Overdetermined Partial Boundary Value Problems on Finite Networks

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Abstract

In this work we define a class of non self–adjoint boundary value problems on finite networks associated with Schrödinger operators. The novelty lies on the fact that on a part of the boundary no data is prescribed, whereas in another part of the boundary both the values of the function as of its normal derivative are given. We show that overdetermined partial boundary value problems are the key for solving inverse boundary value problems on finite networks, since they provide the theoretical foundations of the recovery algorithm. We analyze the uniqueness and existence of solution of overdetermined partial boundary value problems through the non–singularity of the partial Dirichlet–to–Neumann maps. These maps allow us to determine the value of the solution in the part of the boundary where no data was prescribed. Afterwards, we execute a full conductance recovery for spider networks.

Keywords: Boundary value problems, Inverse problem, Dirichlet–to–Neumann map, recovery of conductances

1. Introduction

The *Inverse Boundary Value Problem* arised for the first time around 1950 due to Alberto Calderón's work. However, it was not until 1980 that he published "On an Inverse Boundary Value Problem" [10] detailing his work on the subject. This problem appeared as a consequence of an engineering problem on geophysical electrical prospection in which the objective is to deduce some internal terrain properties from surface electrical measurements.

These works have motivated several developments in the inverse problem field until nowadays. More recently, this problem has been also considered for medical purposes on *Electrical Impedance Tomography* (EIT) [11], which is a medical imaging technique where an image containing visual information of internal parts of the body is obtained from electrical measurements on the boundary.

The mathematical corresponding problem that Calderón proposed is whether it is possible to determine the conductivity of a body by means of current and voltage measurements at its boundary. This problem of recovering conductances from boundary or surface current and potential measurements is a non–linear inverse problem and it is exponentially ill–posed [1, 17], since its solution is highly sensitive to changes in the boundary data.

Since its appearance, Calderón's Inverse Problems have been treated in many ways. For instance, Sylvester and Uhlmann treated in [9, 18] the uniqueness of solution; Curtis, Ingerman and Morrow have worked on critical circular planar networks conductivity reconstruction [12, 13, 14, 16];

Borcea, Druskin, Guevara and Mamonov have gone into EIT problems in depth and their last works on the subject treat numerical conductivity reconstruction [6, 7, 8].

Inverse boundary value problems have been considered both over the continuum and the discrete fields. In this work we define a new class of boundary value problems on finite networks associated with Schrödinger operators. The novelty lies on the fact that on a part of the boundary no data is prescribed, whereas in another part of the boundary both the values of the function as of its normal derivative are given. These problems are not self–adjoint, and hence we worry about the study of existence and uniqueness through the adjoint problem.

We show that overdetermined partial boundary value problems are the key in the framework of inverse boundary value problems on finite networks, since they provide the theoretical foundations of the recovery algorithm. In fact, this type of problems were implicitly considered in some previous works, but only for specific networks and boundary data, see [13, 14]. We analyze the uniqueness and existence of solution of overdetermined partial boundary value problems through the non–singularity of the partial Dirichlet–to–Neumann maps. These maps allow us to determined the value of the solution in the part of the boundary with no prescribed data. Afterwards, we give explicit formulae for the acquirement of boundary spike conductances on critical planar networks and execute a full conductance recovery for spider networks. This algorithm is an adaptation of the one proposed in [14] for the Combinatorial Laplacian and when the corresponding Dirichlet–to–Neumann map is singular.

2. Preliminaries

Let $\Gamma = (V, c)$ be a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V. Let E be the set of edges of the network Γ. Each edge (x, y) has been assigned a *conductance* $c(x, y)$, where $c : V \times V \longrightarrow [0, +\infty)$. Moreover, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $(x, y) \notin E$. Then, $x, y \in V$ are *adjacent*, $x \sim y$, iff $c(x, y) > 0$.

The set of functions on a subset $F \subseteq V$, denoted by $\mathcal{C}(F)$, and the set of non–negative functions on F, $\mathcal{C}^+(F)$, are naturally identified with $\mathbb{R}^{|F|}$ and the nonnegative cone of $\mathbb{R}^{|F|}$, respectively. We denote by $\displaystyle\int_F u(x)dx$ the value $\displaystyle\sum_{x\in F} u(x)$. Moreover, if F is a non empty subset of V, its characteristic function is denoted by $\chi_{_F}$. When $F = \{x\}$, its characteristic function will be denoted by ε_x. If $u \in \mathcal{C}(V)$, we define the *support* of u as $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$.

If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of F. The vertices of $\delta(F)$ are called *boundary vertices* and when a boundary vertex $x \in \delta(F)$ has a unique neighbour in F we call the edge joining them a *boundary spike*. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when F is. Any function $\omega \in \mathcal{C}^+(\bar{F})$ such that $\mathsf{supp}(\omega) = \bar{F}$ and $\displaystyle\int_{\bar{F}} \omega^2(x)\, dx = 1$ is called *weight* on $\bar{F}$. The set of weights is denoted by $\Omega(\bar{F})$. We denote by $\kappa_F \in \mathcal{C}(\delta(F))$ the function $\kappa_F(x) = \sum\limits_{y\in F} c(x, y)$.

We define the *normal derivative of* $u \in \mathcal{C}(\bar{F})$ *on* F as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_{_F}}\right)(x) = \int_F c(x, y)\,\big(u(x) - u(y)\big)\, dy, \quad \text{for any } x \in \delta(F).$$

Any function $K \in \mathcal{C}(F \times F)$ will be called a *kernel on* F. The *integral operator associated with* K is the endomorphism $\mathscr{K}\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$ that assigns to each $f \in \mathcal{C}(F)$, the function $\mathscr{K}(f)(x) = \displaystyle\int_F K(x,y)\, f(y)\, dy$ for all $x \in V$. Conversely, given an endomorphism $\mathscr{K}\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F)$, the associated kernel is given by $K(x,y) = \mathscr{K}(\varepsilon_y)(x)$. Clearly, kernels and operators can be identified with matrices, after giving a label on the vertex set. In addition, a function $u \in \mathcal{C}(F)$ can be identified with the kernel $K(x,x) = u(x)$ and $K(x,y) = 0$ otherwise and hence with a diagonal matrix that will be denoted by D_u.

The *combinatorial Laplacian* of Γ is the linear operator $\mathscr{L}: \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function defined for all $x \in V$ as

$$\mathscr{L}(u)(x) = \int_V c(x,y)\,\big(u(x) - u(y)\big)\, dy.$$

Given $q \in \mathcal{C}(V)$ the *Schrödinger operator* on Γ with *potential* q is the linear operator $\mathscr{L}_q: \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathscr{L}_q(u) = \mathscr{L}(u) + qu$. Since q is real, it is well–known that any Schrödinger operator is self–adjoint. The relation between the values of the Schrödinger operator with potential q on a connected subset $F \subseteq V$ and the values of the normal derivative at $\delta(F)$ is given by the *First Green Identity*, proved in [5]

$$\begin{aligned}\int_F v(x)\,\mathscr{L}_q(u)(x)\,dx = \frac{1}{2}\int_{\bar F}\int_{\bar F} c_F(x,y)(u(x)-u(y))(v(x)-v(y))\,dxdy + \int_F q(x)u(x)v(x)\,dx \\ - \int_{\delta(F)} v(x)\,\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx,\end{aligned}$$

where $u,v \in \mathcal{C}(\bar F)$ and $c_F = c \cdot \chi_{(\bar F \times \bar F)\setminus(\delta(F)\times\delta(F))}$. A direct consequence of the above identity is the so–called *Second Green Identity*: for all $u, v \in \mathcal{C}(\bar F)$,

$$\int_F \Big(v(x)\,\mathscr{L}_q(u)(x) - u(x)\,\mathscr{L}_q(v)(x)\Big)\,dx = \int_{\delta(F)} \left(u(x)\,\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}(x) - v(x)\,\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}(x)\right)\,dx.$$

