Small regular graphs of girth 7

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Abstract

In this paper, we construct new infinite families of regular graphs of girth 7 of smallest order known so far. Our constructions are based on combinatorial and geometric properties of $(q+1, 8)$-cages, for q a prime power. We remove vertices from such cages and add matchings among the vertices of minimum degree to achieve regularity in the new graphs. We obtain $(q+1)$-regular graphs of girth 7 and order $2q^3+q^2+2q$ for each even prime power $q \geqslant 4$, and of order $2q^3+2q^2-q+1$ for each odd prime power $q \geqslant 5$.

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1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. For terminology and notation not explicitly defined here, please refer to [14]. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We denote the subgraph of G induced by a subset $U \subset V(G)$ as $G[U]$, and it is the graph with $V(G[U]) = U$ and for any $u, v \in V(G[U])$ the edge uv belongs to $E(G[U])$ if and only if $uv \in E(G)$. The *girth* of a graph G is the number $g = g(G)$ of edges in a smallest cycle. For every $v \in V$, $N_G(v)$ denotes the *neighbourhood* of v, that is, the set of all vertices adjacent to v, we may denote it simply by $N(v)$. Similarly, for each positive integer t we denote by $N_t(v)$ the neighborhood of v at distance t, i.e. the set $N_t(v) = \{x \in V(G) : d(x, v) = t\}$, and the neighborhood of an edge uv at distance t is the set $N_t(uv) = \{x \in V(G) : d(x, u) = t \text{ or } d(x, v) = t\}$.

The *degree* of a vertex $v \in V$ is the cardinality of $N(v)$. A graph is called *regular* if all the vertices have the same degree. A (k, g)*-graph* is a k-regular graph with girth g. Erdős and Sachs [16] proved the existence of (k, g)-graphs for all values of k and g provided that $k \geqslant 2$. Thus most work carried out has focused on constructing a smallest one [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 17, 19, 20, 22, 23, 27, 28, 31]. A (k, g)*-cage* is a k-regular graph with girth g having the smallest possible number of vertices $n(k, g)$. Cages have been studied intensely since they were introduced by Tutte [35] in 1947, and their construction is a very difficult task.

Counting the numbers of vertices in the distance partition with respect to a vertex yields Moore's lower bound $n_0(k, g)$ (cf. e.g. [18, Eq. (2)]) with the precise form of the bound depending on whether g is even or odd:

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases} \quad (1)$$

Biggs [12] calls the *excess* of a (k, g)-graph G the difference $|V(G)| - n_0(k, g)$. The construction of graphs with small excess is also quite challenging. Biggs is the author of a report on distinct methods for constructing cubic cages [13]. More details about constructions of cages can be found in the survey by Wong [39] or in the book by Holton and Sheehan [21] or in the more recent dynamic cage survey by Exoo and Jajcay [18].

A (k, g)-cage with $n_0(k, g)$ vertices and even girth exists only when $g \in \{4, 6, 8, 12\}$ [19]. If $g = 4$ they are the complete bipartite graph $K_{k,k}$, and for $g = 6, 8, 12$ these graphs are the incidence graphs of generalized $g/2$-gons of order $k-1$. This is the main reason for (k, g)-cages with $n_0(k, g)$ vertices and even girth g are called *generalized polygon graphs* [12]. In particular a 3-gon of order $k-1$ is also known as a *projective plane* of order $k-1$. The 4-gons of order $k-1$ are called *generalized quadrangles* of order $k-1$, and, the 6-gons of order $k-1$, *generalized hexagons* of order $k-1$. All these objects are known to exist for all prime power values of $k-1$, and no example is known when $k-1$ is not a prime power (cf. e.g. [30, p.25], [34]).

On the other hand, a general upper bound for $n(k, g)$ has been given by Sauer (cf.

e.g. [18, Thm. 7], [33]) and it states that for every $k \geqslant 2$ and $g \geqslant 3$

$$n(k,g) \leqslant \begin{cases} 2(k-2)^{(g-2)} & \text{if } g \text{ is odd;} \\ 4(k-1)^{(g-3)} & \text{if } g \text{ is even.} \end{cases} \tag{2}$$

In particular, for $g = 7$, Sauer's bound has been the best known so far, and it gives that $n(k,7) \leqslant 2(k-2)^5$.

In this paper, we construct new infinite families of small regular graphs of girth 7. Our constructions are based on combinatorial and geometric properties of $(q+1,8)$-cages, for q a prime power, which are summarized in Section 2. We remove vertices from such cages and add matchings among the vertices of minimum degree to achieve regularity in the new graphs (cf. Definitions 2 and 9). In Section 3 we construct $(q+1,7)$-graphs of order $2q^3+q^2+2q$ for each even prime power $q \geqslant 4$ (cf. Construction 1 and Theorem 7). In Section 4 we construct $(q+1,7)$-graphs of order $2q^3+2q^2-q+1$ for each odd prime power $q \geqslant 5$ (cf. Construction 2 and Theorem 15). All these graphs are the smallest $(q+1,7)$-graphs known so far, for each prime power $q > 5$.

Indeed, for $q = 4$ our graph matches the order of the known $(5,7)$–graphs of order 152 [26, 32]. For $q = 5$ there are smaller known $(6,7)$–graphs of order 294 [26, 32] (our $(6,7)$–graph has 296 vertices). However, for prime powers $q > 5$, our graphs improve Sauer's upper bound (2), and their order lies within a constant factor of Moore's lower bound which equals $(1+o(1))q^3$. Specifically, we prove that

$$n(k,g) \leqslant \begin{cases} 2(k-1)^3+(k-1)^2+2(k-1) & \text{for each even prime power } k-1 \geqslant 4; \\ 2(k-1)^3+2(k-1)^2-(k-2) & \text{for each odd prime power } k-1 \geqslant 5. \end{cases} \tag{3}$$

2 Preliminaries

It is well known [24, 30] that $Q(4,q)$ and $W(3,q)$ are the only two classical generalized quadrangles with parameters $s = t = q$.

The generalized quadrangle $W(3,q)$ is the dual generalized of $Q(4,q)$, and they are selfdual for q even.

