms of the Christoffel-symbols Γ_{jk}^i of M_E via solving (28) _{$i=r$} as a pathordered exponential,

$$\phi^{(r)}(\varphi^1 \dots \varphi^M, t) = \wp e^{-\int_{\varphi^r}^{\varphi^r + \omega^r} \Gamma_r(\varphi^1 \dots \varphi^r \dots \varphi^M, t) d\varphi_r} \quad (30)$$

It is extremely tempting to speculate that the hidden Lorentz-invariance together with the $(S)\text{Diff}\Sigma$ invariance should allow one to introduce a spectral parameter into (24). This would imply an infinity of conserved quantities by expanding (29) in terms of this parameter (note that the scale-parameter λ , cp (8), on which the Γ_i at first sight seem to depend non-trivially, eventually just leads to a conjugation of $\phi^{(r)}$ by a λ -dependent matrix).

Finally note that the transformation $(\partial_i + \Gamma_i) \rightarrow S(\partial_i + \Gamma_i)S^{-1} = (\partial_i + \tilde{\Gamma}_i)$, with S being the square-root of the $(M+1) \times (M+1)$ matrix (G_{ij}^E) , will yield antisymmetric matrices $\tilde{\Gamma}_i = -\tilde{\Gamma}_i^{tr} = a_{ijk}L_{jk}$, with $(L_{jk})_{mn} := \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, which allows one to formulate an infinite family of equivalent zero-curvature conditions $[\partial_i + \hat{\Gamma}_i, \partial_j + \hat{\Gamma}_j] = 0$,

$\hat{\Gamma}_i = a_{ijk} \hat{L}_{jk}$ (the \hat{L}_{jk} forming $SO(M+1)$ representations of arbitrary (!) dimension), with corresponding non-local conserved quantities \hat{Q}_{rm} .

4. Some Explicit Hypersurface Solutions

As derived in [5], the following hypersurfaces

$$M^{(T)} := \left\{ (t, \vec{x}) \in \mathbb{R}^{1,3} \mid \wp(x)\wp(y)\wp(z) = \wp(t) \right\} \quad (31)$$

$$M^{(S)} := \left\{ (t, \vec{x}) \in \mathbb{R}^{1,3} \mid \wp(x)\wp(y)\wp(t) = \wp(z) \right\}, \quad (32)$$

with $\wp'^2 = 4\wp(\wp^2 - 1)$, i.e. $\wp \geq 1$ being an elliptic Weierstraß function of period $2\omega = \left(\Gamma\left(\frac{1}{4}\right)\right)^2 / \sqrt{8\pi}$, have vanishing mean curvature. Viewed as time-dependent surfaces moving in \mathbb{R}^3 , $M^{(T)}$ gives melting (and forming) “ice-cubes” (at speeds ≥ 1) while $M^{(S)}$ corresponds to “stalactites” growing (and “melting”) with velocities ≤ 1 .

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Singular limit of solutions of some degenerate parabolic equation

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Abstract. We will discuss the singular limit of the solutions of some degenerate parabolic equations including the case $p \rightarrow \infty$ for the equation $u_t = \Delta u^m - \lambda u^p$ and the case $q \rightarrow \infty$ for the equation $u_t = \Delta u^m - A \cdot \nabla(u^q/q)$ in $R^n \times (0, T)$. We will show the appearance of mesa pattern when p and q are very large.

In this talk I will report my recent result on the asymptotic behaviour of non-negative solutions $u = u^{(p)}$ of the equation [3]

$$\begin{cases} u_t = \Delta u^m - \lambda u^p, & (x, t) \in R^n \times (0, T) \\ u(x, 0) = f(x) \geq 0, & f \in C^1(R^n) \cap L^\infty(R^n) \end{cases} \quad (1)$$

where λ is a positive constant and $T > 0$, $m > 1$ as $p \rightarrow \infty$ and the asymptotic behaviour of non-negative solutions $u = u^{(q)}$ of the equation [4]

$$\begin{cases} u_t = \Delta u^m - A \cdot \nabla(u^q/q), & (x, t) \in R^n \times (0, T) \\ u(x, 0) = f(x) \geq 0, & f \in L^1(R^n) \cap L^\infty(R^n) \end{cases} \quad (2)$$

where $0 \neq A = (a_1, a_2, \dots, a_n) \in R^n$ is a constant vector, $T > 0$, $m > 1$, as $q \rightarrow \infty$. We find that as $p \rightarrow \infty$ in (1), the absorption term in (1) disappears. In fact we find that the solution $u = u^{(p)}$ of (1) will converge uniformly on every compact subset of $R^n \times (0, T)$ as $p \rightarrow \infty$. and the limit $u^{(\infty)} = \lim_{p \rightarrow \infty} u^{(p)}$ satisfies the porous medium equation

$$\begin{cases} v_t = \Delta v^m & (x, t) \in R^n \times (0, T) \\ v(\cdot, t) \searrow g \text{ as } t \rightarrow 0 \text{ in } \mathcal{D}'(R^n) \text{ where} \\ g(x) = f(x) \text{ for } f(x) < 1 \text{ and } = 1 \text{ for } f(x) \geq 1 \end{cases}$$

We also find that as $q \rightarrow \infty$ in (2), the convection term in (2) disappears. More precisely, we find that for fixed $m > 1$ the solutions $u = u^{(q)}$ of (2) converges weakly

in $(L^\infty(G))^*$ for any compact subset G of $R^n \times (0, 1)$ as $q \rightarrow \infty$. Moreover the limit $u^{(\infty)} = \lim_{q \rightarrow \infty} u^{(q)}$ satisfies the porous medium equation

$$\begin{cases} v_t = \Delta v^m & (x, t) \in R^n \times (0, T) \\ v(\cdot, t) \searrow g \text{ as } t \rightarrow 0 \text{ in } \mathcal{D}'(R^n) \end{cases}$$

where $g \in L^1(R^n)$, $0 \leq g \leq 1$, satisfies

$$g(x) + (\tilde{g}(x))_{x_1} = f(x) \text{ in } \mathcal{D}'(R^n)$$

for some function $\tilde{g}(x) \geq 0$, $\tilde{g}(x) \in L^1(R^n)$ and $g(x) = f(x)$, $\tilde{g}(x) = 0$ whenever $g(x) < 1$ a.e. $x \in R^n$.

Similar result was obtained by Caffarelli and A.Friedman [2], P.E.Sacks [5], M.Herrero etc. [1], in the case $A = 0$ and $m \rightarrow \infty$, and X.Xu[6] in the case of hyperbolic equations.

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THE GLOBAL BEHAVIOR OF WEAK SOLUTIONS TO SOME NONLINEAR WAVE EQUATIONS

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Abstract. By constructing so called stable set, the global behavior of weak solutions are discussed to some semilinear wave equations with nonlinear damping and source terms.

1. Introduction. Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We are concerned with the following mixed problems:

$$(1.1) \quad u_{tt} - \Delta u + \delta |u_t|^{m-1} u_t = \mu |u|^{p-1} u, \quad x \in \Omega, \quad t \geq 0,$$

$$(1.2) \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega,$$

$$(1.3) \quad u(t, x) \Big|_{\partial\Omega} = 0 \quad \text{for } t \geq 0.$$

Here $p > 1$, $m \geq 1$, $\delta > 0$, $\mu \in \mathbf{R}$ and Δ is the Laplacian in \mathbf{R}^N .

First, for the problems (1.1) – (1.3) Georgiev et al. [1] have shown the following two theorems:

Theorem 1.1. *Suppose $\mu = \delta = 1$ and either $1 < p \leq \frac{N}{N-2}$ ($N \geq 3$) or $p > 1$ ($N \leq 2$). If $p \leq m$ and $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then the problem (1.1) – (1.3) admits a unique global solution $u(t, x)$ such that for any $T > 0$*

$$u(t, \cdot) \in C([0, T]; H_0^1(\Omega)),$$

$$u_t(t, \cdot) \in C([0, T]; L^2(\Omega)) \cap L^{m+1}((0, T) \times \Omega).$$

Theorem 1.2. *Suppose $\mu = \delta = 1$ and either $1 < p \leq \frac{N}{N-2}$ ($N \geq 3$) or $p > 1$ ($N \leq 2$) and let $1 < m < p$. Then the solution of (1.1) – (1.3) blows up for finite time in the L^∞ norm for suitable large initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$.*

In Theorem 1.1 the damping term dominates over the source and the global solution exists for “any” initial data and in Theorem 1.2 the influence of the source is much stronger for sufficiently “large” initial data. So we can ask some questions as follows:

When $1 < m < p$, does the problem (1.1) – (1.3) admit a global solution for sufficiently “small” initial data ?

To answer the question as above, we shall prepare some propositions and notations, where $\|u\|_q$ means the usual $L^q(\Omega)$ -norm. The following Proposition is due to Haraux [2]:

Proposition 1.3. *Suppose $\delta > 0$, $m \geq 1$, $\mu \in \mathbb{R}$ and either $1 < p$ ($N = 1, 2$) or $1 < p \leq \frac{N}{N-2}$ ($N \geq 3$). For any initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, there exists a real number $T_m > 0$ such that the problem (1.1) – (1.3) admits a unique local weak solution*

$$u(t, \cdot) \in C([0, T_m]; H_0^1(\Omega)) \cap C^1([0, T_m]; L^2(\Omega))$$

with $u_t(t, x) \in L^{m+1}((0, T) \times \Omega)$ for any $0 < T < T_m$ and if $T_m < \infty$, then $\lim_{t \uparrow T_m} [\|\nabla u(t, \cdot)\|_2 + \|u_t(t, \cdot)\|_2] = +\infty$.

Let

$$J(u) \equiv \frac{1}{2} \|\nabla u\|_2^2 - \frac{\mu}{p+1} \|u\|_{p+1}^{p+1},$$

$$E(u, v) \equiv \frac{1}{2} \|v\|_2^2 + J(u)$$

and

$$I(u) \equiv \|\nabla u\|_2^2 - \mu \|u\|_{p+1}^{p+1}.$$

Furthermore, let

$$d \equiv \inf_{\lambda \geq 0} \{ \sup J(\lambda u); u \in H_0^1(\Omega), u \neq 0 \}.$$

It is well known (see e.g. Tsutsumi [10]) that the “potential depth” d satisfies either $d > 0$ ($\mu > 0$) or $d = +\infty$ ($\mu \leq 0$).

Now let us define a stable set introduced by Sattinger [9] as follows:

$$W^* \equiv \{ u \in H_0^1(\Omega); J(u) < d, I(u) > 0 \} \cup \{0\}.$$

Note that $W^* = H_0^1(\Omega)$ if $\mu \leq 0$. Then our results read as follows:

Theorem 1.4. *Suppose $\delta > 0$, $m \geq 1$, $\mu > 0$ and either $1 < p$ ($N = 1, 2$) or $1 < p \leq \frac{N}{N-2}$ ($N \geq 3$). Let $u(t, x)$ be a local solution to the problem (1.1) – (1.3) on $[0, T_m)$ as in Proposition 1.3. If there exists a real number $t_0 \in [0, T_m)$ such that $u(t_0, \cdot) \in W^*$ and $E(u(t_0, \cdot), u_t(t_0, \cdot)) < d$, then $T_m = +\infty$.*

Furthermore, if we impose the following conditions:

$$(1.4) \quad \text{either } 1 \leq m \quad (N = 1, 2) \quad \text{or} \quad 1 \leq m \leq \frac{N+2}{N-2} \quad (N \geq 3),$$

then we get the following main

Theorem 1.5. *Suppose $\delta > 0$, $\mu > 0$, (1.4) and either $1 < p$ ($N = 1, 2$) or $1 < p \leq \frac{N}{N-2}$ ($N \geq 3$). Let $u(t, x)$ be a local solution to the problem (1.1) – (1.3) on $[0, T_m)$ as in Proposition 1.3. Then there exists a real number $t_0 \in [0, T_m)$ such that $u(t_0, \cdot) \in W^*$ and $E(u(t_0, \cdot), u_t(t_0, \cdot)) < d$ if and only if $T_m = +\infty$ and $\lim_{t \rightarrow \infty} \|\nabla u(t, \cdot)\|_2 = \lim_{t \rightarrow \infty} \|u_t(t, \cdot)\|_2 = 0$.*

Remark 1.6. When $m = 1$ and $\mu = 1$, this result coincides with that of Ikehata et al. [5]. So Theorem 1.5 will become a kind of extension of [5]. Furthermore, there is no any relation between p and m in comparison with the conditions of Georgiev et al. [1]. Although Theorem 1.4 holds good also when the case of $\delta = 0$ (conservative case), we can not take $\delta = 0$ in Theorem 1.5. Finally, after our work has been completed, I was pointed out kindly by Professor M. Nakao that in Nakao et al. [8] similar results have already studied to the problem (1.1) – (1.3) with a little restricted m (i.e., $1 \leq m \leq \frac{N}{N-2}$). They have obtained the exact decay rate of weak solutions. So our work will become a kind of another proof with respect to the decay property of solutions.

Corollary 1.7. Let δ, m and p be as in Theorem 1.5. Suppose $\mu \leq 0$. Then for any initial data $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$, the problem (1.1) – (1.3) has a unique global solution $u(t, x)$ satisfying $[u(t, \cdot), u_t(t, \cdot)] \rightarrow [0, 0]$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \rightarrow +\infty$.

Remark 1.8. Although the result of Corollary 1.7 is not so new, its proof is new because the arguments are based on the potential well $W^* = H_0^1(\Omega)$.

2. Preliminaries. Throughout this paper the functions considered are all real valued and the notations for their norms are adopted as usual ones (e.g., Lions [7]). Furthermore, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

We shall describe some lemmas which will be used later without proof (see Ikehata [4]).

Lemma 2.1 (Sobolev-Poincaré). If $2 \leq q \leq \frac{2N}{N-2}$, then

$$\|u\|_q \leq C(\Omega, q) \|\nabla u\|_2$$

for $u \in H_0^1(\Omega)$.

The next lemma is due to Haraux [2].

Lemma 2.2. *Let $u(t, x)$ be a local solution to (1.1) – (1.3) on $[0, T_m)$ as in Proposition 1.3. Then the function $t \mapsto E(u(t, \cdot), u_t(t, \cdot))$ is absolutely continuous and*

$$E(u(t, \cdot), u_t(t, \cdot)) + \delta \int_s^t \|u_t(r, \cdot)\|_{m+1}^{m+1} dr \leq E(u(s, \cdot), u_t(s, \cdot))$$

for $0 \leq s \leq t < T_m$.

Lemma 2.3. *If $\mu > 0$, then W^* is a bounded neighbourhood of 0 in $H_0^1(\Omega)$.*

For the proof of Lemma 2.3 we refer to Lions [7].

Lemma 2.4. *Let $u(t, x)$ be a local solution to (1.1)–(1.3) on $[0, T_m)$ as in Proposition 1.3. If there is a real number $t_0 \in [0, T_m)$ such that $u(t_0, \cdot) \in W^*$ and $E(u(t_0, \cdot), u_t(t_0, \cdot)) < d$, then $u(t, \cdot) \in W^*$ and $E(u(t, \cdot), u_t(t, \cdot)) < d$ for all $t \in [t_0, T_m)$.*

When the case $\mu > 0$, the next lemma plays an essential role in deriving the decay estimate of the total energy $E(u(t, \cdot), u_t(t, \cdot))$ as $t \rightarrow \infty$. For the proof we refer to Ikehata [4].

Lemma 2.5. *Let $u(t, x)$ be a local solution to (1.1)–(1.3) on $[0, T_m)$ as in Proposition 1.3. If there exists a real number $t_0 \in [0, T_m)$ such that $u(t_0, \cdot) \in W^*$ and $E(u(t_0, \cdot), u_t(t_0, \cdot)) < d$, then*

$$\|u(t, \cdot)\|_{p+1}^{p+1} \leq (1 - \gamma) \|\nabla u(t, \cdot)\|_2^2 \quad \text{on } [t_0, T_m),$$

where

$$\gamma \equiv 1 - C(\Omega, p+1)^{p+1} \left\{ \frac{2(p+1)}{p-1} \right\}^{\frac{p-1}{2}} E(u(t_0, \cdot), u_t(t_0, \cdot))^{\frac{p-2}{2}} > 0.$$

3. Proof of Corollary 1.7. We shall give only the proof of Corollary 1.7 because the proof of Theorems 1.4 and 1.5 will be appeared in Ikehata [4]. Since the global existence of solutions is easy, we shall devote the final calculations to the decay property of solutions.

Lemma 3.1. *Let $u(t, x)$ be a global solution of (1.1)–(1.3) with $\mu \leq 0$. Then there exists a real number $\xi > 0$ such that*

$$\|u(t, \cdot)\|_{p+1}^{p+1} \leq \xi \|\nabla u(t, \cdot)\|_2^2$$

for all $t \in [0, +\infty)$.

PROOF. It follows from Lemma 2.1 that

$$(3.1) \quad \|u\|_{p+1}^{p+1} \leq C(\Omega, p+1)^{p+1} \|\nabla u\|_2^2 \|\nabla u\|_2^{p-1} \quad \text{for } u \in H_0^1(\Omega).$$

On the other hand, since $J(u) \geq \frac{1}{2} \|\nabla u\|_2^2$ when the case $\mu \leq 0$, by (3.1) we obtain

$$\|u\|_{p+1}^{p+1} \leq C(\Omega, p+1)^{p+1} \{2J(u)\}^{\frac{p-1}{2}} \|\nabla u\|_2^2.$$

So, since $u(t, x)$ is a global solution of (1.1)–(1.3), then from Lemma 2.2 with $s = 0$ we get

$$\begin{aligned} \|u(t, \cdot)\|_{p+1}^{p+1} &\leq C(\Omega, p+1)^{p+1} \{2J(u(t, \cdot))\}^{\frac{p-1}{2}} \|\nabla u(t, \cdot)\|_2^2 \\ &\leq C(\Omega, p+1)^{p+1} \{2E(u_0, u_1)\}^{\frac{p-1}{2}} \|\nabla u(t, \cdot)\|_2^2 \end{aligned}$$

which implies the desired inequality. \square

PROOF OF COROLLARY 1.7. First, by the same argument as in Ikehata [4, Lemma 3.5] we get

$$(3.2) \quad (1+t)E(t) \leq E(0) + \frac{1}{2} \int_0^t \|u'(s)\|_2^2 ds + \int_0^t J(u(s)) ds,$$

where $E(t) \equiv E(u(t, \cdot), u_t(t, \cdot))$, $u(t) \equiv u(t, \cdot)$ and $u'(t) \equiv u_t(t, \cdot)$. So, from (3.2) and Lemma 3.1 it follows that

$$(3.3) \quad (1+t)E(t) \leq E(0) + \frac{1}{2} \int_0^t \|u'(s)\|_2^2 ds + \left\{ \frac{1}{2} + \frac{-\mu\xi}{p+1} \right\} \int_0^t \|\nabla u(s)\|_2^2 ds.$$

Since $I(u(t)) \geq \|\nabla u(t)\|_2^2$ when the case $\mu \leq 0$, from (3.3) we obtain

$$(3.4) \quad (1+t)E(t) \leq E(0) + \frac{1}{2} \int_0^t \|u'(s)\|_2^2 ds + \nu \int_0^t I(u(s)) ds,$$

with some $\nu > 0$. Noting that

$$\int_0^t \|u'(s)\|_{m+1}^{m+1} ds \leq \frac{E(0)}{\delta}$$

because of Lemma 2.2 with $s = 0$, from the same arguments as in Ikehata [4, Lemmas 3.3–3.4] and (3.4) it follows that

$$(1+t)E(t) \leq C_1 + C_2 t^{\frac{m-1}{m+1}} + C_3 t^{\frac{1}{m+1}}$$

which implies the desired statement, where C_i ($i = 1, 2, 3$) are positive constants. \square

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A NEW CONSTRUCTION OF A FUNDAMENTAL SOLUTION
FOR THE FREE WEYL EQUATION
—AN EXAMPLE OF SUPERANALYSIS

ATSUSHI INOUE (ATLUM)

ABSTRACT. We propose, through an example, a new treatise of a system of partial differential equations, by which we may consider that system as if that is a “scalar one” on the superspace.

§1. INTRODUCTION AND RESULT

Let $\psi(t, q) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^2$ satisfy

$$(1.1) \quad \begin{cases} i\hbar \frac{\partial}{\partial t} \psi(t, q) = \mathbb{H} \psi(t, q), & \mathbb{H} = -ic\hbar \sigma_j \frac{\partial}{\partial q_j}, \\ \psi(0, q) = \underline{\psi}(q). \end{cases}$$

Here, $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q))$ and the Pauli matrices $\{\sigma_j\}$ are, for example, represented by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy the following relations (\mathbb{I}_m stands for the $m \times m$ identity matrix):

$$\begin{aligned} \sigma_j \sigma_k + \sigma_k \sigma_j &= 2\delta_{jk} \mathbb{I}_2 \quad \text{for } j, k = 1, 2, 3, \\ \sigma_1 \sigma_2 &= i\sigma_3, \quad \sigma_2 \sigma_3 = i\sigma_1, \quad \sigma_3 \sigma_1 = i\sigma_2. \end{aligned}$$

Applying formally the Fourier transformation w.r.t. $q \in \mathbb{R}^3$ to (1.1), we get

$$(1.2) \quad i\hbar \frac{\partial}{\partial t} \hat{\psi}(t, p) = \hat{\mathbb{H}} \hat{\psi}(t, p)$$

where

$$\hat{\mathbb{H}} = c\sigma_j p_j = c \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \quad \text{with} \quad \hat{\mathbb{H}}^2 = c^2 |p|^2 \mathbb{I}_2.$$

Therefore, we have

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

Proposition. For any $t \in \mathbb{R}$ and $\underline{\psi} \in L^2(\mathbb{R}^3 : \mathbb{C}^2)$,

$$(1.3) \quad e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} dp e^{i\hbar^{-1}qp} e^{-i\hbar^{-1}t\mathbb{H}}\hat{\underline{\psi}}(p).$$

Moreover, for $\underline{\psi} \in S(\mathbb{R}^3 : \mathbb{C}^2)$, we have

$$(1.4) \quad e^{-i\hbar^{-1}t\mathbb{H}}\underline{\psi}(q) = \mathbb{E} * \underline{\psi}(t, q) = \int_{\mathbb{R}^3} dq' \mathbb{E}(t, q - q')\underline{\psi}(q')$$

with

$$(1.5) \quad \mathbb{E}(t, q) = (2\pi\hbar)^{-3} \int_{\mathbb{R}^3} dp e^{i\hbar^{-1}qp} \left[\cos(c\hbar^{-1}t|p|)\mathbb{I}_2 - i \frac{\sin(c\hbar^{-1}t|p|)}{c|p|} \mathbb{H} \right] \in \mathcal{S}'(\mathbb{R}^3 : \mathbb{C}^2).$$

Pauli claimed one day that there exist no classical counter-part for a quantum spinning particle. In spite of this saying, we claim another representation which exhibit the underlying ‘‘Classical Mechanics’’:

Theorem (Path-integral representation of a solution for the Weyl equation).

$$(1.6) \quad \psi(t, q) = \left. \left((2\pi\hbar)^{-3/2} \int_{\mathfrak{R}^{3|2}} d\underline{\xi} d\underline{\pi} \mathcal{D}^{1/2}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}) e^{i\hbar^{-1}S(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})} \mathcal{F}(\#\underline{\psi})(\underline{\xi}, \underline{\pi}) \right) \right|_{\bar{x}_B=q}.$$

Here, $S(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})$ and $\mathcal{D}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})$ are solutions of Hamilton-Jacobi and continuity equations, respectively. \mathcal{F} is the Fourier transformation of functions on $\mathfrak{R}^{3|2}$.

Remark. Unfamiliar notations, such as $\#, \flat, \theta, \pi, x_B, \mathfrak{R}^{3|2}$, will be explained in §3.

§2. GENERAL BACKGROUND

(i) **Derivation of the Weyl equation.**

The Einstein’s relation

$$(2.1) \quad E^2 = c^2|p|^2 + m^2c^4, \quad |p|^2 = p_1^2 + p_2^2 + p_3^2$$

is well known as the energy for a free particle with mass m moving with momentum $p = (p_1, p_2, p_3)$.

The rest energy $E = mc^2$ gives the theoretical foundation of Atomic Energy.

By formal substitution

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad p_j \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial q_j},$$

we have the Klein-Gordon equation

$$(2.2) \quad \hbar^2 \left(\frac{\partial^2}{\partial t^2} - c^2 \Delta \right) u(t, q) + m^2 c^4 u(t, q) = 0.$$

But to apply the Copenhagen interpretation for Quantum Mechanics, we should quantize the Einstein relation so as to obtain a local operator having the first order derivative in time; so explained by many physicists.

To do this, Dirac introduced his matrices to have

$$(2.3) \quad E = \alpha_j p_j + mc^2 \beta$$

and by the substitution above, he derives the Dirac equation

$$(2.4) \quad i\hbar \frac{\partial}{\partial t} \psi(t, q) = \alpha \alpha_j \frac{\partial}{\partial q_j} \psi(t, q) + mc^2 \beta \psi(t, q)$$

Here, $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q), \psi_3(t, q), \psi_4(t, q))$, the summation w.r.t. $j = 1, 2, 3$ is abbreviated, and matrices $\{\alpha_k, \beta\}$ satisfy the Clifford relation:

$$\begin{aligned} \alpha_j \alpha_k + \alpha_k \alpha_j &= 2\delta_{jk} \mathbb{I}_4, \quad j, k = 1, 2, 3, \\ \alpha_k \beta + \beta \alpha_k &= 0, \quad k = 1, 2, 3, \quad \beta^2 = \mathbb{I}_4. \end{aligned}$$

In the following, we use the Dirac representation of matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}.$$

On the other hand, when $m = 0$, Weyl proposed the relation

$$(2.5) \quad E = c \sigma_j p_j$$

with

$$(2.6) \quad i\hbar \frac{\partial}{\partial t} \psi(t, q) = c \sigma_j \frac{\partial}{\partial q_j} \psi(t, q)$$

where $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q))$. But this equation doesn't recognized meaningful until 1956 when Lee-Yang demonstrated the non-conservation of parity under "weak-interaction". Today, "neutrino" is considered to be governed by the above equation, called Weyl equation.

(ii) Feynman's notorious measure.

It is believed to be a nice approximation to use Schrödinger equations for describing the kinematics of non-relativistic "quantum particles". Therefore, one has the desire to solve the following initial value problem:

$$(2.7) \quad \begin{cases} i\hbar \frac{\partial}{\partial t} u(t, q) = \frac{\hbar^2}{2m} \Delta u(t, q) + V(q)u(t, q), \\ u(0, q) = \phi(q). \end{cases}$$

Here, the Hamiltonian is given formally as

$$H = \frac{\hbar^2}{2m} \Delta + V(\cdot) = H_0 + V(\cdot), \quad \Delta = \sum_{j=1}^m \frac{\partial^2}{\partial q_j^2}$$

and m =the mass of the particle.

Assuming suitable conditions on the potential V , H above defines a selfadjoint operator in $L^2(\mathbb{R}^m : \mathbb{C})$ and the solution of (2.7) is written by

$$u(t, q) = (e^{-\frac{i}{\hbar} t H} \phi)(q).$$

On the other hand, by applying formally the Lie-Kato-Trotter formula, we have

$$e^{-\frac{i}{\hbar} t H} = s - \lim_{k \rightarrow \infty} \left(e^{-\frac{i}{\hbar} \frac{t}{k} V} e^{-\frac{i}{\hbar} \frac{t}{k} H_0} \right)^k.$$

If the initial data ϕ belongs to $\mathcal{S}(\mathbb{R}^m : \mathbb{C})$, we get

$$(2.8) \quad (e^{-\frac{i}{\hbar} t H_0} \phi)(q) = \left(2\pi i \frac{\hbar t}{m}\right)^{-m/2} \int_{\mathbb{R}^m} dq' e^{im(q-q')^2/2\hbar t} \phi(q').$$

Therefore, (abusing $\{q_j\}_{j=0}^{k-1}$ as $q_j \in \mathbb{R}^m$)

$$(2.9) \quad \begin{aligned} (e^{-\frac{i}{\hbar} t H} \phi)(q) &= s\text{-}\lim_{k \rightarrow \infty} \left(2\pi i \frac{\hbar t}{m}\right)^{-km/2} \int \dots \int e^{\frac{i}{\hbar} S_t(q_k, \dots, q_0)} \phi(q_0) dq_0 \dots dq_{k-1} \\ &= \int dq' F(t, q, q') \phi(q'). \end{aligned}$$

Here, we put $q_k = q$, $q_0 = q'$,

$$S_t(q_k, \dots, q_0) = \sum_{j=1}^k \left[\frac{m}{2} \frac{(q_j - q_{j-1})^2}{(t/k)^2} - V(q_j) \right] \frac{t}{k},$$

and

$$(2.10) \quad F(t, q, q') = s\text{-}\lim_{k \rightarrow \infty} \left(2\pi i \frac{\hbar t}{m}\right)^{-km/2} \int \dots \int dq_1 \dots dq_{k-1} e^{\frac{i}{\hbar} S_t(q, q_{k-1}, \dots, q_1, q')}.$$

Feynman interpreted this as follows: Consider a path space defined by

$$C_{t, q, q'} = \{\gamma(\cdot) \in AC([0, t] : \mathbb{R}^m) \mid \gamma(0) = q', \gamma(t) = q\}.$$

Here AC stands for absolutely continuous. For any path $\gamma \in C_{t, q, q'}$, putting $\gamma(\tau_j) = q_j$ with $\tau_j = jt/k$, he regarded $S_t(q_k, \dots, q_0)$ as the Riemann sum for the classical action $S_t(\gamma)$ for $\gamma \in C_{t, q, q'}$, i.e.

$$S_t(\gamma) = \int_0^t d\tau L(\gamma(\tau), \dot{\gamma}(\tau)) = \lim_{k \rightarrow \infty} S_t(q_k, \dots, q_0),$$

where

$$L(\gamma, \dot{\gamma}) = \frac{M}{2} \dot{\gamma}^2 - V(\gamma) \in C^\infty(T\mathbb{R}^m : \mathbb{C})$$

When $k \rightarrow \infty$, the 'limit' of the measure $dq_1 \dots dq_{k-1}$ in (2.10) is denoted by

$$d_F \gamma = \prod_{0 < \tau < t} d\gamma(\tau)$$

and considred as the 'measure' on the path space $C_{t, q, q'}$. Using these, he concluded that we may represent (2.10) as

$$(2.11) \quad F(t, q, q') = \int_{C_{t, q, q'}} d_F \gamma e^{\frac{i}{\hbar} \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau)) d\tau}.$$

On the other hand, it is proved unfortunately that there exists no non-trivial 'Feynman measure' on ∞ -dimensional spaces. Therefore, one of our main concern is how to 'justify' the results obtained by using such a notorious measure.

Why it is necessary to do so? Because, even if the usage of the Feynman measure is prohibited in mathematics, we get new insights in "quantum area" by 'using' it, for example, works done by E. Witten and other physicists.

(iii) Fujiwara's procedure.

Fujiwara [7] proposed to construct a fundamental solution of the Schrödinger equation by using the quantities obtained from the corresponding classical mechanics. In some sense, the meaning of (2.11) is given when $L(\gamma, \dot{\gamma})$ is defined as before with $m = 1$, $V \in C^\infty(\mathbb{R}^m)$ satisfying

$$\sup_{x \in \mathbb{R}^m} |D^\alpha V(x)| \leq C_\alpha \quad \text{for any } |\alpha| \geq 2.$$

Then, there exists a unique path γ_0 in $C_{t,q,q'}$ such that

$$(2.12) \quad \inf_{\gamma \in C_{t,q,q'}} S_t(\gamma) = S_t(\gamma_0) = \phi_L(t, q, q').$$

From this, he put

$$(2.13) \quad \mu_L(t, q, q') = (2\pi)^{-m/2} \left(\det \left(\partial_{x_i} \partial_{y_j} \left(\frac{i}{\hbar} \phi_L(t, q, q') \right) \right) \right)^{\frac{1}{2}},$$

and he gave the meaning of the integral transformation

$$(2.14) \quad (F_t u)(q) = \int_{\mathbb{R}^m} dq' \mu_L(t, q, q') e^{\frac{i}{\hbar} \phi_L(t, q, q')} u(q').$$

Theorem (Fujiwara). *Let $T < \infty$ be arbitrary fixed.*

(1) *The operator F_t defines a bounded linear operator in $L^2(\mathbb{R}^m : \mathbb{C})$, i.e. there exists a constant C depending on T such that*

$$\|F_t u\| \leq C \|u\| \quad \text{for any } u \in L^2(\mathbb{R}^m : \mathbb{C}) \quad \text{and } t \in [-T, T].$$

(2) *It satisfies*

$$\begin{aligned} \lim_{t \rightarrow 0} \|F_t u - u\| &= 0 \quad \text{for any } u \in L^2(\mathbb{R}^m : \mathbb{C}), \\ i\hbar \frac{\partial}{\partial t} (F_t u)(q) \Big|_{t=0} &= H(q, D)u(q) \quad \text{where } H(q, D) = \frac{\hbar^2}{2m} \Delta + V(q), \\ \|F_{t+s} - F_t F_s\| &\leq C(t^2 + s^2) \quad \text{for } t, s, t+s \in [-T, T]. \end{aligned}$$

(3) *Moreover, $\lim_{k \rightarrow \infty} (F_{t/k})^k = E_t$ exists as the uniform limit of bounded operators in $L^2(\mathbb{R}^m : \mathbb{C})$ and it defines the fundamental solution of*

$$\begin{cases} i\hbar \frac{\partial}{\partial t} (E_t u)(q) = H(q, D)(E_t u)(q), \\ (E_0 u)(q) = u(q). \end{cases}$$

Remark. Above procedure of Fujiwara was used also by Inoue-Maeda [15] to explain mathematically the origin of the term $(1/12)R$, R = the scalar curvature of the configuration manifold, which appeared when one wants to "quantize" the Lagrangian on a curved manifold.