We define the *energy associated with* F *and* q as the symmetric bilinear form $\mathcal{E}_q^F: \mathcal{C}(\bar F) \times \mathcal{C}(\bar F) \longrightarrow \mathbb{R}$ given for any $u, v \in \mathcal{C}(\bar F)$ by

$$\mathcal{E}_q^F(u,v) = \frac{1}{2}\int_{\bar F}\int_{\bar F} c_F(x,y)\,\big(u(x)-u(y)\big)\,\big(v(x)-v(y)\big)\,dx\,dy + \int_{\bar F} q(x)\,u(x)\,v(x)\,dx.$$

From the First Green Identity, for any $u, v \in \mathcal{C}(\bar F)$ we get that

$$\mathcal{E}_q^F(u,v) = \int_F v(x)\mathscr{L}_q(u)(x)\,dx + \int_{\delta(F)} v(x)\,\left[\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}(x) + q(x)u(x)\right]\,dx.$$

For any weight $\sigma \in \Omega(\bar F)$, the so–called *potential associated with* σ is the function in $\mathcal{C}(\bar F)$ defined as $q_\sigma = -\sigma^{-1}\mathscr{L}(\sigma)$ on F, $q_\sigma = -\sigma^{-1}\,\dfrac{\partial \sigma}{\partial \mathsf{n}_F}$ on $\delta(F)$. It is worth to note that the definition of q_σ is a discrete analogous of the Liouville transform, see [18]. These authors proved in [4] that the Energy is positive semi–definite on $\mathcal{C}(\bar F)$ if there exist $\lambda \geq 0$ and $\sigma \in \Omega(\bar F)$ such that $q = q_\sigma + \lambda\chi_{\delta(F)}$.

In this case, it is positive definite iff $\lambda > 0$. So, through this section, we will suppose that the above condition $q = q_\sigma + \lambda\chi_{\delta(F)}$ holds with $\sigma \in \Omega(\bar{F})$ and $\lambda \geq 0$. Therefore, for any $g \in \mathcal{C}(\delta(F))$ the following Dirichlet problem

$$\mathscr{L}_q(u) = 0 \quad \text{on} \quad F \quad \text{and} \quad u = g \quad \text{on} \quad \delta(F),$$

has a unique solution u_g.

The map $\Lambda_q \colon \mathcal{C}(\delta(F)) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any function $g \in \mathcal{C}(\delta(F))$ the function $\Lambda_q(g) = \dfrac{\partial u_g}{\partial \mathsf{n}_F} + qg$ is called *Dirichlet–to–Robin map.* In [4], the authors proved that the Dirichlet–to–Robin map, Λ_q, is a self–adjoint, positive semi–definite operator whose associated quadratic form is given by

$$\int_{\delta(F)} g(x)\Lambda_q(g)(x)\,dx = \mathcal{E}_q^F(u_g, u_g).$$

Moreover, λ is the lowest eigenvalue of Λ_q and its associated eigenfunctions are multiple of σ. In addition, if $N_q \in \mathcal{C}(\delta(F) \times \delta(F))$ is the kernel of Λ_q, its associated matrix $\mathsf{N_q}$ is an irreducible and symmetric M–matrix. Usually $\mathsf{N_q}$ is called the *response matrix* of the network. Given $A, B \subset \delta(F)$ a pair of disjoint subsets, we consider the submatrix of the response matrix $\mathsf{N_q}(A;B) = \big(N_q(x,y)\big)_{(x,y)\in A\times B}$.

If $A = \{p_1, \dots, p_k\}$ and $B = \{q_1, \dots, q_k\}$, there exist k paths, $\gamma_1, \dots, \gamma_k$, such that γ_i starts at p_i ends at q_i and $\gamma_i \setminus \{p_i, q_i\} \subset F$, since F is connected. The pair $(A;B)$ is called *connected trough* Γ, when there exist k paths connecting A and B that are mutually disjoint.

The network $\Gamma = (\bar{F}, c_F)$ is called a *circular planar network* if it can be embedded in a closed disc D in the plane so that the vertices in F lie in $\overset{\circ}{D}$ and vertices in $\delta(F)$ lie on the circumference $C = \partial D$. In this case, the vertices in $\delta(F)$ can be labelled in the clockwise circular order. The pair $(A;B)$ of boundary vertices is called a *circular pair* if the set $(p_1, \dots, p_k; q_1, \dots, q_k)$ is in circular order. A circular planar network Γ is called *well–connected* if any circular pair $(A;B)$ is connected trough Γ. A *critical circular planar network* $\Gamma = (\bar{F}, c_F)$ is a circular planar network such that the removal of any edge breaks a connection through Γ between pairs of boundary vertices.

In [4], we characterized those M–matrices that are the response matrix of a network, which represent an extension of the previous work by Curtis et al.; see [12].

For any $n = |\delta(F)| \geq 2$, $\sigma \in \Omega(\delta(F))$ and $\lambda \geq 0$, let $\Phi_{\lambda,\sigma}$ be the set of irreducible and symmetric M–matrices of order n, M, for which λ is the lowest eigenvalue and σ is the eigenvector associated with λ, satisfying the following condition

If $\mathsf{M}(A;B)$ is a $k \times k$ circular minor of M, then $-\mathsf{M}(A;B)$ is totally non–negative.

Suppose that Γ is a circular planar network with n boundary vertices, and $\pi = \pi(\Gamma)$ is the set of circular pairs $(A;B)$ which are connected through Γ. A subset $\Phi_{\lambda,\sigma}(\pi)$ of $\Phi_{\lambda,\sigma}$ is defined by the following condition: For each circular pair of indices $(A;B) \in \pi$ iff $(-1)^k \det\mathsf{M}(A;B) > 0$. On the other hand, when $\lambda = 0$ and $\sigma = 1$, $\Phi_{0,1}(\pi)$ will be denoted simply by $\Phi(\pi)$.

Lemma 2.1. [12, Theorem 4] *Suppose* $\Gamma = (\bar{F}, c_F)$ *is a critical planar graph with* m *edges and* $\pi = \pi(\Gamma)$. *Then, the map which sends* c *to* Λ *is a diffeomorphism of* $(\mathbb{R}^+)^m$ *onto* $\Phi(\pi)$.

Given $\mathsf{M} \in \Phi_{\lambda,\sigma}$, we say that $u \in \mathcal{C}(\bar{F})$ is *M–harmonic* iff $\mathscr{L}(u) = 0$ on F, where $\mathscr{L}$ is the combinatorial Laplacian with conductivity given by Lemma 2.1 with $\Lambda = \mathsf{D}_\sigma(\mathsf{M}-\lambda\mathsf{I})\mathsf{D}_\sigma$, where D_σ is the diagonal matrix associated with σ. Observe that, $\mathsf{D}_\sigma(\mathsf{M}-\lambda\mathsf{I})\mathsf{D}_\sigma \in \Phi(\pi)$ since $\mathsf{D}_\sigma(\mathsf{M}-\lambda\mathsf{I})\mathsf{D}_\sigma 1 = 0$ and $\Big(\mathsf{D}_\sigma(\mathsf{M}-\lambda\mathsf{I})\mathsf{D}_\sigma\Big)(A;B) = \mathsf{D}_\sigma(A;A)\mathsf{M}(A;B)\mathsf{D}_\sigma(B;B)$ for any $(A;B)$ circular pair.

Let us recall here that any weight $\omega \in \Omega(\bar{F})$ is always positive on $\bar{F}$. In addition, given a weight on the boundary $\sigma \in \Omega(\delta(F))$, let $\hat{\sigma} : \bar{F} \longrightarrow (0,+\infty)$ be such that $\hat{\sigma} = \sigma$ on $\delta(F)$. Then, we can define a new weight $\omega = k\hat{\sigma}$, where $k = \Big(\int_{\bar{F}} \hat{\sigma}^2(x)\,dx\Big)^{-1}$. Notice that $0 < k < 1$, since $k = \Big(\int_F \hat{\sigma}^2(x)\,dx + 1\Big)^{-1}$. Also, let us use the outer product $\otimes$ given by $u \otimes v\,(x,y) = u(x)v(y)$ for all $u, v \in \mathcal{C}(\bar{F})$ and $x, y \in \bar{F}$.