In 1966 Benson [11] constructed $(q+1,8)$–cages from the generalized quadrangle $Q(4,q)$. He defined the point/line incidence graph Γ_q of $Q(4,q)$ which is a $(q+1)$–regular graph of girth 8 with $n_0(q+1,8)$ vertices, hence Γ_q is a $(q+1,8)$–cage. Note that, Γ_q is also the point/line incidence graph of $W(3,q)$.

For any generalized quadrangle Q of order (s,t) and every point x of Q, let $x^\perp$ denote the set of all points collinear with x. For a nonempty set X of vertices of Q, we define $X^\perp := \bigcap_{x \in X} x^\perp$. Note that $N_2(x)$ in the incidence graph Γ_q, corresponds in the geometry to $x^\perp$ for a point $x \in Q$.

The *span* of the pair (x,y) is $sp(x,y) = \{x,y\}^{\perp\perp} = \{u \in P : u \in z^\perp \ \forall\ z \in x^\perp \cap y^\perp\}$, where P denotes the set of points in Q. If x and y are not collinear, then $\{x,y\}^{\perp\perp}$ is also called the *hyperbolic line* through x and y. If the hyperbolic line through two noncollinear points x and y contains precisely $t+1$ points, then the pair (x,y) is called *regular*. A point x is called *regular* if the pair (x, y) is regular for every point y not collinear with x. It is important to recall that the concept of being regular also exists for a graph. Hence, we will emphasize when the word "regular" refers to a point of a geometry or to a graph.

Remark 1. [30, p.33, dual of 3.3.1(i)] Every point in $W(q)$ is regular (i.e. $|sp(x, y)| = q+1$ for all non-collinear x, y).

There are several equivalent coordinatizations of these generalized quadrangles (cf. [29], [36], [37], see also [24]) each giving a labeling for the graph Γ_q. In Section 4 we present a further labeling of Γ_q, equivalent to previous ones (cf. [1, 2]), which will be central for our constructions since it allows us to keep track of the properties (such as regularity and girth) of the small regular graphs of girth 7 obtained from Γ_q.

3 Construction of small $(q + 1, 7)$-graphs for even prime powers

In this section we construct a family of $(q + 1, 7)$-graphs of order $2q^3 + q^2 + 2q$ obtained from a $(q + 1, 8)$-cage Γ_q for each even prime power $q \geqslant 4$. In general terms, we proceed by removing, from a $(q + 1, 8)$-cage, a subgraph H consisting of a distinguished vertex x, its neighbours, and almost all its second neighbours (the neighbourhoods of all but two of the neighbours of x). The resulting graph is not regular, indeed the neighbours of the subgraph H in the cage, are left with degree q. So we add appropriate matchings among such vertices to restore the $(q + 1)$–regularity of the graph. The constructed graph has girth at most 7 by Equation (1). The details in this Section are devoted to choosing the matchings in an appropriate way to obtain girth exactly 7 in the graph.

Let $x \in V(\Gamma_q)$ and $N(x) = \{x_0, \dots, x_q\}$. Label $X_i = N(x_i) - x = \{x_{i1}, \dots, x_{iq}\}$ for all $i \in \{0, \dots, q\}$. We denote $X_{ij} = N(x_{ij}) \setminus \{x_i\}$ for $i \in \{0, \dots, q\}$ and $j \in \{1, \dots, q\}$ and observe that the sets X_{ij} have even cardinality. Let $\mathcal{Z} = \{X_0, X_1, X_{ij} : i = 2, \dots, q, j = 1, \dots, q\}$. For each set $Z \in \mathcal{Z}$, M_Z will denote a perfect matching of Z. Let

$$H = N(x) \cup \bigcup_{i=2}^{q} N(x_i) \subset V(\Gamma_q) \tag{4}$$

To obtain a small regular graph of girth 7, we consider the graph $\Gamma_q - H$ and then add matchings M_Z between the remaining neighbors of the vertices in H.

Definition 2. Let Γ_q be a $(q+1, 8)$-cage for an even prime power $q \geqslant 4$ and H as in (4). We define Γ_q^1 to be the graph with: $V(\Gamma_q^1) := V(\Gamma_q - H)$ and $E(\Gamma_q^1) := E(\Gamma_q - H) \cup \bigcup_{Z \in \mathcal{Z}} M_Z$.

Remark 3. The graph Γ_q^1 has order $|V(\Gamma_q)| - (q^2+2)$ and all its vertices have degree $q+1$. Furthermore, the girth of Γ_q^1 is at most 7 by Equation (1).

Remark 4. Let u and v be distinct vertices of a graph G of girth 8 such that there is a uv-path P of length $t < 8$. Then every uv-path P' such that $E(P) \cap E(P') = \emptyset$ has length $|E(P')| \geqslant 8 - t$.

Proposition 5. *Let Γ_q be a $(q+1,8)$-cage for an even prime power $q \geqslant 4$ and Γ_q^1 as in Definition 2. Then Γ_q^1 has girth 7 if the following condition holds:*

$$\begin{array}{c}\textit{For each } uv \in M_{X_{ij}} \textit{ and each } X_{kl}, \textit{ where } i,k \in \{0,\ldots,q-2\},\ j,l \in \{1,\ldots,q\} \\ E(\Gamma_q^1[N_2(uv)\cap X_{kl}]) \cap M_{X_{kl}} = \emptyset.\end{array} \tag{*}$$

Proof. By Remark 3 the graph Γ_q^1 has girth at most 7. From Remark 4, the distances in $\Gamma_q - H$ between the elements in the sets $Z \in \mathcal{Z}$ satisfy the following:

(i) If $u, v \in Z$, then $d_{\Gamma_q - H}(u,v) \geqslant 6$ because they have a common neighbor z in Γ_q.

(ii) If $u \in X_0$ and $v \in X_1$, then $d_{\Gamma_q - H}(u,v) \geqslant 4$.

(iii) If $u \in X_i$ and $v \in X_{kj}$, then $d_{\Gamma_q - H}(u,v) \geqslant 3$, for $i \in \{0,1\}$, $k \in \{2,\ldots,q\}$ and $j \in \{1,\ldots,q\}$.

(iv) If $u \in X_{lj}$ and $v \in X_{lk}$, then $d_{\Gamma_q - H}(u,v) \geqslant 4$, for $l \in \{2,\ldots,q\}$ and $j,k \in \{1,\ldots,q\}$.