Problem. *In (2.11), a Lagrangian is used. How should one formulate the above procedure when a Hamiltonian is given?*

(iv) Feynman's problem for spin.

Feynman claimed a problem in p.355 of Feynman & Hibbs [6]:

..... path integrals suffer grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.

(v) The method of characteristics and Hamiltonian path-integral.

Let Ω be a domain of \mathbb{R}^{m+1} . We consider the following initial value problem:

$$(2.15) \quad \begin{cases} \frac{\partial}{\partial t} u(t, q) + \sum_{j=1}^m a_j(t, q) \frac{\partial}{\partial q_j} u(t, q) = b(t, q)u(t, q) + f(t, q), \\ u(\underline{t}, q) = \underline{u}(q). \end{cases}$$

We solve

$$(2.16) \quad \begin{cases} \frac{d}{dt} q_j(t) = a_j(t, q(t)), \\ q_j(\underline{t}) = \underline{q}_j \quad \text{for } j = 1, \dots, m. \end{cases}$$

We denote a solution of above by

$$q(t) = q(t, \underline{t}; \underline{q}) = (q_1(t), \dots, q_m(t)) \in \mathbb{R}^m.$$

The following theorem is well-known:

Theorem. Let $a_j \in C^1(\Omega : \mathbb{R})$ and $b, f \in C(\Omega : \mathbb{R})$. For any fixed point $(\underline{t}, \underline{q}) \in \Omega$, we assume \underline{u} is C^1 near \underline{q} . Then, there exists uniquely a solution $u(t, q)$ near $(\underline{t}, \underline{q})$. Moreover, we put

$$(2.17) \quad U(t, \underline{q}) = e^{\int_{\underline{t}}^t dr B(r, \underline{q})} \left\{ \int_{\underline{t}}^t ds e^{-\int_{\underline{t}}^s dr B(r, \underline{q})} F(s, \underline{q}) + \underline{u}(\underline{q}) \right\}$$

where $B(t, \underline{q}) = b(t, q(t, \underline{t}; \underline{q}))$ and $F(t, \underline{q}) = f(t, q(t, \underline{t}; \underline{q}))$. Then, the solution of (2.15) is given by

$$u(t, \bar{q}) = U(t, x(t, \bar{q}))$$

where $\underline{q} = x(t, \underline{t}; \bar{q})$ is the inverse function of $\bar{q} = q(t, \underline{t}; \underline{q})$.

Remark. $U(t) = U(t, \underline{q}) = u(t, q(t))$ satisfies

$$\frac{dU(t)}{dt} = B(t)U(t) + F(t) \quad \text{where } B(t) = B(t, \underline{q}) \text{ and } F(t) = F(t, \underline{q})$$

near $t = \underline{t}$.

In order to consider more concretely, we take an

Example.

$$\begin{cases} i\hbar \frac{\partial}{\partial t} u(t, q) = a \frac{\hbar}{i} \frac{\partial}{\partial q} u(t, q) + bqu(t, q), \\ u(0, q) = \underline{u}(q). \end{cases}$$

From the right-hand side of above, we have a symbol

$$H(q, p) = e^{-i\hbar^{-1}qp} \left(a \frac{\hbar}{i} \frac{\partial}{\partial q} + bq \right) e^{i\hbar^{-1}qp} = ap + bq.$$

Therefore,

$$\begin{cases} \dot{q}(t) = a, \\ \dot{p}(t) = -b \end{cases} \quad \text{with} \quad \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix},$$

which are solved as

$$q(s) = \underline{q} + as, \quad p(s) = \underline{p} - bs \quad \text{with} \quad \underline{q} = x(t, 0; \bar{q}) = x(t, \bar{q}) = \bar{q} - at.$$

From (2.17), we have

$$U(t, \underline{q}) = \underline{u}(\underline{q}) e^{-i\hbar^{-1}(b\bar{q}t + 2^{-1}abt^2)}.$$

Therefore,

$$u(t, \bar{q}) = \underline{u}(\bar{q} - at) e^{-i\hbar^{-1}(b\bar{q}t - 2^{-1}abt^2)}.$$

On the other hand, we put

$$S_0(t, \underline{q}, \underline{p}) = \int_0^t ds [\dot{q}(s)p(s) - H(q(s), p(s))] = -b\bar{q}t - 2^{-1}abt^2.$$

Defining

$$S(t, \bar{q}, \underline{p}) = \underline{q}\underline{p} + S_0(t, \underline{q}, \underline{p}) \Big|_{\underline{q}=x(t, \bar{q})},$$

we have

$$S(t, \bar{q}, \underline{p}) = \bar{q}\underline{p} - a\underline{p}t - b\bar{q}t + 2^{-1}abt^2.$$

As $\partial^2 S(t, \bar{q}, \underline{p}) / \partial \bar{q} \partial \underline{p} = 1$, we have

$$\begin{aligned} u(t, \bar{q}) &= (2\pi\hbar)^{-1/2} \int d\underline{p} e^{i\hbar^{-1}S(t, \bar{q}, \underline{p})} \hat{u}(\underline{p}) \\ &= (2\pi\hbar)^{-1} \iint d\underline{p} d\underline{q} e^{i\hbar^{-1}(S(t, \bar{q}, \underline{p}) - \underline{q}\underline{p})} \underline{u}(\underline{q}) = \underline{u}(\bar{q} - at) e^{i\hbar^{-1}(-b\bar{q}t + 2^{-1}abt^2)}. \end{aligned}$$

Remark. Above procedure answers the problem posed in (iii) of §2.

Problem. Can we extend the procedure above to systems of PDE (see, Problem 8 of Gelfand [8])?

§3. FUNDAMENTALS OF SUPERANALYSIS

For symbols $\{\sigma_j\}_{j=1}^\infty$ satisfying the Grassmann relation

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 0, \quad j, k = 1, 2, \dots,$$

we put

$$\mathfrak{C} = \{X = \sum_{I \in \mathcal{I}} X_I \sigma^I \mid X_I \in \mathbb{C}\}$$

where

$$\begin{aligned} \mathcal{I} &= \{I = (i_k) \in \{0, 1\}^{\mathbb{N}} \mid |I| = \sum_k i_k < \infty\}, \\ \sigma^I &= \sigma_1^{i_1} \sigma_2^{i_2} \cdots, \quad I = (i_1, i_2, \dots), \quad \sigma^{\bar{0}} = 1, \quad \bar{0} = (0, 0, \dots) \in \mathcal{I}. \end{aligned}$$

WEYL EQUATION

Besides trivially defined linear operations of sums and scalar multiplications, we have a product operation in \mathfrak{C} : For

$$X = \sum_{J \in \mathcal{J}} X_J \sigma^J, \quad Y = \sum_{K \in \mathcal{I}} Y_K \sigma^K,$$

we put

$$XY = \sum_{I \in \mathcal{I}} (XY)_I \sigma^I \quad \text{with} \quad (XY)_I = \sum_{I=J+K} (-1)^{\tau(I;J,K)} X_J Y_K.$$

Here, $\tau(I; J, K)$ is an integer defined by

$$\sigma^J \sigma^K = (-1)^{\tau(I;J,K)} \sigma^I, \quad I = J + K.$$

Proposition. \mathfrak{C} forms a ∞ -dimensional Fréchet-Grassmann algebra over \mathbb{C} , that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.

Remark. (1) Degree in \mathfrak{C} is defined by introducing subspaces

$$\mathfrak{C}_{[j]} = \{X = \sum_{I \in \mathcal{I}, |I|=j} X_I \sigma^I\} \quad \text{for } j = 0, 1, \dots$$

which satisfy

$$\mathfrak{C} = \bigoplus_{j=0}^{\infty} \mathfrak{C}_{[j]}, \quad \mathfrak{C}_{[j]} \cdot \mathfrak{C}_{[k]} \subset \mathfrak{C}_{[j+k]}.$$

(2) Define

$$\text{proj}_I(X) = X_I \quad \text{for } X = \sum_{I \in \mathcal{I}} X_I \sigma^I \in \mathfrak{C}.$$

The topology in \mathfrak{C} is given by; $X \rightarrow 0$ in \mathfrak{C} iff for any $I \in \mathcal{I}$, $\text{proj}_I(X) \rightarrow 0$ in \mathbb{C} .

This topology is equivalent to the one introduced by the metric $\text{dist}(X, Y) = \text{dist}(X - Y)$ where $\text{dist}(X)$ is defined by

$$\text{dist}(X) = \sum_{I \in \mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\text{proj}_I(X)|}{1 + |\text{proj}_I(X)|} \quad \text{with} \quad r(I) = 1 + \frac{1}{2} \sum_{k=1}^{\infty} 2^k i_k \quad \text{for } I \in \mathcal{I}.$$

(3) We introduce parity in \mathfrak{C} by setting

$$p(X) = \begin{cases} 0 & \text{if } X = \sum_{I \in \mathcal{I}, |I|=\text{ev}} X_I \sigma^I, \\ 1 & \text{if } X = \sum_{I \in \mathcal{I}, |I|=\text{od}} X_I \sigma^I, \\ \text{undefined} & \text{if otherwise.} \end{cases}$$

We put

$$\begin{cases} \mathfrak{C}_{\text{ev}} = \bigoplus_{j=0}^{\infty} \mathfrak{C}_{[2j]} = \{X \in \mathfrak{C} \mid p(X) = 0\}, \\ \mathfrak{C}_{\text{od}} = \bigoplus_{j=0}^{\infty} \mathfrak{C}_{[2j+1]} = \{X \in \mathfrak{C} \mid p(X) = 1\}, \\ \mathfrak{C} \cong \mathfrak{C}_{\text{ev}} \oplus \mathfrak{C}_{\text{od}} \cong \mathfrak{C}_{\text{ev}} \times \mathfrak{C}_{\text{od}}. \end{cases}$$

Analogous to \mathfrak{C} , we define

$$\begin{cases} \mathfrak{R} = \{X \in \mathfrak{C} \mid \pi_B X \in \mathbb{R}\}, \quad \mathfrak{R}_{[j]} = \mathfrak{R} \cap \mathfrak{C}_{[j]}, \\ \mathfrak{R}_{\text{ev}} = \mathfrak{R} \cap \mathfrak{C}_{\text{ev}}, \quad \mathfrak{R}_{\text{od}} = \mathfrak{R} \cap \mathfrak{C}_{\text{od}} = \mathfrak{C}_{\text{od}}, \\ \mathfrak{R} \cong \mathfrak{R}_{\text{ev}} \oplus \mathfrak{R}_{\text{od}} \cong \mathfrak{R}_{\text{ev}} \times \mathfrak{R}_{\text{od}}. \end{cases}$$

We introduced the body (projection) map π_B by

$$\pi_B X = \text{proj}_{\bar{0}}(X) = X_{\bar{0}} = X_B \quad \text{for any } X \in \mathfrak{C}.$$

We define the (real) superspace $\mathfrak{R}^{m|n}$ by

$$\mathfrak{R}^{m|n} = \mathfrak{R}_{ev}^m \times \mathfrak{R}_{od}^n.$$

The metric between $X, Y \in \mathfrak{R}^{m|n}$ is defined by,

$$\text{dist}_{m|n}(X, Y) = \text{dist}_{m|n}(X - Y)$$

with

$$\text{dist}_{m|n}(X) = \sum_{j=1}^m \left(\sum_{I \in \mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\text{proj}_I(x_j)|}{1 + |\text{proj}_I(x_j)|} \right) + \sum_{k=1}^n \left(\sum_{I \in \mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\text{proj}_I(\theta_k)|}{1 + |\text{proj}_I(\theta_k)|} \right).$$

We use the following notations:

$$\begin{aligned} X &= (X_A)_{A=1}^{m+n} = (x, \theta) \in \mathfrak{R}^{m|n} \quad \text{with} \\ x &= (X_A)_{A=1}^m = (x_j)_{j=1}^m \in \mathfrak{R}^{m|0}, \quad \theta = (X_A)_{A=m+1}^{m+n} = (\theta_k)_{k=1}^n \in \mathfrak{R}^{0|n}. \end{aligned}$$

We generalize the body map π_B from $\mathfrak{R}^{m|n}$ or $\mathfrak{R}^{m|0}$ to \mathbb{R}^m by putting,

$$X = (x, \theta) \in \mathfrak{R}^{m|n} \longrightarrow \pi_B X = X_B = (x_B, 0) \cong x_B = \pi_B x = (\pi_B x_1, \dots, \pi_B x_m) \in \mathbb{R}^m.$$

We call $x_j \in \mathfrak{R}_{ev}$ and $\theta_k \in \mathfrak{R}_{od}$ as even and odd (alias bosonic and fermionic) variable, respectively.

Supersmooth functions: For $u_a(q) \in C^\infty(\mathbb{R}^m : \mathbb{C})$, we put,

$$u_a(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_q^\alpha u_a(x_B) x_S^\alpha \quad \text{for } x = x_B + x_S$$

which is called the Grassmann continuation of $u_a(q)$. We define a function $u \in \mathcal{C}_{SS}(\mathfrak{R}^{m|n})$ by

$$u(X) = u(x, \theta) = \sum_{|a| \leq n} u_a(x) \theta^a,$$

called a supersmooth function on $\mathfrak{R}^{m|n}$.

Derivations: For a given supersmooth function $u(X)$ on $\mathfrak{R}^{m|n}$, we define its derivatives as follows: For $j = 1, 2, \dots, m$ and $s = 1, 2, \dots, n$, we put

$$\begin{cases} U_j(X) = \sum_{|a| \leq n} \partial_{x_j} u_a(x) \theta^a, \\ U_{s+m}(X) = \sum_{|a| \leq n} (-1)^{l_s(a)} u_a(x) \theta_1^{a_1} \dots \theta_s^{a_s-1} \dots \theta_n^{a_n} \end{cases}$$

where $l_s(a) = \sum_{j=1}^{s-1} a_j$ and $\theta_s^{-1} = 0$. $U_\kappa(X)$ are called the partial derivatives of u with respect to X_κ at $X = (x, \theta)$ and are denoted by

$$\begin{cases} U_j(X) = \frac{\partial}{\partial x_j} u(x, \theta) = \partial_{x_j} u(x, \theta) \quad \text{for } j = 1, 2, \dots, m, \\ U_{m+s}(X) = \frac{\partial}{\partial \theta_s} u(x, \theta) = \partial_{\theta_s} u(x, \theta) \quad \text{for } s = 1, 2, \dots, n \end{cases}$$

or simply by

$$U_\kappa(X) = \partial_{X_\kappa} u(X) \quad \text{for } \kappa = 1, \dots, m+n.$$

For

$$\begin{aligned} \mathbf{a} &= (\alpha, a), \quad \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m, \quad a = (a_1, \dots, a_n) \in \{0, 1\}^n, \\ |\alpha| &= \sum_{j=1}^m \alpha_j, \quad |a| = \sum_{k=1}^n a_k, \quad |\mathbf{a}| = |\alpha| + |a|, \end{aligned}$$

we put

$$\partial_X^{\mathbf{a}} = \partial_x^\alpha \partial_\theta^a \quad \text{with} \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_m}^{\alpha_m}, \quad \partial_\theta^a = \partial_{\theta_1}^{a_1} \dots \partial_{\theta_n}^{a_n}.$$

Example. $\partial_{\theta_2} \theta_1 \theta_2 \theta_3 = -\theta_1 \theta_3$, $\partial_{\theta_1} \partial_{\theta_3} \theta_1 \theta_2 \theta_3 = \theta_2$, etc.

Integration: We define

$$\begin{aligned} \int_{\mathfrak{R}^{m|n}} dx d\theta u(x, \theta) &= \int_{\mathfrak{R}^{m|0}} dx \left\{ \int_{\mathfrak{R}^{0|n}} d\theta u(x, \theta) \right\} \\ &= \int_{\mathbb{R}^m} dX_B (\partial_{\theta_n} \dots \partial_{\theta_1} u)(X_B) \quad (\pi_B(\mathfrak{R}^{m|0}) = \mathbb{R}^m) \\ &= \int_{\mathfrak{R}^{0|n}} d\theta \left\{ \int_{\mathfrak{R}^{m|0}} dx u(x, \theta) \right\} = \int_{\mathfrak{R}^{m|n}} d\theta dx u(x, \theta). \end{aligned}$$

Especially for odd integration, we have the following curious looking but well-known relations

$$\int_{\mathfrak{R}^{0|n}} d\theta_n \dots d\theta_1 \theta_1 \dots \theta_n = 1 \quad \text{and} \quad \int_{\mathfrak{R}^{0|n}} d\theta_n \dots d\theta_1 1 = 0 \quad (\text{Berezin integral}).$$

Remarks for the need of ∞ number of Grassmann generators.

(i) Though, \mathfrak{C} doesn't form a field because $X^2 = 0$ for any $X \in \mathfrak{C}_{0d}$, but if $X, Y \in \mathfrak{C}$ satisfy $XY = 0$ for any $Y \in \mathfrak{C}_{0d}$, then, $X = 0$. This property holds only when the number of generators is infinite. By this, we may determine the derivative $\partial_X^{\mathbf{a}} u(X)$ uniquely.

(ii) If the number of Grassmann generators is finite, then the effect of odd variables may vanish after countable operations.

Scalar products and norms:

$$\begin{aligned} (u, v) &= \int_{\mathfrak{R}^{m|n}} dx d\theta d\bar{\theta} e^{(\bar{\theta}|\theta)} \overline{u(x, \theta)} v(x, \theta) = \sum_{|a| \leq n} \int_{\mathfrak{R}^{m|0}} dx \overline{u_a(x)} v_a(x), \\ ((u, v))_k &= \sum_{|a| \leq k} (\partial_X^{\mathbf{a}} u, \partial_X^{\mathbf{a}} v) = \sum_{|a| + |a| \leq k} (\partial_x^\alpha u_a, \partial_x^\alpha v_a), \\ (((u, v)))_k &= \sum_{|a| + |l| \leq k} ((1 + |X_B|^2)^{l/2} \partial_X^{\mathbf{a}} u, (1 + |X_B|^2)^{l/2} \partial_X^{\mathbf{a}} v) \end{aligned}$$

with

$$\|u\|^2 = (u, u), \quad \|u\|_k^2 = ((u, u))_k, \quad \| \|u\| \|_k^2 = (((u, u)))_k.$$

Fourier transformations (of second kind):

$$\begin{aligned} (F_e v)(\xi) &= (2\pi\hbar)^{-m/2} \int_{\mathfrak{R}^{m|0}} dx e^{-i\hbar^{-1}\langle x|\xi\rangle} v(x), \\ (\bar{F}_e w)(x) &= (2\pi\hbar)^{-m/2} \int_{\mathfrak{R}^{m|0}} d\xi e^{i\hbar^{-1}\langle x|\xi\rangle} w(\xi), \\ (F_o v)(\pi) &= \hbar^{n/2} \int_{\mathfrak{R}^{0|n}} d\theta e^{-i\hbar^{-1}\langle \theta|\pi\rangle} v(\theta), \\ (\bar{F}_o w)(\theta) &= \hbar^{n/2} \int_{\mathfrak{R}^{0|n}} d\pi e^{i\hbar^{-1}\langle \theta|\pi\rangle} w(\pi) \end{aligned}$$

where

$$\langle \eta | y \rangle = \sum_{j=1}^m \eta_j y_j, \quad \langle \rho | \omega \rangle = \sum_{k=1}^n \rho_k \omega_k, \quad \iota_n = i^{n+n^2/2}.$$

We put

$$\begin{aligned} (\mathcal{F}u)(\xi, \pi) &= c_{m,n} \int_{\mathfrak{R}^{m|n}} dX e^{-i\hbar^{-1}\langle X | \Xi \rangle} u(X) = \sum_a [(F_e u_a)(\xi)] [(F_o \theta^a)(\pi)], \\ (\tilde{\mathcal{F}}v)(x, \theta) &= c_{m,n} \int_{\mathfrak{R}^{m|n}} d\Xi e^{i\hbar^{-1}\langle X | \Xi \rangle} v(\Xi) = \sum_a [(\bar{F}_e v_a)(x)] [(\bar{F}_o \pi^a)(\theta)] \end{aligned}$$

where

$$\langle X | \Xi \rangle = \langle x | \xi \rangle + \langle \theta | \pi \rangle \in \mathfrak{R}_{ev}, \quad c_{m,n} = (2\pi\hbar)^{-m/2} \hbar^{n/2} \iota_n.$$

Remark. Though the differential calculus on Fréchet spaces has some difficulties in general, such a calculus on Fréchet-Grassmann algebra holds safely in our case. For example, the implicit and inverse function theorem, and the chain rule of differentials are established as similar as the standard case.

§4. OUTLINE OF OUR PROCEDURE (1)–(6)

(1) We identify a “spinor” $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q)) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}^2$ with an even supersmooth function $u(t, x, \theta) = u_0(t, x) + u_1(t, x)\theta_1\theta_2 : \mathbb{R} \times \mathfrak{R}^{3|2} \rightarrow \mathfrak{C}_{ev}$. Here, $\mathfrak{R}^{3|2}$ is the superspace and $u_0(t, x)$, $u_1(t, x)$ are the Grassmann continuation of $\psi_1(t, q)$, $\psi_2(t, q)$, respectively.

(2) We define operators as

$$(4.1) \quad \begin{cases} \sigma_1\left(\theta, \frac{\partial}{\partial\theta}\right) = \theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2}, \\ \sigma_2\left(\theta, \frac{\partial}{\partial\theta}\right) = i\left(\theta_1\theta_2 + \frac{\partial^2}{\partial\theta_1\partial\theta_2}\right), \\ \sigma_3\left(\theta, \frac{\partial}{\partial\theta}\right) = 1 - \theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}. \end{cases}$$

Then, actions of those on $u(t, x, \theta)$ are identified with those $\{\sigma_j\}$ on vectors $\psi(t, q) = {}^t(\psi_1(t, q), \psi_2(t, q))$.

Example.

$$\left(\theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2}\right)(u_0 + u_1\theta_1\theta_2) = u_1 + u_0\theta_1\theta_2,$$

which maps

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \rightarrow \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

(3) Therefore, we may correspond the differential operator given by

$$(4.2) \quad \begin{aligned} \mathcal{H}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial\theta}\right) &= c\left(\theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2}\right)\frac{\hbar}{i}\frac{\partial}{\partial x_1} + ic\left(\theta_1\theta_2 + \frac{\partial^2}{\partial\theta_1\partial\theta_2}\right)\frac{\hbar}{i}\frac{\partial}{\partial x_2} \\ &\quad + c\left(1 - \theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}\right)\frac{\hbar}{i}\frac{\partial}{\partial x_3}, \end{aligned}$$

which yields the superspace version of the Weyl equation

$$(4.3) \quad \begin{cases} i\hbar\frac{\partial}{\partial t}u(t, x, \theta) = \mathcal{H}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial\theta}\right)u(t, x, \theta), \\ u(0, x, \theta) = u(x, \theta). \end{cases}$$

Moreover, the “complete Weyl symbol” of (4.2) (see, Appendix B of Inoue [13]) is given by

$$(4.4) \quad \begin{aligned} \mathcal{H}(\xi, \theta, \pi) &= c(\theta_1\theta_2 + \hbar^{-2}\pi_1\pi_2)\xi_1 + ic(\theta_1\theta_2 - \hbar^{-2}\pi_1\pi_2)\xi_2 - ic\hbar^{-1}(\theta_1\pi_1 + \theta_2\pi_2)\xi_3 \\ &= c(\xi_1 + i\xi_2)\theta_1\theta_2 + c\hbar^{-2}(\xi_1 - i\xi_2)\pi_1\pi_2 - ic\hbar^{-1}\xi_3(\theta_1\pi_1 + \theta_2\pi_2). \end{aligned}$$

(4) We consider the classical mechanics corresponding to $\mathcal{H}(\xi, \theta, \pi)$ given by

$$(4.5) \quad \begin{cases} \frac{d}{dt}x_j = \frac{\partial\mathcal{H}(\xi, \theta, \pi)}{\partial\xi_j}, & \frac{d}{dt}\xi_k = -\frac{\partial\mathcal{H}(\xi, \theta, \pi)}{\partial x_k} = 0 \quad \text{for } j, k = 1, 2, 3, \\ \frac{d}{dt}\theta_l = -\frac{\partial\mathcal{H}(\xi, \theta, \pi)}{\partial\pi_l}, & \frac{d}{dt}\pi_m = -\frac{\partial\mathcal{H}(\xi, \theta, \pi)}{\partial\theta_m} \quad \text{for } l, m = 1, 2, 3. \end{cases}$$

Proposition (existence). *There exists a unique global solution $(x(t), \xi(t), \theta(t), \pi(t))$ of (4.5) with any initial data $(x(0), \xi(0), \theta(0), \pi(0)) = (\underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}) \in \mathfrak{R}^{6|4} = \mathcal{T}^*\mathfrak{R}^{3|2}$.*

Remark. We also denote the above solution $x(t)$ by $x(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi})$, etc., if necessary.

Moreover, we have

Proposition (inverse). *For any fixed $(t, \underline{x}, \underline{\pi})$, the map defined by*

$$(\underline{x}, \underline{\theta}) \rightarrow (\bar{x} = x(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \bar{\theta} = \theta(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}))$$

gives a supersmooth diffeomorphism from $\mathfrak{R}^{3|2} \rightarrow \mathfrak{R}^{3|2}$. Therefore, there exists the inverse map given by

$$(\bar{x}, \bar{\theta}) \rightarrow (\underline{x} = y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\theta} = \omega(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})),$$

which satisfies

$$(4.6) \quad \begin{cases} \bar{x} = x(t, y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\xi}, \omega(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\pi}), & \bar{\theta} = \theta(t, y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\xi}, \omega(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}), \underline{\pi}), \\ \underline{x} = y(t, x(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \underline{\xi}, \theta(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \underline{\pi}), & \underline{\theta} = \omega(t, x(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \underline{\xi}, \theta(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}), \underline{\pi}). \end{cases}$$

We put

$$(4.7) \quad S_0(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}) = \int_0^t \{ \langle \dot{x}(s) | \xi(s) \rangle + \langle \dot{\theta}(s) | \pi(s) \rangle - \mathcal{H}(x(s), \xi(s), \theta(s), \pi(s)) \} ds,$$

and

$$(4.8) \quad S(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \langle \bar{x} | \underline{\xi} \rangle + \langle \bar{\theta} | \underline{\pi} \rangle + S_0(t, \underline{x}, \underline{\xi}, \underline{\theta}, \underline{\pi}) \Big|_{\substack{\underline{x}=y(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) \\ \underline{\theta}=\omega(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})}}$$

Proposition (Hamilton-Jacobi equation). $S(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$ is given by

$$(4.9) \quad \begin{aligned} S(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) &= \langle \bar{x} | \underline{\xi} \rangle + [|\underline{\xi}| \cos(c\hbar^{-1}t|\underline{\xi}|) - i\underline{\xi}_3 \sin(c\hbar^{-1}t|\underline{\xi}|)]^{-1} [|\underline{\xi}| \langle \bar{\theta} | \underline{\pi} \rangle \\ &\quad - i \sin(c\hbar^{-1}t|\underline{\xi}|) (\underline{\xi}_1 + i\underline{\xi}_2) \bar{\theta}_1 \bar{\theta}_2 + i \sin(c\hbar^{-1}t|\underline{\xi}|) (\underline{\xi}_1 - i\underline{\xi}_2) \underline{\pi}_1 \underline{\pi}_2]. \end{aligned}$$

Moreover, it satisfies the following Hamilton-Jacobi equation:

$$(4.10) \quad \begin{cases} \frac{\partial}{\partial t} S(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) + \mathcal{H}\left(\frac{\partial S}{\partial \bar{x}}, \bar{\theta}, \frac{\partial S}{\partial \bar{\theta}}\right) = 0, \\ S(0, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \langle \bar{x} | \underline{\xi} \rangle + \langle \bar{\theta} | \underline{\pi} \rangle. \end{cases}$$

Now, we put

$$(4.11) \quad \mathcal{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = \text{sdet} \begin{pmatrix} \frac{\partial^2 S}{\partial \bar{x} \partial \underline{\xi}} & \frac{\partial^2 S}{\partial \bar{x} \partial \underline{\pi}} \\ \frac{\partial^2 S}{\partial \bar{\theta} \partial \underline{\xi}} & \frac{\partial^2 S}{\partial \bar{\theta} \partial \underline{\pi}} \end{pmatrix}.$$

Then, we get

Proposition (continuity equation).

$$\mathcal{D}(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = |\underline{\xi}|^{-2} [|\underline{\xi}| \cos(c\hbar^{-1}t|\underline{\xi}|) - i\underline{\xi}_3 \sin(c\hbar^{-1}t|\underline{\xi}|)]^2.$$

It satisfies the following continuity equation:

$$(4.12) \quad \begin{cases} \frac{\partial}{\partial t} \mathcal{D} + \frac{\partial}{\partial \bar{x}} \left(\mathcal{D} \frac{\partial \mathcal{H}}{\partial \underline{\xi}} \right) + \frac{\partial}{\partial \bar{\theta}} \left(\mathcal{D} \frac{\partial \mathcal{H}}{\partial \underline{\pi}} \right) = 0, \\ \mathcal{D}(0, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}) = 1. \end{cases}$$

In the above, the argument of \mathcal{D} is $(t, \bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi})$, those of $\frac{\partial \mathcal{H}}{\partial \underline{\xi}}$ and $\frac{\partial \mathcal{H}}{\partial \underline{\pi}}$ are $(\frac{\partial \mathcal{S}}{\partial \bar{x}}, \bar{\theta}, \frac{\partial \mathcal{S}}{\partial \bar{\theta}})$, respectively.

From here, we change the order of variables $\bar{x}, \underline{\xi}, \bar{\theta}, \underline{\pi}$ to $\bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}$ (this change corresponds to the process from classical to quantum).

We define an operator

$$(4.13) \quad (\mathcal{U}(t)u)(\bar{x}, \bar{\theta}) = (2\pi\hbar)^{-3/2} \hbar \iint d\underline{\xi} d\underline{\pi} \mathcal{D}^{1/2}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi}) e^{i\hbar^{-1}\mathcal{S}(t, \bar{x}, \bar{\theta}, \underline{\xi}, \underline{\pi})} \mathcal{F}u(\underline{\xi}, \underline{\pi}).$$

The function $u(t, \bar{x}, \bar{\theta}) = (\mathcal{U}(t)\underline{u})(\bar{x}, \bar{\theta})$ will be shown as a desired solution for (4.3).

(5) On the other hand, using Fourier transformation, we have readily that

$$\mathcal{H}\left(\frac{\hbar}{i} \frac{\partial}{\partial x}, \theta, \frac{\partial}{\partial \theta}\right) = \hat{\mathcal{H}}$$

where $\hat{\mathcal{H}}$ is a (Weyl type) pseudo-differential operator with symbol $\mathcal{H}(\xi, \theta, \pi)$, that is,

$$(\hat{\mathcal{H}}u)(x, \theta) = (2\pi\hbar)^{-3} \hbar^2 \iint d\xi d\pi dy d\omega e^{i\hbar^{-1}\langle x-y|\xi\rangle + i\hbar^{-1}(\theta-\omega|\pi)} \mathcal{H}\left(\xi, \frac{\theta+\omega}{2}, \pi\right) u(y, \omega).$$

Theorem. (1) For $t \in \mathbb{R}$, $\mathcal{U}(t)$ is well defined unitary operator in $\mathcal{L}_{SS}^2(\mathfrak{A}^{3|2})$.

(2) (i) $\mathbb{R} \ni t \rightarrow \mathcal{U}(t) \in \mathbb{B}(\mathcal{L}_{SS}^2(\mathfrak{A}^{3|2}), \mathcal{L}_{SS}^2(\mathfrak{A}^{3|2}))$ is continuous.

(ii) $\mathcal{U}(t)\mathcal{U}(s) = \mathcal{U}(t+s)$ for any $t, s \in \mathbb{R}$.

