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A characterization of the innovations of first order autoregressive models

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Abstract Suppose that Y_t follows a simple AR(1) model, that is, it can be expressed as $Y_t = \alpha Y_{t-1} + W_t$, where W_t is a white noise with mean equal to μ and variance σ^2. There are many examples in practice where these assumptions hold very well. Consider $X_t = e^{Y_t}$. We shall show that the autocorrelation function of X_t characterizes the distribution of W_t.

Keywords Time series · AR(1) models · Characterization of distributions

1 Introduction

In this paper, we shall consider a general AR(1) time series defined by the classical stochastic difference equation

$$Y_t = \alpha \cdot Y_{t-1} + W_t, \tag{1}$$

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satisfying the condition of stationarity $|\alpha| < 1$, where W_t is the innovation or white noise process, a sequence of iid random variables with mean μ and variance σ^2.

Many authors have studied autoregressive models with non-Gaussian innovations. Several works mainly deal with finding the distribution of the innovations for specified marginals. In Gaver and Lewis (1980), the authors consider the cases where the marginal distribution of Y_t is exponential or gamma. They also show that positive marginals do not allow $\alpha < 0$ in (1), so that the condition of stationarity is restricted to $0 < \alpha < 1$. A good review of autoregressive models with non-Gaussian innovations can be found in Sim (1994), where a model-building methodology is considered focused on parameter estimation and forecasting by means of several models with different marginal distributions for Y_t, such as exponential, logistic, hyperbolic secant and others. The case where the marginal is inverse Gaussian distributed is studied in Abraham and Balakrishna (1999). Another kind of non-Gaussian autoregressive processes different from (1) was introduced in Jayakumar et al. (1995), and some extensions can be found in Kuttykrishnan and Jayakumar (2008). A non-linear AR(1) model with approximate beta marginal is considered in Popović et al. (2013).

In Granger and Newbold (1976), the authors construct non-Gaussian series by taking an instantaneous transformation $T(Z_t)$ of a Gaussian ARMA process Z_t. They study in detail the transformation $T(x) = e^x$, because a huge range of time series of econometric indicators are analysed in logarithmic form, although inference on the original series is the main matter. The study of time series of positive terms of the form $X_t = e^{Y_t}$, where Y_t is a general AR(1) with non-Gaussian innovations, has been hardly considered in the literature, with the exception of McKenzie (1982). In this paper, the case with a Gamma marginal for X_t is studied in detail, proving the surprising result that the Gamma distribution is the only one for which the correlation structures of X_t and Y_t are both the same. From then to the present, as far as we know, there are no new results of characterization of the distribution of the innovations of an AR(1) model using the correlation structure of the exponentiated series X_t.

In Sect. 2 we shall study the autocorrelation function (ACF) of the exponentiated series X_t, which is calculated for some selected marginals. In Sect. 3 we shall show that the ACF of X_t, $\rho_{X_t}(k)$, under mild conditions, characterizes the distribution of the innovations W_t of Y_t in (1).

2 The autocorrelation function of the exponentiated AR(1) process

Consider the general AR(1) Y_t model in (1) and its exponentiated time series $X_t = e^{Y_t}$. Assume that the autocorrelations of X_t exist. McKenzie (1982) pointed out a simple way to calculate the ACF of X_t in terms of the marginal distribution of X_t, indicated by the random variable X. He established the following lemma:

Lemma 1 (McKenzie 1982) *The autocorrelation function of the exponentiated time series X_t has the expression*

$$\rho_{X_t}(k) = \frac{\mathbb{E}[X]\left(\mathbb{E}[X^{\alpha^k+1}] - \mathbb{E}[X^{\alpha^k}]\mathbb{E}[X]\right)}{\mathbb{E}[X^{\alpha^k}]Var[X]}. \tag{2}$$

The following lemma establishes an important relation with the marginal distribution of Y_t (indicated by the random variable Y):

Lemma 2 *The autocorrelation function of X_t can be written as*

$$\rho_{X^t}(k) = \frac{\psi_Y(1)\psi_Y(\alpha^k + 1) - \psi_Y(1)^2\psi_Y(\alpha^k)}{\psi_Y(\alpha^k)(\psi_Y(2) - \psi_Y(1)^2)}, \tag{3}$$

where $\psi_Y(z)$ is the moment generating function of Y.