Theorem 2.2. [4, Theorem 12] *For any $n \geq 2$, $\sigma \in \Omega(\delta(F))$ and $\lambda \geq 0$, suppose M is in $\Phi_{\lambda,\sigma}$. Then, there is a circular planar graph with conductance c such that for any $\omega \in \Omega(\bar{F})$ satisfying $\omega = k\sigma$ on $\delta(F)$, $\mathsf{M} = \Lambda_q$, where Λ_q is the Dirichlet–to–Robin map associated with the operator $\mathscr{L}_q$, with $q = q_\omega + \lambda\chi_{\delta(F)}$ and conductances $c_\omega = \dfrac{c}{\omega \otimes \omega}$. Moreover, if $\mathsf{M} \in \Phi_{\lambda,\sigma}(\pi)$, then there is a unique critical circular planar network with conductance c and a unique $\omega \in \Omega(\bar{F})$, M–harmonic function such that $\mathsf{M} = \Lambda_q$, where Λ_q is the Dirichlet–to–Robin map associated with the operator $\mathscr{L}_q$ with potential $q = q_\omega + \lambda\chi_{\delta(F)}$ and conductances given by c_ω.*

3. Overdetermined Partial boundary value problems

We fix a proper and connected subset $F \subset V$ and $A, B \subset \delta(F)$ non–empty subsets such that $A \cap B = \emptyset$. Moreover we denote by R the set $R = \delta(F) \setminus (A \cup B)$, so $\delta(F) = A \cup B \cup R$ is a partition of $\delta(F)$. We remark that R can be an empty set. We consider a new type of boundary value problems in which the values of the functions and their normal derivatives are known at the same part of the boundary, which represents an overdetermined problem, and there exists another part of the boundary where no data is known. The limit case when $B = R = \emptyset$, the value of the function on the boundary is null and the value of the normal derivative is constant, can be seen as an extension of the so–called discrete Serrin's Problem. The analysis of this problem was carried out by the authors in [3]. For $B = \emptyset$, this kind of problem has been considered in the continuous case as an extension of Serrin's problem, see [15].

For any $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$, the *overdetermined partial Dirichlet–Neumann boundary value problem on F with data f, g, h* consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathscr{L}_q(u) = f \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = h \quad \text{on} \quad A \qquad \text{and} \qquad u = g \quad \text{on} \quad A \cup R. \tag{1}$$

Notice that as the values of u are known in A, the boundary condition $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = h$ is equivalent $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} + q\,u = h + q\,g$.

The *homogeneous overdetermined partial Dirichlet–Neumann boundary value problem on F* consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathscr{L}_q(u) = 0 \quad \text{on} \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = u = 0 \quad \text{on} \quad A \qquad \text{and} \qquad u = 0 \quad \text{on} \quad R. \tag{2}$$

It is clear that the set of solutions of the homogeneous boundary value problem is a subspace of $\mathcal{C}(F\cup B)$ that we will denote by $\mathcal{V}_B$. Moreover, if Problem (1) has solutions and u is a particular one, then $u+\mathcal{V}_B$ describes the set of all its solutions. In addition, if u is a solution of Problem (1), then for any $x\in A$ we get that

$$\int_F c(x,y)u(y)dy = g(x)\kappa_F(x) - h(x).$$

Therefore, if u is a solution of Problem (2), then for any $x\in A$ we get that

$$\int_F c(x,y)u(y)dy = 0.$$

The *adjoint problem of the overdetermined partial Dirichlet–Neumann boundary value problem (2) on F* is given by

$$\mathscr{L}_q(v) = 0 \ \text{ on } \ F, \qquad \frac{\partial v}{\partial \mathsf{n}_\mathsf{F}} = v = 0 \ \text{ on } \ B \quad \text{ and } \quad v = 0 \ \text{ on } \ R. \tag{3}$$

The subspace of solutions of the above problem will be denoted by $\mathcal{V}_A$. It is clear that $\mathcal{V}_A\subset\mathcal{C}(F\cup A)$.

The Second Green Identity leads to the following result.

Proposition 3.1. *Problems (2) and (3) are mutually adjoint; that is*

$$\int_F v(x)\mathscr{L}_q(u)(x)\,dx = \int_F u(x)\mathscr{L}_q(v)(x)\,dx,$$

for any $u,v\in\mathcal{C}(\bar F)$ such that $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = u = 0$ on A, $\dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}} = v = 0$ on B and $u = v = 0$ on R.

Proof. By the Second Green Identity we get that

$$\begin{aligned}\int_F \Big(v(x)\mathscr{L}_q(u)(x) - u(x)\mathscr{L}_q(v)(x)\Big)\,dx &= \int_{\delta(F)} \Big(u(x)\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}(x) - v(x)\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}(x)\Big)\,dx\\ &= \int_B u(x)\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx - \int_A v(x)\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx = 0,\end{aligned}$$

obtaining the result. □

Proposition 3.2 (Fredholm Alternative). *Given $f\in\mathcal{C}(F)$, $g\in\mathcal{C}(A\cup R)$, $h\in\mathcal{C}(A)$, the boundary value problem*

$$\mathscr{L}_q(u) = f, \ \text{ on } \ F, \qquad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = h \ \text{ on } \ A \ \text{ and } \ u = g \ \text{ on } \ A\cup R$$

has solution if and only if

$$\int_F f(x)v(x)\,dx + \int_A h(x)v(x)\,dx = \int_{A\cup R} g(x)\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx, \qquad \text{for each } v\in\mathcal{V}_A.$$

In addition, when the above condition holds, then there exists a unique solution of the boundary value problem in $\mathcal{V}_B^\perp$, i.e. a unique solution u, such that

$$\int_{F\cup B} u(x)z(x)\,dx = 0, \ \ \text{for any } z\in\mathcal{V}_B.$$

Proof. First observe that Problem (1) is equivalent to the boundary value problem

$$\mathscr{L}_q(u) = f - \mathscr{L}(g) \quad \text{on} \quad F, \quad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = h - g\kappa_F \quad \text{on} \quad A \quad \text{and} \quad u = 0, \quad \text{on} \quad A \cup R \tag{4}$$

in the sense that u is a solution of this problem if and only if $u+g$ is a solution of Problem (1). Notice that $\mathscr{L}(g) = \mathscr{L}_q(g)$ since $g = 0$ on F. Consider now the linear operators $\mathcal{F}\colon \mathcal{C}(F \cup B) \longrightarrow \mathcal{C}(F \cup A)$ and $\mathcal{F}^*\colon \mathcal{C}(F \cup A) \longrightarrow \mathcal{C}(F \cup B)$ defined as

$$\mathcal{F}(u) = \begin{cases} \mathscr{L}_q(u), & \text{on} \quad F, \\ \dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}}, & \text{on} \quad A, \end{cases} \qquad \text{and} \qquad \mathcal{F}^*(v) = \begin{cases} \mathscr{L}_q(v), & \text{on} \quad F, \\ \dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}}, & \text{on} \quad B, \end{cases}$$

respectively. Then, for any $u \in \mathcal{C}(F \cup B)$ satisfying Problem (4) and for any $v \in \mathcal{C}(F \cup A)$,

$$\begin{aligned} \int_{F\cup A} v(x)\mathcal{F}(u)(x)\,dx &= \int_F v(x)\mathscr{L}_q(u)(x)\,dx + \int_{\delta(F)} v(x)\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx = \\ &= \int_F u(x)\mathscr{L}_q(v)(x)\,dx + \int_{\delta(F)} u(x)\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx = \int_{F\cup B} u(x)\mathcal{F}^*(v)(x)\,dx. \end{aligned}$$

Clearly, $\ker\mathcal{F}^* = \mathcal{V}_A$. Moreover, Problem (1) has a solution if and only if the function $\tilde{f} \in \mathcal{C}(F \cup A)$ given by $\tilde{f} = f - \mathscr{L}(g)$ on F and $\tilde{f} = h - g\kappa_F$ on A satisfies that $\tilde{f} \in \mathrm{Img}\mathcal{F}$. Therefore, the Fredlhom Alternative for linear operators implies that Problem (1) has solution if and only if for any $v \in \mathcal{V}_A$

$$\begin{aligned} 0 = \int_{F\cup A} \tilde{f}(x)v(x)\,dx &= \int_F f(x)v(x)\,dx + \int_A h(x)v(x)\,dx - \int_F v(x)\mathscr{L}(g)(x)\,dx \\ - \int_A v(x)g(x)\kappa_F(x)\,dx &= \int_F f(x)v(x)\,dx + \int_A h(x)v(x)\,dx - \int_{R\cup A} g(x)\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx. \end{aligned}$$