(iii) For $\underline{u} \in \mathcal{L}_{SS,0}(\mathfrak{A}^{3|2})$, we put $u(t, \bar{x}, \bar{\theta}) = (\mathcal{U}(t)\underline{u})(\bar{x}, \bar{\theta})$. Then, it satisfies

$$\begin{cases} i\hbar \frac{\partial}{\partial t} u(t, \bar{x}, \bar{\theta}) = \hat{\mathcal{H}}u(t, \bar{x}, \bar{\theta}), \\ u(0, \bar{x}, \bar{\theta}) = \underline{u}(\bar{x}, \bar{\theta}). \end{cases}$$

(6) We interpret the above theorem using the identification maps

$$\# : L^2(\mathbb{R}^3 : \mathbb{C}^2) \rightarrow \mathcal{L}_{SS}^2(\mathfrak{A}^{3|2}) \quad \text{and} \quad \flat : \mathcal{L}_{SS}^2(\mathfrak{A}^{3|2}) \rightarrow L^2(\mathbb{R}^3 : \mathbb{C}^2).$$

That is, remarking $\flat\hat{\mathcal{H}}\#\psi = \mathbb{H}\psi$ and putting $\mathbb{U}(t)\psi = \flat\mathcal{U}(t)\#\psi$, we have

Theorem. (1) For $t \in \mathbb{R}$, $\mathbb{U}(t)$ is well defined unitary operator in $L^2(\mathbb{R}^3 : \mathbb{C}^2)$.

(2) (i) $\mathbb{R} \ni t \rightarrow \mathbb{U}(t) \in \mathbb{B}(L^2(\mathbb{R}^3 : \mathbb{C}^2), L^2(\mathbb{R}^3 : \mathbb{C}^2))$ is continuous.

(ii) $\mathbb{U}(t)\mathbb{U}(s) = \mathbb{U}(t+s)$ for any $t, s \in \mathbb{R}$.

(iii) Put $\hbar = i$. For $\underline{\psi} \in C_0^\infty(\mathbb{R}^3 : \mathbb{C}^2)$, we put $\psi(t, q) = \flat(\mathcal{U}(t)\#\underline{\psi})|_{\bar{x}_B=q}$. Then, it satisfies

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \psi(t, q) = \mathbb{H}\psi(t, q), \\ \psi(0, q) = \underline{\psi}(q). \end{cases}$$

A. INOUE

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On KdV Equation and Bore-like Data

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Abstract

We studied the global well-posedness of the initial value problem associated to the Korteweg-de Vries (KdV) equation with bore-like initial data g (see [1, 2] and (1.2) below). Supposing $g' \in H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2) dx)$ we show the existence of a unique solution, says u , for the initial value problem such that $u - g$ belongs to this weighted Sobolev space for any time.

1. Introduction.

In [2] local and global well-posedness was established for the initial value problem (IVP)

$$\begin{cases} \partial_t u + \partial_x^3 u + u \partial_x u = 0, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = g(x) \end{cases} \quad (1.1)$$

with g satisfying

$$\begin{cases} i) g(x) \rightarrow C_{\pm} \text{ as } x \rightarrow \pm\infty, \\ ii) g' \in H^s, \\ iii) (g - C_+) \in L^2([0, \infty)) \text{ and } (g - C_-) \in L^2((-\infty, 0]) \end{cases} \quad (1.2)$$

where $s \geq 0$. More precisely, it was shown that there exists a unique local solution u satisfying that $u(x, t) - g(x) \in C([0, T] : H^s)$, $s > 3/2$. Moreover, this solution remains bounded for any $T > 0$ provided $s \geq 2$. In this note we are interested in the study of the IVP (1.1) in the weighted Sobolev space $H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2) dx)$.

Following the scheme used in [2] we can find a function $\psi \in C^\infty$ with $\psi' \in H^\infty$ satisfying (1.2) and such that $g - \psi \in H^{s+1}$, $s \geq 0$ (see Lemma 2.1 below). Then we define $u(x, t) = v(x, t) + \psi(x)$ and study the (IVP) associated to $v(x, t)$, namely,

$$\begin{cases} \partial_t v + \partial_x^3 v + v \partial_x v + \partial_x(v\psi) + (\psi\psi' + \psi^{(3)}) = 0, \\ v(x, 0) = g(x) - \psi(x) \equiv \phi(x). \end{cases} \quad (1.3)$$

where ϕ is a function in H^r , $r \geq 1$.

To study the IVP (1.3) in $H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2) dx)$ we shall also assume that $g' \in H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2) dx)$. This assumption combined with the construction of ψ gives us a function ϕ in the required space, i.e., $H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2) dx)$ (see Remark 2.1). At this point, we have the IVP (1.3) in the desired setting. In Theorem 3.1 below we will show the global existence of solutions of the IVP (1.3). From this we can deduce our main result concerning the IVP (1.1), that is,

Theorem 1.1. *Let $g' \in H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2) dx)$. Then for any $T > 0$ there exists a unique bounded solution u of the IVP (1.2) such that*

$$u(x, t) - g(x) \in C([0, T]; H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2) dx)).$$

This note is organized as follows. In Section 2, we present some preliminary results used in the proof of global existence of solutions for the IVP (1.3) which will be given in Section 3.

Notation

- We will write $L_1^2(\mathbb{R})$ to refer the space $L^2(\mathbb{R}; (1+x^2) dx)$.
- We denote by $(\cdot, \cdot)_s$ the inner product in H^s , $s \geq 0$ and the inner product in $H^s \cap L_1^2$ will be denoted by $(\cdot, \cdot)_{s,1} \equiv (\cdot, \cdot)_s + (\cdot, \cdot)_{L^2(1+x^2)}$ with $(\cdot, \cdot)_{0,1} \equiv (\cdot, \cdot)_{L^2(1+x^2)}$.
- The norm in H^s will be denoted by $\|\cdot\|_s$, $\|\cdot\|_0$ and $\|\cdot\|_{2r,r} \equiv \|\cdot\|_{2r} + \|(1+x^2)^{r/2} \cdot\|_0$ will denote the L^2 -norm and $H^{2r} \cap L_r^2$ -norm, $r \geq 1$, respectively.

2. Preliminaries.

In this section we give the statements of some results needed in the proof of global existence of solutions for the IVP (1.3). We begin by giving the construction of the function ψ commented in the introduction.

Lemma 2.1. *Let g satisfy conditions (i) and (ii) in (1.2). Then for each $\theta \in (0, \infty)$ there exists a $\psi_\theta \in C^\infty$ such that $\psi'_\theta \in H^\infty$, and $\phi_\theta = g - \psi_\theta \in H^{s+1}$, $s \geq 0$. Moreover,*

$$\begin{cases} \|g - \psi\|_0 \leq \sqrt{2\theta} e^{-1/2} \|g'\|_0 \\ \|\psi'_\theta\|_s \leq C \|g'\|_0, \quad C = C(s, \theta) \end{cases}$$

Finally, if g satisfies (iii) in (1.2) then ψ_θ also has this property and $\lim_{x \rightarrow \pm\infty} \psi_\theta = C_\pm$.

Proof. Define $\psi_\theta(x) = \frac{1}{\sqrt{4\pi\theta}} \exp(-|\cdot|^2/4\theta) * g(x)$ for $\theta \in (0, \infty)$. \square

Remark. Using Fourier transform and the decay properties of the heat kernel it is easy to see that $\phi(x) = g(x) - \psi_\theta(x) \in H^2 \cap L^2_1$ provided $g' \in H^2 \cap L^2_1$.

Next result concerns properties of solutions of the linear problem

$$\begin{cases} \partial_t u + \partial_x^3 u - \mu \partial_x^2 u = 0, & \mu > 0, x \in \mathbb{R}, t > 0 \\ u(x, 0) = f(x). \end{cases} \quad (2.1)$$

Lemma 2.2. Let $\mu \geq 0, \lambda \geq 0$ and $r = 0, 1, 2, \dots$ and define $W_\mu(t) = \exp\{(-\mu \partial_x^2 - i \partial_x^3)t\}$.

Then

$$\|W_\mu(t)f\|_{2r+\lambda, r} \leq K_\lambda \left(1 + \left(\frac{1}{2\mu t}\right)^\lambda\right)^{1/2} \|f\|_{2r, r}$$

for all $t, \mu \in (0, \infty)$, $\lambda \geq 0$ and $f \in H^{2r} \cap L^2_r$. The map $t \in (0, \infty) \rightarrow W_\mu(t)f$ is continuous with respect to the topology of $H^{2r+\lambda} \cap L^2_r$. Moreover, $W_\mu(t)$ defines a C^0 -semigroup in H^s , $s \in \mathbb{R}$, which can be extended to an unitary group if $\mu = 0$ and the map $t \in (0, \infty) \rightarrow W_\mu(t)f$, $\mu \geq 0$ is the unique solution of (2.1).

Proof. See [3]. \square

In the H^s theory for the IVP (1.3), it was established the following global estimate.

Lemma 2.3.

If v satisfies (3.1) with $\phi \in H^2$, then

$$\|v(t)\|_2^2 \leq \left\{ \|\phi\|_2^2 + \int_0^t G(\|\phi\|_2, s, \mu) ds \right\} \cdot \exp\{C(8 + \|\psi\|_{L^\infty})\|\psi'\|_{H^\infty}t\} \equiv F(t; \|\phi\|_2, \mu).$$

where G is a continuous and nondecreasing function in $\|\phi\|_2$.

Proof. See Lemma 4.3 in [2]. \square

3. Proof of main result

As we mentioned in the introduction in order to establish our main result we first show the following theorem concerning the global existence of solutions for IVP (1.3), that is,

Theorem 3.1. *Let $\phi \in H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2)dx)$. Then for any $T > 0$ there exists a unique solution of the IVP (1.3) such that*

$$v \in C([0, T]; H^2(\mathbb{R}) \cap L^2(\mathbb{R}; (1+x^2)dx)).$$

Sketch of the Proof. We first notice that uniqueness follows from the H^s theory since $H^2 \cap L^2_1 \subset H^2$. To show the existence of solution for the IVP (1.3) we use parabolic regularization, that is, we consider for $\mu > 0$ the problem

$$\begin{cases} \partial_t v + \partial_x^3 v + v \partial_x v + \partial_x(v\psi) + (\psi\psi' + \psi^{(3)}) - \mu \partial_x^2 v = 0, \\ v(x, 0) = g(x) - \psi(x) \equiv \phi. \end{cases} \quad (3.1)$$

and establish the existence and uniqueness of solutions for this IVP.

Using Lemma 2.2 it is easy to check that IVP (3.1) is equivalent to the integral equation

$$v_\mu(t) = W_\mu(t)\phi - \int_0^t W_\mu(t-t')(v\partial_x v(t') + \partial_x(\psi v(t')) + (\psi\psi' + \psi^{(3)})) dt'.$$

So combining Lemma 2.2, the construction of ψ and the fact that $H^2 \cap L^2_1$ is a Banach algebra allow us to conclude that the map

$$\Phi(v)(t) = W_\mu(t)\phi - \int_0^t W_\mu(t-t')(v\partial_x v(t') + \partial_x(\psi v(t')) + (\psi\psi' + \psi^{(3)})) dt'$$

is a contraction in the complete metric space

$$Z = \{v \in C([0, T] : H^2 \cap L^2_1) / \|v(t) - W_\mu(t)\phi\|_{2,1} \leq \|\phi\|_{2,1}, t \in [0, T]\}$$

if $T > 0$ is small enough. So this gives us local existence and uniqueness in $H^2 \cap L^2_1$, with $\mu > 0$.

Next we consider $v = v_\mu$ the local solution constructed above and look for a global estimate for the $H^2 \cap L^2_1$ -norm of v . Since we know from the H^s theory that $\|v(t)\|_2$ is globally estimated in terms of $\|\phi\|_2$, it only remains to estimate $\|v(t)\|_{0,1}$.

Let $\rho(x) = (1+x^2)^{1/2}$. Then using integration by parts, the boundedness of $\rho^{(j)}$, $j = 1, 2, 3$ and the construction of ψ we obtain

$$\begin{aligned} \partial_t \|v\|_{0,1}^2 &= 2(v, \partial_t v)_{0,1} = -2\mu \|\partial_x(\rho v)\|_0 + 2\mu \|v\|_0^2 - 2(\rho v, \rho v \partial_x v)_0 \\ &\quad - 2(\rho v, \partial_x^3(\rho v))_0 - (\rho v, [\rho, \partial_x^3]v)_0 \\ &\quad - 2(v, \partial_x(\psi v))_{0,1} - 2(\rho v, \rho \psi \psi')_0 - 2(\rho v, \rho \psi^{(3)})_0 \\ &\leq \|v\|_{0,1}^2 \|\partial_x v\|_0 + c(\rho) \|v\|_{0,1} \|v\|_2 + c\|\psi'\|_{H^\infty} \|v\|_{0,1} \|v\|_0 \\ &\quad + \|\psi'\|_{0,1} \|v\|_{0,1}^2 \|v\|_0 + \|v\|_{0,1}^2 \|v\|_0 + \|\psi^{(3)}\|_{0,1} \|v\|_{0,1}. \end{aligned} \quad (3.2)$$

An application of Sobolev embedding and Lemma 2.3 in (3.2) imply

$$\partial_t \|v\|_{0,1}^2 \leq cF_1(t; \mu, \|\phi\|_2) \|v\|_{0,1}^2 + cF(t; \mu, \|\phi\|_2) \|v\|_{0,1}.$$

Therefore the generalized Gronwall inequality gives us the desired global estimate. \square

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A Note on Uniqueness for the Classical Solution of the MHD Equations in \mathbb{R}^3

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Abstract. The uniqueness for unbounded classical solutions of the magnetohydrodynamic (MHD) equations in the whole space is studied. Under certain growth assumptions, it is shown that the solution to the initial value problem is unique.

1. Introduction and result

We deal with the magnetohydrodynamic (MHD) equations in \mathbb{R}^3 :

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \frac{1}{\rho\mu}(B \cdot \nabla)B + \frac{1}{2\rho\mu}\nabla(|B|^2) + \frac{1}{\rho}\nabla p = \nu\Delta u + f \\ \frac{\partial B}{\partial t} + (u \cdot \nabla)B - (B \cdot \nabla)u = \frac{1}{\mu\sigma}\Delta B \\ \operatorname{div} u = 0 \\ \operatorname{div} B = 0, \end{cases}$$

where the variables u , B and p denote the velocity vector, the magnetic field and the pressure, respectively. The constants ρ , μ , σ and ν represent the unit mass density, the magnetic permeability, the electric conductivity and the kinematic viscosity, respectively. f is the given external volume force to the fluid. For the derivation and other properties of the MHD equations, we refer to, for instance, M.A.Nakamura [5], Z.Yoshida [7].

Our interest is in the uniqueness for classical solutions of the initial value problem to (1). Examples show that no uniqueness property can be expected unless certain restrictions to the solutions are imposed. We want to prove the uniqueness in a class of solutions satisfying

$$(2) \quad \begin{cases} |\nabla_x u(x, t)|, |\nabla_x B(x, t)| \leq A \\ |p(x, t)| \leq A(1 + |x|)^{-1/2}, \end{cases}$$

where A is a constant. Our main theorem is formulated as follows:

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

Theorem. *Let (u_1, B_1, p_1) and (u_2, B_2, p_2) be two classical solutions of (1) in $\mathbf{R}^3 \times [0, T]$, both satisfying the growth condition (2). If $u_1(x, 0) = u_2(x, 0)$ and $B_1(x, 0) = B_2(x, 0)$ in \mathbf{R}^3 , then we have $u_1(x, t) = u_2(x, t)$ and $B_1(x, t) = B_2(x, t)$ in $\mathbf{R}^3 \times [0, T]$.*

This work is motivated by a recent paper by H.Okamoto [6]. He discussed the Navier-Stokes equations (i.e., $B \equiv 0$ in (1)), and established a uniqueness theorem for solutions with

$$\begin{aligned} |u(x, t)| &\leq A(1 + |x|)^\alpha, \quad |\nabla_x u(x, t)| \leq A \\ |p(x, t)| &\leq A(1 + |x|)^{-1/2}, \end{aligned}$$

where $0 \leq \alpha < 1$ and A are constants. Shortly later, N.Kim and D.Chae [3] extended Okamoto's theorem so that it only suffices to assume that

$$|\nabla_x u(x, t)| \leq A, \quad |p(x, t)| \leq A(1 + |x|)^{-1/2}.$$

Moreover, they showed that the growth of the pressure $p(x, t)$ plays an important role in these uniqueness theorems. To be precise, if we admit the linear growth $|p(x, t)| = O(|x|)$ as $|x| \rightarrow \infty$, then the nonuniqueness occurs even in the case that both $u(x, t)$ and $\nabla_x u(x, t)$ are bounded. See Remarks at the end of [3].

The same situation applies to the MHD equations. The linear growth of the pressure leads to nonuniqueness examples of (1). Several formula of exact solutions are already known. See C.C.Lin [4], A.D.D.Craik [1] and the references therein. Some of them can be used to construct nonuniqueness examples. On the other hand, if the pressure decays relatively fast, then the uniqueness follows. Even under these settings, our result seems to be new, though it is rather elementary. To our knowledge, however, it is still an open problem to determine the optimal growth of $p(x, t)$ in order that the uniqueness holds true.

The method of proof is based on the argument originally due to D.Graffi [2], which is also employed in [6] and [3] with generalizations. Computing technically well, we can proceed similiary as in the Navier-Stokes equations.

2. Sketch of proof

We define

$$\begin{aligned} v(x, t) &= u_1(x, t) - u_2(x, t) \\ D(x, t) &= B_1(x, t) - B_2(x, t) \\ q(x, t) &= p_1(x, t) - p_2(x, t). \end{aligned}$$

Then (v, D, q) fulfils

$$(3) \quad \left\{ \begin{aligned} &\frac{\partial v}{\partial t} + (u_1 \cdot \nabla)v + (v \cdot \nabla)u_2 - \frac{1}{\rho\mu}(B_1 \cdot \nabla)D - \frac{1}{\rho\mu}(D \cdot \nabla)B_2 \\ &\quad + \frac{1}{2\rho\mu}\nabla(|B_1|^2 - |B_2|^2) + \frac{1}{\rho}\nabla q = \nu\Delta v \\ &\frac{\partial D}{\partial t} + (u_1 \cdot \nabla)D + (v \cdot \nabla)B_2 - (B_1 \cdot \nabla)v - (D \cdot \nabla)u_2 = \frac{1}{\mu\sigma}\Delta D \\ &\operatorname{div} v = 0 \\ &\operatorname{div} D = 0. \end{aligned} \right.$$

Let $B = B(R)$ be a ball centered at the origin with radius $R \geq R_1$, and let ∂B be its boundary. We write $\|u\|_{2;B}^2 = \int_B |u|^2 dx$.

Multiplying v and $(\rho\mu)^{-1}D$ with the first and the second equations of (3), respectively, integrating by parts and adding term by term, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|v\|_{2;B}^2 + (\rho\mu)^{-1} \|D\|_{2;B}^2) + \frac{1}{2} \int_{\partial B} (u_1 \cdot n)(|v|^2 + (\rho\mu)^{-1}|D|^2) ds \\ & + \int_B ((v \cdot \nabla)u_2 \cdot v) - (\rho\mu)^{-1}((D \cdot \nabla)u_2 \cdot D) dx \\ & - \frac{1}{\rho\mu} \int_B ((D \cdot \nabla)B_2 \cdot v) - ((v \cdot \nabla)B_2 \cdot D) dx \\ & + \frac{1}{\rho\mu} \int_{\partial B} (B_1 \cdot n)(D \cdot v) + 2^{-1}(v \cdot n)(D \cdot (B_1 + B_2)) ds + \frac{1}{\rho} \int_{\partial B} q(v \cdot n) ds \\ & = \int_{\partial B} \nu \left(\frac{\partial v}{\partial n} \cdot v \right) + \frac{1}{\rho\mu^2\sigma} \left(\frac{\partial D}{\partial n} \cdot D \right) ds - \nu \|\nabla v\|_{2;B}^2 - (\rho\mu^2\sigma)^{-1} \|\nabla D\|_{2;B}^2, \end{aligned}$$

where n represents the unit outer normal to ∂B .

Taking into account the growth condition, we further compute

$$\begin{aligned} & \frac{d}{dt} (\|v\|_{2;B}^2 + (\rho\mu)^{-1} \|D\|_{2;B}^2) + (\nu \|\nabla v\|_{2;B}^2 + (\rho\mu^2\sigma)^{-1} \|\nabla D\|_{2;B}^2) \\ & \leq C_1(R+1) (\|v\|_{2;\partial B}^2 + (\rho\mu)^{-1} \|D\|_{2;\partial B}^2) + (\nu \|\nabla v\|_{2;\partial B}^2 + (\rho\mu^2\sigma)^{-1} \|\nabla D\|_{2;\partial B}^2) \\ & \quad + 4C_1 (\|v\|_{2;B}^2 + (\rho\mu)^{-1} \|D\|_{2;B}^2) + 2\rho^{-1} \int_{\partial B} |q(v \cdot n)| ds. \end{aligned}$$

We fix a positive constant $h \leq T$ to be determined later, and integrate the above inequality twice with respect to t . We derive for all sufficiently large R ,

$$(4) \quad \begin{aligned} G(R) & \leq (2C_2Rh + 1)G'(R) + 4C_2hG(R) \\ & \quad + 2\rho^{-1} \int_0^h dt \int_0^t \left(\int_{\partial B} |q(v \cdot n)| ds \right) d\tau, \end{aligned}$$

where we have defined

$$\begin{aligned} G(R) & = \int_0^h (\|v\|_{2;B}^2 + (\rho\mu)^{-1} \|D\|_{2;B}^2) dt \\ & \quad + \int_0^h dt \int_0^t (\nu \|\nabla v\|_{2;\partial B}^2 + (\rho\mu^2\sigma)^{-1} \|\nabla D\|_{2;\partial B}^2) d\tau, \end{aligned}$$

and used the property: $\frac{d}{dR} \int_B |v|^2 dx = \int_{\partial B} |v|^2 ds$.

The last term of the right hand side of (4) is estimated by as follows:

$$\begin{aligned}
& 2\rho^{-1} \int_0^h dt \int_0^t \left(\int_{\partial B} |q(v \cdot n)| ds \right) d\tau \\
& \leq 2\rho^{-1} \sqrt{\int_0^h dt \int_0^t \int_{\partial B} |q|^2 ds d\tau} \sqrt{\int_0^h dt \int_0^t \int_{\partial B} |v|^2 ds d\tau} \\
& \leq 2\sqrt{2\pi}\rho^{-1} h^{3/2} C_1 \sqrt{RG'(R)}.
\end{aligned}$$

Now we choose $h = \min\{(64C_1)^{-1}, T\}$ to discover

$$\begin{aligned}
15G(R) & \leq \left(\frac{1}{2}R + 16 \right) G'(R) + C_2 \sqrt{RG'(R)} \\
& \leq RG'(R) + C_2 \sqrt{RG'(R)},
\end{aligned}$$

where we further take larger R .

Since $G(R) \leq C_3 R^5$ for all large R , the next lemma ensures that $G(R) \equiv 0$. h is independent of R and t . An iteration argument makes sense and we have proved our uniqueness theorem. Note that our argument works also in the case $\nu = 0$.

Lemma. *Let $G(R)$ be a nondecreasing nonnegative function defined on $R \geq 0$ and satisfy*

$$(5) \quad G(R) \leq C_3 R^5$$

$$(6) \quad 15G(R) \leq RG'(R) + C_2 \sqrt{RG'(R)},$$

for all sufficiently large R . Then we have $G(R) = 0$ for all $R \geq 0$.

Proof. Let R_1 be large and $G(R_1) > 0$. If there exists no such R_1 , then we are finished. From the equation, we see

$$15G(R_1) \leq 15G(R) \leq RG'(R) + C_2 \sqrt{RG'(R)}$$

for $R \geq R_1$ and hence

$$\begin{aligned}
\sqrt{RG'(R)} & \geq \frac{1}{2}(-C_2 + \sqrt{C_2^2 + 60G(R_1)}) = C_4(R_1) > 0 \\
G(R) & \leq \frac{1}{15}RG'(R) \left(1 + \frac{15C_2}{\sqrt{RG'(R)}} \right) \leq C_5 RG'(R),
\end{aligned}$$

where $C_5 = (C_4 + 15C_2)/(15C_4) > 0$. This implies

$$(7) \quad G(R) \geq G(R_1) \left(\frac{R}{R_1} \right)^{1/C_5}$$

for $R \geq R_1$.

Back to (6), we estimate

$$(8) \quad \begin{aligned} 15G(R) &\leq \frac{3}{2}RG'(R) + \frac{C_2^2}{2}, \\ 10(G(R) - C_6) &\leq R(G(R) - C_6)', \end{aligned}$$

where $C_6 = C_2^2/30$. By virtue of (7), there is R_0 large enough that $G(R_0) \geq C_6 + 1$. We obtain from (7)

$$G(R) \geq (G(R_0) - C_6) \left(\frac{R}{R_0} \right)^{10} + C_6$$

for $R \geq R_0$, which contradicts with the growth condition (5). Thus we conclude that $G(R) = 0$ for all $R \geq 0$. This completes the proof. \square

Remark. In the two-dimensional case, we merely have to impose that

$$|\nabla_x u(x, t)|, |\nabla_x B(x, t)|, |p(x, t)| \leq A$$

for the uniqueness theorem.

Acknowledgements

We are grateful to Prof. H.Okamoto for informing us [3] and helpful suggestions. The first author was partially supported by Grant-in-Aid for Scientific Research (No. 07740140), Japan Ministry of Education, Science and Culture.

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A Numerical Study of Blow-up Solutions to $u_t = u^\delta(\Delta u + \mu u)$

TETSUYA ISHIWATA AND MASAYOSHI TSUTSUMI

Abstract. Numerical experiments of the initial boundary value problem for a nonlinear degenerate parabolic equation of non-divergent form are presented, under zero Dirichlet boundary condition in the case of dimension $N = 2$. The emphasis is stressed on detailed study of behavior of numerical solutions near the blow-up time. The numerical results indicate that the blow-up set and blow-up rate correspond to those conjectured by several authors.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N . We consider the following initial boundary value problem :

$$(P1) \quad \begin{cases} u_t = u^\delta(\Delta u + \mu u), & x \in \Omega, t > 0, & (1.1) \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, & (1.2) \\ u(x, 0) = u_0(x), & x \in \Omega, & (1.3) \end{cases}$$

here δ, μ are positive constants and $u_0(x)$ is a nonnegative bounded continuous function on Ω .

Several authors have studied problem (P1) from a theoretical point of view. The problem arises in a model for the resistive diffusion of a force-free magnetic field in a plasma

confined between two walls in one dimension [4], when $\delta = 2$. Equation (1.1) also describes the evolution of the curvature of a locally convex plane curve, and it has been studied in [2], [6] under periodic boundary condition.

A. Friedman and B. McLeod [4] considered (P1) in the case $N = 1$ and $\delta = 2$. They showed that the behaviors of solution depend on the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet problem for the Laplacian on the domain Ω . If $\lambda_1(\Omega) > 1$, then there exists a unique global solution which tends to zero as $t \rightarrow \infty$. If $\lambda_1(\Omega) < 1$, then there exists a positive constant T such that we have a unique solution in $0 < t < T$, which blows up as $t \uparrow T$. They also showed that the blow-up set has positive Lebesgue measure. Qi [8] discussed the Cauchy problem for (1.1) and (1.3) with $0 < \delta < 2$. For the case $\delta > 1$, M. Wiegner [11] stated the existence and uniqueness of smooth positive solutions and gave the upper bound of the blow-up time for the positive initial data. When $N = 1$ and $\delta > 0$, K. Anada, I. Fukuda and M. Tsutsumi [1] got more precise information on the blow-up set and an asymptotic behavior near the blow-up time.

In this paper we solve the problem (P1) numerically in the case of $N = 2$ and $\Omega = (-a, a) \times (-a, a)$. Known theoretical results are summarized in ‘‘Appendix.’’

We show the efficiency for our numerical experiments by confirming the behavior of the numerical solutions which agree with those suggested by the theoretical results. We also show the positivity our scheme held. In addition, we estimate the blow-up rate of numerical solutions both theoretically and numerically. Although we do not have the detailed theoretical results about the asymptotic behavior of solutions for a rectangle domain, our numerical results may present useful suggestions to unclear theoretical anticipations.

2 Numerical Scheme

We use the following scheme based on the Alternating Direction Implicit Method:

$$\begin{aligned} \frac{1}{\Delta t_p} (u_{m,n}^{p+1} - u_{m,n}^p) &= (u_{m,n}^p)^\delta \left(\frac{1}{\Delta x^2} (u_{m+1,n}^{p+1} - 2u_{m,n}^{p+1} + u_{m-1,n}^{p+1}) \right. \\ &\quad \left. + \frac{1}{\Delta y^2} (u_{m,n+1}^p - 2u_{m,n}^p + u_{m,n-1}^p) + \mu u_{m,n}^{p+1} \right), \end{aligned} \quad (2.1)$$

$$\begin{aligned} \frac{1}{\Delta t_{p+1}} (u_{m,n}^{p+2} - u_{m,n}^{p+1}) &= (u_{m,n}^{p+1})^\delta \left(\frac{1}{\Delta x^2} (u_{m+1,n}^{p+1} - 2u_{m,n}^{p+1} + u_{m-1,n}^{p+1}) \right. \\ &\quad \left. + \frac{1}{\Delta y^2} (u_{m,n+1}^{p+2} - 2u_{m,n}^{p+2} + u_{m,n-1}^{p+2}) + \mu u_{m,n}^{p+2} \right). \end{aligned} \quad (2.2)$$

where m, n are integers, p is a positive number, $u_{m,n}^p$ is an approximate value of $u(m\Delta x, n\Delta y, t_p)$ and $t_p = \sum_{i=0}^{p-1} \Delta t_i$. In order to get the numerical blow-up solutions, it is important that the larger the solution becomes, the smaller the time interval Δt_n should be chosen such

as

$$\Delta t_p = \left(\frac{\|u^0\|_{L^\infty}}{\|u^p\|_{L^\infty}} \right)^\delta \Delta t_0. \quad (2.3)$$

We note that this method can extensively decrease a computational quantity in comparison with the implicit method. Assume that

$$\frac{\Delta t_0}{\Delta x^2}, \frac{\Delta t_0}{\Delta y^2} < \frac{1}{2} \|u^0\|_{L^\infty}^{-\delta}, \quad (2.4)$$

then we have the two following lemma.

Lemma 2.1 *If $1 - \Delta t_p (u_{m,n}^p)^\delta \mu > 0$ and the initial condition $u_{m,n}^0$ is nonnegative, then $u_{m,n}^p$ is nonnegative for any p .*

proof. Eq.(2.1) may be written as

$$\begin{aligned} & \left(1 + \frac{2\Delta t_p}{\Delta x^2} (u_{m,n}^p)^\delta - \Delta t_p (u_{m,n}^p)^\delta \mu\right) u_{m,n}^{p+1} - \frac{\Delta t_p}{\Delta x^2} (u_{m-1,n}^{p+1} + u_{m+1,n}^{p+1}) = \\ & \left(1 - \frac{2\Delta t_p}{\Delta y^2} (u_{m,n}^p)^\delta\right) u_{m,n}^p + \frac{\Delta t_p}{\Delta y^2} (u_{m,n}^p)^\delta (u_{m,n-1}^p + u_{m,n+1}^p). \end{aligned} \quad (2.5)$$

Let $u_{m,n}^p \geq 0$. Since

$$\begin{aligned} 1 - \frac{2\Delta t_p}{\Delta y^2} (u_{m,n}^p)^\delta &= 1 - \frac{2\Delta t_0}{\Delta y^2} \left(\frac{\|u^0\|_{L^\infty}}{\|u^p\|_{L^\infty}} u_{m,n}^p \right)^\delta \\ &> 1 - \frac{2\Delta t_0}{\Delta y^2} \|u^0\|_{L^\infty}^{-\delta} \\ &> 0, \end{aligned}$$

we have

$$\left(1 + \frac{2\Delta t_p}{\Delta x^2} (u_{m,n}^p)^\delta - \Delta t_p (u_{m,n}^p)^\delta \mu\right) u_{m,n}^{p+1} - \frac{\Delta t_p}{\Delta x^2} (u_{m-1,n}^{p+1} + u_{m+1,n}^{p+1}) \geq 0. \quad (2.6)$$

Take m_0, n_0 such that u_{m_0, n_0}^{p+1} takes its minimum. We can rewrite (2.6) as

$$\left(1 - \Delta t_p (u_{m_0, n_0}^p)^\delta \mu\right) u_{m_0, n_0}^{p+1} \geq \frac{\Delta t_p}{\Delta x^2} (u_{m_0, n_0}^p)^\delta \left((u_{m_0-1, n_0}^{p+1} - u_{m_0, n_0}^{p+1}) + (u_{m_0+1, n_0}^{p+1} - u_{m_0, n_0}^{p+1}) \right).$$

Hence we get

$$\left(1 - \Delta t_p (u_{m_0, n_0}^p)^\delta \mu\right) u_{m_0, n_0}^{p+1} \geq 0,$$

that is, $u_{m,n}^{p+1}$ is nonnegative for any m and n . ▀

In the next lemma, we give an upper bound of blow-up rate of the numerical solution.

We define the sequence $\{M_p\}_{p \geq 0}$ by $M_p = \max_{m,n} u_{m,n}^p$.