Proof Expression (3) is a direct consequence of (2) and the fact that $\psi_Y(z) = \mathbb{E}[e^{zY}] = \mathbb{E}\left[e^{z\log(X)}\right] = \mathbb{E}[X^z]$.

Note that, given the marginal distribution of Y_t, Lemma 2 provides a simple way to calculate the ACF of the exponentiated series.
The relation between the distribution of Y and the distribution of the innovations W_t in (1) can be established by means of their characteristic functions. The characteristic function $\phi_{Y_t}(z)$ of the distribution of Y_t is

$$\phi_{Y_t}(z) = \mathbb{E}[e^{ziY_t}] = \mathbb{E}\left[e^{zi(\alpha Y_{t-1}+W_t)}\right] = \phi_{Y_{t-1}}(\alpha z)\phi_{W_t}(z).$$

From here, because Y_t is stationary, we obtain, $\phi_Y(z) = \phi_Y(\alpha z)\phi_W(z)$, and consequently,

$$\phi_W(z) = \frac{\phi_Y(z)}{\phi_Y(\alpha z)}. \tag{4}$$

However, expression (4) is not always the characteristic function of a proper distribution for the innovations. Random variables Y for which expression (4) is a proper characteristic function have necessarily self-decomposable distributions (its characteristic function $\phi(t)$ satisfies $\phi(t) = \phi(\alpha t)\cdot\phi_\alpha(t)$, $-\infty < t < \infty$, for all $0 < \alpha < 1$, where ϕ_α is also a characteristic function) and the class of self-decomposable distributions is wide. In this paper we are going to work with some of them, that are the logarithm of well known positive random variables X, the marginal distribution of X_t. Table 1 shows the marginals considered here, the moment generating function of $Y = \log(X)$ and the ACF of the time series X_t, expressed as $\rho_X(s)$ where $s = \alpha^k$, that is $\rho_X(s) = \rho_{X^t}(\log(s)/\log(\alpha))$.

Because the innovations of the AR(1) process defined in (1) satisfy $\mathbb{E}[W_t] = \mu$ and $\mathbb{V}[W_t] = \sigma^2$, it is immediate to see that the expectation and variance of the marginal distribution are $\mathbb{E}[Y] = \mu/(1-\alpha)$ and $\mathbb{V}[Y] = \sigma^2/(1-\alpha^2)$. For simplicity, the expressions of $\rho_X(s)$ in Table 1 are those of the standardized marginals of Y, that is, $\mathbb{E}[Y] = 0$ and $\mathbb{V}[Y] = 1$. All the marginal distributions considered in Table 1 have two parameters. Note that the first one corresponds to the classical (Gaussian) AR(1) model. The two first moment generating functions have been expressed in terms of the expectation and variance. For the Gamma distribution, it can be done solving the equations, $\Psi'(\nu) = \frac{\sigma^2}{1-\alpha^2}$ and $\Psi(\nu) - \log(a) = \frac{\mu}{1-\alpha}$, where $\Psi(\cdot)$ is the digamma function and $\Psi'(\cdot)$ its first derivative. For instance, for the standardized marginal the parameters a and ν are calculated solving $\Psi'(\nu) = 1$ and $\Psi(\nu) - \log(a) = 0$,

Table 1 Marginal distributions of X_t with the moment generating function of the marginal distributions of Y_t, $\psi_Y(z)$ ($Y = \log(X)$), and the ACF $\rho_X(s)$ ($s = \alpha^k$) of the standardized marginals

X	$\psi_Y(z)$	$\rho_X(s)$
log-Normal	$e^{\frac{\mu}{1-\alpha}z+\frac{\sigma^2}{2(1-\alpha^2)}z^2}$	$\frac{e^s-1}{e-1}$
Weibull	$e^{\frac{z\sqrt{6}\gamma\sigma}{\pi\sqrt{1-\alpha^2}}+\frac{z\mu}{1-\alpha}}\Gamma\left(1+\frac{z\sigma\sqrt{6}}{\pi\sqrt{1-\alpha^2}}\right)$	$\frac{\Gamma(c)(\Gamma(c)\Gamma(1+sc-s)-\Gamma(sc-s+c))}{\left(-\Gamma(-1+2c)+(\Gamma(c))^2\right)\Gamma(1+sc-s)}$, $c=1+\frac{\sqrt{6}}{\pi}$
Gamma	$\frac{\Gamma(\nu+z)}{\Gamma(\nu)a^z}$	s
Inv. Gaussian	$\frac{K_{z-1/2}(ab)}{K_{-1/2}(ab)}\left(\frac{a}{b}\right)^z$	$c\left(\frac{K_{s+1/2}(c)}{K_{s-1/2}(c)}-1\right)$, $c = 0.608545$