Finally, the Fredholm Alternative also establishes that when the necessary and sufficient condition holds there exists a unique $w \in (\ker\mathcal{F})^\perp$ such that $\mathcal{F}(w) = \tilde{f}$. Therefore, $u = w + g$ is the unique solution of Problem (1) such that for any $z \in \ker\mathcal{F} = \mathcal{V}_B$ satisfies

$$\int_{F\cup B} u(x)z(x)\,dx = 0.$$

□

Observation 3.3. *The Fredholm Alternative establishes the following formula*

$$\dim\mathcal{V}_A - \dim\mathcal{V}_B = |A| - |B|.$$

On the other hand, the existence of solution for any data is equivalent to be $\mathcal{V}_A = \{0\}$*; that is,* $|B| - |A| = \dim\mathcal{V}_B \geq 0$*. Moreover, uniqueness of solutions is equivalent to be* $|A| - |B| = \dim\mathcal{V}_A \geq 0$*. In particular, if* $|A| = |B|$*, the existence of solution of Problem (1) for any data* f, g *and* h *is equivalent to the uniqueness of solution and hence it is equivalent to the fact that the homogeneous problem has* $v = 0$ *as its unique solution.*

Now we give some basic examples in order to show all the possible situations we may find in solving Problem (1). In Figure 1, we show some networks, all of them with sets A and B given by the pictures and with $R = \emptyset$. The conductances are also given in the figures and we consider $q = q_\sigma + 2\chi_{\delta(F)}$, where σ is the weight given in Table 1.

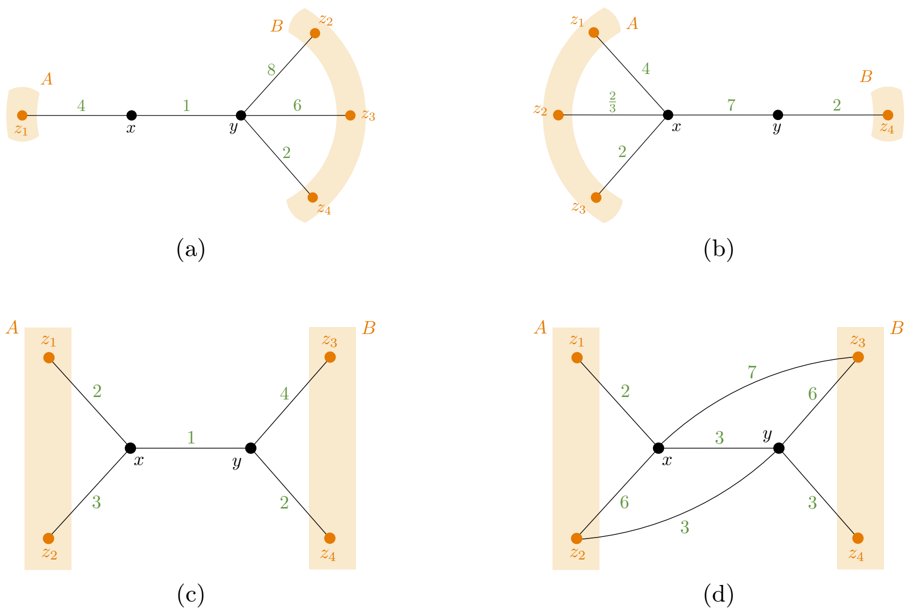


Figure 1: Some examples of networks and sets A and B.

We analyze Problem (1) with different data in the above networks. Network (1a) has sets $\mathcal{V}_A = \{0\}$ and $\dim(\mathcal{V}_B) = 2$. So, on this network Problem (1) has a solution for all data but it is not unique. In fact, all the solutions are given by a particular solution plus all the solutions of the homogeneous problem and as a consequence the set of solutions has also dimension two. Network (1b) has $\mathcal{V}_B = \{0\}$ and $\dim(\mathcal{V}_A) = 2$, which means that there is uniqueness but not necessarily existence. Network (1c) has $|A| = |B|$ but $\mathcal{V}_A, \mathcal{V}_B \neq \{0\}$, and so we do not have existence, in general, and finally Network (1d) has $|A| = |B|$ and also $\mathcal{V}_A = \mathcal{V}_B = \{0\}$, which means that the problem has a unique solution for any data.

On the Table 1 we give some particular examples of these conclusions. The values of all functions are taken in the following order z_1, z_2, z_3, z_4, x, y.

As we can see on the first row of the table, for Network (1a) there is existence but not uniqueness and the set of solutions has dimension 2; rows 2 and 3 show the uniqueness but not necessarily existence on Network (1b); rows 4 and 5 show that we do not have existence in general nor uniqueness and the last row of the table shows us existence and uniqueness of solution for this set of data, as it is expected.

In order to study sufficient and necessary conditions so that $\mathcal{V}_B = \{0\}$ and/or $\mathcal{V}_A = \{0\}$, it is useful to introduce partial Dirichlet–to–Neumann maps. To achieve our purpose we consider again the Dirichlet problem

$$\mathscr{L}_q(u_g) = 0 \ \text{ on } \ F, \quad u_g = g \ \text{ on } \ \delta(F).$$

Network	$\lvert A\rvert$	$\lvert B\rvert$	σ	g	h	f	u
Fig. (1a)	1	3	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{2}{3},\frac{2}{3}\right)$	(3)	(1)	$(0,2)$	$\left(3,a,b,-\frac{149}{8}-4a-3b,\frac{11}{4},-\frac{13}{2}\right)$, with $a,b\in\mathbb{R}$
Fig. (1b)	3	1	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{2}{3},\frac{2}{3}\right)$	$(0,1,-1)$	$(2,1,-1)$	$(4,3)$	$\left(0,1,-1,-\frac{7}{2},-\frac{1}{2},-1\right)$
			$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{2}{3},\frac{2}{3}\right)$	$(3,0,2)$	$(0,1,1)$	$(3,1)$	*none*
Fig. (1c)	2	2	$\left(\frac{1}{6},\frac{2}{3},\frac{2}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$	$(-8,2)$	$(16,54)$	$(0,1)$	$\left(-8,2,a,-\frac{4355}{2}-2a,-16,-230\right)$, with $a\in\mathbb{R}$
			$\left(\frac{1}{6},\frac{2}{3},\frac{2}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$	$(1,2)$	$(0,1)$	$(-2,1)$	*none*
Fig. (1d)	2	2	$\left(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{2}{3},\frac{2}{3}\right)$	$(1,-2)$	$(2,0)$	$(7,-21)$	$(1,-2,3,-9,0,-6)$

Table 1: Some examples of overdetermined partial boundary value problems.

Therefore, we assume that there exist $\sigma\in\Omega(\bar F)$ and $\lambda\ge 0$ such that $q=q_\sigma+\lambda\chi_{\delta(F)}$. Then, the above problem has a unique solution for any data $g\in\mathcal{C}(\delta(F))$.

We define the *partial Dirichlet–to–Neumann map* as the linear operator $\Lambda_{A,B}:\mathcal{C}(A)\longrightarrow\mathcal{C}(B)$, that assigns to any $v\in\mathcal{C}(A)$ the function

$$\Lambda_{A,B}(v)=\frac{\partial u_v}{\partial \mathsf{n}_F}\chi_B.$$

We define $\Lambda_{B,A}$ in an analogous manner. Observe that $u_v=0$ on $B\cup R$ and hence $\Lambda_{A,B}(v)=\Lambda_q(v)\cdot\chi_B$.