Lemma 2.2 *If $\Delta t_0 < 1/\mu\|u^0\|_{L^\infty}^\delta$, then M_p satisfies the inequality*

$$M_{p+1} \leq \frac{1}{1 - \Delta t_0 \mu \|u^0\|_{L^\infty}^\delta} M_p. \quad (2.7)$$

proof. From (2.5), we have

$$\begin{aligned} \left(1 + \frac{2\Delta t_p}{\Delta x^2} (u_{m,n}^p)^\delta - \Delta t_p (u_{m,n}^p)^\delta \mu\right) u_{m,n}^{p+1} - \frac{\Delta t_p}{\Delta x^2} (u_{m,n}^p)^\delta (u_{m-1,n}^{p+1} + u_{m+1,n}^{p+1}) \\ \leq \left(1 - \frac{2\Delta t_p}{\Delta y^2} (u_{m,n}^p)^\delta\right) M_p + \frac{\Delta t_p}{\Delta y^2} (u_{m,n}^p)^\delta (M_p + M_p) \\ = M_p \end{aligned}$$

Take m_0, n_0 such that $u_{m_0, n_0}^{p+1} = M_{p+1}$, then

$$\begin{aligned} M_p &\geq \left(1 + \frac{2\Delta t_p}{\Delta x^2} (u_{m_0, n_0}^p)^\delta - \mu \Delta t_p (u_{m_0, n_0}^p)^\delta\right) M_{p+1} - \frac{\Delta t_p}{\Delta x^2} (u_{m_0, n_0}^p)^\delta (M_{p+1} + M_{p+1}) \\ &= \left(1 - \mu \Delta t_p (u_{m_0, n_0}^p)^\delta\right) M_{p+1} \\ &= \left(1 - \mu \Delta t_0 \left(\frac{\|u^0\|_{L^\infty}}{\|u^p\|_{L^\infty}} u_{m_0, n_0}^p\right)^\delta\right) M_{p+1} \\ &\geq \left(1 - \mu \Delta t_0 \|u^0\|_{L^\infty}^\delta\right) M_{p+1}. \end{aligned} \quad (2.8)$$

Hence, we get

$$M_{p+1} \leq (1 - \mu \Delta t_0 \|u^0\|_{L^\infty}^\delta)^{-1} M_p. \quad (2.9)$$

■

In the sequel we always assume (2.4) and

$$\Delta t_0 < \frac{1}{\mu \|u^0\|_{L^\infty}^\delta}. \quad (2.10)$$

Next, we give a lower bound of the numerical blow-up time T and an upper bound of M_p .

From lemma 2.2, we have

$$M_p \leq (1 - \mu \Delta t_0 \|u^0\|_{L^\infty}^\delta)^{-p} M_0. \quad (2.11)$$

Since

$$T = \sum_{p=0}^{\infty} \Delta t_p = \sum_{p=0}^{\infty} \left(\frac{M_0}{M_p}\right)^\delta \Delta t_0 = M_0^\delta \Delta t_0 \sum_{p=0}^{\infty} \frac{1}{M_p^\delta}, \quad (2.12)$$

we have

$$T \geq M_0^{\delta-1} \Delta t_0 \frac{1}{1 - (1 - \mu \Delta t_0 \|u^0\|_{L^\infty}^\delta)^\delta}. \quad (2.13)$$

By making use of this inequality, we get

$$1 - \mu \Delta t_0 \|u^0\|_{L^\infty}^\delta \leq \left(1 - \frac{M_0^{\delta-1} \Delta t_0}{T}\right)^{\frac{1}{\delta}}. \quad (2.14)$$

Hence, we obtain

$$M_p \leq \left(1 - \frac{M_0^{\delta-1} \Delta t_0}{T}\right)^{-\frac{p}{\delta}}. \quad (2.15)$$

3 Numerical Results

We have done numerical experiments for a initial value:

$$u_0 = \cos \frac{\pi}{2a} x \cos \frac{\pi}{2a} y \quad (3.1)$$

on domain $\Omega = (-2.0, 2.0) \times (-2.0, 2.0)$ ($a = 2.0$). Theoretical prospect shows that if $a > \frac{\pi}{\sqrt{2\mu}}$, then the solution for problem (P1) blows up. Therefore, we take $\mu = 2.0$. In fact, if $a < \frac{\pi}{\sqrt{2\mu}}$, then our numerical solutions decay to zero, and if $a = \frac{\pi}{\sqrt{2\mu}}$, then our numerical solutions grow up. (In this paper, we don't show these results.)

We take $\Delta x, \Delta y = 4.0/30$ and $\Delta t_0 = 0.005$ which satisfy the two condition (2.4) and (2.10). We computed for $\delta = 1.0, 1.5, 2.0, 2.5$.

Fig. 1, 2 and 3 show the time evolution of numerical solutions for each time step $N = 0, 10000, 40000$ when $\delta = 1$. We see that the solution blows up with keeping its shape and symmetry in whole domain. We plot the graph of L^∞ norm of the solutions in Fig. 4. Table 1, 2, 3 and 4 show t_n, L^∞ norm of the solutions and Δt_n for time step $N = 0, 10000, 20000, 30000, 40000$, respectively.

Fig. 5, 6 and 7 show the time evolution of the solutions in case $\delta = 1.5$. We see that the solution changes its shape and the blow up set of the solution becomes regionally. Fig. 8 shows the graph of L^∞ norm of the solutions.

Fig. 9, 10 and 11 show the time evolution of the solutions in case $\delta = 2$. It is seen that the solution blows up regionally and it seems that its blow up set converges to a radially symmetric domain. Fig. 12 yields the graph of L^∞ norm of the solutions.

Fig. 13, 14 and 15 show the time evolution of the solutions when $\delta = 2.5$. Fig. 16 shows the graph of L^∞ norm of the solutions.

In Fig 9-11, 13-15 the blow up sets look like the support of the solutions.

In the case $\delta \geq 2$, Theorem 7.5 below yields that if the domain is radially symmetric, then the shape of $u^n / \|u^n\|_{L^\infty}$ near the blow-up time will be the first eigenfunction of the following eigenvalue problem:

$$-\Delta v = \lambda v \quad , \quad v|_{\partial U_R} = 0. \quad (3.2)$$

Here $R = r_0/\sqrt{2}$, $U_R = \{(x, y) \mid x^2 + y^2 \leq R^2\}$ where r_0 is the first positive zero of the Bessel function of order 0. We plot the first eigenfunction in Fig. 19. In Fig. 17 and 18, we plot the graph of $u^n/\|u^n\|_{L^\infty}$ for time step $n = 40000$ in the case of $\delta = 2$ and 2.5. We can see the shape and the blow up set of the numerical solution are similar to the eigenfunction and the domain U_R in Fig. 19 though the domain is rectangular. Fig. 20 is the graph of the projection to x-z plane of that in Fig. 18.

Table 1. $\delta = 1.0$: $t_n, \|u^n\|_{L^\infty}$ and Δt_n for each step.

n	t_n	$\ u^n\ _{L^\infty}$	Δt_n
10000	2.018381689186295	2893877097.685297	$1.731521527814143 \times 10^{-12}$
20000	2.018381689986517	$6.952215328402329 \times 10^{18}$	$7.207501859846734 \times 10^{-22}$
30000	2.018381689986517	$1.670190659031886 \times 10^{28}$	$3.000142806363422 \times 10^{-31}$
40000	2.018381689986517	$4.012443091241728 \times 10^{37}$	$1.248817834173743 \times 10^{-40}$

Table 2. $\delta = 1.5$: $t_n, \|u^n\|_{L^\infty}$ and Δt_n for each step.

n	t_n	$\ u^n\ _{L^\infty}$	Δt_n
10000	1.719148010680321	177380.1169453133	$6.703926089865509 \times 10^{-11}$
20000	1.719148051356698	9572160818.839784	$5.347643203042103 \times 10^{-18}$
30000	1.719148051356698	497252343580906.3	$4.516603155897372 \times 10^{-25}$
40000	1.719148051356698	$2.580561156641957 \times 10^{19}$	$3.820374588123436 \times 10^{-32}$

Table 3. $\delta = 2.0$: $t_n, \|u^n\|_{L^\infty}$ and Δt_n for each step.

n	t_n	$\ u^n\ _{L^\infty}$	Δt_n
10000	1.625380387548987	346.5623343033312	$4.166315338588180 \times 10^{-8}$
20000	1.625434352585319	13095.19650627358	$2.917708575688688 \times 10^{-11}$
30000	1.625434395875942	345924.9327773206	$4.181002702386943 \times 10^{-14}$
40000	1.625434395942738	7317251.144430679	$9.343969670642357 \times 10^{-17}$

Table 4. $\delta = 2.5$: $t_n, \|u^n\|_{L^\infty}$ and Δt_n for each step.

n	t_n	$\ u^n\ _{L^\infty}$	Δt_n
10000	1.615436390094144	29.943886948927126	$1.019487443064002 \times 10^{-6}$
20000	1.618048922071174	119.45743335856288	$3.206754591569421 \times 10^{-8}$
30000	1.618155837077613	351.79547840655403	$2.154531125435835 \times 10^{-9}$
40000	1.618163939168007	906.07403710799724	$2.023759410516244 \times 10^{-10}$

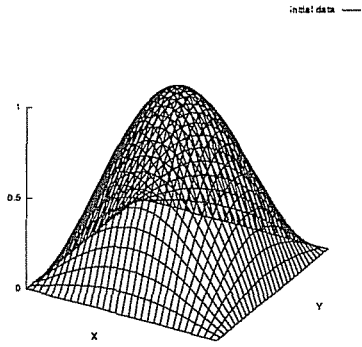


Fig 1. initial data.

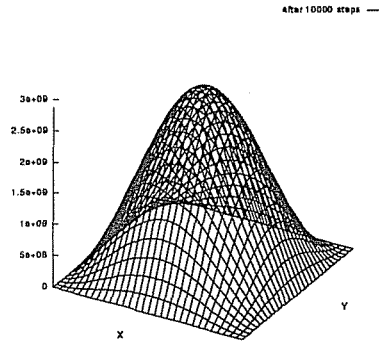


Fig 2. u^{10000} for $\delta = 1$.

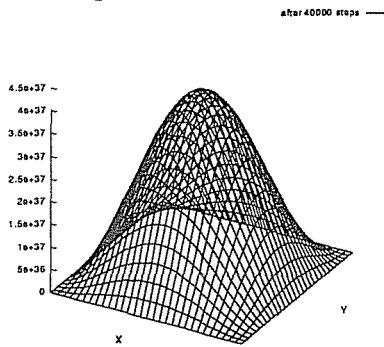


Fig 3. u^{40000} for $\delta = 1$.

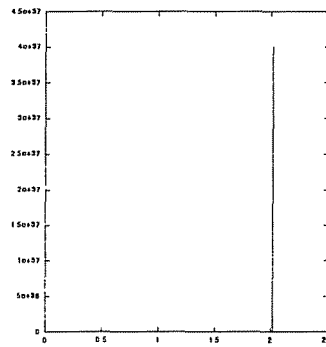


Fig 4. time - $\|u\|_{L^\infty}$ for $\delta = 1$.

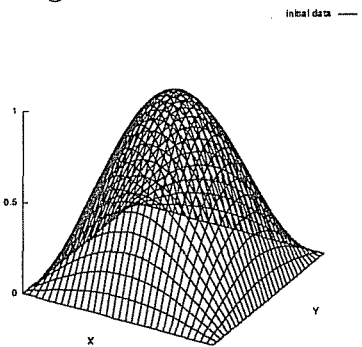


Fig 5. initial data.

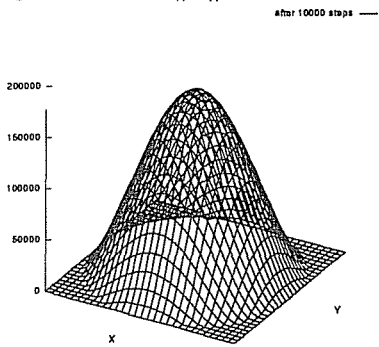


Fig 6. u^{10000} for $\delta = 1.5$.

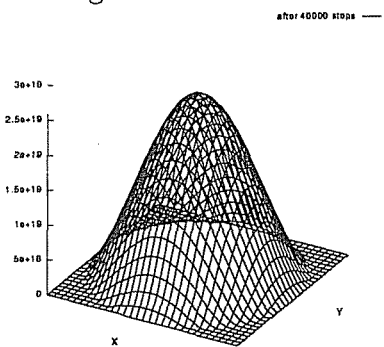


Fig 7. u^{40000} for $\delta = 1.5$.

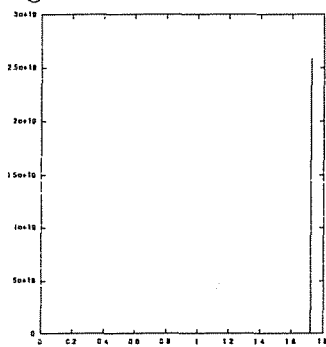


Fig 8. time - $\|u\|_{L^\infty}$ for $\delta = 1.5$.

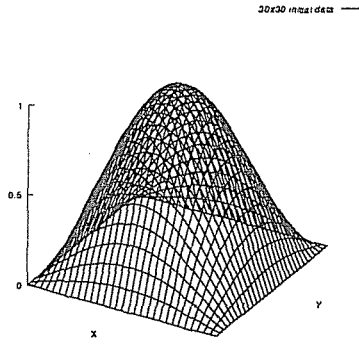


Fig 9. initial data.

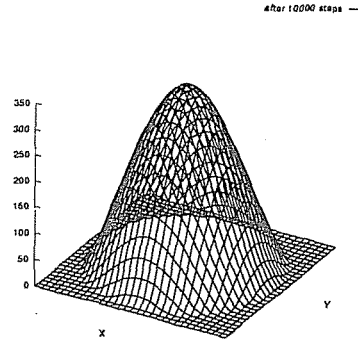


Fig 10. u^{10000} for $\delta = 2$.

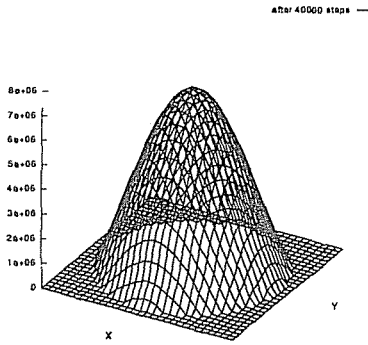


Fig 11. u^{40000} for $\delta = 2$.

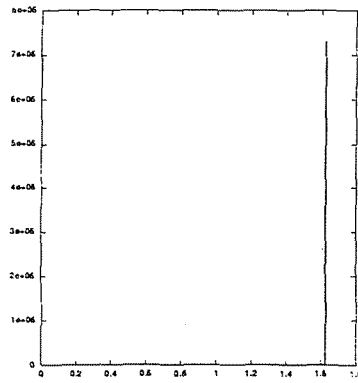


Fig 12. time - $\|u\|_{L^\infty}$ for $\delta = 2$.

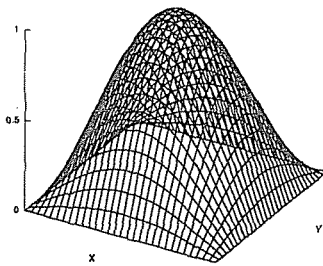


Fig 13. initial data.

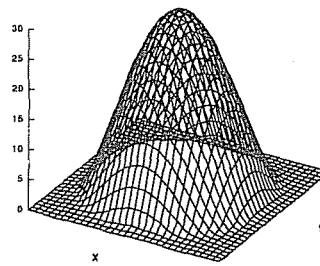


Fig 14. u^{10000} for $\delta = 2.5$.

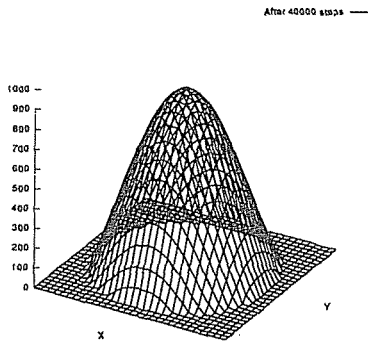


Fig 15. u^{40000} for $\delta = 2.5$.

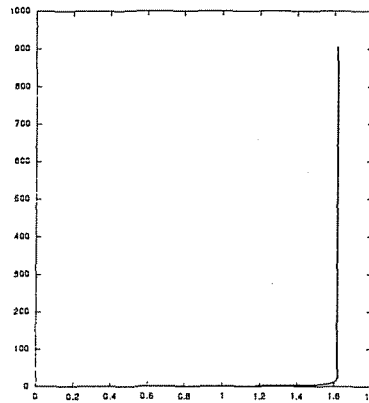


Fig 16. time - $\|u\|_{L^\infty}$ for $\delta = 2.5$.

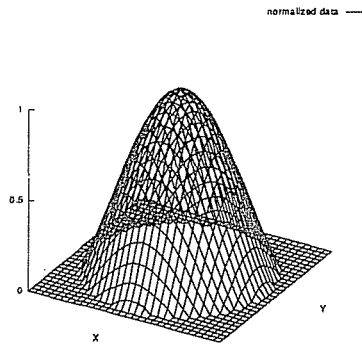


Fig 17. $\frac{u^{40000}}{\|u^{40000}\|_{L^\infty}}$ for $\delta = 2$.

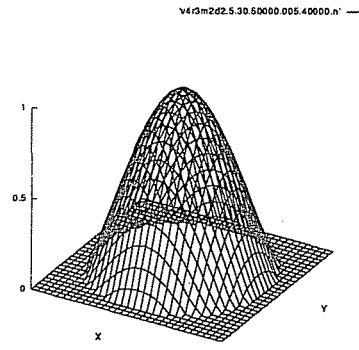


Fig 18. $\frac{u^{40000}}{\|u^{40000}\|_{L^\infty}}$ for $\delta = 2.5$.

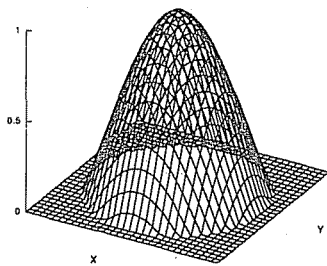


Fig 19. eigenfunction for eq.(3.2).

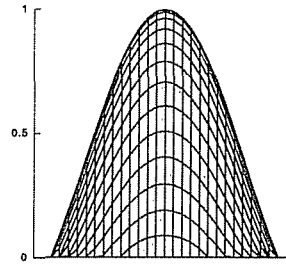


Fig 20. section for Fig 18.

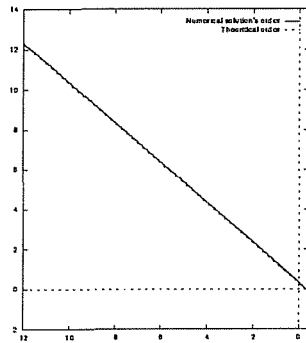


Fig 21. $\log(T-t) - \log(\|u\|_{L^\infty})$
for $\delta = 1$.

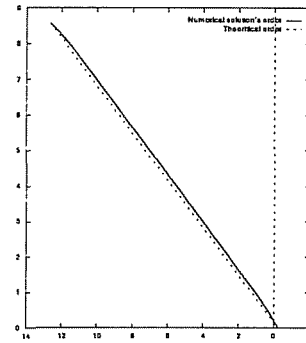


Fig 22. $\log(T-t) - \log(\|u\|_{L^\infty})$
for $\delta = 1.5$.

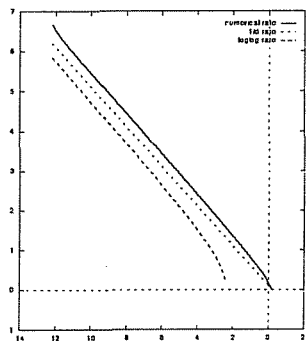


Fig 23. $\log(T-t) - \log(\|u\|_{L^\infty})$.
for $\delta = 2$.

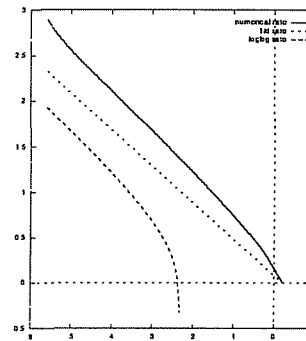


Fig 24. $\log(T-t) - \log(\|u\|_{L^\infty})$.
for $\delta = 2.5$.

4 Estimates of the blow-up time

Theorem 7.1, and 7.2 say that if $\delta < 2$, the solutions blow up like $\|u\|_{L^\infty} = C(T-t)^{-1/\delta}$, where T is blow-up time. Let $\{N_n\}$ be the sequence defined by $N_n = \|u(\cdot, t_n)\|_{L^\infty}$, $N_0 = \|u(\cdot, 0)\|_{L^\infty}$, then

$$N_n = \frac{T^{1/\delta} N_0}{(T - t_n)^{1/\delta}}. \quad (4.1)$$

Hence we get

$$T = \frac{N_n^\delta}{N_n^\delta - N_0^\delta} t_n. \quad (4.2)$$

For the case $\delta = 2$, A. Friedman and B. McLeod conjectured in [4] that if the dimension $N = 1$, then

$$\|u\|_{L^\infty}^2 \sim \frac{1}{T-t} \log \log \frac{1}{T-t}. \quad (4.3)$$

Therefore, we guess that if $\delta \geq 2$, then

$$\|u\|_{L^\infty} \sim (T-t)^{-1/\delta} (\log \log \frac{1}{T-t})^{1/\delta}. \quad (4.4)$$

Using this formula, we estimate the blow-up time T .

Table 5 shows the blow-up time for each δ , which are calculated with these method. Here \tilde{T} means the numerical blow up time corresponding T .

Table 5. Blow-up Time for each δ .

δ	\tilde{T}
1.0	2.018381689986517
1.5	1.719148051356698
2.0	1.625434395943160
2.5	1.618166119030117

5 Blow-up Rate

In order to clear the blow-up rate for each δ , we plot the graph of $\log_{10} \|u\|_{L^\infty}$ and $\log_{10}(\tilde{T} - t)$ for each δ in Fig. 21, 22, 23 and 24.

In Fig. 21, 22, the graph in case $\delta = 1, 1.5$ are plotted, where a solid line means the blow-up rate of the numerical solution and a dotted line means theoretical blow-up rate which are calculated by the following method:

Let N be a function defined by $N(t) = \|u(\cdot, t)\|_{L^\infty}$. When $\delta < 2.0$, then

$$N(t) = \left(\frac{T}{T-t}\right)^{1/\delta} N(0). \quad (5.1)$$

Therefore, we have the following formula.

$$\log N(t) = \frac{1}{\delta} \log T - \frac{1}{\delta} \log(T - t) + \log N(0). \quad (5.2)$$

We can see that the numerical results give good agreement with the values that had been obtained by theoretical calculation. This shows that, our numerical scheme has the appropriate property which the theory suggests, in particular, concerning the blow-up rate. So we can consider that our method is suitable.

The graphs of $\log_{10} \|u\|_{L^\infty}$ and $\log_{10}(\tilde{T} - t)$ in the case $\delta = 2, 2.5$ are plotted in Fig. 23 and 24. A solid line means the blow-up rate of numerical solution. The straight dotted line means the theoretical blow-up rate by formula (5.2). The curved dotted line means the blow-up rate based on formula (4.4). These two graphs show that if $\delta \geq 2$, then the solution blows up faster than $(T - t)^{1/\delta}$. These results agree with the assertion of Lemma 7.7. Comparing the slope of solid line with that of dotted line when $T - t$ is small, we can see the blow-up rate of the numerical solution is as same as a rate shown by formula (4.4). These results show that the blow-up rate of numerical solutions in case $\delta \geq 2$ are different from those of numerical solution in case $\delta < 2$.

6 Conclusions

In this paper, we show the detailed numerical experiments of the initial boundary value problem for a nonlinear degenerate parabolic equation in dimension $N = 2$. We also estimate the blow-up time and asymptotic behavior near blow-up time. These results will give information to the theoretical study.

7 Appendix : Summary of theoretical results

The following theoretical results for problem (P1) are discussed in [9]. In this section, we consider only the case $\mu = 1$, but these results stand up for any $\mu > 0$ by changing of variable.

When $0 < \delta < 2$, we can get upper and lower bounds of blow-up rate.

Theorem 7.1 *Suppose that $0 < \delta < 2$. Then*

$$u(x, t) \leq C(T - t)^{-1/\delta}, \quad \forall x \in \Omega, \quad 0 \leq t < T. \quad (7.1)$$

Here and in the sequel by C we denote various positive constants changeable from line to line.

Theorem 7.2 *Let $a > 0$ and $B = \{x \in \Omega \mid \lim_{t_n \nearrow T} \frac{u(x, t_n)}{\max_{x \in \Omega} u(x, t_n)} > a\}$. For any $\kappa > 0$ there exists a closed set $F \subset \Omega$ with $\text{meas}(B) - \kappa \leq \text{meas}(B \cap F)$ such that*

$$u(x, t) \geq C(T - t)^{-1/\delta}, \quad \forall x \in B \cap F, \quad 0 \leq t < T. \quad (7.2)$$

Next, we consider the radially symmetric case. Let $R > 0$, and $\Omega = U_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$. We assume that

(A1) $u_0(x)$ is nonnegative, radially symmetric, non-increasing about $|x|$ ($u_0(z) \geq u_0(x)$ for $|z| \leq |x|$) and $\Delta u_0 + u_0 \geq 0$ on Ω .

Since the solution $u(x, t)$ of (P1) is radially symmetric, the Cauchy problem can be reduced to a problem in one spatial dimension. Let $Q_T = \{(r, t) \mid 0 < r < R, 0 < t < T\}$. If $r = |x|$, then $v(r, t) = u(x, t)$ is well-defined on Q_T and satisfies

$$(PR1) \quad \begin{cases} v_t = v^\delta (v_{rr} + \frac{1}{r} v_r + v), & r \in (0, R), t > 0, & (7.3) \\ v(0, t) = 0, v(R, t) = 0, & t > 0, & (7.4) \\ v(r, 0) = v_0(r), & r \in [0, R]. & (7.5) \end{cases}$$

We define blow-up set S and monotone blow-up set S_* by the following:

$$S = \{x \in \Omega \mid \exists (x_n, t_n) \in \Omega \times [0, t] \text{ s.t. } x_n \rightarrow x, t_n \rightarrow t \text{ and } u(x_n, t_n) \rightarrow \infty\}$$

$$S_* = \{x \in \Omega \mid \exists t_n \text{ s.t. } t_n \rightarrow t \text{ and } u(x, t_n) \rightarrow \infty\}$$

Let r_0 be the first positive zero of the Bessel function of order 0.

Theorem 7.3 *Suppose that $0 < \delta < 2$. We have*

$$[0, r_0] \subset S_*. \quad (7.6)$$

Theorem 7.4 *Let $\delta \geq 2$. Then, we have*

$$[0, r_0] = S. \quad (7.7)$$

Under the assumptions (A1) we can investigate the behavior of the solutions near the blow-up time.

Theorem 7.5 *Suppose that $\delta \geq 2$ and u_0 satisfies (A1). Let $v(r, t) = u(x, t)$ blow up at $t = T$. Then,*

$$\frac{v(r, t)}{v(0, t)} \longrightarrow J_0(r) \quad \text{for } r \in [0, r_0) \quad (7.8)$$

uniformly as $t \rightarrow T$. Moreover, if $r_0 < r < R$, then

$$\frac{v(r, t)}{v(0, t)} \longrightarrow 0 \quad \text{for } r_0 < r < R. \quad (7.9)$$

Theorem 7.6 *suppose that $1 < \delta < 2$, and u_0 satisfies (A1). Let v blow up at $t = T$. Then,*

$$(T - t)^{1/\delta} v(r, t) \longrightarrow \begin{cases} z(r) & \text{for } r \in S, \\ 0 & \text{for } r \in S^c \end{cases} \quad (7.10)$$

uniformly as $t \rightarrow T$, where $z(r)$ is the unique positive solution of the boundary value problem

$$z_{rr} + \frac{1}{r}z_r - \frac{1}{\delta z^{\delta-1}} = 0, \quad r \in S, \quad (7.11)$$

$$z(r) = 0, \quad r \in \partial S. \quad (7.12)$$

Lemma 7.7 *Under the same assumption as in Theorem 7.5, we have*

$$(T - t)^{1/\delta} v(0, t) \longrightarrow \infty \quad \text{as } t \longrightarrow T. \quad (7.13)$$

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GLOBAL EXISTENCE FOR NONLINEAR WAVE EQUATIONS IN TWO SPACE DIMENSIONS

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Abstract. We study the Cauchy problem for nonlinear wave equations in two space dimensions with cubic nonlinear terms which depend on both the unknown and its gradients. Global existence of classical solutions with small data will be proved under the null condition on the cubic part of the nonlinear term.

1. Introduction We consider the Cauchy problem for nonlinear wave equations

$$(1.1)_\varepsilon \quad \begin{cases} \square u \left(= (\partial_t^2 - \Delta_x)u \right) = F(u, Du, D_x Du) & \text{in } (0, \infty) \times \mathbf{R}^n, \\ u(0, x) = \varepsilon \phi(x), (\partial_t u)(0, x) = \varepsilon \psi(x) & \text{in } \mathbf{R}^n, \end{cases}$$

where $Du = (\partial_a u)_{0 \leq a \leq n}$, $D_x Du = (\partial_i \partial_a u)_{\substack{1 \leq i \leq n \\ 0 \leq a \leq n}}$, $\partial_0 = \partial_t$ and $\partial_i = \partial_{x_i}$ ($1 \leq i \leq n$). We suppose that the nonlinear term F is a smooth function in its arguments and

$$(1.2) \quad F(u, \mathbf{v}, \mathbf{w}) = O(|u|^\lambda + |\mathbf{v}|^\lambda + |\mathbf{w}|^\lambda) \quad \text{near } (u, \mathbf{v}, \mathbf{w}) = 0$$

with some integer $\lambda \geq 2$. We also suppose that ϕ and ψ are C_0^∞ -functions and that ε is a small positive parameter. For any positive integer s and any smooth function G , we define

$$(1.3) \quad G^{(s)}(z) = \sum_{|\alpha|=s} \frac{1}{\alpha!} \partial_z^\alpha F(z) \Big|_{z=0} z^\alpha,$$

where $z = (z_1, \dots, z_m) \in \mathbf{R}^m$, $\partial_z^\alpha = \partial_{z_1}^{\alpha_1} \dots \partial_{z_m}^{\alpha_m}$, $z^\alpha = z_1^{\alpha_1} \dots z_m^{\alpha_m}$.

Here we want to mention some known results briefly, restricting our attention to the cases $n = 3$ and 2 . First we consider the case $n = 3$. When $F(u, \mathbf{v}, \mathbf{w}) = \tilde{F}(\mathbf{v}, \mathbf{w})$ with some function \tilde{F} , Klainerman ([8]) introduced the method of invariant norms to

get uniform decay estimates, and combining them with the classical energy estimate, he proved global existence of small solutions when $\lambda \geq 3$ and almost global existence when $\lambda = 2$. Here we note that we cannot get the global existence in general when $\lambda = 2$, because it is known that every non-trivial classical solution to $\square u = u_t^2$ must blow up in finite time (see John [5]).

When F depends both on u and derivatives of u , the problem becomes more complicated, because the classical energy inequality for linear wave equations provides the estimate of L^2 -norms of gradients of u , but not a good estimate for L^2 -norms of u itself. Klainerman ([9]) and Christodoulou ([1]) proved independently that when $n = 3$ and $\lambda \geq 3$, there exists a global classical solution for any small data. Klainerman used a certain conformal energy combined with the method of invariant norms. In addition, concerning the case $\lambda = 2$, they introduced a sufficient condition (the null condition) for global existence of small solutions. We say a smooth function $G = G(u, \mathbf{v}, \mathbf{w})$ with $u \in \mathbf{R}$, $\mathbf{v} = (v_a)_{0 \leq a \leq n} \in \mathbf{R}^{n+1}$ and $\mathbf{w} = (w_{ja})_{1 \leq j \leq n, 0 \leq a \leq n} \in \mathbf{R}^{n(n+1)}$ satisfies the null condition when $G(p, (qX_a)_{0 \leq a \leq n}, (rX_jX_a)_{1 \leq j \leq n, 0 \leq a \leq n}) \equiv 0$ holds for any $p, q, r \in \mathbf{R}$ and any $(X_0, X_1, \dots, X_n) \in \mathbf{R}^{n+1}$ satisfying $X_0^2 - X_1^2 - \dots - X_n^2 = 0$. In [9] and [1], it was proved that if $F^{(2)}$ (defined by (1.3) with $z = (u, \mathbf{v}, \mathbf{w})$) satisfies the null condition, then global existence of solutions with small data is assured.