obtaining $\nu_0 = 1.4263$ and $a_0 = 0.9658$. For the inverse Gaussian distribution, to obtain an expression in terms of $\frac{\mu}{1-\alpha}$ and $\frac{\sigma^2}{1-\alpha^2}$ is more difficult due to the modified Bessel function of the second kind that appears in $\psi_Y(z)$. However, it can be done numerically using an appropriate software like Maple.

Most of the moment generating functions considered in Table 1 are well defined for $z \in (-1, 1)$. This is important because it allows AR(1) models (1) with $\alpha \in (-1, 1)$, and not just with $\alpha \in (0, 1)$ since due to expression (3) the autocorrelation function $\rho_{X_t}(k)$ is well defined for all $\alpha \in (-1, 1)$. However, for the Gamma distribution we have $\psi_Y(z) = \frac{\Gamma(\nu+z)}{\Gamma(\nu)a^z}$, which is well defined for $z \in (-1, 1)$ only if $\nu > 1$.

Note that the ACF of the AR(1) model in (1) is always the same independently of the marginal distribution or the distribution of the innovations. Unlike Y_t, the ACF of X_t depends on the marginal distribution of Y_t.

In the next section we shall show that the ACF of X_t characterizes the marginal distribution of Y_t and consequently, the distribution of the innovations.

3 Characterization of the distribution of the innovations

Taking logarithms in expression (3) of Lemma 2, and using some algebra we obtain

$$\kappa_Y(s+1) - \kappa_Y(s) = \log\left(\frac{\psi_Y(2) - \psi_Y(1)^2}{\psi_Y(1)} \cdot \rho_X(s) + \psi_Y(1)\right), \quad (5)$$

where $s = \alpha^k$ and $\kappa_Y(s) = \log(\psi_Y(s))$ is the cumulant generating function of Y. Note that $\psi_Y(1) = \mathbb{E}[X]$ and $\psi_Y(2) - \psi_Y(1)^2 = \mathbb{V}[X]$. Writing $h(s)$ for the right-hand side of (5), we can see that $\kappa_Y(s)$ is a solution of a first order functional equation of the form

$$f(s+1) - f(s) = h(s), \quad (6)$$

with the initial condition $f(0) = 0$. This kind of functional equations have been widely studied in Kuczma (1968). A slight modification of his Theorem 5.11 allows us to state the following proposition:

Proposition 1 (Kuczma 1968) *Let a function $h : I \to \mathbb{R}$ in (6), $I = (a, \infty)$, $-\infty \leq a < \infty$ satisfy the condition*

$$\lim_{s\to\infty} [h(s+1) - h(s)] = 0.$$

Suppose that we have a convex solution of (6), $\varphi(s)$ defined in I, fulfilling the condition $\varphi(s_0) = \eta_0$, for $s_0 \in I$ and $\eta_0 \in \mathbb{R}$. Then, $\varphi(s)$ is the unique convex solution of the functional equation (6), and it can be expressed as

$$\varphi(s) = \eta_0 + (s - s_0)h(s_0) + \sum_{n=0}^{\infty}(s - s_0)(h(s_0 + n + 1) - h(s_0 + n)) - (h(s + n) - h(s_0 + n)). \tag{7}$$

From this proposition we can state the following theorem which characterizes the distribution of the innovations.

Theorem 1 *Suppose that the distribution of the innovations W_t in (1) is such that,*

1. *The marginal distribution of Y_t (indicated by the random variable Y) has a moment generating function $\psi_Y(s)$ defined for $s \in (a, \infty)$, $a < 0$.*
2. *The distribution of $X = \exp(Y)$ is well defined, with an ACF of the exponentiated time series $X_t = \exp(Y_t)$ given by $\rho_{X_t}(s)$, and $\lim_{s\to\infty} h(s+1) - h(s) = b$, $0 \leq b < \infty$, where*

$$h(s) = \log\left(\frac{\psi_Y(2) - \psi_Y(1)^2}{\psi_Y(1)} \cdot \rho_X(s) + \psi_Y(1)\right).$$

3. *The function $\log(\psi_Y(s)) - bs(s-1)/2$ is convex.*

Then, the distribution of W_t is the unique having an ACF of the exponentiated series equal to $\rho_{X_t}(s)$.