Proposition 3.4. $\Lambda^*_{A,B}=\Lambda_{B,A}$ *and, in addition,* $\mathsf{ker}\Lambda_{A,B}=\mathcal{V}_A\cdot\chi_A$ *and* $\mathsf{ker}\Lambda_{B,A}=\mathcal{V}_B\cdot\chi_B$. *Moreover, the kernels of* $\Lambda_{A,B}$ *and* $\Lambda_{B,A}$ *are* $N_q\cdot\chi_{B\times A}$ *and* $N_q\cdot\chi_{A\times B}$*; and hence the associated matrices are* $\mathsf{N}_q(B;A)$ *and* $\mathsf{N}_q(A;B)$*, respectively.*

Proof. Given $v\in\mathcal{C}(A)$ and $w\in\mathcal{C}(B)$, then from the Second Green Identity

$$\begin{aligned}\int_B w(x)\Lambda_{A,B}(v)(x)\,dx &= \int_B u_w(x)\frac{\partial u_v}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx=\int_{\delta(F)} u_w(x)\frac{\partial u_v}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx=\\ &=\int_{\delta(F)} u_v(x)\frac{\partial u_w}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx=\int_A u_v(x)\frac{\partial u_w}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx=\int_A v(x)\Lambda_{B,A}(w)(x)\,dx,\end{aligned}$$

where we have taken into account that $\mathscr{L}_q(u_v)=\mathscr{L}_q(u_w)=0$ on F.

Clearly, if $v\in\mathsf{ker}\Lambda_{A,B}$, then $u_v\in\mathcal{V}_A$ and $v=u_v\cdot\chi_A$. Conversely, if $u\in\mathcal{V}_A$ then $\mathscr{L}_q(u)=0$ on F, $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}}=0$ on B and $u=0$ on $B\cup R$. Therefore, if we consider $v=u\cdot\chi_A$, then $u=u_v$ and clearly $v\in\mathsf{ker}\Lambda_{A,B}$. The equality for $\mathsf{ker}\Lambda_{B,A}$ follows analogously. $\square$

Corollary 3.5. *Problem (1) has solution for any data if and only if* $\Lambda_{A,B}$ *has maximum rank. Moreover, Problem (1) has uniqueness of solutions for any data if and only if* $\Lambda_{B,A}$ *has maximum* *rank These problems are not self–adjoint, and hence we worry about the study of existence and uniqueness through the adjoint problem. In particular, when $|A| = |B|$, $\Lambda_{A,B}$ is non–singular if and only if $\Lambda_{B,A}$ is, and in this case Problem (1) has a unique solution for any data.*

Let us study the kernel of the application $\Lambda_{A,B}$ in the above examples. For Networks (1a) and (1b) the partial Dirichlet–to–Neumann map has maximum rank. As a consequence, on Network (1a) there is existence and on Network (1b) there is uniqueness of solutions for Problem (1). For Network (1c), we get that

$$N_q(z_3,z_1) = -\frac{2}{71}, \quad N_q(z_3,z_2) = -\frac{3}{71}, \quad N_q(z_4,z_1) = -\frac{1}{71} \quad \text{and} \quad N_q(z_4,z_2) = -\frac{3}{142},$$

which implies that $\Lambda_{A,B}$ and $\Lambda_{B,A}$ are singular. Therefore, Problem (1) may have no solution for any data and when it has, the solution is non unique. On the other hand, for Network (1d),

$$N_q(z_3,z_1) = -\frac{80}{21}, \quad N_q(z_3,z_2) = -\frac{121}{7}, \quad N_q(z_4,z_1) = -\frac{4}{7} \quad \text{and} \quad N_q(z_4,z_2) = -\frac{51}{14},$$

which implies that $\Lambda_{A,B}$ and $\Lambda_{B,A}$ are non–singular. Therefore, Problem (1) has a unique solution for any data.

Next result tell us which is the value on B of a solution of Problem (1) with data $f = 0$.

Proposition 3.6. *Suppose that $\Lambda_{A,B}$ has maximum rank and let $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$. If u is a solution of problem*

$$\mathscr{L}_q(u) = 0 \ \ on \ \ F, \qquad u = g \ \ on \ \ A \cup R \quad \ and \quad \frac{\partial u}{\partial \mathsf{n}_F} = h \ \ on \ \ A,$$

then the values of u on B are determined by the identity

$$\Lambda_{B,A}(u) = h - \Lambda_{A \cup R,A}(g).$$

In addition, if $|A| = |B|$, then

$$u = \Lambda^{-1}_{B,A}(h) - \Lambda^{-1}_{B,A} \circ \Lambda_{A \cup R,A}(g).$$

Proof. If $\psi = u\chi_B$ and $\varphi = \psi + g$, then u is the unique solution of the Dirichlet problem

$$\mathscr{L}_q(u) = 0 \ \text{ on } \ F \quad \text{ and } \quad u = \varphi \ \text{ on } \ \delta(F).$$

Moreover, from the superposition principle $u = u_\psi + u_g$ and hence

$$\frac{\partial u}{\partial \mathsf{n}_F} = \frac{\partial u_\psi}{\partial \mathsf{n}_F} + \frac{\partial u_g}{\partial \mathsf{n}_F}.$$

Therefore,

$$h = \frac{\partial u}{\partial \mathsf{n}_F}\chi_A = \Lambda_{B,A}(u) + \Lambda_{A \cup R,A}(g)$$

and the result follows from the injectivity of $\Lambda_{B,A}$. $\square$

E.B. Curtis and J.A. Morrow proved in [14, Corollary 3.14] that for a circular planar network $\Lambda_{A,B}$ is non–singular iff the pair $(A;B)$ is connected through Γ. In particular, this happens if Γ is well–connected. In the same reference they found the 4 types of basic well–connected planar networks and then, described an algorithm for recovering the conductances. One of the keys of the algorithm is the boundary spike formula, see [14, Corollary 3.16]; that allows to recover the conductance on a boundary spike edge. The proof of this result can be easily adapted to the case of Schrödinger operators and Dirichlet–to–Neumann maps.

Lemma 3.7. *Let $\Gamma = (\bar{F}, c)$ be a connected circular planar network. Suppose that Γ has a boundary spike xy with $x \in \delta(F)$ and $y \in F$. If contracting xy to a unique boundary vertex results in breaking the connection through Γ between a circular pair (A, B), then*

$$c(x,y) = \frac{\omega(x)}{\omega(y)} \big(\mathsf{N}_q(x;x) - \mathsf{N}_q(x;B)\cdot \mathsf{N}_q(A;B)^{-1}\cdot \mathsf{N}_q(A;x) - \lambda\big).$$

We would like to remark that all the results of this section (except for Lemma 3.7) hold for general networks, not necessarily planar.

4. Recovering the conductances on Spider Networks

Now the goal is the complete recovery of the conductivity function of a family of networks using only the information provided by the Dirichlet–to–Robin map. This algorithm represents an extension of the one developed in [13, 14], since we consider not only the Laplacian but Schrödinger operators. Although the guidelines are very similar to the ones in the above reference, we describe the algorithm entirely to show the relevance of the above results on overdetermined partial boundary value problems associated with positive semi–definite Schrödinger operators. Therefore, we will assume that $q = q_\sigma + \lambda\chi_{\delta(F)}$, $\sigma \in \Omega(\bar{F})$ and $\lambda \geq 0$.

Well–connected spider networks are a subfamily of critical circular planar networks and were first introduced in [14] because of their remarkable properties. However, we use the same definitions and notations as in [2], where the Green function of this type of networks was given.