(v) If $u \in X_{tj}$ and $v \in X_{lk}$, then $d_{\Gamma_q - H}(u,v) \geqslant 2$, for $t \neq l$, $t,l \in \{2,\ldots,q\}$ and $j,k \in \{1,\ldots,q\}$.

Let C be a shortest cycle in Γ_q^1. If $E(C) \subset E(\Gamma_q - H)$, then $|C| \geqslant 8$. Suppose C contains edges in $M = \bigcup_{Z \in \mathcal{Z}} M_Z$. If C contains exactly one such edge, then by (1), $|C| \geqslant 7$. If C contains exactly two edges $e_1, e_2 \in M$, we have the following cases:

- If both e_1, e_2 lie in the same M_Z, then by (i), $|C| \geqslant 14 > 7$.
- If $e_1 \in M_{X_0}$ and $e_2 \in M_{X_1}$, then $|C| \geqslant 10 > 7$ by (ii).
- If $e_1 \in M_{X_i}$, $i = 0,1$, and $e_2 \in M_{X_{kj}}$, then $|C| \geqslant 8 > 7$ by (iii).
- If $e_1 \in M_{X_{lj}}$ and $e_2 \in M_{X_{lk}}$, then $|C| \geqslant 10 > 7$ by (iv).
- If $e_1 \in M_{X_{tj}}$ and $e_2 \in M_{X_{lk}}$, then $|C| \geqslant 7$ for $t \neq l$, by condition (*).

If C contains at least three edges of M, then $|C| \geqslant 9 > 7$ since $d_{\Gamma_q}(u,v) \geqslant 2$ for all $u, v \in \{X_0, X_1, X_{ij}\}$ with $i \in \{2,\ldots,q\}$ and $j \in \{1,\ldots,q\}$ by (i)–(v). From the above and Remark 3 it follows that Γ_q^1 has girth 7. □

Lemma 6. *There exist $q^2 - q$ matchings $M_{X_{ij}}$ satisfying condition (*) in Proposition 5.*

Proof. Let $\Omega_j := \bigcap_{i=2}^{q} N(X_{ij})$ for $j = 1,\ldots,q$. Let $w_{j1} \in \Omega_j$. As $d_{\Gamma_q}(x, w_{j1}) = 4$ there exist other $q-1$ elements in Ω_j mutually at distance four, since Γ_q is the incidence graph of a generalized quadrangle $W(q)$. Label $\Omega_j = \{w_{j1}, \ldots, w_{jq}\}$.

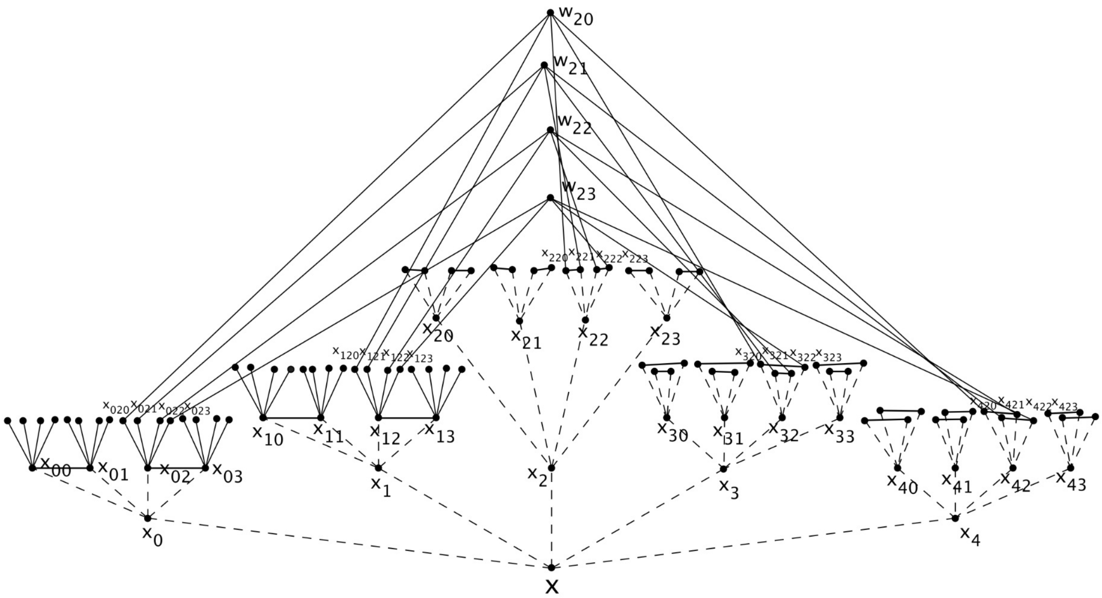

Figure 1: The dashed edges and their lower end-vertices illustrate the subgraph H, removed from Γ_q, when $q = 4$. Above, find the choice of the matchings that lead to a $(5, 7)$–graph of girth exactly 7 and order 152.

Every vertex w_{jh} is adjacent to exactly one vertex in X_{ij} since the girth of Γ_q is 8. Denote $\{x_{ijh}\} = N(X_{ij}) \cap N(w_{jh})$ for each $i \in \{2, \ldots, q\}$ and $j, h \in \{1, \ldots, q\}$ (see Figure 1). Note that x_{ijh} is well labeled, because if x_{ijh} had two neighbors $w_{jh}, w_{jh'} \in \bigcap_{i=2}^{q} N(X_{ij})$, then Γ_q would contain the cycle $x_{ijh}w_{jh'}x_{i'jh'}x_{i'j}x_{i'jh}w_{jh}$ of length 6.

Therefore, take the complete graph K_q label its vertices as $h \in \{1, \ldots, q\}$. We know that it has a 1-factorization with $q-1$ factors $F_1, \ldots, F_{q-1}$ since q is even (cf. e.g. [38]). For each $i = 2, \ldots, q$, let $x_{ijh}x_{ijh'} \in M_{X_{ij}}$ if and only if $hh' \in F_{i-1}$.

To prove that the matchings $M_{X_{ij}}$ defined in this way fulfill condition (*), suppose that $x_{ijh}x_{ijh'} \in M_{X_{ij}}$ and $x_{i'jh}x_{i'jh'} \in M_{X_{i'j}}$ for $i' \neq i$. Then F_i and $F_{i'}$ would have the edge hh' in common contradicting that they come from a 1-factorization.

