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Domino tilings of the Aztec Diamond

Juanjo Rué [1]

Imagine you have a cutout from a piece of squared paper and a pile of dominoes, each of which can cover exactly two squares of the squared paper. How many different ways are there to cover the entire paper cutout with dominoes? One specific paper cutout can be mathematically described as the so-called *Aztec Diamond*, and a way to cover it with dominoes is a *domino tiling*. In this snapshot we revisit some of the seminal combinatorial ideas used to enumerate the number of domino tilings of the Aztec Diamond. The existing connection with the study of the so-called *alternating-sign matrices* is also explored.

1 Introduction

In the last years, tilings of large discrete structures have become a central area of research in combinatorics. They connect different research areas such as statistical mechanics, combinatorial representation theory of groups, and enumerative combinatorics.

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In this snapshot we will address the most basic question in this domain: in how many ways can we tile a certain structure (the *Aztec Diamond*, defined in the following section) in terms of simpler blocks? The approach we use is the one developed by Elkies, Kuperberg, Larsen, and Propp in [2], and it combines elementary bijective arguments with some properties of the so-called *alternating-sign matrices* (see Section 4).

2 Domino tilings of the Aztec Diamond

An *Aztec Diamond of size n* is the union of the integer lattice squares (also called *cells*) of the form $[a, a+1] \times [b, b+1] \subseteq \mathbb{R}^2$ with $a, b \in \mathbb{Z}$ which lie completely inside the tilted square $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq n + 1\}$. We call the lattice points (and the straight horizontal and vertical lines between them) inside or on the boundary of this frontier square the *vertices* (respectively *edges*) of the Aztec Diamond.

A *domino* is the union of two adjacent cells (either horizontally or vertically). A *domino tilling* of the Aztec Diamond is a set of dominoes whose interior is disjoint and whose union is the whole Aztec Diamond – that is, the domino tiles don't overlap and they cover the whole Aztec Diamond. Figure 1 gives an illustrative example of the Aztec Diamond of size 5 together with a domino tiling.

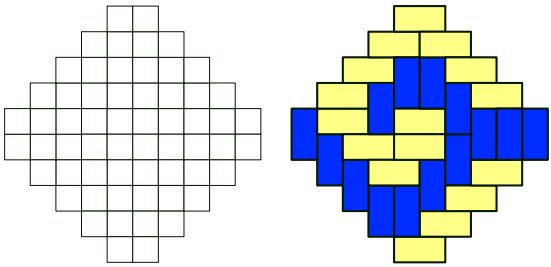

Figure 1: A domino tilling of the Aztec Diamond of size 5.

The natural question we may address is the following: how many domino tilings does an Aztec Diamond of size n admit? When $n = 1$, it is defined in terms of 4 cells, and trivially the number of tilings is equal to 2 (either consisting of two horizontal or two vertical dominoes). Less obvious is the enumeration for $n = 2$. In Figure 2, all the possible 8 configurations are drawn.

From this point on (when $n \geq 3$), the number of distinct cases becomes too large to enumerate them here, and, additionally, a case-by-case analysis seems very complicated. So instead of deriving a direct enumeration formula

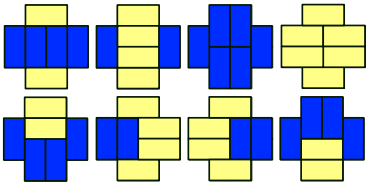

Figure 2: All 8 tilings of the Aztec Diamond of size 2.

we exploit the discrete structure of a tiling and relate it with some family of matrices with interesting properties. We will show in this snapshot a very nice (and simple!) formula: the number of domino tilings of the Aztec Diamond of size n is equal to $2^{\binom{n}{2}}$.[2]

3 Constructing a height function

Our aim in this section is to build a convenient function over the vertices of a given domino tiling T. This will provide all the structural information needed to encode the combinatorial information. We call such a function a *height function*.