Now we turn our attention to the case $n = 2$. Applying the method in [8], we can see that solutions with small data exist globally when $F(u, \mathbf{v}, \mathbf{w}) = \tilde{F}(\mathbf{v}, \mathbf{w})$ with some \tilde{F} and $\lambda \geq 4$. This result was extended to general $F(u, \mathbf{v}, \mathbf{w})$ with $\lambda \geq 4$ by Li - Zhou ([11]). In [10], they also proved the almost global existence for F with $\lambda = 3$ under the condition that $F(u, 0, 0) = O(|u|^5)$ near $u = 0$. Since it is known that we are not able to get the global existence for $(1.1)_\varepsilon$ with general cubic nonlinearity, we need some restriction on the cubic part of F for that purpose. According to the works of Godin ([2]) and Hoshiga ([4]), we can show that when $n = 2$ and $\lambda \geq 3$, if we assume

(H1) $F^{(3)}(u, \mathbf{v}, \mathbf{w})$ (defined by (1.3) with $z = (u, \mathbf{v}, \mathbf{w})$) satisfies the null condition,

and if F does not depend on u , then there exists a global solution to the problem $(1.1)_\varepsilon$ for any small data. Their proof is based on the *a priori* estimate of the classical energy and some decay estimates for the solution. When F depends also on u , the author ([6]) proved the global existence of small solutions under the assumptions (H1) and

(H2) $F(u, 0, 0) = O(|u|^5)$ near $u = 0$,

by developing the estimate for L^2 -norms of u itself similar to Klainerman's energy in [9]. The difficulty in two space-dimensional cases which does not appear in higher space dimensions is that the L^2 -norm of the solution even for linear wave equations with C_0^∞ -data does not stay bounded in general (of course, L^2 -norms of its gradients stay bounded). Hence the term u^4 prevents us to get a suitable *a priori* estimate for L^2 -norms of u , and this is the reason why we needed (H2) in [6]. But (H2) seems removable, if we compare the result of [6] with Klainerman's global existence result for $n = 3$, or Li - Zhou's result for $n = 2$. Our aim is to show that this observation is true.

Theorem 1.1. *Let $n = 2$ and $\lambda = 3$ in (1.2). Assume (H1) is fulfilled. Then for any $\phi, \psi \in C_0^\infty(\mathbf{R}^2)$, there exists a positive constant ε_0 such that $(1.1)_\varepsilon$ admits a unique solution $u \in C^\infty([0, \infty) \times \mathbf{R}^2)$ for any $\varepsilon \leq \varepsilon_0$.*

We will get a priori estimates of $L^{\frac{2}{1-\gamma}}$ -norms of u with some $\gamma > 0$ instead of L^2 -norms. The decay of the solution with respect to $1 + |t - |x||$ plays an important role in our proof.

Remark. Concerning the semi-linear wave equations of the form $\square u = F(u, Du)$ in $(0, \infty) \times \mathbf{R}^2$ with $F(u, v) = O(|u|^2 + |v|^2)$, we can prove the global existence of small solutions, provided that both $F^{(2)}$ and $F^{(3)}$ satisfy the null condition. In fact, it is known that when $F^{(2)}$ satisfies the null condition, we can eliminate the quadratic term of F by a certain transformation in this case (see [2]), and then Theorem 1.1 implies the assertion.

Concerning the almost global existence result in [10], we conjecture that the condition $F(u, 0, 0) = O(|u|^5)$ is also relaxed to $F(u, 0, 0) = O(|u|^4)$, but this seems to remain open. For the equations with nonlinear terms which may contain u^4 , we have:

Theorem 1.2. *Let $n = 2$ and $F(u, Du, D_x Du) = \sum_{a=0}^2 \partial_a G_a(u, Du) + H(u, Du, D_x Du)$, where $G_a(u, v) = O(|u|^3 + |v|^3)$ ($a = 0, 1, 2$), $H(u, v, w) = O(|u|^3 + |v|^3 + |w|^3)$ and $H^{(3)}$ satisfies the null condition. Then for any $\phi, \psi \in C_0^\infty(\mathbf{R}^2)$, there exist positive constants ε_0 and A such that $(1.1)_\varepsilon$ admits a classical solution for $0 \leq t < T_\varepsilon$ provided $\varepsilon \leq \varepsilon_0$, where $T_\varepsilon \geq \exp(A\varepsilon^{-2})$.*

2. Outlines of the proof We sketch the proof of Theorem 1.1 here. See [7] for full details of the proof. Following Klainerman [8], we introduce $\Gamma_0 = t\partial_t + \sum_{i=1}^2 x_i \partial_i$, $\Gamma_1 = t\partial_1 + x_1 \partial_t$, $\Gamma_2 = t\partial_2 + x_2 \partial_t$, $\Gamma_3 = x_1 \partial_2 - x_2 \partial_1$, $\Gamma_4 = \partial_t$, $\Gamma_5 = \partial_1$ and $\Gamma_6 = \partial_2$. We write $\Gamma^\alpha = \Gamma_0^{\alpha_0} \cdots \Gamma_6^{\alpha_6}$ for any multi-index $\alpha = (\alpha_0, \dots, \alpha_6)$. With these operators, Klainerman's decomposition of the null forms (see [9]) as well as Hörmander's $L^1 - L^\infty$ estimate (see [3]) is available. For any non-negative integer s , we define

$$|v(t, x)|_s = \sum_{0 \leq |\alpha| \leq s} |\Gamma^\alpha v(t, x)|,$$

$$\|v(t)\|_{s,p} = \|v(t, \cdot)\|_{s,p} = \left\| |v(t, \cdot)|_s \right\|_{L^p(\mathbf{R}^2)} \quad \text{for } 1 \leq p \leq \infty.$$

We also write $\|\cdot\|_p$ for $\|\cdot\|_{L^p(\mathbf{R}^2)}$.

Now let $u(t, x)$ be a smooth solution of $(1.1)_\varepsilon$ for $0 \leq t < T$ with some $T > 0$. We note that $\text{supp } u(t, \cdot) \subset \{x \in \mathbf{R}^2; |x| \leq t + R\}$ for $t \geq 0$ with some $R > 0$, because initial data are compactly supported. We want to get an a priori estimate of

$$E(T) = \sup_{0 \leq t < T} \left\{ \left\| W_\kappa(t, \cdot) |u(t, \cdot)|_{k+2} \right\|_\infty + (1+t)^{-\mu} \left(\|u(t, \cdot)\|_{2k, \frac{2}{1-\gamma}} + \|Du(t, \cdot)\|_{2k+1, 2} \right) \right\},$$

where k is a sufficiently large integer, $W_\kappa(t, x) = (1 + t + |x|)^{1/2} (1 + |t - |x||)^{(1-\kappa)/2}$ and κ is a constant with $0 < \kappa < 1/3$, while $\mu > 0$ and $\gamma \in (0, 1)$ are constants small enough. Here we note that $\|Du(t, \cdot)\|_{2k+1,2}$ cannot be replaced by $\|Du(t, \cdot)\|_{2k,2} + \|D^2u(t, \cdot)\|_{2k,2}$ in our proof, because we use the decomposition of the null forms in the estimates of the first and second terms of $E(T)$. From the definition of $E(T)$, direct calculations yield

$$(2.1) \quad \left\| |u(t, \cdot)|_{k+2}^q \right\|_{L^p(\mathbf{R}^2)} \leq (1+t)^{-\frac{q}{2} + \frac{1}{p}} E(T)^q \quad \text{for } 0 \leq t < T$$

if $qp(1-\kappa) > 2$. This is one of the essential estimates in our proof.

Since we are only concerned with small solutions, we may assume $E(T) \leq 1$. Define $F_1(u, v, w) = F(u, v, w) - F(u, 0, 0)$ and $F_2(u) = F(u, 0, 0)$. Clearly we have $F = F_1 + F_2$ and $F_1(u, 0, 0) = 0$. From the assumption (H1), it follows that $F_1^{(3)}$ satisfies the null condition and $F_2(u, 0, 0) = O(|u|^4)$ near $u = 0$. Now we prepare some lemmas.

Lemma 2.1. *Let $v(t, x)$ be a smooth function satisfying $\text{supp } v(t, \cdot) \subset \{|x| \leq t + R\}$ with some $R > 0$. Then we have*

$$(2.2) \quad \left\| \frac{v(t, \cdot)}{1 + |t - |\cdot||} \right\|_{L^2(\mathbf{R}^n)} \leq C_R \|Dv(t, \cdot)\|_{L^2(\mathbf{R}^n)}.$$

Lemma 2.2. *For any smooth function $v(t, x)$ vanishing sufficiently fast at spatial infinity, we have*

$$(2.3) \quad (1 + |t - |x||) |v(t, x)| \leq C |v(t, x)|_1 \quad \text{for } (t, x) \in [0, \infty) \times \mathbf{R}^n.$$

For the proof of Lemmas 2.1 and 2.2, see Lindblad [12].

Since the cubic part of F_1 satisfies the null condition, using the decomposition of the null forms (see [9]), and then by Hölder's inequality with the help of Lemmas 2.1 and 2.2 we get

$$(2.4) \quad \begin{aligned} \|F_1(u, Du, D_x Du)(t)\|_{s,p} &\leq C(1+t)^{-1} \left\| |u(t, \cdot)|_{[s/2]+2}^2 \right\|_q \|Du(t)\|_{s+1,2} \\ &\quad + C \left\| |u(t, \cdot)|_{[s/2]+2}^3 \right\|_q \left(\|Du(t)\|_{s,2} + \|D^2u(t)\|_{s,2} \right), \end{aligned}$$

where $1 \leq p \leq 2$ and $1/q = 1/p - 1/2$. For F_2 , it is easy to see that

$$(2.5) \quad \|F_2(u)(t)\|_{s,p} \leq C \left\| |u(t, \cdot)|_{[s/2]}^3 \right\|_q \|u(t)\|_{s, \frac{2}{1-\gamma}},$$

where $1 \leq p \leq 2/(1-\gamma)$ and $1/q = 1/p - (1-\gamma)/2$.

Now we are going to estimate $E(T)$. In the following, C_k represents various constants which may change line by line, but are independent of T . Applying Hörmander's $L^1 - L^\infty$ estimate ([3]) to (1.1) $_\varepsilon$, we have

$$(2.6) \quad W_\kappa(t, x) |u(t, x)|_{k+2} \leq C_k \left(\varepsilon + \int_0^t (1+\tau)^{-\frac{\kappa}{2}} \|F(\tau, \cdot)\|_{k+3,1} d\tau \right).$$

When k is large enough, by (2.4) and (2.5) we get

$$(2.7) \quad \|F(\tau, \cdot)\|_{k+3,1} \leq C_k \left\{ (1+\tau)^{\mu-\frac{3}{2}} E(T)^3 + (1+\tau)^{\mu-1} E(T)^4 \right\} \\ + C_k (1+\tau)^{\mu-1+\frac{\gamma}{2}} E(T)^4 \leq C_k (1+t)^{\mu-1+\frac{\gamma}{2}} E(T)^3.$$

Here we used (2.1) which is applicable for small γ . Observing that $\mu-1+\gamma/2-\kappa/2 < -1$ for sufficiently small μ and γ , from (2.6) and (2.7) we obtain

$$(2.8) \quad \|W_\kappa(t, \cdot)|u(t, \cdot)|_{k+2}\|_\infty \leq C_k (\varepsilon + E(T)^3) \quad \text{for } 0 \leq t < T.$$

According to Li - Zhou [11], we have

$$(2.9) \quad \|u(t, \cdot)\|_{2k,p} \leq C_k \left(\varepsilon + \int_0^t \|F(\tau, \cdot)\|_{2k,q} d\tau \right),$$

where $p > 2$ and $1/q = 1/p + 1/2$. Since (2.4), (2.5) and (2.1) imply $\|F(\tau)\|_{2k,2/(2-\gamma)} \leq C_k (1+t)^{\mu-1} E(T)^3$ as in (2.7), from (2.9) we obtain

$$(2.10) \quad \|u(t, \cdot)\|_{2k, \frac{2}{1-\gamma}} \leq C_k (1+t)^\mu (\varepsilon + E(T)^3) \quad \text{for } 0 \leq t < T.$$

Finally let $|\alpha| \leq 2k+1$. Then we have from (1.1) $_\varepsilon$

$$\square(\Gamma^\alpha u) - \sum_{j,a} \frac{\partial F_1}{\partial w_{ja}} \partial_j \partial_a (\Gamma^\alpha u) = \left(\tilde{\Gamma}^\alpha F_1 - \sum_{j,a} \frac{\partial F_1}{\partial w_{ja}} \partial_j \partial_a (\Gamma^\alpha u) \right) + \tilde{\Gamma}^\alpha F_2(u) \equiv I_1 + I_2,$$

where $\tilde{\Gamma}^\alpha = (\Gamma_0 + 2)^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_6^{\alpha_6}$. With the help of Lemmas 2.1 and 2.2, we have

$$(2.11) \quad \|I_1(t)\|_2 \leq C_k \|u(t)\|_{k+2,\infty}^2 (\|Du(t)\|_{2k+1,2} + \|D^2u(t)\|_{2k+1,2}) \\ \leq C_k (1+t)^{\mu-1} E(T)^3.$$

Note that the decomposition of the null forms is not used here. Since $\tilde{\Gamma}^\alpha F_2(u) \leq |u(t, x)|_k^3 |u(t, x)|_{2k+1}$, Hölder's inequality implies

$$(2.12) \quad \|I_2(t)\|_2 \leq C_k \left\| \left(1 + |t - |\cdot|| \right)^{\frac{1}{3}} |u(t, \cdot)|_k \right\|_\infty^3 \left\| \frac{|u(t, \cdot)|_{2k+1}}{1 + |t - |\cdot||} \right\|_2 \\ \leq C_k (1+t)^{-\frac{3}{2}} E(T)^3 \|Du(t)\|_{2k+1,2} \leq C_k (1+t)^{\mu-\frac{3}{2}} E(T)^4.$$

Here we have used Lemma 2.1 to estimate the last factor in the right-hand side of the first line. Now from the classical energy estimate for hyperbolic equations, we obtain

$$(2.13) \quad \|Du(t, \cdot)\|_{2k+1,2} \leq C_k (1+t)^\mu (\varepsilon + E(T)^3) \quad \text{for } 0 \leq t \leq T.$$

Summing up, we have proved $E(T) \leq C_k (\varepsilon + E(T)^3)$ provided $E(T) \leq 1$. By the local existence theorem and the standard arguments, this implies Theorem 1.1 immediately.

The proof of Theorem 1.2 can be done by getting an *a priori* estimate of

$$\sup_{0 \leq t < T} \left(\|W_0(t, \cdot)|u(t, \cdot)|_{k+2}\|_\infty + \|Du(t, \cdot)\|_{2k,2} \right).$$

In the estimate, the systematic usage of (2.1) and Lemma 2.1 is essential.

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INTERACTION OF ANALYTIC SINGULARITIES FOR SEMILINEAR WAVE EQUATIONS

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Abstract. In this article, we study the interaction of analytic singularities for nonlinear wave equations in the interior. We show that if three analytic conormal singularities hit at a single point, then new singularities may occur at that point.

1. Introduction.

In this article, we study the interaction of analytic singularities of solutions to semilinear wave equations in the interior. We consider the semilinear wave equations,

$$(1) \quad \square u = f(u, Du) \quad \text{in } \Omega \subset R_t \times R_x^2,$$

where u is a real valued function, $\square = \partial^2/\partial t^2 - \Delta$ with $\Delta = \sum_{j=1}^2 \partial^2/\partial x_j^2$, $Du = (\partial_t u, \nabla u)$, Ω is a bounded domain which contains the origin and $f(u, v)$ is an real analytic function of u and v .

We assume that all solutions u considered here are in $H^s(\Omega)$ with $s > 5/2$ where $H^s(\Omega)$ denotes a Sobolev space of order s in Ω . In 1982, J. Rauch and M. Reed [4] have made an example in which three singularities produce new singularities. In 1984, J. M. Bony [2] and R. Melrose and N. Ritter [3] have had a general result of this phenomenon for C^∞ singularity independently. We denote $\Sigma_j = \{(t, x) \in R^3; t = \omega_j \cdot x\}$ ($j = 1, 2, 3$) with $\omega_j \in S^1$. Their result for the equation (1) is as follows.

Theorem 1 (J. M. Bony [2], R. Melrose and N. Ritter [3]). *If $u \in H^s(\Omega)$ with $s > 5/2$, u satisfies (1) and u is conormal with respect to $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ in $\Omega_- = \Omega \cup \{t < 0\}$, then the solution u is C^∞ in $K \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \{t^2 = |x|^2\})$ where K is a domain of determine with respect to Ω_- .*

In this paper, we shall show the analytic singularity version of this result.

To state our results precisely, we introduce some notation and some function spaces. For $m \in R$ and for an open set $\omega \in R^n$, we denote the usual Sobolev space of order m on Ω by $H^m(\Omega)$. We denote $S = \inf_{(t,x) \in \Omega} t$, $T = \sup_{(t,x) \in \Omega} t$ and $\Omega(t) = \{x \in R^2; (t, x) \in \Omega\}$. $C((S, T); H^m(\Omega(t)))$ (or $C^1((S, T); H^m(\Omega(t)))$) denotes a set of $H^m(\Omega(t))$ valued continuous functions (or a set of $H^m(\Omega(t))$ valued continuously differentiable function)

in (S, T) respectively. Let Σ be a analytic submanifolds with codimension 1 in Ω or a union of two analytic submanifolds with codimension 1 which intersect transversally.

Definition 1 (Analytic conormal distribution). For $s \in \mathbb{R}$, we call that $u \in H^s(\Sigma, \omega; \Omega)$ if for any compact set $K \subset \Omega$ and for any analytic vector fields V_1, \dots, V_l which are tangent to Σ with any integer l , there exist constants $C, A > 0$ such that

$$(2) \quad \sup_{\substack{t \in [S, T] \\ j=1, 2}} \|\partial_t^j V_1^{\alpha_1} \cdots V_l^{\alpha_l} u\|_{H^s(K(t))} \leq CA^{|\alpha|} |\alpha|!$$

for any non negative integers $\alpha_1, \dots, \alpha_l$ with $|\alpha| = \alpha_1 + \cdots + \alpha_l$, where $K(t) = \{x \in \mathbb{R}^2; (t, x) \in K\}$.

Assumption 1. The nonlinear term $f \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ satisfies that for any positive number $M > 0$, there exist positive constants C_1 and A_1 such that

$$(3) \quad \sup_{|u| < M, |v| < M} |\partial_u^k \partial_v^\alpha f(u, v)| \leq C_1 A_1^{k+|\alpha|} k! |\alpha|!$$

Theorem 2. Suppose that the nonlinear term f satisfies the assumption 1, u is in $C((S, T); H^m(\Omega)) \cap C^1((S, T); H^{m-1}(\Omega(t)))$ with some integer $m \geq 3$, u satisfies the equation (1) and $u \in H^m(\Sigma_1, \omega; \Omega_-)$. Then the solution u satisfies that

$$(4) \quad u \in H^m(\Sigma_1, \omega; K),$$

where $S = \inf_{(t,x) \in \Omega} t$, $T = \sup_{(t,x) \in \Omega} t$, $\Omega_- = \Omega \cap \{(t, x); t < 0\}$ and K is a domain of determine with respect to Ω_- .

Theorem 3. Suppose that the nonlinear term f satisfies the assumption 1, u is in $C((S, T); H^m(\Omega)) \cap C^1((S, T); H^{m-1}(\Omega(t)))$ with some integer $m \geq 3$, u satisfies the equation (1) and $u \in H^m(\Sigma_1 \cup \Sigma_2, \omega; \Omega_-)$. Then the solution u is real analytic in $K \setminus \Sigma_1 \cup \Sigma_2$, where $S = \inf_{(t,x) \in \Omega} t$, $T = \sup_{(t,x) \in \Omega} t$, $\Omega_- = \Omega \cap \{(t, x); t < 0\}$ and K is a domain of determine with respect to Ω_- .

Theorem 4 (Main result). Suppose that the nonlinear term f satisfies the assumption 1, u is in $C((S, T); H^m(\Omega(t))) \cap C^1((S, T); H^{m-1}(\Omega(t)))$ with some integer $m \geq 3$, u satisfies the equation (1) and

$$(5) \quad u \in H^m(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3, \omega; \Omega_-).$$

Then u is real analytic in $K \setminus \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_+$, where $\Gamma_+ = \{t^2 = |x|^2, t > 0\}$, $\Omega_- = \Omega \cap \{t < 0\}$ and K is a domain of determine with respect to Ω_- .

In what follows, C and C_j denote constants. They may be changed from line to line.

2. Preliminaries.

Let S and T be real numbers with $S \leq T$ and let K be a compact set in $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}_x^2$ such that each subset $K \cap \{(t, x); s \leq t \leq T\}$ is a domain of determine with respect to $K \cap \{(t, x); t \leq s\}$ for $S \leq s \leq T$. For $m \geq 1$ and $f \in C([S, T]; H^m(K(t))) \cap C^1([S, T]; H^{m-1}(K(t)))$, we denote

$$(6) \quad E_m(t)[f] = \sqrt{\|f(t)\|_{H^m(K(t))}^2 + \|\partial_t f(t)\|_{H^{m-1}(K(t))}^2}$$

with $K(s) = K \cap \{(t, x); t = s\}$. We recall the energy estimate for the operator, $\square v + g \cdot Dv + hv$.

Proposition 1 (Energy estimate). *Let g and h be C^∞ functions in K . For $v \in C([S, T]; H^m(K))$, we have*

$$(7) \quad E_m(t_2)[v] \leq C(T, \|g\|_{L^\infty}, \|h\|_{L^\infty}) (E_m(t_1)[v] + \sqrt{\int_{t_1}^{t_2} \|\square f + g \cdot Dv + hv\|_{H^{m-1}(K(\tau))}^2 d\tau})$$

for $S \leq t_1 < t_2 \leq T$.

For Σ_1 and Σ_2 , we write $\tilde{\omega}_j = (1, -\omega_j)$, $\tilde{\omega}_j^* = (1, \omega_j)$, $\nabla = (\partial_t, \partial_{x_1}, \partial_{x_2})$ and write

$$(8) \quad X_1 = \omega_1^{(1)} \partial_t + \partial_{x_1},$$

$$(9) \quad X_2 = \omega_1^{(2)} \partial_t + \partial_{x_2},$$

$$(10) \quad X_3 = t \partial_t + x \cdot \nabla,$$

$$(11) \quad X_4 = (\tilde{\omega}_1 \times \tilde{\omega}_2) \cdot \nabla.$$

Proposition 2. *We have*

$$(12) \quad [X_j, X_3] = X_j \quad \text{for } j = 1, 2, 4,$$

$$(13) \quad [\square, X_3] = 2\square,$$

$$(14) \quad \square X_3^l = (X_3 + 2)^l \square \quad \text{for } l \in \mathbb{N}$$

$$(15) \quad X_3^l \partial_t = \partial_t (X_3 - 1)^l \quad \text{for } l \in \mathbb{N}$$

3. Lemmas.

In this section, we prepare several lemmas which are used to prove the theorems.

For $\tilde{T}, t_0 \in \mathbb{R}$ with $t_0 < T$ and for $x_0 \in \mathbb{R}^2$, we write

$$\Xi = \{(t, x) \in \mathbb{R}_t \times \mathbb{R}_x^2; |x - x_0| < \tilde{T} - t, t_0 < t < T \leq \tilde{T}\},$$

and

$$\Gamma(t) = \{x \in \mathbb{R}^2; |x - x_0| < \tilde{T} - t\} \quad \text{for } t_0 \leq t \leq T.$$

Lemma 1. *Suppose that $u \in H^m(\Xi)$ with a integer $m \geq 3$, that u satisfies the equation (1) in Ξ , that the nonlinear term f satisfies the assumption 1 and that there exist positive constants C_6 and A_6 such that*

$$(16) \quad E[X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} u](t_0) \leq C_6 A_6^{|\alpha|} |\alpha|! \quad \text{for } \forall \alpha_1, \forall \alpha_2, \forall \alpha_3 \in \mathbb{N},$$

with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. Then there exist positive constants C_7 and A_7 such that

$$(17) \quad \sup_{t \in [t_0, T]} E[X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} u](t) \leq C_7 A_7^{|\alpha|} |\alpha|! \quad \text{for } \forall \alpha_1, \forall \alpha_2, \forall \alpha_3 \in \mathbb{N}.$$

Lemma 2. *Suppose that $u \in H^m(\Xi)$ with a integer $m \geq 3$, that u satisfies the equation (1) in Ξ , that the nonlinear term f satisfies the assumption 1 and that there exist positive constants C_8 and A_8 such that*

$$(18) \quad E[X_3^{\alpha_3} X_4^{\alpha_4} u](t_0) \leq C_8 A_8^{|\alpha|} |\alpha|! \quad \text{for } \forall \alpha_3, \forall \alpha_4 \in \mathbb{N},$$

with $|\alpha| = \alpha_3 + \alpha_4$. Then there exist positive constants C_9 and A_9 such that

$$(19) \quad \sup_{t \in [t_0, T]} E[X_3^{\alpha_3} X_4^{\alpha_4} u](t) \leq C_9 A_9^{|\alpha|} |\alpha|! \quad \text{for } \forall \alpha_3, \forall \alpha_4 \in \mathbb{N}.$$

Lemma 3. Suppose that $u \in H^m(\Xi)$ with a integer $m \geq 3$, u satisfies the equation (1) in Ξ , the nonlinear term f satisfies the assumption 1 and there exist positive constants C_{10} and A_{10} such that

$$(20) \quad E[X_3^l u](t_0) \leq C_8 A_8^l l! \quad \text{for } \forall l \in N.$$

Then there exist positive constants C_{11} and A_{11} such that

$$(21) \quad \sup_{t \in [t_0, T]} E[X_3^l u](t) \leq C_{11} A_{11}^l l! \quad \text{for } \forall l \in N.$$

Lemma 4. Suppose that

$$(22) \quad E(t)[X^\alpha u] \leq C A^{|\alpha|-1} (|\alpha| - 2)_+! \quad \text{for } \forall \alpha \geq 0,$$

and that u satisfies the equation (1). Then there exists a positive constant \widehat{C} such that

$$(23) \quad \|X^\alpha u\|_{H^{m-1}(\Gamma(t))} \leq \widehat{C} A^{|\alpha|-1} (|\alpha| - 2)_+! \quad \text{for } \forall \alpha \geq 0.$$

4. Regularity in the interior of the cone

In this section, we show that the solution u is analytic in the forward light cone from the origin if u is "analytic" with respect to X_3 . We denote $P = t\partial_t + x \cdot \partial_x = X_3$ and $\Gamma_+ = \{(t, x) \in R^3; t^2 > |x|^2, t > 0\}$. The following method which uses the operator P is due to M. Beals [1]. Let B_R be a ball in Γ_+ with radius R and the center (t_0, x_0) .

Proposition 3. If we take R sufficiently small, we have for a integer $s \geq 0$

$$(24) \quad \|f\|_{H^{s+2}(B_R)} \leq C_{18} \left(\|\square f\|_{H^s(B_R)} + \|P^2 f\|_{H^s(B_R)} \right),$$

for $f \in H_0^{s+2}(B_R)$ where $H_0^1(B_R)$ is a completion of $C_0^\infty(B_R)$ with the norm of $H^1(B_R)$.

The following is the key lemma to prove the theorem 3 and 4.

Lemma 5. Suppose that $u \in H^m(K)$, u satisfies (1) and that

$$(25) \quad \|P^l u\|_{H^m(K)} \leq C_{19} A_{19}^l l! \quad \text{for } \forall l \in N \cup \{0\},$$

for some $C_{19} > 0$ and $A_{19} > 0$. Then u is real analytic in $\Gamma_+ \cap K$.

5. Proof of the theorem 3.

In this section, we give a proof of the theorem 3. We divide $K \setminus (\Sigma_1 \cup \Sigma_2)$ into two parts \mathcal{O}_1 and \mathcal{O}_2 :

$$\begin{aligned} \mathcal{O}_1 &= \{(t, x) \in K \setminus (\Sigma_1 \cup \Sigma_2); t < \omega_1 \cdot x \text{ or } t < \omega_2 \cdot x\}, \\ \mathcal{O}_2 &= \{(t, x) \in K \setminus (\Sigma_1 \cup \Sigma_2); t > \omega_1 \cdot x, t > \omega_2 \cdot x\}. \end{aligned}$$

First, we show that u is analytic in \mathcal{O}_1 . Let (t_0, x_0) be a point in \mathcal{O}_1 . We write $\Xi_{(t_0, x_0)}^\epsilon = \{(t, x); |x - x_0| < t_0 - t + \epsilon, -\epsilon < t < t_0 + \epsilon\}$. Since (t_0, x_0) is under Σ_1 or Σ_2 in space-time, we can take $\epsilon > 0$ so small that $\Xi_{(t_0, x_0)}^\epsilon \subset \mathcal{O}_1$ and $\Xi_{(t_0, x_0)}^\epsilon$ has no intersection with Σ_1 or Σ_2 . Hence the theorem 2 yields that u is analytic in $\Xi_{(t_0, x_0)}^\epsilon \setminus (\Sigma_1 \cup \Sigma_2)$.

Next, we show that u is analytic in \mathcal{O}_2 . We denote $S = \Sigma_1 \cap \Sigma_2 \cap K$. For a while, we fix $(t_0, x_0) \in S$. We denote $P_{(t_0, x_0)} = (t - t_0)\partial_t + (x - x_0) \cdot \partial_x$ and $\Gamma_{(t_0, x_0)}^+ = \{(t, x) \in R^3; |t - t_0|^2 > |x - x_0|^2, t > t_0\}$. Since $t_0\partial + x_0 \cdot \partial_x = kX_4$ with a real number k ,

$P_{(t_0, x_0)} = X_3 - kX_4$. We fix a cone shaped domain $\Xi \subset K$. From the lemma 2 and the lemma 4, there exist positive constants C and A such that

$$\|X_3^{l_1} X_4^{l_2} u\|_{H^{m-1}(\Xi)} \leq C A^{l_1+l_2} (l_1 + l_2)!,$$

for $\forall l_1, \forall l_2 \in N\{0\}$. Until the end of the proof, we write $\|\cdot\|_{H^{m-1}(\Xi)} = \|\cdot\|$ for abbreviation.

$$\begin{aligned} \|P_{(t_0, x_0)}^l u\| &= \|(X_3 - kX_4)^l u\| \\ &\leq \sum_{2^l \text{ terms}} k^l \|X_{i(1)} \cdots X_{i(l)} u\|, \end{aligned}$$

where $i(j) = 3$ or 4 ($1 \leq j \leq l$).

$$\begin{aligned} \|X_{i(1)} \cdots X_{i(l)} u\| &\leq \|(X_3 + l_2)^{l_1} X_4^{l_2} u\| \quad (\text{with } l = l_1 + l_2) \\ &\leq \sum_{l'=0}^{l_1} \binom{l_1}{l'} l_2^{l_1-l'} \|X_3^{l'} X_4^{l_2} u\| \\ &\leq \sum_{l'=0}^{l_1} \binom{l_1}{l'} l_2^{l_1-l'} C A^{l'+l_2} (l' + l_2)! \\ &\leq C A^{l_1} l! \sum_{l'=0}^{l_1} \binom{l_1}{l'} A^{-(l_1-l')} \\ &\leq C A^{l_1} l! (1 + A^{-1})^{l_1} \\ &\leq C (1 + A)^{l_1} l!. \end{aligned}$$

Hence we have

$$(26) \quad \|P_{(t_0, x_0)}^l u\| \leq C (2k(1 + A))^{l_1} l!.$$

From the lemma 5 and (26), u is analytic in $\Gamma_{(t_0, x_0)}^+ \cap \Xi$. Since

$$\bigcup_{(t_0, x_0) \in S} \bigcup_{\Xi \subset K} (\Gamma_{(t_0, x_0)}^+ \cap \Xi) = \mathcal{O}_2,$$

u is analytic in \mathcal{O}_2 .

6. Proof of the theorem 4.

In this section, we complete the proof of the theorem 4. We divide $K \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_+)$ into two parts \mathcal{O}_1 and \mathcal{O}_2 :

$$\begin{aligned} \mathcal{O}_1 &= \{(t, x) \in K \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_+); t^2 < |x|^2 \text{ or } t < 0\}, \\ \mathcal{O}_2 &= \{(t, x) \in K \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_+); t^2 > |x|^2, t > 0\}. \end{aligned}$$

Let (t_0, x_0) be a point in \mathcal{O}_1 . We write $\Xi_{(t_0, x_0)}^\epsilon = \{(t, x); |x - x_0| < t_0 - t + \epsilon, -\epsilon < t < t_0 + \epsilon\}$. We can take $\epsilon > 0$ so small that $\Xi_{(t_0, x_0)}^\epsilon \subset \mathcal{O}_1$ and $\Xi_{(t_0, x_0)}^\epsilon$ has intersection with only two sets between Σ_1, Σ_2 and Σ_3 . The theorem 3 yields that u is analytic in $\Xi_{(t_0, x_0)}^\epsilon \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$. Combining the lemma 3 and the lemma 5, u is analytic in \mathcal{O}_2 .

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ON THE SMOOTHING PROPERTIES OF SOME DISPERSIVE HYPERBOLIC SYSTEMS

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Abstract. We prove that dispersive hyperbolic systems have in general worse regularizing properties than dispersive elliptic systems.

1. Introduction and statement of the results.

Consider the initial value problem

$$(1.1) \quad \begin{cases} i\partial_t u + Q(\partial_x)u = P(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}) & x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^n, \end{cases}$$

where $Q(\partial_x)$ denotes any real non degenerate quadratic form (i.e. $Q(\partial_x) = \sum_{j=1}^n \pm \partial_{x_j}^2$) and P is any polynomial of degree at least two.

