Blocking the k-holes of point sets in the plane

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Abstract

Let P be a set of n points in the plane in general position. A subset H of P consisting of k elements that are the vertices of a convex polygon is called a k-hole of P, if there is no element of P in the interior of its convex hull. A set B of points in the plane blocks the k-holes of P if any k-hole of P contains at least one element of B in the interior of its convex hull. In this paper we establish upper and lower bounds on the sizes of k-hole blocking sets, with emphasis in the case $k = 5$.

1 Introduction

Let P be a set of n points in the plane in general position, i.e., such no three of them are collinear. All point sets considered in this paper are assumed to be in general position, and therefore this assumption is mentioned only occasionally hereafter. The *convex hull* of P, denoted as $CH(P)$, is the smallest convex set containing all of the elements of P. A set of points is in *convex position*, if its elements are the vertices of a convex polygon. A

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subset H of P with k elements is called a k-hole of P if it is in convex position, and no element of P belongs to the interior of $CH(H)$.

Counting and finding k-holes of point sets has been a very active area of research since Erdős and Szekeres [9, 10] asked about the existence of k-holes in planar point sets. Harborth proved that any point set with at least ten points always contains at least one 5-hole [13]. Horton [14] proved that for $k \geq 7$ there are point sets containing no k-holes. Recently Nicolás [17] and independently Gerken [12] proved that any point set with sufficiently many points contains at least one 6-hole.

Let $f_k(n)$ be the minimum number of k-holes that every point set has. Katchalski and Meir [16] proved that $\binom{n}{2} \leq f_3(n) \leq kn^2$ for some $k < 200$; see also Purdy [20]. Their lower bounds were improved by Dehnhardt [6] to $n^2 - 5n + 10 \leq f_3(n)$, who also proved that $\binom{n-3}{2} + 6 \leq f_4(n)$. Point sets with few k-holes for $3 \leq k \leq 6$ were obtained by Bárány and Valtr [4].

Chromatic variants of the Erdős-Szekeres problem were introduced by Devillers, Hurtado, Károly, and Seara [7]. They proved among other results that any bichromatic point set contains at least $\frac{n}{4} - 2$ compatible monochromatic empty triangles (i.e., having pairwise disjoint interiors). Aichholzer *et al.* [1] proved that any bichromatic point set always contains $\Omega(n^{5/4})$ empty monochromatic triangles; this bound was improved by Pach and Tóth [18] to $\Omega(n^{4/3})$. For a thorough survey on this topic, the reader is referred to B. Vogthenhuber's doctoral's thesis [2], where new variations on these and other problems (e.g. dropping the convexity condition on holes) are studied.

In this paper we consider the problem of, given a point set P, finding a second set of points, as small as possible, that *pierce*, *stab*, or *block* all the holes of a certain size in P. More precisely: A point $q \notin P$ *blocks* a hole H of P if it belongs to the interior of $CH(H)$. A set of points B such that $B \cap P = \varnothing$ is called a *k-hole blocking set* of P, for short a *k-blocking set of* P, if for any k-hole H of P, there at least one element of B in the interior of $CH(H)$. In the rest of this paper, P will always be a point set in general position with n elements, $n \geq 3$.

Given a point set P, let c_P be the number of elements of P on the boundary of $CH(P)$. The problem of finding 3-blocking sets has been studied for some time now. It is known that any point set P always has a 3-blocking set with exactly $2n - c_P - 2$ elements, and since any triangulation of P contains exactly $2n - c_P - 2$ elements, this bound is tight; see Katchalski and Meir [16], and Czyzowicz, Kranakis and Urrutia [5].

Sakai and Urrutia proved in [21] that there are point sets for which $2n - o(n)$ points are necessary to block all their 4-holes; as $2n - c_P - 2$ points are always sufficient to block all the 3-holes of any point set, and thus its 4-holes, this bound is essentially tight. In fact, we believe that in general, the number of points needed to block the 4-holes of any point set P is essentially the same as the number of points needed to block the 3-holes of P (i.e., that the asymptotically dominating terms are the same). In Section 2, we prove that this is the case for point sets in convex position: We prove that to block the 4-holes of any set of n points in convex position, we need at least $n - O(\sqrt{n})$ points, while it is known that $n-2$ points are sufficient and necessary to block the 3-holes.