To conclude, notice that for $uv \in M_{X_{ij}}$ and $a, b \in X_{kl}$ with $l \neq j$ and possibly $k = i$, the distances $d(u, a)$ and $d(v, b)$ are at least 4.

Therefore, there exist $q^2 - q$ matchings $M_{X_{ij}}$ with the desired property. □

Construction 1: Let $q \geqslant 4$ be an even prime power. Let Γ_q^1 be the $(q+1)$-regular graph of order $2q^3 + q^2 + 2q$ from Definition 2 with $M_{X_{ij}}$ as in the proof of Lemma 6, for $i \in \{2, \ldots, q\}$ and $j \in \{1, \ldots, q\}$; and with M_{X_0} and M_{X_1} matchings of X_0 and X_1 respectively, chosen arbitrarily. Then, the graph Γ_q^1 obtained with such a choice of matchings has girth 7 by Proposition 5.

As a consequence we have the following theorem.

Theorem 7. *Let $q \geqslant 4$ be an even prime power. Then, there is a $(q+1)$-regular graph of girth 7 and order $2q^3 + q^2 + 2q$.* $\square$

Figure 1 illustrates this construction for $q = 4$. Note that this $(5, 7)$–graph has 152 vertices, as the two found in 2001 by McKay and Yang [26, 32].

4 Constructions of small $(q + 1, 7)$–graphs for an odd prime power.

In this section we construct an infinite family of $(q+1, 7)$-graphs of order $2q^3+2q^2-q+1$ for odd prime power $q \geqslant 5$. Analogously to Section 3, we will delete a set H of vertices from a $(q + 1, 8)$-cage Γ_q and add matchings M_Z between the remaining neighbors of H to obtain a small regular graph of girth 7. In general terms, the subgraph H consists of two distinguished vertices x and y at distance 4 in Γ_q; their neighbours and all but three of the common second neighbours of x and y (see Figure 2). The removal of H from Γ_q leaves a non regular graph. Thus, we add appropriate matchings among the vertices of lesser degree, and three sporadic 2–paths, to restore the $(q + 1)$–regularity of the graph (cf. Definition 9). The constructed graph has girth at most 7 by Equation (1). As in Section 3, the details that follow are devoted to choosing the matchings in an appropriate way to obtain girth exactly 7 in the graph. In particular, some of the matchings can be chosen combinatorially, as for the even case (cf. Lemma 12). However, for the remaining ones, we rely on an algebraic coordinatization of the $(q+1, 8)$-cage (cf. Definition 13 and Lemma 14).

Specifically, the set H and matchings M_Z, are defined as follows.

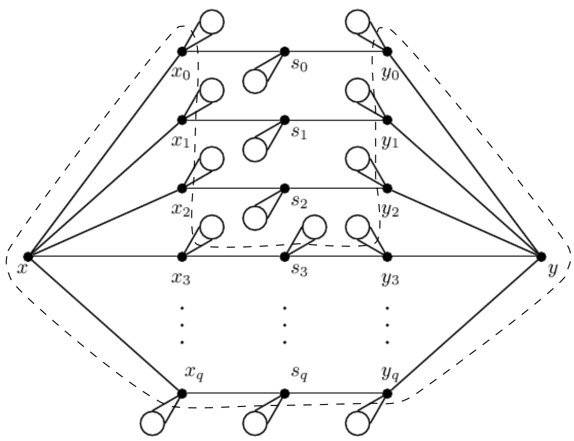

Figure 2: Subgraph of Γ_q used to define H and $\mathcal{Z}$. The subgraph H is highlighted by the dashed line.

Definition 8. Let $x, y \in V(\Gamma_q)$ be vertices at distance 4 in Γ_q, and let $xx_is_iy_iy$ be the internally (vertex) disjoint xy-paths for $i = 0, \ldots, q$ (which exist since Γ_q is $(q+1)$-connected, see [25]). We define the following sets (see Figure 2):

$$\begin{array}{rcl} H &=& \{x, y, s_3, s_4, \ldots, s_q\} \cup N(x) \cup N(y) \subset V(\Gamma_q); \\ X_i &=& N(x_i) \cap V(\Gamma_q - H), \quad i = 0, \ldots, q; \\ Y_i &=& N(y_i) \cap V(\Gamma_q - H), \quad i = 0, \ldots, q; \\ S_i &=& N(s_i) \cap V(\Gamma_q - H), \quad i = 3, \ldots, q. \end{array}$$

Notice that the vertices of $\Gamma_q - H$ have degrees $q-1, q$ and $q+1$. The vertices s_0, s_1, s_2 have degree $q-1$, those in $X_i \cup Y_i \cup S_i$ have degree q and all the remaining vertices of $\Gamma_q - H$ have degree $q+1$. Therefore, in order to obtain a $(q+1)$-regular graph, we need to add edges to $\Gamma_q - H$ in such a way that cycles of length smaller than 7 are avoided.

Similarly as before, let $\mathcal{Z}$ be the family of all the sets X_i, Y_i, S_i. Note that, all sets in $\mathcal{Z}$ have even cardinality. For each $Z \in \mathcal{Z}$, M_Z will denote a perfect matching of $V(Z)$.

Definition 9. Let Γ_q be a $(q+1, 8)$-cage for odd prime power $q \geqslant 5$.

- Let Γ_q^1 be the graph with: $V(\Gamma_q^1) := V(\Gamma_q - H)$ and $E(\Gamma_q^1) := E(\Gamma_q - H) \cup \bigcup_{Z \in \mathcal{Z}} M_Z$.
- Define Γ_q^2 as $V(\Gamma_q^2) := V(\Gamma_q^1)$ and $E(\Gamma_q^2) := (E(\Gamma_q^1) \setminus \{u_0v_0, u_1v_1, u_2v_2\}) \cup \{s_0u_0, s_0v_0, s_1u_1, s_1v_1, s_2u_2, s_2v_2\}$, the deleted edges u_iv_i belong to M_{X_i} in Γ_q^1 and they are replaced by the paths of length two $u_is_iv_i$, $i \in \{0, 1, 2\}$.

Remark 10. Note that $|V(\Gamma_q^1)| = |V(\Gamma_q^2)| = |V(\Gamma_q)| - 3(q+1) + 1$. All vertices in Γ_q^1 have degree $q+1$ except for s_0, s_1, s_2 which remain of degree $q-1$. Hence, by Definition 9, in Γ_q^2 all vertices are left with degree $q+1$. From Equation (1) the girth of both Γ_q^1 and Γ_q^2 is at most 7.