Let us first introduce a canonical *orientation* on each lattice square of the plane, that is, we give each edge a 'direction' by attaching an arrow to it, as seen in Figure 3. To do so, colour the cells of the Aztec Diamond using two colours (say, black and white) in a chessboard way (see Figure 3). Once this is done, we orient white and black cells in clockwise and counterclockwise order, respectively. This orientation induces an orientation on the edges of the Aztec Diamond: for two vertices v_1 and v_2, we write $v_1 \to v_2$ if there is an (oriented) edge starting at v_1 and finishing at v_2. See the full construction in Figure 3 for the case $n = 5$. For technical reasons is it useful to also consider the four vertices $(\pm(n+1), 0)$ and $(0, \pm(n+1))$ (even though they do not play an active role in a tiling).

Observe that the same orientation on the outer edges as in our example is naturally induced for every tiling of the Aztec Diamond. Keeping this observation in mind, we build a height function H_T associated to a given tiling T in the following way. For the vertex $v = (n+1, 0)$, we define its height function as $H_T(v) = 0$. Then, we extend this function: for two vertices v_1, v_2 linked by an edge $v_1 \to v_2$ which is not covered by a domino of the tiling, we define $H_T(v_2) := H_T(v_1) + 1$. One may imagine this like piling stacks of coins on the corners of the cutout of squared paper: if we already have a stack

[2] Remember that the *binomial coefficient* $\binom{n}{k}$ is defined as $\binom{n}{k} := \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k\cdot(k-1)\cdot\ldots\cdot2\cdot1}$.

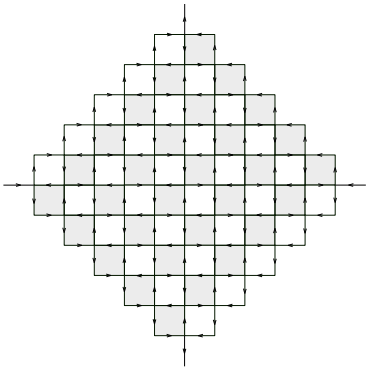

Figure 3: The orientation of the edges in the Aztec Diamond.

of, say, 3 coins on one corner, we put 4 coins on every neighbouring corner that is linked to the first corner by an edge pointing towards it. Thus, we successively obtain a relief that covers the whole Aztec Diamond (justifying the name 'height function').[3] By our construction, the sequence of values taken by this height function when moving around the Aztec Diamond along its boundary (starting at v) is $0, 1, \ldots, 2n+2$ (when reaching the point $(0, n+1)$), then $2n+1, \ldots, 0$ (when reaching the point $(-n-1, 0)$), and so on. As an example, the construction of the height function for the domino tiling T of Figure 1 is shown in the left picture in Figure 4. In the right picture we only show the height function *without* the tiling.

Each domino tiling T defines a height function H_T, which satisfies the following properties:

(a) H_T takes successive values $0, 1, \ldots, 2n+1, 2n+2, 2n+1, \ldots, 0, \ldots, 2n+1, 2n+2, 2n+1, \ldots, 0$ when moving counterclockwise along the boundary.

(b) If $v_1 \to v_2$, then $H_T(v_2)$ is either $H_T(v_1) + 1$ or $H_T(v_1) - 3$ (depending on whether v_1 and v_2 are adjacent in the tiling or not).

Clearly, one can reconstruct the domino tiling from the height function by looking at vertices which are adjacent in the Aztec Diamond and whose heights differ by 3.

Indeed, each function which satisfies the previous two conditions defines a domino tiling. This claim is true because when turning around each cell following its corresponding orientation, necessarily we find four transitions

[3] The reader may try to convince themselves that this height function is well-defined, that is, it can never happen that two different neighbours of one vertex can force us to assign different values to this one vertex.