Remarkably, blocking the k-holes of a point set changes substantially for $k \geq 5$, a problem that, to the best of our knowledge, had not been considered before. In Section 3, the core of this paper, we show that there are point sets, both in general and in convex position, for which the number of points needed to block their 5-holes is as low as a fifth of the number of triangles in a triangulation of the respective point set. We also prove the somehow surprising fact that the number of points needed to block the 5-holes of a point set depends on the geometry of the specific point set, unlike the case of blocking its triangles which only depends on the number of points in the convex hull: We show point sets of the same cardinality, with the same number of points on their convex hulls, for which their 5-blocking sets with minimum cardinality have different sizes. What is more, we show that even for point sets in convex position the size of the 5-blocking sets may be different and depends on the specific geometry.

Finally, in Section 4, we give results on blocking the k-holes of point sets in convex position, for general values of k, and we conclude in Section 5 with some observations and open problems.

As a final remark in this introduction, it is worth mentioning that the case $k = 2$, *i.e.*, blocking the visibility between pairs of points, has also received attention recently, see [19] and the references therein.

2 Blocking the 4-holes of convex point sets

Is is well known that $n-2$ points are sufficient and necessary to block the 3-holes of any set of n points in convex position [16, 5]. In this first section we show that for 4-holes the same amount is essentially needed, in the sense that $n-o(n)$ blocking points are always necessary. More precisely, our main goal in this section is to prove the next result[1]:

[1] Another proof of this result has independently been found recently by P. Valtr, inspired by discussions during a meeting in Spain in May 2011 (personal communication).

Theorem 2.1. *Let P any set of n points in convex position. Then, any 4-blocking set for P has at least $n - O(\sqrt{n})$ elements.*

To prove this, we use a result on the chromatic number of a certain geometric type Kneser graph. Araujo, Dumitrescu, Hurtado, Noy, and Urrutia [3] introduced the following graph: Let P be a set of n points in convex position. The *convex segment disjointness graph* of P, denoted by D_n, is the graph whose vertex set is the set of all line segments with endpoints in P, two of which are adjacent if they are disjoint. Clearly D_n does not depend on the choice of P.

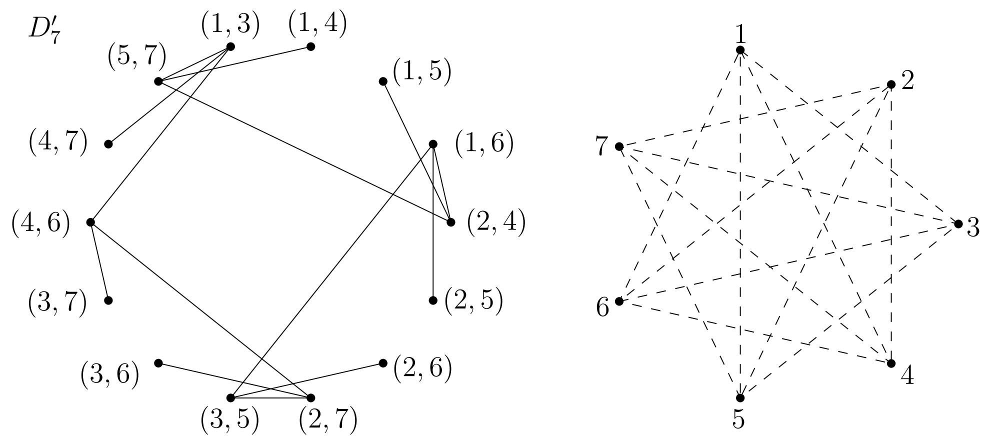

Figure 1: Graph D'_7.

Let $\chi(D_n)$ denote the chromatic number of D_n. A lower bound on this value was obtained by Fabila-Monroy and Wood in [11], while an upper bound was obtained by Dujmović and Wood in [8]. Both bounds combine into the following theorem:

Theorem 2.2 ([11, 8])**.**

$$n - \sqrt{2n + \frac{1}{4}} + \frac{1}{2} \leq \chi(D_n) < n - \sqrt{\frac{1}{2}n} - \frac{1}{2}(\log n) + 4.$$