Proposition 11. *Let Γ_q be a $(q+1, 8)$-cage for odd prime power $q \geqslant 5$ and Γ_q^1, Γ_q^2 be as in Definition 9.*

(a) Γ_q^1 has girth 7 if the matchings M_{S_i}, M_{X_i} and M_{Y_i} have the following properties:

- *(a1) For all $uv \in M_{S_i}$, $E(\Gamma_q^1[N_2(uv) \cap S_j]) \cap M_{s_j} = \emptyset$.*
- *(a2) For all $uv \in M_{X_i}$, $E(\Gamma_q^1[N_2(uv) \cap Y_j]) \cap M_{Y_j} = \emptyset$.*

(b) If both conditions (a1) and (a2) hold, the graph Γ_q^2 also has girth 7.

Proof. By Remark 10 the graphs Γ_q^1 and Γ_q^2 have girth at most 7. From Remark 4, the distances in $\Gamma_q - H$ between the elements in the sets $Z \in \mathcal{Z}$ satisfy the following (see Figure 2):

(i) If $u, v \in Z$, then $d_{\Gamma_q - H}(u, v) \geqslant 6$.

(ii) If $u \in X_i$ and $v \in X_j$, then $d_{\Gamma_q - H}(u, v) \geqslant 4$.

(iii) If $u \in Y_i$ and $v \in Y_j$, then $d_{\Gamma_q - H}(u,v) \geqslant 4$.

(iv) If $u \in S_i$ and $v \in S_j$, then it may exist $w \in \Gamma_q - H$ such that $u, v \in N(w)$, that is, $d_{\Gamma_q - H}(u,v) \geqslant 2$.

(v) If $u \in S_i$ and $v \in X_j \cup Y_j$, then $d_{\Gamma_q - H}(u,v) \geqslant 3$.

(vi) If $u \in X_i$ and $v \in Y_j$, then $d_{\Gamma_q - H}(u,v) \geqslant 2$.

Let C be a shortest cycle in Γ_q^1. If $E(C) \subset E(\Gamma_q - H)$, then $|C| \geqslant 8$. Suppose C contains edges in $M = \bigcup_{Z \in \mathcal{Z}} M_Z$. If C contains exactly one such edge, then by (i), $|C| \geqslant 7$. If C contains exactly two edges $e_1, e_2 \in M$, the following cases arise:

- If both e_1, e_2 lie in the same M_Z, then by (i), $|C| \geqslant 14 > 7$.
- If $e_1 \in M_{X_i}$ and $e_2 \in M_{X_j}$ for $i \neq j$, by (ii), $|C| \geqslant 10 > 7$.
- If $e_1 \in M_{Y_i}$ and $e_2 \in M_{Y_j}$ for $i \neq j$, by (iii), $|C| \geqslant 10 > 7$.
- If $e_1 \in M_{S_i}$ and $e_2 \in M_{X_j} \cup M_{Y_j}$, by (v), $|C| \geqslant 8 > 7$.
- If $e_1 \in M_{S_i}$ and $e_2 \in M_{S_j}$ for $i \neq j$, by item $(a1)$, $|C| \geqslant 7$.
- If $e_1 \in M_{X_i}$ and $e_2 \in M_{Y_j}$, by item $(a2)$, $|C| \geqslant 7$.

If C contains at least three edges of M, since $d(u,v) \geqslant 2$ for all $u, v \in \{X_i \cup Y_i\}_{i=1}^k \cup \{S_i\}_{i=4}^k$, $|C| \geqslant 9 > 7$. Hence Γ_q^1 has girth 7, concluding the proof of (a).

To prove (b), let C be a shortest cycle in Γ_q^2. If $E(C) \subset E(\Gamma_q - H) \cup M$, then $|C| \geqslant 7$.

- If C contains exactly one edge $s_i u_i$ or $s_i v_i$, then $|C| \geqslant 7$ since $d_{\Gamma_q}(s_i, u_i) = d_{\Gamma_q}(s_i, v_i) = 2$ which implies $d_{\Gamma_q^1}(s_i, u_i) \geqslant 6$ and $d_{\Gamma_q^1}(s_i, v_i) \geqslant 6$.
- If C contains a path $u_i s_i v_i$, then $(C - \{s_i\}) + u_i v_i$ is a cycle in Γ_q^1 with one vertex less than C, therefore $|C| \geqslant 8$.
- If C contains two edges $s_i u_i$, $s_j u_j$, for $i \neq j$, their distances $d_{\Gamma_q^1}(s_i, u_j) \geqslant 4$, $d_{\Gamma_q^1}(s_i, s_j) \geqslant 4$, and $d_{\Gamma_q^1}(u_i, u_j) \geqslant 4$.

Since in either case C has length at least 7 and by Remark 10, the result holds. □

The following lemma states the existence of the matchings M_{S_i} for the sets S_i, which fulfill condition $(a1)$ from Proposition 11. Notice that in the incidence graph of a generalized quadrangle $\{x, y\}^{\perp\perp} = \bigcap_{s \in N_2(x) \cap N_2(y)} N_2(s)$, thus Remark 1 implies that $|\bigcap_{i=0}^{q} N(S_i)| = q - 1$, recalling that $\{s_i\}_{i=0}^q = N_2(x) \cap N_2(y)$. Since $|\bigcap_{i=0}^{q} N(S_i)|$ is contained in $|\bigcap_{i=3}^{q} N(S_i)|$, and $|\bigcap_{i=3}^{q} N(S_i)| = q - 1$, then the condition for the following lemma holds.

Lemma 12. *There exist matchings M_{S_i}, for $i = 3, \ldots, q$, such that condition (a1) in Proposition 11 holds.*

Proof. From the regularity of $W(q)$ we know that $\bigcap_{i=0}^{q} N(S_i) = \{w_1, \ldots, w_{q-1}\}$, and since S_i has $q-1$ vertices, every vertex w_j is adjacent to exactly one vertex in $s_{ij} \in S_i$. Moreover, note that s_{ij} is well labeled, because if s_{ij} had two neighbors $w_j, w_{j'} \in \bigcap_{i=0}^{q} N(S_i)$, Γ_q would contain the cycle $(s_{ij} w_j s_{kj} s_k s_{kj'} w_{j'})$ of length 6.