In [6] a local well posedness result in, possibly weighted, Sobolev spaces is proved for small initial data. Existence and uniqueness of the solution is established using the contraction principle on a suitable function space. The main difficulty comes from the nonlinear terms which involved derivatives of u . These can be handled thanks to the following inequality which plays a crucial role in the whole argument.

$$(1.2) \quad \sup_{x_0, R} \frac{1}{R} \int_{B(x_0, R)} \int_{-\infty}^{\infty} |\nabla_x u(x, t)|^2 dx dt \leq C \|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^n)}^2 + \|F\|,$$

where u is a solution of the free lineal equation

$$(1.3) \quad \begin{cases} i\partial_t u + Q(\partial_x)u = F(x, t) & x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^n, \end{cases}$$

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$$\dot{H}^{1/2}(\mathbb{R}^n) = \left\{ u : \int |\hat{u}(\xi)|^2 |\xi| d\xi < +\infty \right\},$$

and

$$\|F\| = \sum_Q \|F\|_{L^2(Q; L_t^2)} = \sum_Q \left(\int_Q \int_{-\infty}^{\infty} |F(x, t)|^2 dt dx \right)^{1/2},$$

where the sum is made over all the unit cubes Q of disjoint interiors of a grid of \mathbb{R}^n . Above $\hat{\cdot}$ denotes the Fourier transform. For the homogeneous equation (i.e. $F \equiv 0$) inequality (1.2) was already known in the elliptic case. See [2], [9], and [10].

Shortly afterwards the size restriction on the data was removed by Hayashi and Ozawa [5], in the case of one space variable. They used a simple change of variable to eliminate the term in u_x . Hence, an equation with just terms in \bar{u}_x and of zero order is obtained and can be treated by the energy method.

More recently H. Chihara [1] has been able to avoid the smallness condition in the data in any dimension in the elliptic case (i.e. $Q(\partial_x) = \Delta_x$). He also transforms the equation using a pseudo-differential operator K of order zero in such way (see [4]) that the commutator $[K; i\Delta]$ basically eliminates the term in $\nabla_x u$ (see also [3]). Unfortunately the new terms obtained in $\nabla_x \bar{u}$ do not have the symmetry property $K\bar{u} \neq \overline{Ku}$ to be treated by the energy method. Chihara overcomes this difficulty by first diagonalizing the corresponding system in (u, \bar{u}) , so that the first order terms in $\nabla_x \bar{u}$ disappear.

It is in this diagonalization where the argument brakes down when the laplacian is changed by more general quadratic forms. The interest in these quadratic forms comes from the possibility to extend the method to more complicated situations as Davey–Stewartson, Zakharov and Zakharov–Schulman systems.

In a forthcoming paper the authors are able to bypass the diagonalization procedure and hence to avoid the size restriction for general quadratic forms. They combine with some modifications the ideas in [1], [4], and [6] already mentioned, together with a freezing argument in the temporal variable of the coefficients in $\nabla_x u$ and $\nabla_x \bar{u}$. This freezing reduces the question to prove the inequality (1.2) for finite time T to the variable coefficient equation

$$(1.4) \quad \begin{cases} i\partial_t u + Q(\partial_x)u + b_1(x) \cdot \nabla_x u + b_2(x) \cdot \nabla_x \bar{u} = f & (x, t) \in \mathbb{R}^{n+1}, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^n, \end{cases}$$

where b_1, b_2 are, say, functions in the Schwartz class. Once (1.2) is established the arguments in [6] can be applied. Our purpose in this paper is to illustrate that the regularizing properties of (1.4) for general (b_1, b_2) are quite better in the elliptic case (i.e. $Q(\partial_x) = \Delta_x$) than in the positive signature setting. To fix the ideas we reduce the problem to the simpler case $b_1 \equiv 0$ and to the homogeneous equation $F \equiv 0$. In fact, in [8] a pseudo-differential operator K is used to eliminate the term $b_1 \cdot \nabla_x u$ with the extra, and fundamental, property $K\bar{u} = \overline{Ku} + \text{error terms}$. Hence $b_1 \equiv 0$ can be considered as a model problem for our interests.

We have the following results.

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Theorem 1. Let $b \in \mathbb{C}^n$ with $|b| = 1$, $\square_x = \partial_{x_1}^2 - \Delta_{\bar{x}}$ for $x = (x_1, \bar{x}) \in \mathbb{R}^n$, and u the solution of the I.V.P.

$$(1.5) \quad \begin{cases} i\partial_t u + \square_x u + b \cdot \nabla_x \bar{u} = 0 & x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^n. \end{cases}$$

Then if $D_x^s = (-\Delta_x)^{s/2}$ and $0 \leq s \leq 1/4$,

$$(1.6) \quad \sup_{x_0} \int_{B(x_0, R)} \int_{-T}^T |D_x^s u(x, t)|^2 dx dt \leq CRT \|u_0\|_{L^2}^2.$$

Moreover (1.6) does not hold if $s > 1/4$.

However the results in the elliptic case are much stronger.

Theorem 2. Given $u_0 \in L^2(\mathbb{R}^n)$ and $b(x) = (b_j(x))_{j=1, \dots, n}$ such that $|D^\alpha b_j| \leq C$ if $|\alpha| \leq 2$, there exists a unique solution to

$$(1.7) \quad \begin{cases} i\partial_t u + \Delta_x u + b(x) \cdot \nabla_x \bar{u} = 0 & x \in \mathbb{R}^n, |t| \leq T, \\ u(x, 0) = u_0(x), \end{cases}$$

such that $u \in \mathcal{C}([-T, T] : L^2(\mathbb{R}^n))$.

Moreover

$$(1.8) \quad \sup_{x_0, R} \frac{1}{R} \int_{B(x_0, R)} \int_{-T}^T |D_x^{1/2} u(x, t)|^2 dt dx \leq C(T) \|u_0\|_{L^2}^2.$$

2. Proof of theorem 1.

Define

$$(2.1) \quad A = \begin{pmatrix} \square_x & b \cdot \nabla_x \\ -\bar{b} \cdot \nabla_x & -\square_x \end{pmatrix},$$

with $\omega = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$ and $\omega_0 = \begin{pmatrix} u_0 \\ \bar{u}_0 \end{pmatrix}$. Then the I.V.P. (1.5) can be written as:

$$(2.2) \quad \begin{cases} i\partial_t \omega + A\omega = 0 \\ \omega(x, 0) = \omega_0(x). \end{cases}$$

Now observe that

$$A^2 = \begin{pmatrix} \square_x^2 - |b \cdot \nabla_x|^2 & 0 \\ 0 & \square_x^2 - |\bar{b} \cdot \nabla_x|^2 \end{pmatrix}$$

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where $|b \cdot \nabla_x|^2 = |\sum_j b_j \partial_j|^2 = \sum_{j,k} b_j \bar{b}_k \partial_j \partial_k$. Hence from (2.2) and applying the operator $-i\partial_t + A$ we obtain that ω also solves

$$(2.3) \quad \begin{cases} \partial_t^2 u + A^2 \omega = 0 \\ \omega(x, 0) = \omega_0(x) \\ \omega_t(x, 0) = i A \omega_0(x). \end{cases}$$

Therefore we are reduced to study the scalar I.V.P.

$$(2.4) \quad \begin{cases} \partial_t^2 v + (\square_x^2 - |b \cdot \nabla_x|^2) v = 0 \\ v(x, 0) = f \\ v_t(x, 0) = g. \end{cases}$$

Notice that the solution of (2.4) is written as

$$(2.5) \quad v(x, t) = \cos t B(f) + \sin t B(B^{-1}g),$$

where

$$(2.6) \quad (\cos t B(f))^\wedge(\xi) = \cos t (Q^2(\xi) + |b \cdot \xi|^2)^{1/2} \hat{f}(\xi),$$

$$(2.7) \quad (\sin t B(g))^\wedge(\xi) = \sin t (Q^2(\xi) + |b \cdot \xi|^2)^{1/2} \hat{g}(\xi),$$

and $Q(\xi) = \eta^2 - |\bar{\xi}|^2$ with $\xi = (\eta, \bar{\xi})$.

In particular the theorem follows from the inequality

$$(2.8) \quad \sup_{x_0} \int_{B(x_0, R)} \int_{-T}^T |D_x^{1/4} e^{itB} v_0|^2 dt dx \leq CRT \|v_0\|_{L^2(\mathbb{R}^n)}^2,$$

since

$$(2.9) \quad \frac{|Q(\xi)| + |b \cdot \xi|}{(Q^2(\xi) + |b \cdot \xi|^2)^{1/2}} \leq C.$$

We write

$$(2.10) \quad e^{itB} u_0(x) = \int_{\mathbb{R}^n} e^{it\Phi(\xi) + ix \cdot \xi} \hat{u}_0(\xi) d\xi$$

where $\Phi(\xi) = (Q^2(\xi) + |b \cdot \xi|^2)^{1/2}$.

Hence

$$(2.11) \quad \nabla_x \Phi(\xi) = \frac{1}{\Phi(\xi)} (Q(\xi)\xi_Q + |b \cdot \xi|b),$$

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where $Q(\xi)\xi_Q = \frac{1}{2}\nabla Q^2(\xi)$.

Next we consider two different regions. Write $\xi = (\eta, \bar{\xi})$,

$$\begin{aligned}\Gamma_1 &= \left\{ \xi : \left| |\bar{\xi}| - |\eta| \right| \leq |\xi|^{-1/2} \right\}, \\ \Gamma_2 &= \left\{ \xi : \left| |\bar{\xi}| - |\eta| \right| \geq |\xi|^{-1/2} \right\},\end{aligned}$$

and u_{0j} as $\widehat{u_{0j}}(\xi) = \chi_{\Gamma_j}(\xi) \hat{u}_0(\xi)$, $j = 1, 2$.

Now notice that

$$(2.12) \quad \frac{|\widehat{u_{02}}(\xi)|^2}{|\nabla_x \Phi(\xi)|} \leq \frac{|\hat{u}_0(\xi)|^2}{|\xi|^{1/2}}.$$

Thus (1.6) for u_{02} follows from theorem 4.1 in [7].

Let us just consider the inequality for u_{01} . First notice that if $(\eta, \bar{\xi}) = \xi \in \Gamma_1$ then

$$(2.13) \quad |\eta| \sim |\bar{\xi}| \simeq |\xi|.$$

Now we have

$$\begin{aligned}e^{itB}u_{01}(x) &= \int e^{it(Q^2(\xi)+|b \cdot \xi|^2)^{1/2}} e^{ix \cdot \xi} \hat{u}_{01}(\xi) d\xi \\ &= \int_{\bar{\xi}} e^{i\bar{x} \cdot \bar{\xi}} \left(\int_{\eta} e^{it\Phi(\xi)+ix_1\eta} \widehat{u_0}(\eta, \bar{\xi}) d\eta \right) d\bar{\xi}.\end{aligned}$$

Using Plancherel's theorem in \bar{x} we have

$$\begin{aligned}\int_{B(x_0, R)} \int_{-T}^T |D^{1/4} e^{itB} u_{01}(x)|^2 dt dx_1 d\bar{x} \\ \leq 2R \sup_{x_1} \int_{t=-T}^T \int_{\mathbb{R}^{n-1}} |D^{1/4} e^{itB} \hat{u}_{01}(x)|^2 d\bar{x} dt \\ = CR \sup_{x_1} \int_{-T}^T \int_{\bar{\xi}} \left| \int_{\eta} e^{it\Phi(\xi)+ix_1\eta} |\xi|^{1/4} \hat{u}_{01}(\eta, \bar{\xi}) d\eta \right|^2 d\bar{\xi} dt \\ \leq CTR \int_{\bar{\xi}} (\|\chi_{\Gamma_1}(\cdot, \bar{\xi})\|_{L^2}^2 \int_{\eta} |\xi|^{1/2} |\hat{u}_0(\eta, \bar{\xi})|^2 d\eta) d\bar{\xi} \\ \leq CTR \|u_0\|_{L^2(\mathbb{R}^n)}^2\end{aligned}$$

as desired.

Let us now prove the sharpness of the inequality (1.6). Consider the I.V.P.

$$(2.14) \quad \begin{cases} i\partial_t u + \partial_{xy}^2 u - \partial_x \bar{u} = 0 & (x, y) \in \mathbb{R}^2, t \in \mathbb{R} \\ u(x, 0) = u_0(x) & (x, y) \in \mathbb{R}^2. \end{cases}$$

Hence u also solves the I.V.P.

$$(2.15) \quad \begin{cases} \partial_t^2 u + (\partial_{xy}^2)^2 u - \partial_x^2 u = 0 \\ u(x, 0) = u_0(x) \\ u_t(x, 0) = i(\partial_{xy}^2 u_0 - \partial_x \bar{u}_0). \end{cases}$$

Write

$$\begin{aligned} u &= \int_{\mathbb{R}^2} \cos(t|\xi|\sqrt{1+\eta^2}) e^{ix\xi+iy\eta} \hat{u}_0(\xi, \eta) d\xi d\eta \\ &+ \int_{\mathbb{R}^2} \sin(t|\xi|\sqrt{1+\eta^2}) e^{ix\xi+iy\eta} \left(\frac{-i\xi\eta}{|\xi|\sqrt{1+\eta^2}} \hat{u}_0(\xi, \eta) \right. \\ &\left. + \frac{\xi}{|\xi|\sqrt{1+\eta^2}} \bar{\hat{u}}_0(-\xi, -\eta) \right) d\xi d\eta. \end{aligned}$$

Now choose for a given $N > 1$

$$u_0(x, y) = (1-i)\psi_N(x, y)$$

with $\hat{\psi}_N(\xi, \eta) = \varphi(\xi - N)\varphi(N^{1/2}\eta)$ and φ an even, real, C^∞ function supported in $[-1, -1/4] \cup [1/4, 1]$. Then straightforward computations give

$$\begin{aligned} u(x, y, t) &= \int_{\mathbb{R}^2} e^{it\xi\sqrt{1+\eta^2}+ix\xi+iy\eta} \hat{u}_0(\xi, \eta) d\xi d\eta + O(N^{-1}) \\ &= \int_{\mathbb{R}^2} e^{it\xi(1+\frac{\eta^2}{2})+ix\xi+iy\eta} \hat{u}_0(\xi, \eta) d\xi d\eta + O(N^{-1}) \\ &= (1-i)e^{itN} \int e^{itN\frac{\eta^2}{2}+iy\eta} \hat{\varphi}\left(x + t(1 + \frac{\eta^2}{2})\right) \varphi(N^{1/2}\eta) d\eta + O(N^{-1}). \end{aligned}$$

Using the stationary phase lemma we trivially have that if $|t - 1/2| \leq 1/100$

$$|u(x, y, t)| \geq CN^{-1/2} \chi_{[-1/10, 1/10]}(x) \chi_{[-N^{1/2}/2, N^{1/2}/2]}(y),$$

and therefore for large N ,

$$\left(\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 |(u(x, y, t))|^2 dx dy dt \right)^{1/2} \geq CN^{-1/2}.$$

On the other hand $\|D^{-s}u_0\|_{L^2} \leq CN^{-s-1/4}$. Hence $s \leq 1/4$.

3. Proof of theorem 2.

As in the proof of Theorem 1 we write the I.V.P.

$$(3.1) \quad \begin{cases} i\partial_t u + \Delta_x u + b(x) \cdot \nabla_x \bar{u} = 0 & x \in \mathbb{R}^n, |t| \leq T, \\ u(x, 0) = u_0(x), \end{cases}$$

as

$$(3.3) \quad \begin{cases} i\partial_t \omega + A\omega = 0 \\ \omega(x, 0) = \omega_0(x), \end{cases}$$

where in this case we have $\omega = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$, $\omega_0 = \begin{pmatrix} u_0 \\ \bar{u}_0 \end{pmatrix}$ and

$$(3.4) \quad A = \begin{pmatrix} \Delta_x & b \cdot \nabla_x \\ -\bar{b} \cdot \nabla_x & -\Delta_x \end{pmatrix}.$$

Now observe that

$$(3.5) \quad A^2 = \begin{pmatrix} \Delta_x^2 - b \cdot \nabla_x \bar{b} \cdot \nabla_x & [\Delta_x; b \cdot \nabla_x] \\ -[\bar{b} \cdot \nabla_x; \Delta_x] & \Delta_x^2 - \bar{b} \cdot \nabla_x b \cdot \nabla_x \end{pmatrix},$$

where $[C; D] = CD - DC$. Assume that u is a solution of (3.1) such that

$$(3.6) \quad \sup_{[-T, T]} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq C(T) \|u_0\|_{L^2(\mathbb{R}^n)}.$$

Then ω is a solution of

$$(3.7) \quad \begin{cases} (\partial_t^2 + A^2)\omega = 0 \\ \omega(x, 0) = \omega_0(x) \\ \omega_t(x, 0) = iA\omega_0(x). \end{cases}$$

We shall prove that solutions of (3.7) verify (1.8) and, therefore so does ω . Write

$$(3.8) \quad \begin{aligned} A^2 &= \Delta_x^2 \omega + \begin{pmatrix} -(b \cdot \nabla_x)(\bar{b} \cdot \nabla_x) & [\Delta_x; b \cdot \nabla_x] \\ -[\Delta_x; \bar{b} \cdot \nabla_x] & -\bar{b} \cdot \nabla_x b \cdot \nabla_x \end{pmatrix} \omega \\ &= \Delta_x^2 + B\omega. \end{aligned}$$

Then

$$(3.9) \quad \begin{aligned} \omega(t) &= \cos t \Delta_x \omega_0 + \sin t \Delta_x (\Delta_x^{-1} \partial_t \omega_0) \\ &\quad + \int_0^t \sin(t-r) \Delta_x (\Delta_x^{-1} B \omega(\cdot, r)) dr. \end{aligned}$$

Notice that B is a differential matrix operator with bounded coefficients. On the other hand we trivially have that

$$(3.10) \quad \|\cos t \Delta_x \omega_0\|_{L^2} \leq \|\omega_0\|_{L^2},$$

$$(3.11) \quad \|\sin t \Delta_x (\Delta_x^{-1} F)\|_{L^2} \leq |t| \|F\|_{H^{-2}}.$$

Hence

$$(3.12) \quad \sup_{|t| \leq T} \|\omega(t)\|_{L^2} \leq C\|\omega_0\|_{L^2} + CT^2 \sup_{|t| \leq T} \|\omega(t)\|_{L^2},$$

which allows us to solve (3.9) for T small enough depending just on the size of b . By repeating the argument we solve (3.9) for arbitrary large T .

Moreover

$$(3.13) \quad \sup_{|t| \leq T} \|\omega(t)\|_{L^2} \leq C(T)\|\omega_0\|_{L^2}.$$

Now use (3.9) and theorem 4.1 in [7] to obtain

$$\begin{aligned} \sup_{x_0, R} \frac{1}{R} \int_{B(x_0, R)} \int_{-T}^T |D_x^{1/2} \omega(t)|^2 dt dx \\ \leq C\|\omega_0\|_{L^2} + T \int_{-T}^T \|B\omega(\cdot, r)\|_{H^{-2}} dr \\ \leq C(T)\|\omega_0\|_{L^2}. \end{aligned}$$

Finally we must establish the solvability of (3.1).

Define

$$(3.14) \quad B_s = \begin{pmatrix} 0 & b \cdot \nabla_x + \frac{1}{2} \nabla_x \cdot b \\ -\bar{b} \cdot \nabla_x - \frac{1}{2} \nabla_x \cdot \bar{b} & 0 \end{pmatrix},$$

$$(3.15) \quad B_b = \begin{pmatrix} 0 & -\frac{1}{2} \nabla_x \cdot b \\ \frac{1}{2} \nabla_x \cdot \bar{b} & 0 \end{pmatrix},$$

$$(3.16) \quad H = \begin{pmatrix} \Delta_x & 0 \\ 0 & -\Delta_x \end{pmatrix}.$$

Now notice B_s is a symmetric operator with $H^2 \subset D(B_s)$. Moreover

$$\|B_s \omega\|_{L^2} \leq \|\omega\|_{H^1} \leq 1/2 \|\omega\|_{H^2} + C\|\omega\|_{L^2}.$$

By Kato–Rellich's theorem $H + B_s$ is a self adjoint operator and therefore

$$(3.17) \quad \|e^{it(H+B_s)} \omega_0\|_{L^2} = \|\omega_0\|_{L^2}.$$

Now consider the integral equation

$$(3.18) \quad \omega(t) = e^{it(H+B_s)} \omega_0 + i \int_0^t e^{i(t-r)(H+B_s)} B_b \omega(r) dr.$$

Then

$$\sup_{|t| \leq T} \|\omega\|_{L^2} \leq \|\omega_0\|_{L^2} + CT \sup_{|t| \leq T} \|\omega\|_{L^2}.$$

Taking T small enough depending just on the size of b we can find an inverse for (3.18) written as a Neuman series. Hence the solvability of

(3.1) is established.

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Vortex filament equation and semilinear Schrödinger equation

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Abstract: We consider an initial value problem of the vortex filament equation:

$$\gamma_t = \gamma_x \times \gamma_{xx} \quad (|\gamma_x| \equiv 1) \quad (\times \text{ is the exterior product}).$$

In this paper, we will prove the existence and the uniqueness of a classical solution for the initial value problem, and generalize it to the case of curves in 3-dimensional space forms. We also consider related semilinear Schrödinger equations for curves in Kähler manifolds. It is remarkable that we need symmetric spaces as manifolds for infinite time existence of solutions.

0. Introduction

Let $\gamma(x, t)$ be a solution of the vortex filament equation. If we introduce a complex valued function $u(x, t)$ by

$$u(x, t) = \kappa \exp(\sqrt{-1} \int \tau dx),$$

where κ (resp. τ) is the curvature (resp. torsion) of γ , then u satisfies $u_t = \sqrt{-1}(u_{xx} + (1/2)|u|^2u)$. This is called Hasimoto's transformation. However, we should note that this transformation is not defined when κ vanishes at some point. In fact, even $\int \tau dx$ may be discontinuous.

In section 1, we will give another transformation using not Frenet-Serret's formula but 'development of curve'. Our transformation is well defined even when κ vanishes at some point, and it gives same result when κ does not vanish (Proposition 1.3, Theorem 1.4). As a result, we have a unique global existence result for the vortex filament equation (Theorem 1.6).

More precisely, we will show that the following 3 PDEs are equivalent.

$$\begin{aligned} (0.1) \quad & \gamma_t = \gamma_x \times \gamma_{xx} \quad (|\gamma_x| \equiv 1) && \text{in } \mathbf{R}^3, \\ (0.2) \quad & \xi_t = J \nabla_x \xi_x && \text{in } S^2, \\ (0.3) \quad & u_t = \sqrt{-1}(u_{xx} + (1/2)|u|^2u) && \text{in } \mathbf{C}. \end{aligned}$$

Equation (0.1) has a natural generalization to a 3-dimensional riemannian manifold. We will give a unique global existence result in a 3-dimensional space form (Theorem 2.2). We will consider equation (0.2) in a Kähler manifold, and will get a unique short-time existence result (Theorem 3.1).

We will find that equation (0.2) in a hermitian symmetric space has very similar property with that in S^2 . In fact, equations (0.1), (0.2) and (0.3) are generalized and each has a unique global solution for any initial data (Theorem 4.2, Proposition 4.3, Proposition 4.4).

We use following notations: On a riemannian manifold, we denote by ∇ the covariant derivation and by R the curvature tensor. The partial derivation is denoted by ∂ or the subscript, e.g., $\partial_x \gamma$, γ_x . We denote by $(*, *)$ the pointwise inner product, by $\langle *, * \rangle$ the L_2 inner product for x -direction, by $\| * \|$ the L_2 -norm for x -direction.

We mainly consider closed curves defined on $S^1 = \mathbf{R}/\mathbf{Z}$ and quasi-periodic curves defined on \mathbf{R} . When curves are not closed, we should set some appropriate boundedness condition or boundary condition. We only treat C^∞ -objects.

After this work was done, the author received a preprint [7] by T. Nishiyama and A. Tani. They prove the existence and uniqueness of a vortex filament equation containing γ_{xxx} , which is more general than equation (0.1). However, their method can be applied only on the case of \mathbf{R}^3 . (Compare with Theorem 2.2).

1. Vortex filament equation in the euclidean space

If we set $\xi = \gamma_x$, then ξ becomes a family of curves in S^2 . We rewrite the equation by means of ξ and get an equation:

$$\xi_t = (\gamma_x \times \gamma_{xx})_x = \xi \times \xi_{xx}.$$

Using the covariant derivation ∇ and the complex structure J on S^2 , this equation is expressed as

$$(1.1) \quad \xi_t = J \nabla_x \xi_x,$$

and locally as

$$(1.2) \quad z_t = \sqrt{-1} \left(z_{xx} - \frac{2\bar{z}}{1+|z|^2} z_x^2 \right).$$

Remark 1.1. We will see that initial value problem (1.1) has a unique all-time solution for any initial data. However, a solution of equation (1.2) may diverge at finite time, because z may cross the point ∞ .

We transform solutions of equation (1.1) by means of ‘development of curve’.

Definition 1.2. Let c be a curve in a riemannian manifold M and $F = \{e_i\}$ a parallel orthonormal frame field along c . We call such a pair a *curve c with frame field F* . For a curve with frame field, we represent its velocity vector as

$$c'(x) = u^i(x) e_i(x).$$

The integral $\int u^i(x) dx$ is called *the development of c to the euclidean space*. In this paper, we do not use the development itself, but *the differential u of the development*. If c is closed, u is quasi-periodic.

Let ξ be a solution of (1.1). We attach to it a frame field $\{e_i\}$, and seek conditions for the differential u of its development. We fix the orientation of the frame by $Je_1 = e_2$. From $\nabla_x e_i \equiv 0$, we see

$$\partial_x(e_2, \nabla_t e_1) = -\frac{1}{2}\partial_x|u|^2.$$

Thus we can choose the frame field $\{e_i\}$ so that $\nabla_t e_1 = -(1/2)|u|^2 e_2$, hence $\nabla_t e_i = -(1/2)|\xi_x|^2 J e_i$. Then, we can check that

$$(1.3) \quad u_t^i e_i = J(u_{xx}^i e_i + \frac{1}{2}|u|^2 u^i e_i).$$

We can reverse this procedure.

Proposition 1.3. *The above transformation $\xi \leftrightarrow u$ gives one-to-one correspondence between the solutions of initial value problem (1.1) and the solutions of initial value problem:*

$$(1.4) \quad u_t = J(u_{xx} + \frac{1}{2}|u|^2 u).$$

If we regard the \mathbf{R}^2 -valued function u as a complex valued function $u^1 + \sqrt{-1}u^2$, then u satisfies a so-called non-linear Schrödinger equation:

$$(1.5) \quad u_t = \sqrt{-1}(u_{xx} + \frac{1}{2}|u|^2 u).$$

This transformation of solutions coincides with a transformation found by Hasimoto ([3]). We can restate Proposition 1.3 as follows.

Theorem 1.4. *Hasimoto's transformation is well defined, even when the curvature vanishes at some point.*

Since equation (1.5) is well understood ([1]), we have

Theorem 1.5. *The initial value problem of the semilinear Schrödinger equation (1.1) $\xi_t = J\nabla_x \xi_x$ for closed curves in S^2 has a unique solution on $-\infty < t < \infty$ for any initial data.*

Theorem 1.6. *The initial value problem $\gamma_t = \gamma_x \times \gamma_{xx}$ ($|\gamma_x| \equiv 1$) for closed curves in the euclidean space has a unique solution on $-\infty < t < \infty$.*

2. Vortex filament equation in 3-dimensional space forms

In this section, we generalize results in section 1 to oriented 3-dimensional riemannian manifolds (M, g) with constant curvature c . We consider initial value problem:

$$(2.1) \quad \gamma_t = \gamma_x \times \nabla_x \gamma_x \quad (|\gamma_x| \equiv 1).$$

Let γ be a solution of equation (2.1). By a similar way to the case on S^2 , we find that the differential v of its development to \mathbf{R}^3 satisfies

$$(2.2) \quad v_t = v \times v_{xx}.$$

This equation has just same expression with the case of euclidean space.

Proposition 2.1. *Let M be an oriented 3-dimensional riemannian manifold with constant curvature c . The above transformation $\gamma \leftrightarrow v$ gives one-to-one correspondence between the solutions of initial value problem (2.1) and the solutions of initial value problem (2.2).*

Theorem 2.2. *Let M be an oriented 3-dimensional riemannian manifold with constant curvature c . Initial value problem (2.1) $\gamma_t = \gamma_x \times \nabla_x \gamma_x$ ($|\gamma_x| \equiv 1$) for closed curves in M has a unique solution on $-\infty < t < \infty$ for any initial data.*

3. A semilinear Schrödinger equation in a Kähler manifold

In section 1, we introduced a semilinear Schrödinger equation (1.1) in S^2 . This equation can be defined in general Kähler manifolds (M, g) . We consider a PDE:

$$(3.1) \quad \xi_t = J \nabla_x \xi_x,$$

which has just same expression as in S^2 . Here, ∇ is the riemannian connection and J is the complex structure, both defined on M . This equation is locally expressed as

$$(3.2) \quad \xi_t^\alpha = \sqrt{-1}(\xi_{xx}^\alpha + \Gamma_{\beta\gamma}^\alpha(\xi) \xi_x^\beta \xi_x^\gamma),$$

using a complex coordinate system.

By perturbing this equation to a parabolic equation

$$(3.3) \quad \xi_t = (J + \varepsilon) \nabla_x \xi_x,$$

we get

Theorem 3.1. *Let M be a Kähler manifold. Initial value problem (3.1) $\xi_t = J \nabla_x \xi_x$ for closed curves in M has a unique short time solution for any initial data.*

4. A semilinear Schrödinger equation in a hermitian symmetric space

In a hermitian symmetric space, we can show the all-time existence of a solution of equation (3.1). We can prove it by a way similar to the case of S^2 , but we give here a proof which uses results in the previous section. Therefore, we will give another proof for results in section 1. By direct computation, we have

Lemma 4.1. *Let M be a locally hermitian symmetric space and ξ a solution of equation (3.1) for closed curves. Then the quantity*

$$(4.1) \quad \|\nabla_x \xi_x\|^2 + \frac{1}{4} \langle R(\xi_x, J \xi_x) \xi_x, J \xi_x \rangle$$

is preserved.

Let ξ be a maximal solution of initial problem (3.1). From the above equality, we see that $\|\nabla_x \xi_x\|$ is time-independently bounded. This implies, by induction, that ξ is uniformly C^∞ -ly bounded in any finite time interval. Therefore, we get

Theorem 4.2. *Let M be a complete locally hermitian symmetric space. Equation (3.1) $\xi_t = J\nabla_x \xi_x$ for closed curves has a unique all time solution ($-\infty < t < \infty$) for any initial value.*

Now, we compare this with the case of S^2 . We can generalize the transformation defined in Proposition 1.3 as follows.

Proposition 4.3. *Let M be a locally hermitian symmetric space. The above transformation $\xi \leftrightarrow u$ gives one-to-one correspondence between the solutions of initial value problem (3.1) and the solutions of initial value problem:*

$$(4.2) \quad u_t = Ju_{xx} - \frac{1}{2}R(u, Ju)u.$$

We also can construct a vortex filament type equation. This generalization is based on the identification $(\mathbb{R}^3, * \times *) = (\mathfrak{so}(3), [* , *])$. Let M be a hermitian symmetric space G/K , where G is the isometry group of M and K is the isotropy group. We use standard decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where \mathfrak{g} (resp. \mathfrak{k}) is the Lie algebra of G (resp. K), and the vector space \mathfrak{m} is canonically identified with the tangent space of M at the origin.

There is an element Z of the center of \mathfrak{k} such that $\text{ad}_Z|_{\mathfrak{m}} \cong J$, and M is locally isomorphic to the orbit $\text{Ad}_G Z \subset \mathfrak{g}$. We assume that M and the orbit are isomorphic, and identify them. Then, a curve ξ in M is regarded as a curve in \mathfrak{g} , and we have $J\nabla_x \xi_x = [\xi, \xi_{xx}]$. Thus we have the following

Proposition 4.4. *Consider a PDE for a curve γ in \mathfrak{g}*

$$(4.3) \quad \gamma_t = [\gamma_x, \gamma_{xx}] \quad (\gamma_x \in M).$$

There is a one-to-one correspondence between solutions of (4.3) and solutions of (3.1) by putting $\xi = \gamma_x$.

Remark 4.5. Irreducible hermitian symmetric spaces are classified into four classical types and two exceptional types. Classical types are (AIII) $SU(p+q)/S(U_p \times U_q)$, (DIII) $SO(2n)/U(n)$, (BDI) $SO(n+2)/SO(n) \times SO(2)$ and (CI) $Sp(n)/U(n)$. Their corresponding nonlinear Schrödinger equations are expressed as follows. (c is a real number.)

Type	\mathfrak{m}	Equation
AIII	$\{(p, q) \text{ matrix}\}$	$u_t = \sqrt{-1}(u_{xx} + cu^t \bar{u}u)$
DIII	$\mathfrak{so}(n, \mathbb{C})$	$u_t = \sqrt{-1}(u_{xx} + cu^t \bar{u}u)$
BDI	\mathbb{C}^n	$u_t = \sqrt{-1}(u_{xx} + c(2 u ^2u - {}^t u u \bar{u}))$
CI	$\{\text{symmetric } n\text{-matrix}\}$	$u_t = \sqrt{-1}(u_{xx} + cu \bar{u}u)$

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STRONG PERIODIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN UNBOUNDED DOMAINS

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1. Introduction.