A *well–connected spider network* with $n \equiv 3 \,(\mathrm{mod}\ 4)$ radii and $m = \dfrac{n-3}{4}$ circles is a circular planar network $\Gamma = (\bar{F}, c)$ with n boundary vertices given by $\delta(F) = \{v_1, \dots, v_n\}$ placed in the circular order provided by ∂D. The vertices in F are distributed in the following way: place a vertex x_{00} in the center of the boundary circle ∂D and draw a straight line from x_{00} to each v_j. This line is called the *radius* j. Now draw m different concentric circumferences with center x_{00} and such that all lie in $\mathring{D}$. We call each one of them the *circle* i, where the circles are labeled from less to most diameter. Finally, place a vertex x_{ji} in the intersection of every circle i and radius j. Then, $F = \{x_{ji}\}_{i=1,\dots,m,\, j=1,\dots,n} \cup \{x_{00}\}$. The edges are the ones given by the radius and the circles, see Figure 2. For the sake of simplicity, we define $x_{j0} = x_{00}$ and $x_{j\,m+1} = v_j$ for all $j = 1, \dots, n$. Also, we take the notation $x_{ji} = x_{j-n\,i}$ for any $j > n$. For each $j = 1, \dots, n$ we consider the boundary sets $A_j = \{v_{1+j}, \dots, v_{\frac{n-1}{2}+j}\} \subset \delta(F)$, $B_j = \{v_{\frac{n+1}{2}+j}, \dots, v_{n-1+j}\} \subset \delta(F)$ and $R_j = \{v_j\} \subset \delta(F)$. Notice that $|A_j| = |B_j| = \dfrac{n-1}{2}$. Moreover, these boundary configurations on a well–connected spider network guarantee that A_j and B_j are always connected through Γ and indeed the set formed by the vertices in the paths that connect A_j and B_j coincides with $V \setminus \{v_j\}$.

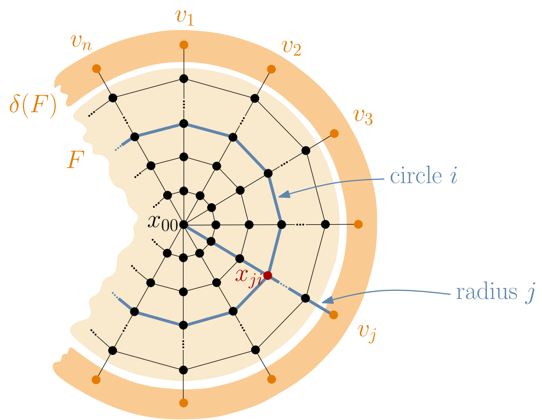


Figure 2: Structure of a spider network.

Therefore, if we contract the boundary spike $v_j x_{jm}$ to a single boundary vertex then we break the connection between A_j and B_j. Given an index $i \in \{0, \dots, m+1\}$, we consider the circular layers of vertices $D_i = \{x_{li} \in V : l = 1, \dots, n\} \subset V$. In particular, $D_0 = \{x_{00}\}$ and $D_{m+1} = \delta(F)$. Therefore, the set π defined in Section 2, coincides with the whole set of circular pairs.

The recovery of conductances on a well–connected spider network is an iterative process, for we are not able to give explicit formulae for all the conductances at the same time but we can give a recovery algorithm instead. Hence, we describe the algorithm in steps, each of them requiring the information obtained in the last one.

To start with let $\mathsf{N}_q \in \Phi_{\lambda,\sigma}$. Then, $\lambda \geq 0$ is the lowest eigenvalue of N_q and $\sigma \in \Omega(\delta(F))$ is the eigenvector associated with λ. In addition, we choose $\omega \in \Omega(\bar{F})$ such that $\omega = k\sigma$ on $\delta(F)$, $0 < k < 1$.

Let us clarify here that the inputs of the recovery algorithm are given by the known information, such as N_q, λ and ω, and the outputs are the conductances c such that Λ_q is the Dirichlet–to–Robin map associated with the Schrödinger operator with potential $q = q_\omega + \lambda \chi_{\delta(F)}$ and c_ω, see Theorem 2.2. Also, it is important to remark that this recovery algorithm is equivalent to recovering q and the data used is different from the one in [13], since in this reference the Dirichlet–to–Robin matrix is assumed to be in $\Phi_{0,1}$. Notice, that the matrices in the last set are singular and weakly diagonally dominant.

Step 0

In this step we do not recover any conductance. However, we set the necessary tools to obtain them in future steps. We fix the index $j \in \{1, \dots, n\}$ and consider the overdetermined partial boundary value problem that consists in finding $u_j \in \mathcal{C}(\bar{F})$ such that

$$\mathscr{L}_q(u_j) = 0 \;\text{ on }\; F, \quad u_j = \varepsilon_{v_j} \;\text{ on }\; A_j \cup R_j \;\text{ and }\; \frac{\partial u_j}{\partial \mathsf{n}_F} = 0 \;\text{ on }\; A_j. \tag{5}$$

There exists a large set of vertices of the well–connected spider network Γ where $u_j = 0$. We denote this set by

$$Z(u_j) = \big\{x \in \bar{F} : u_j(x) = 0\big\} = \bar{F} \setminus \mathsf{supp}(u).$$

Clearly, $A_j \subseteq Z(u_j)$. The size of $Z(u_j)$, however, is much bigger than the size of A_j.

Proposition 4.1. *It is satisfied that*

$$Z(u_j) = \big\{x_{li} \in V : i = 0, \dots, m+1,\ l = i+j-m, \dots, 3m+2+j-i\big\}.$$

Proof. We divide the proof into different stages that lead to the result. So, we have to prove the following claims:

(i) $H_k = \big\{x_{\ell\, m-k} \in V\ :\ \ell = 1+j+k, \dots, \frac{n-1}{2}+j-k\big\} \subseteq Z(u_j)$ for all $k = 0, \dots, m$,

(ii) $T_1 = \big\{x_{j+k-\ell\, m-k} \in V\ :\ \ell = 0, \dots, 2m-2,\ k = \left\lceil \frac{\ell}{2} \right\rceil, \dots, m-1\big\} \subseteq Z(u_j)$,

(iii) $T_2 = \Big\{x_{\frac{n+1}{2}+j-k+\ell\, m-k} \in V\ :\ \ell = 0, \dots, 2m-2,\ k = \left\lceil \frac{\ell}{2} \right\rceil, \dots, m-1\Big\} \subseteq Z(u_j)$,

and let $Z_j = \bigcup\limits_{k=0}^{m} H_k \cup T_1 \cup T_2$.

To prove (i), we perform induction on k. For $k = 0$ and $\ell = 1+j, \dots, \frac{n-1}{2}+j$, it is satisfied that $v_\ell \in A_j$ and hence

$$0 = \frac{\partial u_j}{\partial \mathsf{n}_F}(v_\ell) = -c(v_\ell, x_{\ell m}) u_j(x_{\ell m}),$$

which means that $u_j(x_{\ell m}) = 0$ for all $\ell = 1+j, \dots, \frac{n-1}{2}+j$. Then, $H_0 \subseteq Z(u_j)$. Let us assume that (i) is true for any index $l < k$ and we want to see that the result holds for k. If $\ell \in \{1+j+k, \dots, \frac{n-1}{2}+j-k\}$, then by induction hypothesis

$$0 = \mathscr{L}_q(u_j)(x_{\ell\, m-k+1}) = -c(x_{\ell\, m-k+1}, x_{\ell\, m-k}) u_j(x_{\ell\, m-k}),$$

which means that $u_j(x_{\ell\, m-k}) = 0$ and so (i) follows.

To prove (ii) we use double induction on ℓ and k. For $\ell = 0$ and $k = m-1$, using (i) we get that

$$0 = \mathscr{L}_q(u_j)(x_{j+m\, 1}) = -c(x_{j+m\, 1}, x_{j+m-1\, 1}) u_j(x_{j+m-1\, 1})$$

and hence $u_j(x_{j+m-1\, 1}) = 0$. Now we assume that the result holds for $\ell = 0$ and any index $l > k$ and we want to see that it also holds for k. Using (i) and the induction hypothesis,

$$0 = \mathscr{L}_q(u_j)(x_{j+k+1\, m-k}) = -c(x_{j+k+1\, m-k}, x_{j+k\, m-k}) u_j(x_{j+k\, m-k})$$

and so $u_j(x_{j+k\, m-k}) = 0$. Therefore, the case $i = 0$ holds. The next phase is to suppose that (ii) holds for any index $l < \ell$ and any $k = \left\lceil \frac{l}{2} \right\rceil, \dots, m-1$, and to prove that in this case it also holds for ℓ and $k \in \big\{\left\lceil \frac{\ell}{2} \right\rceil, \dots, m-1\big\}$. By induction hypothesis,

$$0 = \mathscr{L}_q(u_j)(x_{j+k-\ell+1\, m-k}) = -c(x_{j+k-\ell+1\, m-k}, x_{j+k-\ell\, m-k}) u_j(x_{j+k-\ell\, m-k})$$

and hence $u_j(x_{j+k-\ell\, m-k}) = 0$, completing the double induction. In consequence, $T_1 \subseteq Z(u_j)$. The result in (iii) is proved analogously.