Therefore, take the complete graph K_{q-1}, label its vertices as $j \in \{1, \ldots, q-1\}$. We know that it has a 1-factorization with $q-2$ factors $F_1, \ldots, F_{q-2}$ since $q-1$ is even. For each $i = 3, \ldots, q+1$, let $s_{ij}s_{il} \in M_{S_i}$ if and only if $jl \in F_{i-2}$.

To prove that the matchings M_{S_i} defined in this way fulfill the desired property suppose that $s_{ij}s_{il} \in M_{S_i}$ and $s_{i'j}s_{i'l} \in M_{S_i'}$ for $i' \neq i$. Then F_i and $F_{i'}$ would have the edge jl in common contradicting that they were a factorization. $\square$

So far, our construction has been independent from the coordinatization of the chosen $(q+1, 8)$-cage, however, in order to define M_{X_i} and M_{Y_i} satisfying condition $(a2)$ of Proposition 11, we need to fix all the elements chosen so far. To this purpose we use the following convenient description of a $(q+1, 8)$-cage.

Definition 13. [1, 2] Let $\mathbb{F}_q$ be a finite field with $q \geqslant 2$ a prime power and ϱ a symbol not belonging to $\mathbb{F}_q$. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be a bipartite graph with vertex sets $V_i = \mathbb{F}_q^3 \cup \{(\varrho, b, c)_i, (\varrho, \varrho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_i\}$, $i = 0, 1$, and edge set defined as follows:

For all $a \in \mathbb{F}_q \cup \{\varrho\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_{\Gamma_q}((a,b,c)_1) = \begin{cases} \{(w,\ aw+b,\ a^2w+2ab+c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, a, c)_0\} & \text{if } a \in \mathbb{F}_q; \\ \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, c)_0\} & \text{if } a = \varrho. \end{cases}$$

$$N_{\Gamma_q}((\varrho, \varrho, c)_1) = \{(\varrho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}$$
$$N_{\Gamma_q}((\varrho, \varrho, \varrho)_1) = \{(\varrho, \varrho, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}.$$

Or equivalently

For all $i \in \mathbb{F}_q \cup \{\varrho\}$ and for all $j, k \in \mathbb{F}_q$:

$$N_{\Gamma_q}((i,j,k)_0) = \begin{cases} \{(w,\ j-wi,\ w^2i-2wj+k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, j, i)_1\} & \text{if } i \in \mathbb{F}_q; \\ \{(j, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, j)_1\} & \text{if } i = \varrho. \end{cases}$$

$$N_{\Gamma_q}((\varrho, \varrho, k)_0) = \{(\varrho, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\};$$
$$N_{\Gamma_q}((\varrho, \varrho, \varrho)_0) = \{(\varrho, \varrho, w)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\}.$$

Lemma 14. *There exist matchings M_{X_i} and M_{Y_i}, for $i = 0, \ldots, q$, such that condition (a2) in Proposition 11 holds.*

Proof. Let $x = (\varrho, \varrho, \varrho)_1$ and $y = (0, 0, 0)_1$.

We will distinguish two cases, when $q = p$ is a prime or when $q = p^a$, $a > 1$ is a prime power.

Case 1: $q = p$ a prime.

Let $x_i = (\varrho, \varrho, i)_0$, $y_i = (i, 0, 0)_0$ for $i = 0, \ldots, p-1$, and $x_p = (\varrho, \varrho, \varrho)_0$, $y_p = (\varrho, 0, 0)_0$.

$N(x_i) = \{(\varrho, t, i)_1 : t = 0, \ldots, p-1\} \cup \{x\}$, $i = \{0, \ldots, p-1\}$,

$N(x_p) = \{(\varrho, \varrho, t)_1 : t = 0, \ldots, p-1\} \cup \{x\}$,

$N(y_i) = \{(t, -it, it^2)_1 : t = 0, \ldots, p-1\} \cup \{(\varrho, 0, i)_1\}$, $i = \{0, \ldots, p-1\}$,

and $N(y_p) = \{(0, t, 0)_1 : t = 0, \ldots, p-1\} \cup \{(\varrho, \varrho, 0)_1\}$.

Hence, we have the sets:

$X_i = \{(\varrho, t, i)_1 : t = 1, \ldots, p-1\}$ and $Y_i = \{(t, -it, it^2)_1 : t = 1, \ldots, p-1\}$ for $i = 0, \ldots, p-1$.

$X_p = \{(\varrho, \varrho, t)_1 : t = 1, \ldots, p-1\}$ and $Y_p = \{(0, t, 0)_1 : t = 1, \ldots, p-1\}$.

Define the following matchings, depicted in Figure 3:

$M_{X_i} = \{(\varrho, \ell, i)_1(\varrho, -(\ell+2), i)_1 : \ell = 1, \ldots, (p-3)/2\} \cup \{(\varrho, -2, i)_1(\varrho, -1, i)_1\}$, and

$M_{Y_i} = \{(t, -it, it^2)_1(-t, it, it^2)_1 : t = 1, \ldots, (p-1)/2\}$, for $i = 0, \ldots, p-1$.

$M_{X_p} = \{(\varrho, \varrho, \ell)_1(\varrho, \varrho, -(\ell+2))_1 : \ell = 1, \ldots, (p-3)/2\} \cup \{(\varrho, \varrho, -2)_1(\varrho, \varrho, -1)_1\}$, and

$M_{Y_p} = \{(0, t, 0)_1(0, -t, 0)_1 : t = 1, \ldots, (p-1)/2\}$.

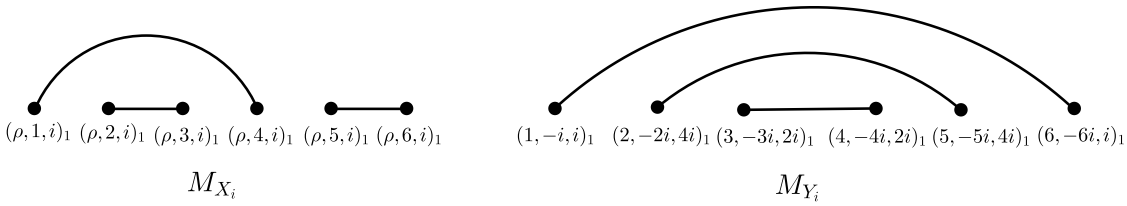

Figure 3: The matchings M_{X_i} and M_{Y_i} for $i = 0, \ldots, p-1$ and $p = 7$, determined by the 2^{nd} and the 1^{rst} coordinates of the vertices in X_i and Y_i respectively.