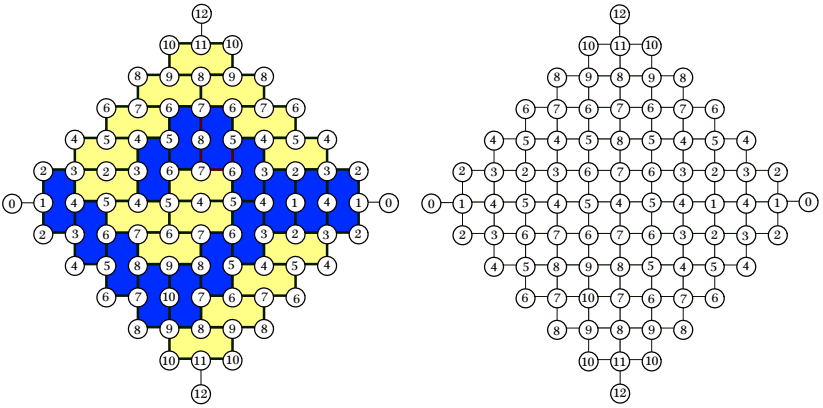

Figure 4: The construction of the height function H_T.

$v \to w$ where exactly three of them satisfy $H_T(w) = H_T(v) + 1$, and just one satisfies $H_T(w) = H_T(v) - 3$. The last transition indicates a missing edge in the domino tiling, which shows how to place the dominoes over the Aztec Diamond.

Hence, it is equivalent that a function satisfies (a) and (b), and that it is a height function of some domino tiling. Our next objective is to understand the combinatorial structure of such functions and to relate them with well-known discrete structures.

4 Alternating-sign matrices

We will now relate the functions studied in the previous section with a very special (and combinatorial) class of matrices. A function over the Aztec Diamond can be thought of as a pair of square matrices of order $n+1$ and $n+2$, respectively, by the following geometric procedure: First, we rotate the Aztec Diamond clockwise by 45 degrees. Then, we construct the pair of matrices by keeping rows of even and odd order, respectively. In Figure 5 this process is shown, with the function from Figure 4 as the starting point.

In this example we obtain the following pair of matrices:

$$A' = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 3 & 5 & 3 & 5 & 7 & 9 \\ 5 & 7 & 5 & 7 & 5 & 7 \\ 7 & 9 & 7 & 5 & 3 & 5 \\ 9 & 7 & 5 & 3 & 1 & 3 \\ 11 & 9 & 7 & 5 & 3 & 1 \end{pmatrix}, \; B' = \begin{pmatrix} 0 & 2 & 4 & 6 & 8 & 10 & 12 \\ 2 & 4 & 2 & 4 & 6 & 8 & 10 \\ 4 & 6 & 4 & 6 & 8 & 6 & 8 \\ 6 & 8 & 6 & 4 & 6 & 4 & 6 \\ 8 & 10 & 8 & 6 & 4 & 2 & 4 \\ 10 & 8 & 6 & 4 & 2 & 4 & 2 \\ 12 & 10 & 8 & 6 & 4 & 2 & 0 \end{pmatrix}.$$

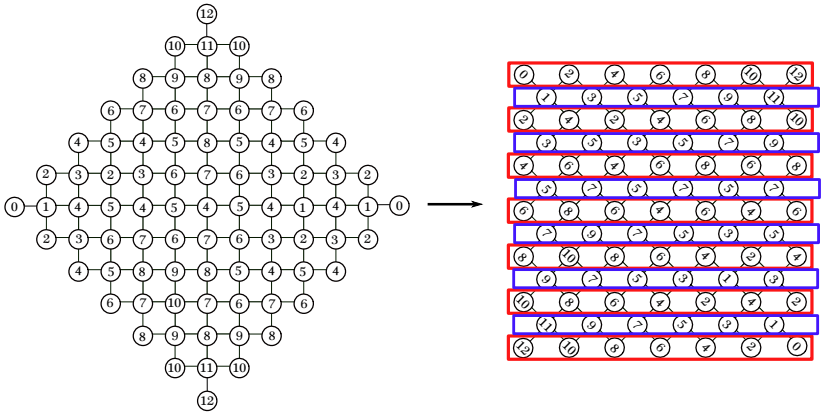

Figure 5: Building two matrices from a height function over the Aztec Diamond.