The inclusion $Z_j \subseteq Z(u_j)$ is a direct consequence of (i), (ii) and (iii) if we rearrange the indices. Moreover, suppose that there exists a vertex $x \in \delta(Z_j) \cap Z(u_j)$. Then, using the same techniques for equation $\mathscr{L}_q(u_j) = 0$ on F as in the proofs of (i) and (ii), we see that $u_j = 0$ on $\bar{F}$. This is a contradiction with $u(v_j) = 1$ and hence $\delta(Z_j) \subset \mathsf{supp}(u_j)$.

Finally, notice that the k–connection between A_j and B_j covers any vertex of $\bar{F} \setminus \{v_j\}$ and hence keeping in mind the strong alternating property, that was proved in [4, Theorem 8], we conclude that $\bar{F} \setminus Z_j \subset \mathsf{supp}(u_j)$. □

Actually, the set $Z(u_j)$ has a very characteristic shape. In Figure 3(a) we show this pattern. In particular, there are exactly $n-2$ vertices on D_1 for which $u_j = 0$ and exactly two vertices on D_1 for which $u_j \neq 0$.

Step 1

Let us fix the index $j \in \{1, \ldots, n\}$ and let us consider the unique solution $u_j \in \mathcal{C}(\bar{F})$ of Problem (5). We already know that $u_j = 0$ on A_j and $u_j = 1$ on R_j. Moreover, we know the values of u_j on B_j are given by Proposition 3.6

$$\mathsf{u}_{B_j} = -\mathsf{N}_q(A_j; B_j)^{-1} \cdot \mathsf{N}_q(A_j; v_j).$$

In consequence, we know u_j on $\delta(F)$. In Figure 3(b) we show all the information obtained at the end of this step.

Step 2

Applying the boundary spike formula given in Lemma 3.7 and keeping in mind that all the boundary edges are spikes, we know the values of the conductances of all the edges joining vertices from D_{m+1} and D_m. In Figure 3(c) we show all the information obtained at the end of this step.

Step 3

Again, let us fix the index $j \in \{1, \ldots, n\}$ in this step and let us consider the unique solution $u_j \in \mathcal{C}(\bar{F})$ of Problem (5). Then, we know all the values of u_j on D_m, as the following result shows.

Lemma 4.2. *The values of u_j on D_m are given by*

$$u_j(x_{km}) = \frac{1}{c(v_k, x_{km})}\Big(\lambda u_j(v_k) - \mathsf{N}_q(v_k; v_j) - \mathsf{N}_q(v_k; B_j) \cdot \mathsf{u}_{B_j}\Big) + \frac{\omega(x_{km})}{\omega(v_k)} u_j(v_k)$$

for all $k = 1, \ldots, n$.

Proof. As before, we can express Problem (5) as the Dirichlet problem

$$\mathscr{L}_q(u) = 0 \text{ on } F \quad \text{and} \quad u = \varepsilon_{v_j} + u_{B_j} \text{ on } \delta(F)$$

with the additional condition $\dfrac{\partial u}{\partial \mathsf{n}_F} = 0$ on A_j. Therefore, by the definition of the Dirichlet–to–Robin map, for all $v_k \in \delta(F)$ it is satisfied that

$$\begin{aligned}\mathsf{N}_q(v_k; v_j) + \mathsf{N}_q(v_k; B_j) \cdot \mathsf{u}_{B_j} &= \Lambda_q\Big(\varepsilon_{v_j} + u_{B_j}\Big)(v_k) = \frac{\partial u_j}{\partial \mathsf{n}_F}(v_k) + q(v_k) u_j(v_k) \\ &= \left(\lambda + \frac{\omega(x_{km})}{\omega(v_k)} c(v_k, x_{km})\right) u_j(v_k) - c(v_k, x_{km}) u_j(x_{km}).\end{aligned}$$

Observe that all the terms of this equality, except the value $u_j(x_{km})$, are already known. Therefore, we get the result. □

In Figure 3(d) we show all the data gathered from a well–connected spider network at the end of this step.

Let us define the linear operator $\wp\colon \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(F \setminus \{x_{00}\})$ given by the values

$$\wp(z)(x_{lk}) = c(x_{lk}, x_{l\,k+1})z(x_{l\,k+1}) + c(x_{lk}, x_{l+1\,k})z(x_{l+1\,k}) + c(x_{lk}, x_{l-1\,k})z(x_{l-1\,k})$$

for all $z \in \mathcal{C}(\bar{F})$ and $x_{lk} \in F \setminus \{x_{00}\}$. This operator will be useful in the following steps.

Step 4

In this step we give the conductances of all the edges with both ends in D_m. However, we state a more general result.

Proposition 4.3. *Let $i \in \{0, \ldots, m-1\}$. For every $j = 1, \ldots, n$, let us suppose that we know the values of u_j on D_{i+2} and D_{i+1}. Also, we suppose that the conductances of all the edges joining vertices from D_{i+2} and D_{i+1} are known. Then, the conductances $c(x_{i+j-m+1\,i+1}, x_{i+j-m\,i+1})$ are given by*

$$c(x_{i+j-m+1\,i+1}, x_{i+j-m\,i+1}) = -\frac{u_j(x_{i+j-m+1\,i+2})}{u_j(x_{i+j-m\,i+1})}\, c(x_{i+j-m+1\,i+1}, x_{i+j-m+1\,i+2}).$$

Proof. We fix the indices $i \in \{0, \ldots, m-1\}$ and $j \in \{1, \ldots, n\}$. Then, by Proposition 4.1,

$$u_j(x_{i+j-m+1\,i+1}) = u_j(x_{i+j-m+2\,i+1}) = u_j(x_{i+j-m+1\,i}) = 0.$$

Hence,

$$\begin{aligned} 0 = \mathscr{L}_q(u_j)(x_{i+j-m+1\,i+1}) &= -c(x_{i+j-m+1\,i+1}, x_{i+j-m+1\,i+2})u_j(x_{i+j-m+1\,i+2}) \\ &- c(x_{i+j-m+1\,i+1}, x_{i+j-m\,i+1})u_j(x_{i+j-m\,i+1}). \end{aligned}$$

The value $c(x_{i+j-m+1\,i+1}, x_{i+j-m\,i+1})$ is the only unknown term of this equality and by Proposition 4.1 we know that $u_j(x_{i+j-m\,i+1}) \neq 0$. □

When $i = m-1$, Propositions 4.1 and 4.3 show that $c(x_{jm}, x_{j-1\,m})$ is known for all $j = 1, \ldots, n$. See Figure 3(e) in order to see all the known information at the end of this step.

Step 5

In this step we give the conductances of all the edges joining the vertices from D_m and D_{m-1}. Furthermore, we state a more general result.