Claim: Matchings M_{X_j} and M_{Y_i} for $i, j \in \{0, \ldots, p\}$, satisfy property $(a2)$.
To prove it we must analyze the intersection of the second neighborhood of X_j with Y_i.

Recall that the vertices in Y_i are in $N((i, 0, 0)_0) = N(y_i)$, the vertices in X_j have coordinates $(\varrho, \ell, j)_1$ for $\ell = 1, \ldots, p-1$. We will show that there exists a unique vertex

$$w_{i\ell j} \in Y_i \cap N_2(X_j) \cap N_2(X_p)$$

and to this purpose we describe the coordinates of the vertices in X_p in terms of the subscripts of $w_{i\ell j}$ as $(\varrho, \varrho, s_{i\ell j})_1$ where $s_{i\ell j} \in \{1, \ldots, p-1\}$ and where the relationship between $w_{i\ell j}$ and $s_{i\ell j}$ is highlighted in what follows:

Note that $w_{i\ell j} = (a, b, c)_1 \in N((i, 0, 0)_0) \cap N_2((\varrho, \ell, j)_1) \cap N_2((\varrho, \varrho, s_{i\ell j})_1)$.

Since $N((i, 0, 0)_0) = \{(t, -it, it^2)_1 : t = 0, \ldots, p-1\} \cup (\varrho, 0, i)_1$, then $a = t$, $b = -it$, and $c = it^2$. Also, $N_2((\varrho, \varrho, s_{i\ell j})_1) = N(\{(\varrho, s_{i\ell j}, x)_0 : x \in 0, \ldots, p-1\}) = \{(s_{i\ell j}, u, x)_1 : x, u \in 0, \ldots, p-1\}$. Hence, $s_{i\ell j} = a = t$ and $(a, b, c)_1 = (s_{i\ell j}, -is_{i\ell j}, is_{i\ell j}^2)_1$. Moreover,

$N((\varrho,\ell,j)_1)=\{(j,\ell,s)_0: s=0,\dots,p-1\}\cup\{(\varrho,\varrho,j)_0\}$; and$N_2((\varrho,\ell,j)_1)=N(\{(j,\ell,s)_0: s=0,\dots,p-1\})\cup\{N(\varrho,\varrho,j)_0)\}=\{(x,\ell-xj,\ell x^2-2xj+s)_1\}\cup\{(\varrho,\ell,j)_1\}$. By substitution in the first and second coordinates we obtain the following equation:

$$s_{i\ell j}(j-i)=\ell \text{ for } i,\ell,j\in\mathbb{F}_p, \ell\neq 0, j-i\neq 0. \tag{5}$$

Notice that this equation is undefined for $j=i$, otherwise it would mean that y_i has a neighbor at distance 3 from x_j and this would imply the existence of a cycle of length 6 in Γ_q.

By equation (5) we have $s_{i(-\ell)j}=-s_{i\ell j}$ implying, for fixed i and j, that the vertices $w_{i\ell j}=(s_{i\ell j},-is_{i\ell j},is^2_{i\ell j})_1$ and $w_{i-\ell j}=(-s_{i\ell j},is_{i\ell j},is^2_{i\ell j})_1$ in Y_i are at distance two, respectively, only from the vertices $(\varrho,\ell,j)_1$ and $(\varrho,-\ell,j)_1$ in X_j for $j=\{1,\dots,p\}$; and to the vertices (ϱ,ϱ,ℓ) and $(\varrho,\varrho,-\ell)$ in X_p. Therefore the matchings M_{Y_i} and M_{X_j} for $i\in\{1,\dots,p-1\}$ and $j\in\{1,\dots,p\}$ satisfy property $(a2)$.

Finally, also the matchings M_{X_j} and M_{Y_p} satisfy property $(a2)$, since for all $(\varrho,\ell,j)_1\in X_j$ it holds that $Y_p\cap N_2((\varrho,\ell,j)_1)=(0,\ell,0)_1$; and the edge $(\varrho,\ell,j)_1(\varrho,-\ell,j)_1\notin M_{X_j}$. This ends the proof of the claim.

Case 2: q a prime power.

Let α be a primitive root of unity in $GF(q)=\{\alpha^i : i=0,\dots,q-2\}\cup\{0\}$.

In this case, $x_i=(\varrho,\varrho,\alpha^{i-1})_0$, $y_i=(\alpha^{i-1},0,0)_0$, for $i=1,\dots,q-1$, $x_0=(\varrho,\varrho,0)_0$, $y_0=(0,0,0)_0$, $x_q=(\varrho,\varrho,\varrho)_0$ and $y_q=(\varrho,0,0)_0$.

$N(x_i)=\{(\varrho,\alpha^t,\alpha^{i-t})_1 : t=0,\dots,q-2\}\cup(\varrho,0,\alpha^{i-1})_1\cup x$,

$N(x_0)=\{(\varrho,\alpha^t,0)_1 : t=0,\dots,q-2\}\cup(\varrho,0,0)_1\cup x$,

$N(x_q)=\{(\varrho,\varrho,\alpha^t)_1 : t=0,\dots,q-2\}\cup(\varrho,\varrho,0)_1\cup x$.

$N(y_i)=\{(\alpha^t,-\alpha^{i-1+t},\alpha^{i-1+2t})_1 : t=0,\dots,q-2\}\cup(\varrho,0,\alpha^{i-1})_1$,

$N(y_0)=\{(\alpha^t,0,0)_1 : t=0,\dots,q-2\}\cup(\varrho,0,0)_1$,

$N(y_q)=\{(0,\alpha^t,0)_1 : t=0,\dots,q-2\}\cup(\varrho,\varrho,0)_1\cup y$.

Hence, we have the sets:

$X_i=\{(\varrho,\alpha^t,\alpha^{i-1})_1 : t=0,\dots,q-2\}$ and $Y_i=\{(\alpha^t,-\alpha^{i-1+t},\alpha^{i-1+2t})_1 : t=0,\dots,q-2\}$ for $i=1,\dots,q-1$.