By the definition of the height function, the resulting matrices A' and B' have the following property: adjacent entries differ by ± 2. For our purposes, we scale both matrices by dividing by 2 (after subtracting 1 in each entry in the case of A').

In our example, we get the matrices A^* and B^*:

$$A^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 & 2 & 3 \\ 3 & 4 & 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}, \; B^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 2 & 3 & 4 & 3 & 4 \\ 3 & 4 & 3 & 2 & 3 & 2 & 3 \\ 4 & 5 & 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 & 2 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}.$$

4.1 The connection with alternating-sign matrices

What can we say about this pair of matrices? Let us relate them with a well-known family of combinatorial structures. An *alternating-sign matrix* is a square matrix whose entries are ± 1 and 0 such that every row total and column total is equal to 1 and such that the nonzero entries in each row and column alternate in sign. Such objects, with a strong combinatorial taste, have played an important role in enumerative combinatorics (see for instance [1, 3]) and have deep connections with discrete models in statistical physics.

Given an alternating-sign matrix $A = (a_{i,j})_{1 \le i,j \le n}$ of order n, we define its *extended matrix* $A^* = (a^*_{r,s})_{0 \le r,s \le n}$ of order $n+1$ in the following way: we write

$$a^*_{r,s} = r + s - 2\sum_{i=1}^{r}\sum_{j=1}^{s} a_{i,j}. \tag{1}$$

In particular, this extended matrix is a square matrix of order $n+1$. The operation in Equation (1) is reversible, and indeed

$$a_{r,s} = \frac{1}{2}\left(a^*_{r-1,s} + a^*_{r,s-1} - a^*_{r-1,s-1} - a^*_{r,s}\right). \tag{2}$$

In other words, there is a one-to-one correspondence (a *bijection*) between alternating-sign matrices of order n and their extended matrices. Some immediate consequences can be deduced from the alternating-sign condition. In particular, by applying Equation (1), one gets

(α) $a^*_{i,0} = a^*_{0,i} = i$ and $a^*_{n,i} = a^*_{i,n} = n - i$, and
(β) adjacent entries in A^* differ by ± 1.

It is also true that each matrix satisfying (α) and (β) is an extended matrix arising from an alternating-sign matrix, which can be easily checked by exploiting Equation (2).

Observe now that both matrices in our example satisfy properties (α) and (β). This is true for every matrix derived in the described way from a domino tiling of the Aztec Diamond. So by starting from a domino tiling of the Aztec diamond we have constructed a unique a pair of alternating-sign matrices, from which we can reconstruct the domino tiling.

5 The final argument

The last remaining point to be discussed is how the two alternating-sign matrices in a pair are related with each other. It is clear that the pair of alternating-sign matrices should have some kind of compatibility relation, so that we can merge their (conveniently scaled) extended matrices to build a correct height function. Denote by $\mathcal{A}_n$ the set of all alternating-sign matrices of order n, and write for alternating-sign matrices $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_{n+1}$ that $A \sim B$ if such a pair of alternating-sign matrices can be used to define a valid height function and thus to build a domino tiling. We then say that A and B are *compatible*.

Let us have another look at Figure 5, keeping in mind relation (b). It is obvious that the coefficient $b'_{i,j}$ of the unscaled matrix B' depends (locally!) on the six possible choices for the submatrix

$$\begin{pmatrix} a'_{i-1,j-1} & a'_{i-1,j} \\ a'_{i,j-1} & a'_{i,j} \end{pmatrix}$$