Proposition 4.4. *Let $i \in \{0, \ldots, m-1\}$. For every $j = 1, \ldots, n$, let us suppose that we know the values of u_j on D_{i+2} and D_{i+1}. Also, let us suppose that we know the conductances of all the edges joining vertices from D_{i+2} and D_{i+1}, and the ones of the edges with both ends in D_{i+1}. Then, the conductances $c(x_{i+j-m\,i}, x_{i+j-m\,i+1})$ are given by*

$$c(x_{i+j-m\,i}, x_{i+j-m\,i+1}) = \left(\frac{\wp(u_j)(x_{i+j-m\,i+1})}{u_j(x_{i+j-m\,i+1})} - \frac{\wp(\omega)(x_{i+j-m\,i+1})}{\omega(x_{i+j-m\,i+1})}\right) \cdot \frac{\omega(x_{i+j-m\,i+1})}{\omega(x_{i+j-m\,i})}.$$

Proof. We fix the indices $i \in \{0,\dots,m-1\}$ and $j \in \{1,\dots,n\}$. Observe that $\wp(\omega)(x_{i+j-m\,i+1})$ and $\wp(u_j)(x_{i+j-m\,i+1})$ are already known. Then,

$$0 = \mathscr{L}_q(u_j)(x_{i+j-m\,i+1}) = \frac{u_j(x_{i+j-m\,i+1})}{\omega(x_{i+j-m\,i+1})}\wp(\omega)(x_{i+j-m\,i+1}) - \wp(u_j)(x_{i+j-m\,i+1})$$
$$+ \frac{\omega(x_{i+j-m\,i})}{\omega(x_{i+j-m\,i+1})}c(x_{i+j-m\,i+1}, x_{i+j-m\,i})u_j(x_{i+j-m\,i+1})$$

and hence $c(x_{i+j-m\,i+1}, x_{i+j-m\,i})$ is the only unknown term of this equality. Notice that Proposition 4.1 assures that $u_j(x_{i+j-m\,i+1}) \neq 0$. □

In particular, when $i = m-1$, Propositions 4.1 and 4.4 show that $c(x_{j-1\,m}, x_{j-1\,m-1})$ is known for all $j = 1,\dots,n$. See Figure 3(f) in order to see all the information gathered at the end of this step.

Step 6

In this step we are able to obtain the values of u_j on D_{m-1} for all $j = 1,\dots,n$. In fact, let us state a more general result.

Proposition 4.5. *Let $i \in \{0,\dots,m-1\}$. For every $j = 1,\dots,n$, let us suppose that we know the values of u_j on D_{i+2} and D_{i+1}. Also, let us suppose that we know the conductances of all the edges joining vertices from D_{i+2} and D_{i+1}, from D_{i+1} and D_i and the ones of the edges with both ends in D_{i+1}. Then, the values of u_j on D_i are given by*

$$u_j(x_{k\,i}) = -\frac{u_j(x_{k\,i+1})\wp(\omega)(x_{k\,i+1})}{\omega(x_{k\,i+1})c(x_{k\,i+1}, x_{k\,i})} - \frac{\wp(u_j)(x_{k\,i+1})}{c(x_{k\,i+1}, x_{k\,i})} - \frac{\omega(x_{k\,i})}{\omega(x_{k\,i+1})}u_j(x_{k\,i+1})$$

for all $k = 1,\dots,n$.

Proof. Fixed two indices $i \in \{0,\dots,m-1\}$ and $j \in \{1,\dots,n\}$, let $x_{ki} \in D_i$ with $k \in \{1,\dots,n\}$. Observe that $\wp(\omega)(x_{k\,i+1})$ and $\wp(u_j)(x_{k\,i+1})$ are known. Then,

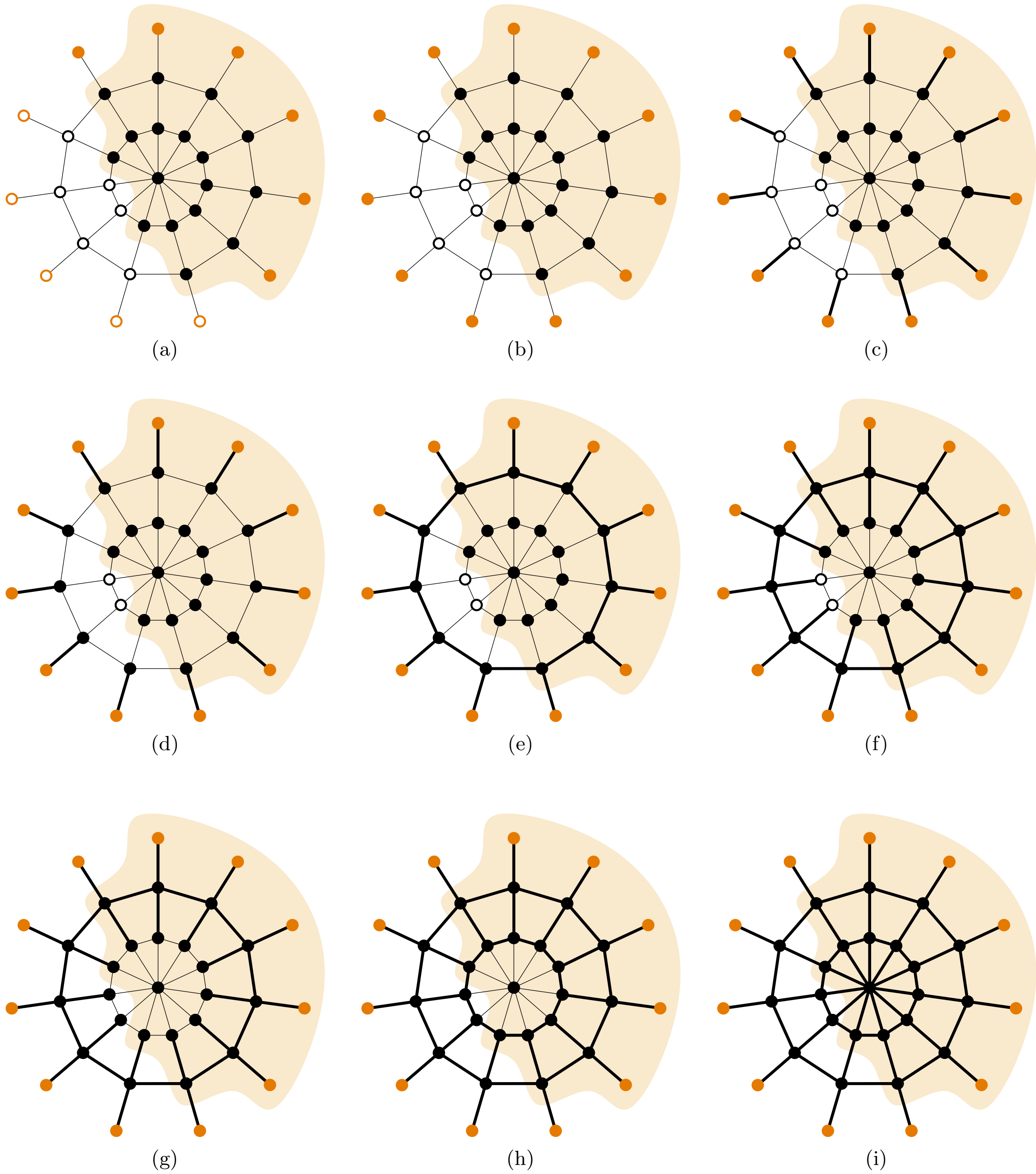
$$0 = \mathscr{L}_q(u_j)(x_{k\,i+1}) = -\frac{u_j(x_{k\,i+1})}{\omega(x_{k\,i+1})}\wp(\omega)(x_{k\,i+1}) - \wp(u_j)(x_{k\,i+1})$$
$$- c(x_{k\,i+1}, x_{k\,i})u_j(x_{k\,i}) - \frac{\omega(x_{k\,i})}{\omega(x_{k\,i+1})}c(x_{k\,i+1}, x_{k\,i})u_j(x_{k\,i+1})$$

and hence $u_j(x_{k\,i})$ is the only unknown term of this equality. ∎

In particular, when $i = m-1$, Propositions 4.1 and 4.5 show that u_j is known on D_{m-1} for all $j = 1,\dots,n$. Observe that we already knew some of the values of u_j on D_{m-1}, which are those of the vertices in $Z(u_j)$. Figure 3(g) shows the information obtained until this step.

Step 7 and beyond

We keep repeating the same process to obtain more conductances; that is, we keep applying Proposition 4.3 from Step 4, then Proposition 4.4 from Step 5 and then Proposition 4.5 from Step 6 for each $i = m-2,\dots,0$. We stop when applying Proposition 4.5 from Step 6 for $i = 0$. In fact, we obtain the value $u_j(x_{00}) = 0$ for all $j = 1, \ldots, n$, which is already known because $x_{00} \in Z(u_j)$. This is the last step of the process, since all the conductances are known at this point.



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Figure 3: The bold items are the ones known at the end of each step for the case $n = j = 11$.

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