$X_0=\{(\varrho,\alpha^t,0)_1 : t=0,\dots,q-2\}$ and $Y_0=\{(\alpha^t,0,0)_1 : t=0,\dots,q-2\}$.

$X_q=\{(\varrho,\varrho,\alpha^t)_1 : t=0,\dots,q-2\}$ and $Y_q=\{(0,\alpha^t,0)_1 : t=0,\dots,q-2\}$.

In order to define the matchings M_{X_i} and M_{Y_i} we proceed as above, but in this case we obtain the following equations:

$$\begin{aligned} \alpha^{s_{i\ell j}}(\alpha^{j-1}-\alpha^{i-1})&=\alpha^\ell && \text{for } i,j\geqslant 1;\\ \alpha^{s_{0\ell j}}(\alpha^{j-1})&=\alpha^\ell && \text{for } i=0;\\ \alpha^{s_{i\ell 0}}(-\alpha^{i-1})&=\alpha^\ell && \text{for } j=0.\end{aligned} \tag{6}$$

Notice that equation (6) is undefined for $j=i$, otherwise it would mean that y_i has a neighbor at distance 3 from x_j and this would imply the existence of a cycle of length 6 in Γ_q.

Define the following matchings, depicted in Figure 4:

$M_{X_i}=\{(\varrho,\alpha^{2\ell},i)_1(\varrho,\alpha^{2\ell+1},i)_1 : \ell=0,\dots,(q+1)/2\}$; and

$M_{Y_i} = \{(\alpha^{2t}, -\alpha^{i-1+2t}, \alpha^{i-1+4t})_1(\alpha^{2t+3}, -\alpha^{i-1+(2t+3)}, \alpha^{i-1+2(2t+3)})_1 : t = 0, \ldots, (q-1)/2\} \cup \{(\alpha, -\alpha^i, \alpha^{i+1})_1(\alpha^{q-3}, -\alpha^{i+q-4}, \alpha^{i+2q-7})\}$, for $i = 0, \ldots, q-1$.

$M_{X_q} = \{(\varrho, \varrho, \alpha^{2\ell})_1(\varrho, \varrho, \alpha^{2\ell+1})_1 : \ell = 0, \ldots, (q+1)/2\}$; and

$M_{Y_q} = \{(0, \alpha^{2t}, 0)_1(0, \alpha^{2t+3}, 0)_1 : t = 0, \ldots, (q-1)/2\} \cup \{(0, \alpha, 0)_1, (0, \alpha^{q-3}, 0)_1\}$.

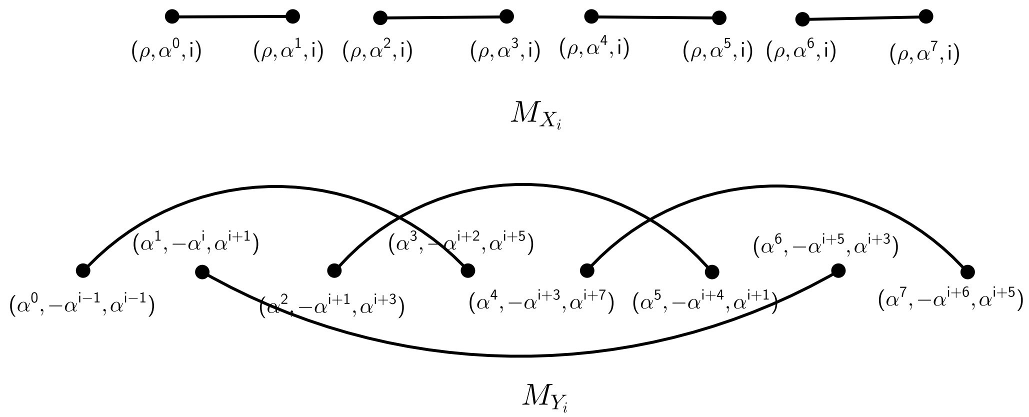

Figure 4: The matchings M_{X_i} and M_{Y_i} for $i = 0, \ldots, q-1$ and $q = 9$, determined by the 2^{nd} and the 1^{rst} coordinates of the vertices in X_i and Y_i respectively.

On the other hand, we obtain that $s_{i\ell+1j} = s_{i\ell j} + 1$, multiplying the equation (6) by α, which implies that the matchings M_{X_i} and M_{Y_i} satisfy property $(a2)$, because the two vertices $(\alpha^{s_{i\ell j}}, -\alpha^{i-1+s_{i\ell j}}, \alpha^{i-1+2s_{i\ell j}})_1$ and $(\alpha^{s_{i\ell j}+1}, -\alpha^{i-1+(s_{i\ell j}+1)}, \alpha^{i-1+2(s_{i\ell j}+1)})_1$ in Y_i, are at distance two in X_j only from the vertices $(\varrho, \alpha^{\ell}, j)_1$ and $(\varrho, \alpha^{\ell+1}, j)_1$, concluding the proof. □

Construction 2: For $q \geqslant 5$ an odd prime power let Γ_q^2 be the graph of order $2q^3 + 2q^2 - q + 1$ given in Definition 9, with the choice of matchings as in Lemmas 12 and 14. Then, the graph Γ_q^2 is a $(q+1)$-regular graph of girth 7 with $2q^3 + 2q^2 - q + 1$ vertices as we prove in the following theorem.

Theorem 15. *Let $q \geqslant 5$ be an odd prime power. Then, there is a $(q+1)$-regular graph of girth 7 and order $2q^3 + 2q^2 - q + 1$.*

Proof. Consider the graph in Construction 2, obtained with the choice of matchings from Lemmas 12 and 14. Then, condition (a) in Proposition 11 holds, and by Proposition 11(b) the graph Γ_q^2 has girth 7. By Remark 10, the order of Γ_q^2 is $2(q^3 + q^2 + q + 1) - (q - 3 + 2(q+2)) = 2q^3 + 2q^2 - q + 1$, as required. □

Note that for $q = 5$, McKay and Yang found eighty-seven $(6,7)$–graph has 294 vertices, in 2001 [26, 32]. Our $(6,7)$–graph has 296 vertices, but for $q > 5$, Theorem 15 improves the previously known upper bounds from Sauer (cf. [33]).

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