In this note, we show that if the incompressible fluid in *unbounded* domains is governed by the periodic external force, the Navier-Stokes equations have a *periodic strong* solution with the same period as the external force. Let Ω be a domain in \mathbb{R}^n ($n \geq 3$), not necessarily bounded, with smooth boundary $\partial\Omega$. Consider the following Navier-Stokes equations in Ω :

$$(N-S) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f & x \in \Omega, t \in \mathbb{R}, \\ \operatorname{div} u = 0 & x \in \Omega, t \in \mathbb{R}, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $u = u(x, t) = (u^1(x, t), \dots, u^n(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and pressure of the fluid at point $(x, t) \in \Omega \times \mathbb{R}$, respectively; while $f = f(x, t) = (f^1(x, t), \dots, f^n(x, t))$ is the given periodic external force.

Under some restrictive conditions, Serrin [20] gave a criterion for the existence of periodic solutions of (N-S) when Ω is a three-dimensional bounded domain whose boundary moves periodically in time. Kaniel-Shinbrot [11] considered a simpler case such as bounded domains whose boundary is fixed in time and realized the criterion of Serrin. Having introduced the notion of reproductive property, they showed the existence of periodic strong solutions with periodic small forces f . In two-dimensional bounded domains, Takeshita [23] obtained the same result as Kaniel-Shinbrot [11] without assuming the smallness of f . The original problem posed by Serrin had been

treated by Morimoto [19] and Miyakawa-Teramoto [18] who showed the existence of periodic weak solutions. Later on, Teramoto [25] constructed periodic strong solutions in a situation such that the boundary moves slowly in time.

All of these results are obtained in two- or three-dimensional *bounded* domains. On the other hand, few results are known in *unbounded* domains. Recently, Maremonti [15], [16] showed the existence of periodic strong solutions in the three-dimensional whole space \mathbb{R}^3 and the half space \mathbb{R}_+^3 , respectively. However, the result corresponding to exterior domains has not been obtained up to the present. The main difficulty in unbounded domains stems from the lack of exponential decay in time for solutions to the initial value problem of (N-S). Indeed, Serrin [20] and Kaniel-Shinbrot [11] made full use of the fact that $\|u(t)\|_2$ and $\|\nabla u(t)\|_2$ decay exponentially in t provided the initial data at $t = 0$ are prescribed. Such a decay property is due to the Poincaré inequality in bounded domains, and invertibility of the Stokes operator in L^2 makes it easy to obtain better asymptotic behaviour of solutions as $t \rightarrow \infty$.

To overcome this difficulty, Maremonti [15], [16] first showed the algebraic decay rates in time of strong solutions for initial value problem of (N-S) in \mathbb{R}^3 and in \mathbb{R}_+^3 . As a by-product, he constructed periodic strong solutions for periodic small external forces. His method is based on the skillful energy estimates in L^2 for higher derivatives of solutions. Although our results are not altogether new, our approach is different and gives more results than those by Maremonti [15], [16]. We do not employ the energy estimates in L^2 but the L^p -theory of the Stokes operator. Making use of L^p - L^r estimates for the semigroup generated by the Stokes operator, we shall show the existence and uniqueness of *periodic strong* solutions more directly than Maremonti [15], [16]. Compared with the energy estimates in L^2 , our L^p method can cover also the higher dimensional cases. Unfortunately, we cannot obtain the same result in three-dimensional exterior domains because the corresponding L^p - L^r estimate is still an open problem.

We shall first reduce our problem to an integral equation, the solution of which is necessarily periodic with the same period as the external force. The solution will be constructed in the class of functions defined on the whole interval \mathbb{R} with values in $L^n(\Omega)$. Then by a regularity criterion similar to Serrin's [21], we shall show that our solution is actually a strong solution. For that purpose, we shall estimate a time-interval of the existence of local strong solutions for the initial-boundary value problem to (N-S) in terms of the given data. Our estimate extends the result obtained by Giga [7, Theorem 4].

2. Results.

Before stating our results, we need to impose the following assumption on the domain Ω :

Assumption 1.

- (Case I) Ω is the whole space \mathbb{R}^n and the half-space \mathbb{R}_+^n , where $n \geq 3$.
 (Case II) Ω is an exterior domain in \mathbb{R}^n with $C^{2+\mu}$ -boundary $\partial\Omega$, where $n \geq 4$.

The reason why we exclude three-dimensional exterior domains in (Case II) is due to the restriction on gradient bounds for the Stokes semigroup in L^p (see Lemma 2.1 (2) below).

We shall next introduce some notation and function spaces. Let $C_{0,\sigma}^\infty$ denote the set of all real vector C^∞ -functions $\phi = (\phi^1, \dots, \phi^n)$ with compact support in Ω such that $\operatorname{div} \phi = 0$. L_σ^r is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r -norm $\|\cdot\|_r$; (\cdot, \cdot) denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r -space over Ω , $1 < r < \infty$. When X is a Banach space, its norm is denoted by $\|\cdot\|_X$. Then $C^m([t_1, t_2]; X)$ is the usual Banach space, where $m = 0, 1, 2, \dots$ and t_1 and t_2 are real numbers such that $t_1 < t_2$. $BC^m([t_1, t_2]; X)$ is the set of all functions $u \in C^m([t_1, t_2]; X)$ such that $\sup_{t_1 < t < t_2} \|d^m u(t)/dt^m\|_X < \infty$.

Let us recall the Helmholtz decomposition:

$$L^r = L_\sigma^r \oplus G^r \text{ (direct sum), } \quad 1 < r < \infty,$$

where $G^r = \{\nabla p \in L^r; p \in L_{\text{loc}}^r(\overline{\Omega})\}$. For the proof, see Fujiwara-Morimoto [4], Miyakawa [17] and Simader-Sohr [22]. P_r denotes the projection operator from L^r onto L_σ^r along G^r . The Stokes operator A_r on L_σ^r is then defined by $A_r = -P_r \Delta$ with domain $D(A_r) = \{u \in H^{2,r}(\Omega); u|_{\partial\Omega} = 0\} \cap L_\sigma^r$. It is known that the dual space $(L_\sigma^r)^*$ of L_σ^r and the adjoint operator A_r^* of A_r are respectively

$$(L_\sigma^r)^* = L_\sigma^{r'}, \quad A_r^* = A_{r'},$$

where $1/r + 1/r' = 1$. Moreover, we have:

Proposition 1. (Giga [5], Giga-Sohr [9]) *Let $1 < r < \infty$. Then $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}\}_{t \geq 0}$ of class C^0 in L_σ^r .*

Applying the projection operator P_r to both sides of the first equation of (N-S), we have

$$(E) \quad \frac{du}{dt} + A_r u + P_r(u \cdot \nabla u) = P_r f \quad \text{on } L_\sigma^r, t \in \mathbb{R}.$$

The above (E) can be further transformed to the following integral equation:

$$(I.E.) \quad u(t) = \int_{-\infty}^t e^{-(t-s)A_r} P_r f(s) ds - \int_{-\infty}^t e^{-(t-s)A_r} P_r (u \cdot \nabla u)(s) ds.$$

Concerning the external force f , we impose the following assumption:

Assumption 2. Let the exponents r and q be according to the (Case I) and (Case II) of Assumption 1 as

$$(Case I) \quad 2 < r < n, \quad n/2 < q < n;$$

$$(Case II) \quad 2n/(n-1) \leq r < n, \quad n/2 < q < n.$$

For such r and q , we assume that f belongs to the class

$$(1.1) \quad f \in BC(\mathbb{R}; L^p \cap L^l)$$

for $1 < p, l < \infty$ with $1/r + 2/n < 1/p$, $1/q < 1/l < 1/q + 1/n$ provided $n \geq 4$ in both (Case I) and (Case II).

If $n = 3$ in (Case I), assume moreover that

$$(1.2) \quad P_p f(s) = A_p^\delta g(s) (s \in \mathbb{R}) \quad \text{with some } g \in BC(\mathbb{R}; D(A_p^\delta)) \text{ and } f \in BC(\mathbb{R}; L^l)$$

for $1 < p < \min\{r, q\}$ and $\delta > 0$ satisfying $3/2p + \delta > 1 + \max\{1 + 3/2r, 1/2 + 3/2q\}$ and for $1/q < 1/l < 1/q + 1/3$.

Our result now reads:

Theorem 1. *Let Ω and f satisfy Assumption 1 and Assumption 2, respectively. Suppose that $f(t) = f(t + \omega)$ for all $t \in \mathbb{R}$ with some $\omega > 0$. Then there is a constant $\eta = \eta(n, r, q, p, l, \delta) > 0$ such that if*

$$\begin{aligned} \sup_{s \in \mathbb{R}} \|P_p f(s)\|_p + \sup_{s \in \mathbb{R}} \|P_l f(s)\|_l &\leq \eta \quad \text{for } n \geq 4 \text{ in (Case I) and (Case II),} \\ \sup_{s \in \mathbb{R}} \|g(s)\|_p + \sup_{s \in \mathbb{R}} \|P_l f(s)\|_l &\leq \eta \quad \text{for } n = 3 \text{ in (Case I),} \end{aligned}$$

we have a periodic solution u of (I.E.) with the same period ω as f in the class $u \in BC(\mathbb{R}; L^r_\sigma)$ with $\nabla u \in BC(\mathbb{R}; L^q)$.

Such a solution u is unique within this class provided $\sup_{s \in \mathbb{R}} \|u(s)\|_r + \sup_{s \in \mathbb{R}} \|\nabla u(s)\|_q$ is sufficiently small.

Concerning the existence of solutions to (E), we have:

Theorem 2. *In addition to the hypotheses of Theorem 1, let us assume furthermore that f is a Hölder continuous function on \mathbb{R} with values in L^n . Then the periodic solution u given by Theorem 1 has the following additional properties:*

- (i) $u \in BC(\mathbb{R}; L_\sigma^n) \cap C^1(\mathbb{R}; L_\sigma^n)$;
- (ii) $u(t) \in D(A_n)$ for all $t \in \mathbb{R}$ and $A_n u \in C(\mathbb{R}; L_\sigma^n)$;
- (iii) u satisfies (E) in L_σ^n for all $t \in \mathbb{R}$.

Remarks. (1) Taking $n = 3$, $2 < r < 3$ and $q = 2$ in (Case I), our theorems include Maremonti [15, Theorem 1] and [16, Theorem 2].

(2) The first condition of (1.2) seems to be artificial, but it may be replaced by $f(s) = \operatorname{div} F(s)$ with some $F = \{F_{i,j}\}_{i,j=1,2,3} \in BC(\mathbb{R}; H^{1,p}(\Omega))$ for $1 < p < \infty$ satisfying $1/r + 1/3 < 1/p$.

(3) When Ω is a bounded domain in \mathbb{R}^n ($n \geq 2$), the above results also hold and we can relax the assumption on the external force. Indeed, it suffices to assume that $f \in BC(\mathbb{R}; L^r)$ with $\sup_{s \in \mathbb{R}} \|Pf(s)\|_r$ small for $r > n/2$. Under such a hypothesis, there is a periodic solution u of (I.E.) in the class $u \in BC(\mathbb{R}; D(A_r^{1/2}))$.

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**THE CAUCHY PROBLEM IN THE LORENTZ SPACE
FOR THE NAVIER-STOKES EQUATION
IN EXTERIOR DOMAINS**

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Abstract: The Cauchy problem is considered for the Navier-Stokes equation in exterior domains with initial data in the space $L^{n,\infty}$ or $L^{n,\infty} + L^q$ for some $q > n$. Given are some conditions on initial data for the local solvability and the global solvability of the above Cauchy problem. The results are generalizations of well-known results for initial data in the L^n space, but the asymptotic behavior of some solutions are different from that of the solutions with initial data in the L^n space.

1. Introduction.

Let Ω be an exterior domain in \mathbb{R}^n with smooth boundary $\partial\Omega$, where $n \geq 2$. We consider the following initial-boundary value problem for the n -dimensional nonstationary Navier-Stokes equation in Ω :

$$\begin{aligned} (1) \quad & \frac{\partial u}{\partial t} - \Delta_x u + (u \cdot \nabla_x)u + \nabla_x p = 0 && \text{in } (0, \infty) \times \Omega, \\ (2) \quad & && \nabla_x \cdot u = 0 && \text{in } (0, \infty) \times \Omega, \\ (3) \quad & && u = 0 && \text{on } (0, \infty) \times \partial\Omega \end{aligned}$$

with the initial condition

$$(4) \quad u(0, x) = a(x) \text{ in } \Omega.$$

Here we assume that the initial data $a(x)$ satisfy the conditions $\nabla_x \cdot a(x) = 0$ in Ω and $\nu \cdot a(x) = 0$ on $\partial\Omega$, where ν denotes the unit normal vector of $\partial\Omega$.

For the above Cauchy problem, there are a large amount of references, mainly with initial data in the space $L^n(\Omega)$. For example, see Borchers and Sohr [5], Giga and Sohr [11], Iwashita [13], Borchers and Miyakawa [2], [3], Borchers and Varnhorn [6], Kozono and Ogawa [17], [18] and the references cited therein.

On the other hand, there are a number of recent works on the Navier-Stokes equations in spaces larger than the space $L^n(\Omega)$. See Cottet [7], Giga, Miyakawa and Osada [10], Giga and Miyakawa [9], Grubb [12], Kobayashi and Muramatsu [16], Miyakawa and Yamada [25], Taylor [26], Kato [14], [15], Federbush [8], Michaux and Rakotoson [24] and Kozono and Yamazaki [20]. In particular, Borchers and Miyakawa [4] and Kozono and Yamazaki [23] showed that the space $L^{n,\infty}(\Omega)$ is the most suitable one for the Navier-Stokes exterior problem, particularly for the uniqueness and the stability of the stationary solutions.

The first result of this paper on the problem (1)–(4) is the unique existence of a time-local strong solution with a bound near $t = 0$ for the initial data $a(x) \in (L^{n,\infty}(\Omega) + L^q(\Omega))^n$ with some $q \in (n, \infty)$, provided that the local singularity of $a(x)$ in $L^{n,\infty}(\Omega)$ is sufficiently small. We also show that, in the case $n = 2$, the above solution exists globally in time for initial data $a(x) \in L^{2,\infty}(\Omega)$ satisfying the above condition on the local singularity. In particular, we can construct 2-dimensional smooth global solutions for initial data in a class strictly larger than $L^2(\Omega)$, which covers the Leray-Hopf weak solutions.

The second result on (1)–(4) is the unique existence of a time-global strong solution with a bound near $t = 0$ for initial data sufficiently small in the space $L^{n,\infty}(\Omega)$. We also obtain the asymptotic behavior of this solution as $t \rightarrow \infty$.

Part of the results are announced in [21], and the proof is given in [22].

2. Local solutions.

We first recall that the space $L^{r,\infty}(\Omega)$ is defined to be the set of measurable functions $u(x)$ on Ω such that $\sup_{R>0} R\mu(\{x \in \Omega \mid |u(x)| > R\})^{1/r} < \infty$ for $r \in (1, \infty)$, where $\mu(X)$ denotes the Lebesgue measure of the set X . Then there exists a norm on $L^{r,\infty}(\Omega)$ with which the space $L^{r,\infty}(\Omega)$ becomes a Banach space. Furthermore, the space $L^{r,\infty}(\Omega)$ coincides with the real interpolation space $(L^{r_0}(\Omega), L^{r_1}(\Omega))_{\theta, \infty}$, where $r_0, r_1 \in [1, \infty]$, $r_0 \neq r_1$, $\theta \in (0, 1)$ and $1/r = (1 - \theta)/r_0 + \theta/r_1$. In the sequel we abbreviate $L^r(\Omega)$, $L^{r,\infty}(\Omega)$ and $C_0^\infty(\Omega)$ to L^r , $L^{r,\infty}$ and C_0^∞ respectively.

We next recall that there exists a projection P in $(\sum_{1 < r < \infty} L^r)^n$ such that $Pu = 0$ holds if u can be written as $u = \nabla f$ with some scalar function $f \in L_{\text{loc}}^1(\bar{\Omega})$ with $\nabla f \in (\sum_{1 < r < \infty} L^r)^n$, and that $Pu = u$ holds if u satisfies $\nabla \cdot u = 0$ in Ω and $\nu \cdot u = 0$ on $\partial\Omega$. Putting $L_\sigma^r = P(L^r)^n$ and $L_\sigma^{r,\infty} = P(L^{r,\infty})^n$ respectively, we have $(L_\sigma^{r_0}, L_\sigma^{r_1})_{\theta, \infty} = L_\sigma^{r,\infty}$, where r_0, r_1, θ and r be the same as above. (See Miyakawa and Yamada [25]). In the sequel let $\|\cdot\|_r$ and $\|\cdot\|_{r,\infty}$ the norms of $(L^r)^n$ and $(L^{r,\infty})^n$ respectively.

Then the results on time-local solutions are as follows.

Theorem 1. *Suppose that $n < q < \infty$. Then there exists a positive constant*

$M = M(n, q)$ such that, for every $a(x) \in L_\sigma^{n, \infty} + L_\sigma^q$ and every $T \in (0, \infty]$, there exists at most one strong solution $u(t, x)$ of (1)–(3) on $(0, T) \times \Omega$ satisfying the conditions

$$(5) \quad \sup_{0 < t \leq T'} t^{(q-n)/2q} \|u(t, \cdot)\|_q < \infty \text{ for every } T' \in (0, T);$$

$$(6) \quad \sup_{0 < t \leq T'} \|u(t, \cdot)\|_{L^{n, \infty} + L^q} < \infty \text{ for every } T' \in (0, T),$$

$$(7) \quad \limsup_{t \rightarrow +0} t^{(q-n)/2q} \|u(t, \cdot)\|_q < M$$

and the initial condition (4) in the following sense:

$$(8) \quad u(t, \cdot) \rightarrow a \text{ in the weak-* topology of } L_\sigma^{n, \infty} + L_\sigma^q \text{ as } t \rightarrow +0.$$

Theorem 2. Let q be the same as in Theorem 1. Then there exists a positive constant $\varepsilon = \varepsilon(n, q)$ such that, for every $a(x) \in L_\sigma^{n, \infty} + L_\sigma^q$ satisfying

$$(9) \quad \limsup_{R \rightarrow \infty} R\mu(\{x \in \Omega \mid |a(x)| > R\})^{1/n} \leq \varepsilon,$$

there exist $T \in (0, \infty]$ and a smooth solution $u(t, x)$ of (1)–(3) on $(0, T) \times \Omega$ satisfying (5)–(8) with $q = 2n$.

Remark 1. The condition (9) asserts that $a(x)$ can be written as the sum of a function in L_σ^q and a function in $L_\sigma^{n, \infty}$ which is sufficiently near the closure of the set $L_\sigma^{n, \infty} \cap (L^\infty)^n$ in the space $L_\sigma^{n, \infty}$. Hence this condition implies that the local singularity of $a(x)$ in the space $(L^{n, \infty})^n$ is sufficiently small. In particular, since $L_\sigma^n \subset L_\sigma^{n, \infty}$ and since $L_\sigma^n \cap (L^\infty)^n$ is dense in L_σ^n , all functions in L_σ^n enjoy the condition (9) for arbitrary $\varepsilon > 0$.

3. Global solutions.

Theorem 3. There exists a positive constant $\delta = \delta(n)$ such that, if $a(x) \in L_\sigma^{n, \infty}$ satisfies $\|a\|_{n, \infty} < \delta$, there exists a smooth solution $u(t, x)$ of (1)–(3) on $(0, \infty) \times \Omega$ satisfying (5)–(8) with $T = \infty$ and $q = 2n$. Moreover, we have $\sup_{t > 0} \|u(t, \cdot)\|_{n, \infty} < \infty$ and $\sup_{t > 0} t^{(q-n)/2q} \|u(t, \cdot)\|_q < \infty$ for every $q \in (n, \infty)$.

Remark 2. Contrary to the case $a \in L_\sigma^n$, the norm $\|u(t, \cdot)\|_{n, \infty}$ does not decay in general as $t \rightarrow \infty$. In fact, in the case $n = 2$, we construct in Example 3 an example of global solution whose $L^{2, \infty}$ norm is bounded from below by a positive constant. This example implies that our theory can be used to show the time-global existence and the regularity of some solutions which essentially differ from the Leray-Hopf solutions.

In the case $n = 2$, the smallness of the initial data can be replaced by the smallness of the distance between the initial data and the closure of the set $L_\sigma^{2, \infty} \cap (L^\infty)^2$ in the space $L_\sigma^{2, \infty}$. In fact, we have the following theorem.

Theorem 4. Suppose that $n = 2$ and that $a(x) \in L_\sigma^{2, \infty}$ satisfies (9) with $n = 2$ and $q = 4$. Then there exists a smooth solution $u(t, x)$ of (1)–(3) on $(0, \infty) \times \Omega$ satisfying (5)–(8) with $T = \infty$ and $q = 4$.

Remark 3. As is remarked in Remark 1, the condition (9) is fulfilled for every $a(x) \in L_\sigma^n$. Hence, in the case $n = 2$, our class of the initial data with global smooth solutions is larger than that of Leray-Hopf. So far we can tell nothing about the asymptotic behavior of the above time-global solutions as $t \rightarrow \infty$ with large initial data.

Remark 4. The Cauchy problem with finite Radon measures as initial vorticity is a special case of the above problem. This fact is verified as follows: Since the space of finite Radon measures are contained in the homogeneous Besov space $\dot{B}_{1,\infty}^0$, (See Bergh and Löfström [1] or Triebel [27],) the condition that $\nabla \times a$ is a finite Radon measure and the Biot-Savard law imply that $a \in \left(\dot{B}_{1,\infty}^1\right)^2$. On the other hand, by taking the real interpolation of the Sobolev type imbeddings $\dot{B}_{1,1}^{2-2/p} \subset \dot{B}_{p,1}^0 \subset L^p$, we conclude that $\dot{B}_{1,\infty}^{2-2/p} \subset L^{p,\infty}$ for every $p \in (1, \infty)$. Choosing $p = 2$ we have $a \in L_\sigma^{2,\infty}$.

For this case, Cottet [7], Giga, Miyakawa and Osada [10] and Kato [15] obtained the existence of time-global solutions of the Navier-Stokes equation on \mathbb{R}^2 with no smallness conditions on the initial data. (For the uniqueness some smallness of the atomic part is necessary.) On the other hand, our theorems require no extra assumption on $\nabla \times a$, and hence is applicable to more general situations, as we shall see in Example 2.

Miyakawa and Yamada [25] and Michaux and Rakotoson [24] considered the Cauchy problem with finite Radon measures as initial vorticity in bounded domains under boundary conditions different from ours.

4. Examples.

Example 1. Suppose that $0 \in \Omega$, and put $v(y) = (-y_2, y_1)$ for $n = 2$ and $v(y) = (v_1(y), \dots, v_n(y))$ for $n \geq 3$, where $v_1(y) = y_2 - y_n$, $v_j(y) = y_{j+1} - y_{j-1}$ for $j = 2, \dots, n-1$ and $v_n(y) = y_1 - y_{n-1}$. Then $v(y)$ is a homogeneous linear vector such that $\nabla \cdot v(y) = 0$, $y \cdot v(y) = 0$ and $\nabla \times v(y) \neq 0$. Next, let $\chi(t)$ be a smooth function on \mathbb{R} satisfying $0 \leq \chi(t) \leq 1$ for every $t \in \mathbb{R}$, $\chi(t) \equiv 1$ near $t = 0$ and $\chi(t) \equiv 0$ for $t \geq \text{dist}(0, \partial\Omega)/2$. Then the function $a_0(x) = \chi(|x|)v(x)|x|^{1-n}$ satisfies $\nabla \cdot a_0(x) = 0$ on Ω , and the support of $a_0(x)$ is compact in Ω . It follows that $a_0(x) \in L_\sigma^{n,\infty} \setminus L_\sigma^n$. Hence we can take $a(x) = c_0 a_0(x) + b(x)$ in Theorem 2 for every $b(x) \in L_\sigma^n$ and $a(x) = c_0 a_0(x)$ in Theorem 3, provided that $|c_0|$ is sufficiently small.

Example 2. Suppose that $n = 2$ and $0 \in \Omega$. Suppose moreover that $\varphi(x) \in C_0^\infty(\Omega)$ satisfies $\varphi(x) \equiv 1$ on a neighborhood U of 0 , and put

$$\psi_0(x) = \frac{x_1}{|x|} = \frac{\partial}{\partial x_1}|x|, \quad \psi(x) = \varphi(x)\psi_0(x) \quad \text{and} \quad a_0(x) = \left(-\frac{\partial \psi}{\partial x_2}(x), \frac{\partial \psi}{\partial x_1}(x) \right).$$

Then we have $\psi_0(x) \in L^\infty(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$, $\psi(x) \in L^\infty(\Omega) \cap C^\infty(\Omega \setminus \{0\})$, $\text{supp } a_0 \subset \text{supp } \varphi \subset \Omega$ and $a_0(x) \in L_\sigma^{2,\infty}(\Omega) \setminus L_\sigma^2(\Omega)$. It follows that we can take $a(x) = c_0 a_0(x) + b(x)$ in Theorem 4 for every $b(x) \in L_\sigma^2(\Omega)$, provided that $|c_0|$ is sufficiently small. On the other hand, we have the following equality

$$(\nabla_x \times a_0)(x) = \Delta_x \psi(x) = \frac{\partial}{\partial x_1} \Delta_x |x| = \frac{\partial}{\partial x_1} \frac{1}{|x|}$$

on U . It follows that the equality $(\nabla_x \times a_0)(x) = -x_1/|x|^3$ holds on $U \setminus \{0\}$, and $-x_1/|x|^3 \notin L^1_{\text{loc}}(U)$. This implies that the initial vorticity $(\nabla_x \times a)(x)$ is not a Radon measure on Ω in the case $b(x) \equiv 0$.

Example 3. We construct an example of solution of (1)–(3) on $(0, \infty) \times \Omega$ whose $L^{2,\infty}$ norm is bounded from below by a positive constant, where $\Omega = \{x \in \mathbb{R}^2 \mid |x| > 1\}$. Let (r, θ) denote the polar coordinate of \mathbb{R}^2 , and seek the solution of (1)–(4) of the form $u(t, x) = f(t, r)(-\sin \theta, \cos \theta)$ and $p(t, x) = \pi(t, r)$. Then this $u(t, x)$ and $p(t, x)$ solve (1)–(3) if and only if $f(t, r)$ enjoys the equation

$$\frac{\partial f}{\partial t}(t, r) = \frac{\partial^2 f}{\partial r^2}(t, r) + \frac{1}{r} \frac{\partial f}{\partial r}(t, r) - \frac{1}{r^2} f(t, r)$$

in $(0, \infty) \times (1, \infty)$, and the boundary condition $f(t, 1) = 0$ on $r = 1$. Here we assume that the initial data $f(0, r)$ is a C^1 -function on $[1, \infty)$ such that $f(0, r) \equiv 0$ on $[1, 2]$, $-1/r \leq f(0, r) \leq 0$ on $[2, 3]$ and $f(0, r) \equiv -1/r$ on $[3, \infty)$. Since the function $(-x_2, x_1)/(x_1^2 + x_2^2)$ does not belong to the closure of the set $L^{2,\infty}_\sigma \cap (L^\infty)^2$ in the space $L^{2,\infty}_\sigma$, the required property of the solution $u(t, x)$ will follow if the function $g(t, r) = \sqrt{r}f(t, r) + 1/\sqrt{r}$ enjoys the inequality

$$\frac{1}{\sqrt{2\pi}} \left\| u(t, x) + \frac{(-x_2, x_1)}{x_1^2 + x_2^2} \right\|_{L^2(\mathbb{R}^2)} = \left\| \sqrt{r}f(t, r) + \frac{1}{\sqrt{r}} \right\|_{L^2([0, \infty))} = \|g(t, r)\|_{L^2([0, \infty))} < \infty$$

for every $t > 0$. But this inequality follows from the maximum principle, since the function $g(t, r)$ enjoys the equation

$$\frac{\partial g}{\partial t}(t, r) = \frac{\partial^2 g}{\partial r^2}(t, r) - \frac{3}{4r^2}g(t, r)$$

in $(0, \infty) \times (1, \infty)$ and the boundary condition $g(t, r) = 1$ on $r = 1$, and since the initial data $g(0, r)$ is nonnegative and compactly supported.

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Asymptotic behaviors of radially symmetric solutions of $\square u = |u|^p$ for super critical values p in high dimensions

HIDEO KUBO AND KÔJI KUBOTA

Abstract: In this note we shall consider wave equations with power nonlinearity, for example, $|u|^p$ or $|u|^{p-1}u$ with $p > 1$. In particular, we shall derive the asymptotic behaviors of radially symmetric solutions of it, which guarantee the existence of the scattering operator, for $p > p_0(n)$ and $n \geq 4$. Here $p_0(n)$ is a so-called critical exponent. This work essentially depends on space-time decay estimate for a fundamental solution concerning L^∞ -norm.

1. Introduction

We study asymptotic behaviors as $t \rightarrow \pm\infty$ of radially symmetric solutions of the nonlinear wave equation

$$(1.1) \quad u_{tt} - \Delta u = F(u) \quad \text{in } x \in \mathbb{R}^n, t \in \mathbb{R},$$

where $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$ with $p > 1$ and $n \geq 2$.

Let $p_0(n)$ be the positive root of the quadratic equation in p :

$$(1.2) \quad \Phi(n, p) \equiv \frac{n-1}{2}p^2 - \frac{n+1}{2}p - 1 = 0.$$

Note that $p_0(n)$ is strictly decreasing with respect to n and $p_0(4) = 2$. If $1 < p < p_0(n)$, it is known that the Cauchy problem for (1.1) with initial data prescribed on $t = 0$ does not admit global (in time) solutions, provided the initial data are chosen appropriately, even if they are sufficiently small. (See [6], [8] and [19]). The same is true for $p = p_0(n)$ if $n = 2$ or $n = 3$. (See [18]).

On the other hand, the case where $p > p_0(n)$ seems to be more complicated. When $2 \leq n \leq 4$, it is known that the problem admits a global solution for small initial data. (See [7], [8] and [24]). When $n \geq 5$, for $p \geq (n+3)/(n-1)$ a global weak solution of the problem obtained by [13] and [20]. (See also [3], [4], [11] and [12]). Recently, the case where p is between $p_0(n)$ and $(n+3)/(n-1)$ is treated by [5] and [14], independently.

Moreover, when $p > p_0(n)$ and either $n = 2$ or $n = 3$, it has been shown that the scattering operator for (1.1) exists on a dense set of a neighborhood of 0 in the energy space. (See [10], [17] and [23]). Namely, let $u_-(x, t)$ be the solution of the homogeneous wave equation

$$(1.3) \quad u_{tt} - \Delta u = 0 \quad \text{in } x \in \mathbb{R}^n, t \in \mathbb{R},$$

with small initial data

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \quad \text{for } x \in \mathbb{R}^n.$$

Then there exists a solution $u(x, t)$ of (1.1) such that $\|u(t) - u_-(t)\|_e \rightarrow 0$ as $t \rightarrow -\infty$, where

$$(1.4) \quad \|v(t)\|_e = \left\{ \int_{\mathbb{R}^n} (|\nabla v(x, t)|^2 + |v_t(x, t)|^2) dx \right\}^{1/2},$$

and there exists another solution $u_+(x, t)$ of (1.3) such that $\|u(t) - u_+(t)\|_e \rightarrow 0$ as $t \rightarrow \infty$. The analogous results have been obtained also for the high dimensional case, provided $p > p_1(n)$, where $p_1(n)$ is the largest root of the quadratic equation in p :

$$(n^2 - n)p^2 - (n^2 + 3n - 2)p + 2 = 0.$$

(See [13], [15], [16], and [20]).

The purpose of this note is to search the asymptotic behaviors of radially symmetric solutions of (1.1), which guarantee the existence of the scattering operator, for any $p > p_0(n)$ in high dimensions $n \geq 5$.

2. Statements of main results

Throughout this section, we assume $n \geq 5$ (unless stated otherwise). First we shall consider the Cauchy problem for the homogeneous wave equation:

$$(2.1)_0 \quad u_{tt} - u_{rr} - \frac{n-1}{r}u_r = 0 \quad \text{in } \Omega,$$

$$(2.1)_1 \quad u(r, 0) = f(r), \quad u_t(r, 0) = g(r) \quad \text{for } r > 0,$$

where $\Omega = \{(r, t) \in \mathbb{R}^2; r > 0\}$ and $u(r, t)$ a real valued function. Then we have

Theorem 1. Assume $f \in C^2([0, \infty))$ and $g \in C^1([0, \infty))$ satisfy

$$(2.2) \quad |f(r)|\langle r \rangle^{-1} + \sum_{j=0}^1 (|f^{(j+1)}| + |g^{(j)}(r)|) \leq \varepsilon \langle r \rangle^{-\kappa - (n+1)/2} \quad \text{for } r > 0,$$

where ε and κ are positive numbers and $\langle r \rangle = \sqrt{1+r^2}$. Here if n is even number, we further assume $\kappa < (n-1)/2$. Then (2.1) admits uniquely a weak solution $u(r, t) \in C^1(\Omega)$ such that for $(r, t) \in \Omega$ and $|\alpha| \leq 1$ we have

$$(2.3) \quad |D_{r,t}^\alpha u(r, t)| \leq C\varepsilon r^{1-m-|\alpha|} \langle r \rangle^{-1+|\alpha|} \Psi(r, |t|),$$

where we have set $m = [(n-2)/2]$ and

$$\Psi(r, t) = \langle r + |t| \rangle^{-\chi(n)} \langle r - t \rangle^{-\kappa}$$

with

$$\chi(n) = \begin{cases} 1/2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

and C is a constant depending only on m and κ .