Let D'_n be the graph obtained from D_n by removing the vertices of D_n corresponding to the edges of the convex hull of P, see Figure 1. Then D'_n has $\binom{n}{2} - n$ vertices. It is easy to see from the proof of Theorem 2.2 in [11], that the chromatic number of D'_n satisfies:

$$\chi(D'_n) \geq n - \sqrt{4n + \frac{1}{4}} + \frac{1}{2}\,.$$

We now use this bound to obtain a lower bound on the number of points blocking all the 4-holes of P that have two edges on the boundary of the convex hull of P. We call 2-*quadrilateral of* P any convex quadrilateral having two sides that are non-consecutive edges of the convex hull of P (see Figure 2)

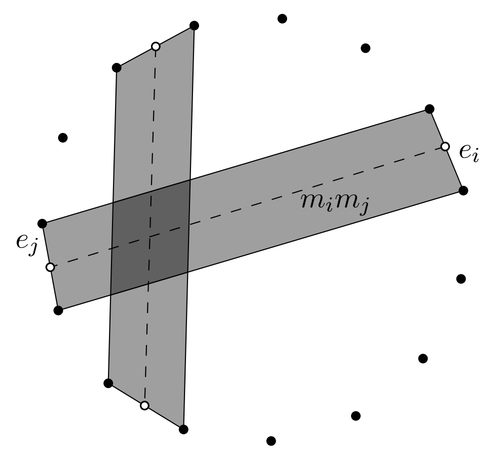

Figure 2: Two intersecting 2-quadrilaterals of P.

Let e_i be an edge in the convex hull of P, and m_i be its mid-point. Let P' the set of all mid-points of the edges of the convex hull of P. Let e_i and e_j be two *non-consecutive* edges of the convex hull of P. We denote by $Q(i,j)$ the 2-quadrilateral of P induced by e_i and e_j. It is obvious that $Q(i,j) \cap Q(r,s) \neq \varnothing$ if and only if the line segments $m_i m_j$ and $m_r m_s$ intersect. Clearly, two 2-quadrilaterals of P can be simultaneously blocked by a point if and only if their interiors intersect.

Let $G'(P)$ be the graph whose vertex set is the set of the 2-quadrilaterals of P, two of which are adjacent if their interiors do not intersect. Observe that D'_n and $G'(P)$ are isomorphic graphs: if the elements of P are the points $p_1, ..., p_n$, labelled as they appear clockwise ordered on the convex hull of P, diagonal $p_i p_j$ (with $j \neq i+1$) corresponds to the 2-quadrilateral $Q(i,j)$ defined by the edges $e_i = p_i p_{i+1}$ and $e_j = p_j p_{j+1}$.

Suppose that we can block all the 4-holes of P using a set of points $S = \{q_1, \ldots, q_t\}$ with less than $t < \chi(D'_n) = \chi(G'(P))$ points. For each

2-quadrilateral C of P, pick a point $q_r \in S$ that blocks C, and assign color r to C. This induces a valid coloring of D'_n, and hence $t \geq n - \sqrt{4n + \frac{1}{4}} + \frac{1}{2}$. Theorem 2.1 follows.

3 Blocking 5-holes

Given a set of n points P in general position, let us recall that we denote by c_P the number of elements of P that are vertices of $CH(P)$. In this section we study the problem of blocking the 5-holes of point sets in the plane. As announced in the introduction, 5-holes behave, both for convex and general position, quite differently that 4-holes and 3-holes do.

3.1 Point sets in convex position

3.1.1 Piercing the 5-holes

The main objective of this section is to prove the following result, which requires several intermediate lemmas:

Theorem 3.1. $\frac{n}{2} - 2$ *points are always necessary and sometimes sufficient to block the* 5*-holes of a point set with* n *elements in convex position and* $n = 4k$*.*

We start by proving a more general result:

Lemma 3.2. *Let* P *a set of* n *points in convex position. Then any* 5*-blocking set for* P *has at least* $2\lceil \frac{n}{4} \rceil - 3$ *elements.*

Proof. Let B be a 5-blocking set of P with r elements and $\mathcal{M}$ a crossing-free geometric matching of maximum cardinality of the elements of B; that is, a set of disjoint pairs of elements of B such that the line segments $\{\ell_1, \ldots, \ell_{\lfloor \frac{r}{2} \rfloor}\}$ joining them do not intersect. Note that if r is odd, we are left with an isolated element of B. One at a time, extend $\ell_1, \ldots, \ell_{\lfloor \frac{r}{2} \rfloor}$ until they hit a line segment in M or a previously extended segment. Observe that some ℓ_i's might be extended to semi-lines or lines. When r is odd, start with a tiny line segment containing the unmatched element of B and extend it as before; see Figure 3.