Proof Let $\kappa_Y(s) = \log(\psi_Y(s))$, the cumulant generating function of Y. Taking into account (5), it is evident that $f(s) = \kappa_Y(s)$ satisfies the functional equation (6) with initial condition $f(0) = 0$. Because any cumulant function is convex, if the limit b is equal to 0, Proposition 1 directly shows that (6) has only one solution. If the limit b is greater than 0, we define $h^*(s) = h(s) - bs$ and it is immediate to see that $f(s) = \kappa_Y(s) - bs(s-1)/2$ is a solution of the functional equation, $f(s+1) - f(s) = h^*(s)$. Because $\lim_{s\to\infty} h^*(s+1) - h^*(s) = 0$ and by hypothesis $\kappa_Y(s) - bs(s-1)/2$ is convex, Proposition 1 shows that this solution is unique.

The values of the limit $b = \lim_{s\to\infty} h(s+1) - h(s)$ for the examples shown in Table 1 are 0 for the Gamma and inverse Gaussian marginals, and 1 for the log-Normal (Gaussian innovations) and the Weibull marginals.

Remark 1 In general, the functional equation (6) has infinitely many solutions. Given any solution $f(s)$ and any periodic function with period 1 $\varphi(s)$, it is readily seen that

$f(s)+\varphi(s)$ is also a solution of (6). To ensure the uniqueness, additional assumptions on the solution are required, like those of Proposition 1. Therefore, the assumptions stated in Theorem 1 are necessary in order to ensure the regularity conditions required in Proposition 1. In our context, the solution of the functional equation (6) is a cumulant generating function and consequently this is a convex solution. The uniqueness of this solution is assured by Proposition 1 if the three assumptions stated in Theorem 1 are fulfilled.

Given the function $\rho_X(s)$, and the values of $\mathbb{E}[X]$ and $\mathbb{V}[X]$, Theorem 1 combined with Proposition 1 not only allow to ensure the uniqueness of a solution, but also to construct such a solution. To illustrate this point, suppose that for example $\rho_X(s) = s$, the case covered by McKenzie (1982), and $\mathbb{E}[X] = 1$, $\mathbb{V}[X] = 2$. Then, $\log(\psi_Y(s))$ would be the solution $\varphi(s)$ of the functional equation (6), with the initial condition $\varphi(0) = 0$, i.e., $\eta_0 = s_0 = 0$, and $h(s) = \log(2s + 1)$. According to (7) the solution has the form

$$\varphi(s) = \sum_{n=0}^{\infty} s \log\left(\frac{2n+3}{2n+1}\right) - \log\left(\frac{2s+2n+1}{2n+1}\right) \tag{8}$$

It can be numerically checked that $\varphi(s)$ is just the logarithm of $\psi_Y(s) = \frac{\Gamma(\nu+s)}{\Gamma(\nu)a^s}$, as stated in Table 1, with $\nu = a = 0.5$.

The characterization of the family of functions that are autocorrelation functions of X_t and lead to proper distributions for the innovations of an AR(1) process, is an open problem. Of course, some relevant examples of such autocorrelation functions are those considered in Table 1.

4 Concluding notes

Following the ideas introduced in McKenzie (1982), we have characterized the distribution of the innovations W_t of an AR(1) model according to the ACF of its exponentiated series. The novelty of this result sets out new problems and lines of research. For instance, could similar results be obtained using other transformations different than the exponential? How could the characterization of the innovations be extended to higher order AR models? The characterization proposed here could be used in order to develop a goodness of fit test, maybe using the techniques shown in Anderson et al. (2004), calculating a kind of Cramér-von Mises statistic measuring the difference between the sample and theoretical spectral densities. It would also be interesting to study how the techniques introduced in this work can be generalized to higher order AR models or even ARMA models. Further research would be required to solve these challenges.

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