of the unscaled matrix A'. For example, when this submatrix is equal to

$$\begin{pmatrix} 2r+1 & 2r-1 \\ 2r-1 & 2r+1 \end{pmatrix}$$

for some $r \in \mathbb{N}$, then it can be shown using (b) that necessarily $b'_{i,j} = 2r$. Moreover, we can scale this submatrix of A' to obtain a submatrix of A^* and then use Equation (2) to obtain the entry $a_{i,j} = -1$ of the corresponding alternating-sign matrix A. Indeed, in all but one of the six possibilities the value of $b'_{i,j}$ is uniquely determined, and $a_{i,j} \neq 1$. The only case with an ambiguity is the configuration

$$\begin{pmatrix} a'_{i-1,j-1} & a'_{i-1,j} \\ a'_{i,j-1} & a'_{i,j} \end{pmatrix} = \begin{pmatrix} 2r-1 & 2r+1 \\ 2r+1 & 2r-1 \end{pmatrix}.$$

In such cases, $a_{i,j} = 1$, but $b'_{i,j}$ can be either $2r-2$ or $2r+2$. In other words: once the matrix A is fixed, the coefficients of B' are uniquely determined except for the ones where we have $a_{i,j} = 1$.[4] From B', we can calculate first the scaled matrix B^* and then the corresponding alternating-sign matrix B. Hence, fixing $A \in \mathcal{A}_n$, the number of matrices $B \in \mathcal{A}_{n+1}$ which are compatible with A is equal to $2^{N_+(A)}$, where $N_+(A)$ is the number of coefficients in A which are equal to 1. By a similar case-by-case analysis, the following statement also holds: fixing $B \in \mathcal{A}_{n+1}$, the number of matrices $A \in \mathcal{A}_n$ which are compatible with B is equal to $2^{N_-(B)}$, where $N_-(B)$ is the number of coefficients in B which are equal to -1.

We are now ready to conclude the argument. As we have shown, the number of domino tilings of the Aztec Diamond of size n is equal to the size of the set

$$\{(A, B) : A \in \mathcal{A}_n,\, B \in \mathcal{A}_{n+1}, A \sim B\}.$$

Observe that for each alternating-sign matrix A of size n, the equality $N_+(A) = N_-(A) + n$ holds. This is obviously true because in each row of A the number of $+1$'s minus the number or -1's is exactly equal to 1.

Let us apply this observation. Once fixed an alternating-sign matrix $A \in \mathcal{A}_n$, the number of alternating-sign matrices $B \in \mathcal{A}_{n+1}$ which are compatible with A is equal to $2^{N_+(A)}$. Going in the opposite direction, fixing $B \in \mathcal{A}_{n+1}$, the number of $A \in \mathcal{A}_n$ which are compatible with B is equal to $2^{N_-(B)}$. In other words, the number d_n of possible domino tilings of the Aztec Diamond of size n can be calculated in two ways:

$$d_n = \sum_{A \in \mathcal{A}_n} 2^{N_+(A)} = \sum_{B \in \mathcal{A}_{n+1}} 2^{N_-(B)}.$$

[4] Strictly speaking, we have phrased the argumentation not including a proper treatment of the entries in the first and last row and column of B', but the reader may check that the same arguments apply.

We are now almost done. We now use the property $N_+(A) = N_-(A) + n$ for alternating-sign matrices of order n, joint with a convenient index shift:

$$d_n = \sum_{A \in \mathcal{A}_n} 2^{N_+(A)} = \sum_{A \in \mathcal{A}_n} 2^{N_-(A)+n} = 2^n \sum_{B \in \mathcal{A}_n} 2^{N_-(B)} = 2^n d_{n-1}$$

Now we can just iterate this relation to get

$$d_n = 2^n d_{n-1} = 2^{n+(n-1)} d_{n-2} = \ldots = 2^{n+(n-1)+\ldots+2} d_1.$$

Remember that we already know that there are exactly two possible tilings for the Aztec Diamond of size 1, that is, we know the initial condition $d_1 = 2$ for the above sequence. This proves (by mathematical induction) the initial claim that there are $d_n = 2^{\binom{n}{2}}$ domino tilings of the Aztec Diamond of size n.

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