Next we shall consider the nonlinear wave equation

$$(2.4) \quad u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u) \quad \text{in } \Omega,$$

where $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$. Here we assume

$$(2.5) \quad p_0(n) < p < (n+3)/(n-1).$$

We shall introduce a function space X , in which we will look for solutions of (2.4), defined by

$$X = \{u(r, t) \in C^0(\Omega) : D_r u(r, t) \in C^0(\Omega), \|u\| < \infty\},$$

and

$$\|u\| = \sup_{(r,t) \in \Omega} \{(|u(r, t)|r^{m-1}\langle r \rangle + |D_r u(r, t)|r^m)\Psi^{-1}(r, |t|)\},$$

where Ψ is the same function as in (2.3). As for the parameter κ , we assume

$$(2.6) \quad \frac{1}{2} < \kappa \quad \text{and} \quad \frac{p+1}{p-1} - \frac{n+1}{2} < \kappa \leq q,$$

where we have set

$$q = (1 + \Phi(n, p))/p = \frac{n-1}{2}p - \frac{n+1}{2}$$

with $\Phi(n, p)$ in (1.2). Note that there exist really numbers κ satisfying (2.6) for $p > p_0(n)$, because

$$\Phi(n, p) = (p-1)\left\{q - \left(\frac{p+1}{p-1} - \frac{n+1}{2}\right)\right\} > 0 \quad \text{for } p > p_0(n).$$

We are now in a position to state the main theorem in this note. Let $u_-(r, t)$ be the solution of (2.1) which is obtained in Theorem 1. Note that $u_- \in X$ and

$$\|u_-\| \leq C\varepsilon \quad \text{for any } \varepsilon > 0.$$

Then we have

Theorem 2. (Main theorem). Assume conditions (2.2), (2.5) and (2.6) hold. Then there is positive constant ε_0 (depending only on p, n and κ) such that, if $0 < \varepsilon \leq \varepsilon_0$, there exists uniquely a weak solution $u(r, t)$ of the nonlinear wave equation (2.4) such that $u \in C^1(\Omega) \cap X$,

$$(2.7) \quad \|u\| \leq 2\|u_-\|$$

and for $(r, t) \in \Omega$ and $|\alpha| \leq 1$ we have

$$(2.8) \quad |D_{r,t}^\alpha(u(r, t) - u_-(r, t))| \leq C\|u\|^p r^{1-m-|\alpha|} \langle r \rangle^{-1+|\alpha|} \Psi(r, t)$$

and

$$(2.9) \quad \|u(t) - u_-(t)\|_\varepsilon \leq C\|u\|^p \langle t \rangle^{-\theta} \quad \text{if } t \leq 0,$$

where $\|\cdot\|$ is defined by (1.4) and we have set

$$\theta = \min\{q, \chi(n)p + p\kappa - 1\},$$

and C is a constant depending only on p, n and κ .

Moreover there exists uniquely a weak solution $u_+(r, t)$ of (2.1)₀ which belongs to $C^1(\Omega) \cap X$, such that for $(r, t) \in \Omega$ and $|\alpha| \leq 1$ we have (2.8) and (2.9) with $u_-(r, t)$, $\Psi(r, t)$ and “if $t \leq 0$ ” by $u_+(r, t)$, $\Psi(r, -t)$ and “if $t \geq 0$ ”, respectively.

Remarks. 1) If n is odd, in Theorems 1 and 2, one can replace $u \in C^1(\Omega)$ by $u \in C^2(\Omega)$. Moreover in (2.6) we can replace $\kappa > 1/2$ by $\kappa > 0$. In this case, we interpret (2.9) as follows. When $\kappa > 1/2p$, (2.9) is still valid. When $0 < \kappa \leq 1/2p$, it holds with $\theta = \kappa$. (See [9]).

2) For $n \geq 2$, consider the following Cauchy problem

$$(2.10) \quad \begin{cases} u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u) & \text{in } r > 0, t > 0, \\ u(r, 0) = 0, \quad u_t(r, 0) = g(r) & \text{for } r > 0. \end{cases}$$

It is known that, if $g(r) \geq Mr^{-\mu}$ for $r \geq 1$ with some positive constants M, μ and $\mu < (p+1)/(p-1)$, then (2.10) does not admit global solutions. (See [1], [2], [21] and [22]). Therefore condition (2.6) is partially necessary to obtain Theorem 2.

3) One can also show that the Cauchy problem for the nonlinear wave equation (2.4) admits a unique global solution, provided the hypotheses of Theorems 1 and 2 are fulfilled.

In the proof of Theorem 1, it is very important to represent a weak solutions $u(\cdot, t) \in C^0(\mathbb{R}; L^2_{loc}(\mathbb{R}^n))$ of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = 0, \quad u_t(x, 0) = c_n g(|x|) & \text{for } x \in \mathbb{R}^n \end{cases}$$

where

$$c_n = \begin{cases} 2 \Gamma(\frac{n-1}{2}) & \text{if } n \text{ is odd,} \\ \sqrt{\pi} \Gamma(\frac{n-1}{2}) & \text{if } n \text{ is even.} \end{cases}$$

The representation is given in the following lemma with $u(x, t) = \Theta(g)(|x|, t)$. Moreover Theorem 2 is obtained by considering the associated integral equation with the differential equation (2.4). So the lemma below is very essential in our work.

Lemma 3. Let $g \in C^0((0, \infty))$ and $g(r) = O(r^{-m-1})$ as $r \downarrow 0$. For $r > 0$ and $t \geq 0$ we define a function $\Theta(g)$ as follows.

(1) n is odd : $n = 2m + 3$ ($m = 1, 2, \dots$).

$$\Theta(g)(r, t) = \int_{|t-r|}^{t+r} g(\lambda) K(\lambda, r, t) d\lambda,$$

where we have set

$$K(\lambda, r, t) = r^{2-n} \lambda^{2m+1} H_m(\lambda, r, t),$$

$$H_m(\lambda, r, t) = \left(\frac{\partial}{\partial \lambda} \frac{-1}{2\lambda}\right)^m (r^2 - (\lambda - t)^2)^{(n-3)/2}.$$

(2) n is even : $n = 2m + 2$ ($m = 1, 2, \dots$).

$$\Theta(g)(r, t) = \int_{|t-r|}^{t+r} g(\lambda) K_1(\lambda, r, t) d\lambda + \int_0^{\max(t-r, 0)} g(\lambda) K_2(\lambda, r, t) d\lambda,$$

where we have set

$$K_1(\lambda, r, t) = r^{2-n} \lambda^{2m+1} \int_{\lambda}^{t+r} \frac{H_m(\rho, r, t)}{\sqrt{\rho^2 - \lambda^2}} d\rho,$$

$$K_2(\lambda, r, t) = r^{2-n} \lambda^{2m+1} \int_{t-r}^{t+r} \frac{H_m(\rho, r, t)}{\sqrt{\rho^2 - \lambda^2}} d\rho,$$

and

$$H_m(\rho, r, t) = \left(\frac{\partial}{\partial \rho} \frac{-1}{2\rho}\right)^m (r^2 - (\rho - t)^2)^{(n-3)/2}.$$

And we extend $\Theta(g)(r, t)$ as an odd function with respect to t . Then $\Theta(g) \in C^0(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

$$|\Theta(g)(r, t)| \leq C_B r^{-m} \quad \text{for } (r, t) \in B.$$

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The phase function of the wave-front of coherent optical radiation determination: modern methods and means

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Introduction

In calculating the wave fields presented in the form of Guegens-Kirgoff integral in its $E(\bar{r}) = A \int_D e^{i\varphi(\bar{r})} d\bar{r}$ linear approximation any attempt to solve a inverse problem of optics analytically appears to be of no success due to its incorrectness. It is connected with the fact that the optical wave field values fixation is possible only in the form of $|E(\bar{r})|^2$, that is intensity determined by the photometric technics in contrast to the radio-range of the electromagnetic field, where the tension of the field can be measured.

As for as the optical systems are concerned, particularly the adaptive optical systems of laser radiation, for the wave front phase function determination two widespread methods are usually used : the phase conjugation and the aperture sounding.

The former is based on the phase front inclination determination within the subaperture limits by the diffraction spot shift with the further local inclinations joining along all the aperture, for example by applying the least square method.

The latter is based on widely used in the radio-range modulation with the following detection of the intensity values, in the result of which the average phase value in the subaperture limits is found out.

Both of the methods are characterized by the insufficient accuracy of the phase wave determination in the aperture limits through the necessity of sectioning into the subapertures of the forming optical elements.

The growing demand to the optical systems quality cause the research necessity of the coherent optical radiation wave front functions determinations new high-effective methods based on the non-traditional ways of measuring and the light field phase functions calculating.

1. The problem statement

Let the laser beam with the wave function in Fraungoffer approximation

$$(1.1) \quad \psi(x_1, y_1) = A \int_D e^{-iK(x x_1 + y y_1)/L} dx dy \quad ,$$

where

A - an amplitude multiplier from the point of view of a wave function in its linear approximation;

D - beam aperture in a radiation plane;

$K = 2\pi/\lambda$ - a wave number (λ - wave length);

$r(x, y)$ - plane of a laser beam radiation;

$r_1(x_1, y_1)$ - beam measurement plane;

L - a distance of a measurement plane location, scatter through the distorting medium.

The amplitude multiplier A is a constant, that is why further it could be not taken into consideration since this A multiplier doesn't contain the information about a phase function distribution along the aperture D , and influences only on the intensity level. Here and further, if it is not pointed out otherwise, the expressions written for i -subapertures are also true for j -subapertures.

A radiation plane is conventionally divided into n number of the square configured subapertures D_i , having a square form with a side length a , (x_i, y_i) - the coordinates of these squares centers, the sides of the squares are parallel to the coordinates axes, the tip of the coordinates coincides with a center of a central square.

No restrictions are put on the subaperture quantity n , that allows to increase considerably the accuracy of measuring.

The radiation and registration (of measurement) planes are parallel in the space and at the projection to each other the tips of the coordinates of both planes coincide, the appropriate axes are colinear as minimum.

Let's examine a case of laser radiation interference in parallel beams :

$$(1.2) \text{Int}_k(x_1, y_1) = \sum_{i=1}^n \sum_{j=1}^n \text{Re} \left\{ \int_{D_i} \psi_i(x_1, y_1) dx dy \cdot \int_{D_j} \psi_j(x_1, y_1) dx dy \right\} \cos(\varphi_i - \varphi_j)$$

where

$\text{Int}_k(x_1, y_1)$ - intensity in a measurement plane within photoreceiving cell D_k ;

D_i - i -subaperture of a radiation plane;

φ_i - wave phase in the appropriate subaperture D_i .

In the limits of subaperture D_i wave phase φ_i is considered to be a constant, i.d. that is function's piecewise-step approximation is considered.

The task is to find out the wave function's phase distribution $\psi(x, y)$ along the radiating aperture D using the intensity values $\text{Int}_k(x_1, y_1)$.

2. The determination of laser beam wave function distribution along the radiated aperture

According to (1.1) the intensity of the electromagnetic field in the radiation plane of i -subaperture could be presented as an integral :

$$(2.1) \quad E_i = \int_{y_i - \frac{a}{2}}^{y_i + \frac{a}{2}} dy \int_{x_i - \frac{a}{2}}^{x_i + \frac{a}{2}} e^{-iK(x x_1 + y y_1)/L} dx$$

Its solution is the function of the electromagnetic field intensity in a measurement plane $r_1(x_1, y_1)$ from the radiating subaperture D_1 , the coordinates of which are

$r_1(x_1, y_1)$:

$$(2.2) \quad E_i = \left(2\frac{L}{K}\right)^2 \frac{1}{x_1 y_1} \sin\left(\frac{Ka}{2L}x_1\right) \sin\left(\frac{Ka}{2L}y_1\right) e^{-iK(x x_1 + y y_1)/L}$$

In the investigated case of a laser radiation interference in parallel beams the intensity of k photoreceiving matrix cell is to be determined as the intensity sum of radiating subapertures pairs

$$(2.3) \quad Int_k(x_1, y_1) = \sum_{i=1}^n \sum_{j=1}^n \iint_{D_k} (E_i + E_j)(\overline{E_i + E_j}) dx_1 dy_1 \cdot \cos(\varphi_i - \varphi_j)$$

where

$$(2.4) \quad (E_i + E_j)(\overline{E_i + E_j}) = 2C^2 + 2C^2 \cos[K((x_i - x_j)x_1 + (y_i - y_j)y_1)/L]$$

$$(2.5) \quad C = \left(2\frac{L}{K}\right)^2 \frac{1}{x_1 y_1} \sin\left(\frac{Ka}{2L}x_1\right) \sin\left(\frac{Ka}{2L}y_1\right)$$

In the connection with the fact that the first item of the expression (2.4) is the magnitude influencing only on the average value of the intensity in a measurement plane, it could be excluded from the further computations.

For the next numerical integrating we'll write the formula (2.4) as

$$(2.6) \quad (E_i + E_j)(\overline{E_i + E_j}) = \alpha_i \alpha_j + \beta_i \beta_j$$

where

$$(2.7) \quad \alpha_i = C \cos[K(x_i x_1 + y_i y_1)/L] \quad ,$$

$$(2.8) \quad \beta_i = C \sin[K(x_i x_1 + y_i y_1)/L] \quad .$$

Then the expression (2.3) is to be

$$(2.9) \quad Int_k(x_1, y_1) = \sum_{i=1}^n \sum_{j=1}^n B_{ijk} \cos(\varphi_i - \varphi_j)$$

where

$$(2.10) \quad B_{ijk} = \iint_{D_k} (\alpha_i \alpha_j + \beta_i \beta_j) dx_1 dy_1 \quad .$$

Extending the equation (2.9) to m photoreceiving cells the matrix equation is obtained

$$(2.11) \quad [Int]_m = [B]_{m \times m} [\cos(\varphi_i - \varphi_j)]_m \quad .$$

Matrix B is composed of the constant coefficients B_{ijk} .

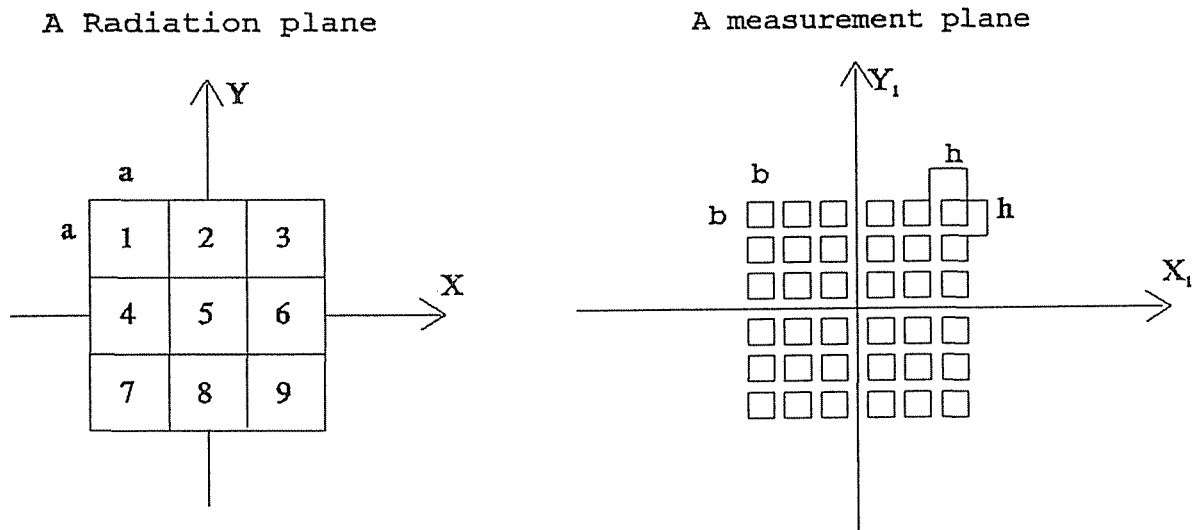
The number of the photoreceiving cells m , where intensity $Int_k(x_1, y_1)$ is measured, determines the matrices dimensions and is to be equal to n^2 in accordance with the solution singularity. However, taking into account as it has shown above the constitutions B_{ij} and B_{ji} being equal [(2.4),(2.6)-(2.11)], the number of the photoreceiving cells could be reduced to $n(n+1)/2$, where n - subapertures number in radiation plane. It's enough to take the double values of the constant coefficients B_{ij} , $i < j$, without the coefficients by $i > j$. From the matrix equation (2.11) the values $\cos(\varphi_i - \varphi_j)$ can be obtained but, though it's impossible to determine the values of the proper phases φ in every of the radiation plane n -subapertures. This uncertainty might be eliminated with the help of some excessive information. Such an excessiveness is obtained by forming two pictures in a measurement plane, but under the condition that any of the apertures phase would be shifted to $-\pi/2$ by a phase-shifter plate. Then, subtracting the distribution with a phase shift in one of the subapertures from the initial intensity distribution the required for the determination of the average phase front values in which results in the matrix equation (2.11) dimensions reduction to $m = n - 1$.

In this case, having assumed to be equal to zero the phase value in the aperture, where a shift to $-\pi/2$ in a measurement plane takes place, the phase values in the other subapertures by using their differences to it are possible.

The choice of $-\pi/2$ shift value caused by the necessity to ensure a sign sensitivity to the phase values in the subapertures that could be fulfilled by means of a substitution of the related to the phase difference trigonometric function for a co-function Sin. This substitution is produced by argument shifting.

3. The results of the numerical experiment

Let the number of subapertures in a radiation plane be equal to 9 ($n = 9$). And let's assume the central subaperture numbered 5 as a "shifting" one.



In the measurement plane there is the matrix with the photoreceiving square-shaped

cells with their side length b ; forming a square grid 32×32 with a measured pace h ; (x_k, y_k) - coordinates of cell's centers, cell sides are parallel to the coordinates' axes, the tip of the coordinates coincides with photomatrix center. According to (2.9) the initial intensity distribution in a measurement plane, when a central ("shifting") subaperture isn't shifted but is assumed to be a zero, can be presented as follows

$$(3.1) \quad \underbrace{Int_k(x_1, y_1)}_{\varphi_5=0} = 2 \sum_{j=1, j \neq 5}^n B_{5jk} \cos(-\varphi_j) + \sum_{j=1}^n B_{jjk} + \sum_{i=1}^n \sum_{j=1, i \neq j \neq 5}^n B_{ijk} \cos(\varphi_i - \varphi_j) \quad .$$

The next expression is for the formed intensity distribution in a measurement plane, when a "shifting" subaperture is shifted to $-\pi/2$

$$(3.2) \quad \underbrace{Int_k(x_1, y_1)}_{\varphi_5=-\pi/2} = 2 \sum_{j=1, j \neq 5}^n B_{5jk} \cos\left(\frac{-\pi}{2} - \varphi_j\right) + \sum_{j=1}^n B_{ijk} + \sum_{i=1}^n \sum_{j=1, i \neq j \neq 5}^n B_{ijk} \cos(\varphi_i - \varphi_j) \quad .$$

After simple transformations the unknown average phase front values in the limits of the radiation plane subapertures which are defined from the matrix equation

$$(3.3) \quad \left[\sin\left(\varphi_j + \frac{\pi}{4}\right) \right]_m = [B_{5j}]_{m \times m}^{-1} \left[\frac{Int_{\varphi_5=0} - Int_{\varphi_5=-\pi/2}}{2\sqrt{2}} \right]_m, \quad j \neq 5,$$

where

j - subaperture number (except the fifth, where on the way of beam distribution a phase-shifter plate is placed);

$[B_{5j}]_{m \times m}^{-1}$ - inverted coefficient's matrix;

$B_{5jk}, j = \overline{1, n}, j \neq 5, k = \overline{1, n-1}$.

Basing on the values φ_j we could determine the intensity distribution along an aperture in a focal plane, and, vice a versa, its possible to calculate the phase function values distribution in each subaperture using the intensity values in the focal plane, corresponding to the definite phase function (a solution of a direct and reversed matrix equation (3.3)).

The conducted numerical experiments demonstrated the following results : the error of a phase determination by every square subapertures ($j = \overline{1, 9}$) appeared to be only 2.2%. Even a small subapertures number increasing results in an essential increase of a phase determination accuracy. The form of the subapertures, the analyzed aperture is divided into, could be arbitrary, this will influence only on the type of the expression's subintegral function (2.3).

Conclusion

The effectiveness of a wave front sensor using based on the described method is in its high space resolution, and the application of chips-multipliers permits to develop a high-resolution sensor for the adaptive optical systems operatng in real time.

A compound retro-reflector mirror on a corner reflector as an adaptive system mirror allows to obtain a greater effect. Here in both phase and amplitude (of an odd row) distortions compensation takes place. Japanese scientists Takuso Sato, Yochihide Nagura,

Osamu Ikeda and Takeshi Hatsuzawa published their reports on such systems in the Applied Optics v. 10, 1982 .

On the base of one of the research-production associations the production of high quality corner reflectors was organized and the experimental stand for the realisation of the adaptive optical systems of different versions with using the retro-reflector mirrors made of corner reflectors was set up in the university laboratory.

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THE STABILITY OF ROLL SOLUTIONS OF THE 2-D SWIFT-HOHENBERG EQUATION AND THE PHASE DIFFUSION EQUATION

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Abstract: A stability criterion of roll solutions of the 2-D Swift-Hohenberg equation is presented. It clarifies the effect of the system size on the primary instability of rolls. An interpretation of the phase diffusion equation is also given from the view point of the spectral analysis. The key to carry out the spectral analysis is that the infinite dimensional system of linear equations naturally induced by the Fourier decomposition for the linearized eigenvalue problem of the roll solution can be reduced to the three dimensional one.

1. Introduction.

Let us consider a fluid contained in a rectangular cell whose aspect ratio of the depth of the fluid to the horizontal width is sufficiently small. For a critical temperature gradient between the upper and lower plates, buoyancy forces overcome the dissipative effects of viscous shear and thermal conduction, and the motionless fluid spontaneously breaks up into convective rolls of upward- and downward- moving regions of fluid. In order to study this phenomena, the following simple model equation is proposed, which was first derived by Swift and Hohenberg [10]:

$$(1.1) \quad u_t = (\alpha - (1 + \partial_x^2 + \partial_y^2)^2)u - u^3,$$

where $u(x, y, t)$ represents the rescaled fluid field in a given horizontal plane, e.g., the vertical velocity component in the midplane of the convective rolls, and α is the (reduced) Rayleigh number. The reader should consult Cross and Hohenberg [2], Greenside and

Coughran, Jr. [3] and Newell [8] for the physical background of the Swift-Hohenberg equation. (1.1) describes the onset of thermal convection which forms roll patterns. In fact, Collet and Eckmann [1] proved the existence of roll solutions of (1.1):

Suppose that ω satisfies

$$2/5 < \omega^2 < 2.$$

Then, there exists a positive constant ε_0 independent of ω such that for $0 < \varepsilon < \varepsilon_0$, the equation

$$(\alpha - (1 + \partial_x^2 + \partial_y^2)^2)u - u^3 = 0$$

has a unique solution of the form

$$(1.2) \quad \alpha = 3\varepsilon^2 + (1 - \omega^2)^2,$$

and

$$(1.3) \quad \bar{u}_\varepsilon(\omega x) = \varepsilon 2 \cos(\omega x) + O(\varepsilon^3).$$

In this article, we study the stability of roll solutions and the dynamics near the roll solutions.

2. Mathematical formulation and main results.

We consider (1.1) on the rectangle domain

$$\Omega = (-L/2, L/2) \times (-M/2, M/2), \quad 0 < L < \infty, \quad 0 < M < \infty$$

with the periodic boundary conditions. Here, we assume $L = (2N)\lambda$, where $\lambda = 2\pi/\omega$ is the wavelength of $\bar{u}_\varepsilon(\omega x)$, and N is a positive integer which corresponds to the number of rolls. In other words, the length of the side in the x -direction is a integer multiple of the basic wavelength of roll pattern. Notice that M is the length of the axis of rolls. We know that (1.1) generates a semi-flow on $H_{per}^{4\beta}(\Omega)$ for $0 \leq \beta < 1$, which denotes the scale of the usual Sobolev spaces with the periodic boundary conditions. For more details, see Henry [4] and Temam [11].

Notice that $\bar{u}_\varepsilon(\omega x)$ is also a stationary solution of (1.1) on the rectangle domain Ω . By (1.2) and (1.3), $\bar{u}_\varepsilon(\omega x)$ is determined by two parameters ε and ω which represent the amplitude and wavenumber, respectively. Therefore, the stability of $\bar{u}_\varepsilon(\omega x)$ is determined by these parameters. However, we use a new parameter W instead of ω to investigate the stability of the equilibrium $\bar{u}_\varepsilon(\omega x)$, which is defined by

$$W = \frac{\omega^2 - 1}{\sqrt{3}\varepsilon}.$$

In what follows, we regard ε and W as independent parameters. ω is determined by ε and W in terms of $\omega^2 = 1 + \sqrt{3}\varepsilon W$. The linear stability criterion of roll solutions is as follows:

Theorem 2.1. *Let $A : L^2_{per}(\Omega) \longrightarrow L^2_{per}(\Omega)$ be the linearized operator of the right-hand side of (1.1) at $\bar{u}_\varepsilon(\omega x)$ defined by*

$$(2.1) \quad Av = (\alpha - (1 + \partial_x^2 + \partial_y^2)^2)v - 3\bar{u}_\varepsilon(\omega x)^2v, \quad v \in H^4_{per}(\Omega)$$

Then, we have

(1) *If $0 \leq W < 1/\sqrt{2}$, then for sufficiently small $\varepsilon > 0$, the spectrum of A lies in the closed left half-plane in \mathbf{C} . This is independent of L and M .*

(2) *If $-1/\sqrt{2} < W < 0$, then for sufficiently small $\varepsilon > 0$,*

(i) *the spectrum of A lies in the closed left half-plane in \mathbf{C} provided*

$$0 < \varepsilon < \frac{2\pi^2}{\sqrt{3}|W|M^2},$$

(ii) *the spectrum of A intersects the right half-plane in \mathbf{C} provided*

$$\varepsilon > \frac{2\pi^2}{\sqrt{3}|W|M^2},$$

where $\varepsilon M^2 = O(1)$ as $\varepsilon \downarrow 0$ and $M \rightarrow \infty$.

(3) *If $|W| > 1/\sqrt{2}$, then for sufficiently small $\varepsilon > 0$ and large L , the spectrum of A intersects the right half-plane in \mathbf{C} .*

When the rectangle domain Ω is sufficiently large and the roll pattern is stable, we can give an accurate characterization of the critical eigenvalues and the associated eigenfunctions.

Theorem 2.2. *When L and M are sufficiently large, for $0 \leq W < 1/\sqrt{2}$ and sufficiently small $\varepsilon > 0$, there exist $\delta > 0$ which depend only on ε and W (independent of L and M) such that*

$$(i) \quad \lim_{\varepsilon \downarrow 0} \delta(\varepsilon, W) = 0$$

(ii) *The eigenvalues of A which belong to the interval $[-\delta, 0]$ are given by*

$$\mu_{mn} = -D_\perp \nu_m - \nu_m^2 - D_{//} \kappa_n^2 + O((\kappa_n + \nu_m)^3)$$

$$\text{for } 0 \leq \nu_m < \sqrt{3}\varepsilon\rho \text{ and } |\kappa_n| \leq \sqrt{3}\varepsilon\rho/2,$$

where $D_{//}$ and D_{\perp} are given

$$D_{//} = 4 - 8W^2 + O(\varepsilon), \quad D_{\perp} = 2\sqrt{3}\varepsilon W,$$

and

$$\nu_m = \left(\frac{2\pi m}{M}\right)^2, \quad \kappa_n = \frac{2\pi n}{L},$$

and $\rho > 0$ is a constant independent of ε, W, L and M . The associated eigenfunctions are given by

$$\psi_{mn} = \partial_x \bar{u}_\varepsilon(\omega x) \exp(2\pi i m y / M) \exp(2\pi i n x / L) + O(\kappa_n) + O(\nu_m).$$

(iii) The other eigenvalues belong to the interval $(-\infty, -\delta)$.

The above theorem says that there are many eigenvalues near zero when the system size is sufficiently large. However, these eigenvalues are discrete because L and M are finite. Therefore, we can take an eigenspace whose dimension is finite but sufficiently large as follows:

Theorem 2.3. When $0 < W < 1/\sqrt{2}$ and ε is sufficiently small, for sufficiently large L and M , one can choose $\beta > 0$ and $\gamma > 0$ which depend on ε, W, L and M such that

(i) β and γ satisfy

$$\begin{aligned} 0 < \beta < \gamma \\ \lim_{L, M \rightarrow \infty} \beta(\varepsilon, W, L, M) &= 0 \\ \lim_{L, M \rightarrow \infty} (\gamma(\varepsilon, W, L, M) - \beta(\varepsilon, W, L, M)) &= 0 \end{aligned}$$

(ii) The eigenvalues of A which belong to the interval $[-\beta, 0]$ are given by

$$\mu_{mn} = -D_{\perp} \left(\frac{2\pi m}{M}\right)^2 - D_{//} \left(\frac{2\pi n}{L}\right)^2 + o\left(\left(\frac{1}{M} + \frac{1}{L}\right)^2\right)$$

for $|m| \leq \rho_1(M)$ and $|n| \leq \rho_2(L)$, where $\rho_1(M)$ and $\rho_2(L)$ are integers such that

$$\begin{aligned} \lim_{M \rightarrow \infty} \rho_1(M) &= \infty, & \lim_{M \rightarrow \infty} \frac{\rho_1(M)}{\sqrt[3]{M}} &= 0, \\ \lim_{L \rightarrow \infty} \rho_2(L) &= \infty, & \lim_{L \rightarrow \infty} \frac{\rho_2(L)}{\sqrt[3]{L}} &= 0, \end{aligned}$$

and the associated eigenfunctions are given by

$$\psi_{mn} = \partial_x \bar{u}_\varepsilon(\omega x) \exp(2\pi i m y / M) \exp(2\pi i n x / L) + O(1/M) + O(1/L).$$

(iii) The eigenvalues μ which belong to the interval $(-\infty, -\beta)$ satisfy $\mu < -\gamma$.

The choice of the eigenspace in the last theorem is not unique because it depends on the choice of β and γ . Using an argument in the same spirit as the one in the inertial manifold theory, the dynamics near the roll solutions can be well approximated by the

dynamics projected on this space [6].

When the domain is square (i.e. $L = M$), we determine a scaling parameter ν by

$$(2.2) \quad \nu = \frac{\lambda}{L} \left(= \frac{\lambda}{M} \right),$$

where λ is the basic wavelength of roll patterns. The dynamics on the above eigenspace are described by the following system of the ordinary differential equations:

$$(2.3) \quad \frac{da_{mn}}{dT} = -(D_{\perp}(m\omega)^2 + D_{//}(n\omega)^2)a_{mn},$$

where a_{mn} is the coefficient of the eigenfunction

$$\psi_{mn} = \partial_x \bar{u}_\varepsilon(\omega x) \exp(im\omega Y) \exp(in\omega X) + O(1/M) + O(1/L),$$

and $X = \nu x$, $Y = \nu y$ and $T = \nu^2 t$. Recalling the Taylor formula and the Fourier series expansion of the solutions of the diffusion equation, we find that under the scaling (2.2), the dynamics near the roll solutions can be well approximated by

$$(2.4) \quad \begin{cases} \bar{u}(\omega x + \phi(X, Y, T)) \\ \phi_T = D_{//}\phi_{XX} + D_{\perp}\phi_{YY}. \end{cases}$$

The diffusion equation in (2.4) is called *the phase diffusion equation* which describes the dynamics near the roll solutions through the modulation in the phase of \bar{u} . The phase diffusion equation is obtained by the formal perturbation method from the view point of physics (Kuramoto [5], Pomeau and Manneville [9]). Thus, we know that the dynamics near the roll solutions are described by the solutions of the diffusion equation for the phase modulation with respect to the rescaled spatio-temporal variables.

3. The strategy for the proof of Theorems 2.1 and 2.2.

We apply the separation of the variables to the eigenvalue problem corresponding to (2.1). The y -component of the eigenvalue problem is easily solved. In order to solve the x -component, we apply the Bloch transformation which was introduced by Collet-Eckmann [1] to study the 1-dimensional case. This technique convert the eigenvalue problem in $L^2(-L/2, L/2)$ into the one in $L^2(0, \lambda)$. Next, we deal with the system of linear equations naturally induced by the Fourier decomposition of the eigenvalue problem in the same line of arguments as given in [1]. At first glance, it seems to be difficult to solve our problem since the dimension of the system is infinite. Our system, however, can be reduced to three dimensional one which consists of the Fourier components with the wavenumbers $\pm\omega$ and 0. This is the most outstanding property of our system which enables us to carry out the spectral analysis precisely. For more details, see [7]

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