This process yields a decomposition of the plane into exactly $\lceil \frac{r}{2} \rceil + 1$ convex regions. If one of these regions contains five or more points, it would contain a 5-hole of P not blocked by B. Thus each of these regions contains at most 4 elements of P, and therefore $|B| = r \geq 2 \lceil \frac{n}{4} \rceil - 3$. □

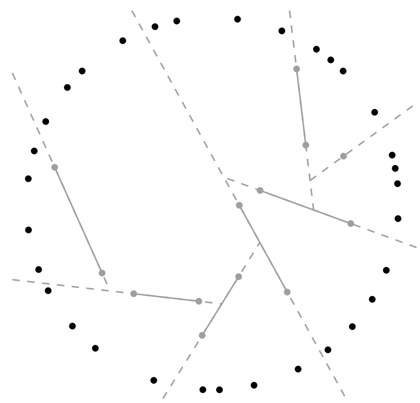

Figure 3: Illustration of Theorem 3.2.

For $n = 4k$, we can improve slightly on the previous bound:

Lemma 3.3. *Let P a set of n points in convex position with $n = 4k$. Then any 5-blocking set for P has at least $\frac{n}{2} - 2$ elements.*

Proof. Suppose that we have a 5-blocking set B for P with $\frac{n}{2} - 3$ points and $n = 4k$. Obtain a decomposition of the plane as in the proof of Lemma 3.2 by an almost perfect geometric matching of the elements of B. Clearly each cell of such decomposition contains exactly 4 elements of P. Since $|B|$ is odd, there is one element b of B unmatched and then, there is an edge ℓ of the decomposition that only contains b, rotate ℓ around b until it hits one element of P, now there are 5 points in one of the cells incident to ℓ that contains 5 elements of P in its closure, and clearly those 5 points define a 5-hole that does not contains b in its interior, so we need at least one more point to block all the 5-holes of P. We conclude that any 5-blocking set of P contains at least $\frac{n}{2} - 2$ points. $\square$

A point set P is called *almost convex* if any triangle whose vertices are in P contains at most one element of P in its interior. Almost convex sets were introduced by Károlyi, Pach and Tóth in [15]. They constructed a family $\mathcal{X}_j$ of almost convex point sets as follows.

Let $\mathcal{Z}_1$ be the end-points of a horizontal line segment ℓ_1 of length two, and define $\mathcal{X}_1 = \mathcal{R}_1$. Let $\mathcal{R}_2$ be the set of endpoints of two vertical line segments ℓ_2 and ℓ_3 of length one whose mid-points are very close to the endpoints of ℓ_1, and let $\mathcal{X}_2 = \mathcal{R}_1 \cup \mathcal{R}_2$. See Figure 4(a).

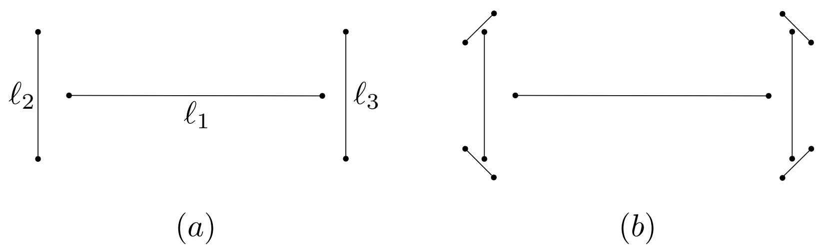

Figure 4: In (a) we show point set $\mathcal{X}_2$, in (b) point set $\mathcal{X}_3$.

Assume that we have already defined $\mathcal{R}_1, \ldots, \mathcal{R}_j$, $\mathcal{X}_1, \ldots, \mathcal{X}_j$,$j \geq 2$, such that they satisfy the following conditions:

(1) $\mathcal{X}_j := \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_j$ is in general position,

(2) the vertices of $CH(\mathcal{X}_j)$ are the elements of $\mathcal{R}_j$, and

(3) any triangle determined by three elements of $\mathcal{R}_j$ contains precisely one point of $\mathcal{X}_{j-1}$ in its interior.

Clearly $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy the preceding conditions. Observe that condition (3), implies that $\mathcal{X}_{j-1}$ is a 3-blocking set of $\mathcal{R}_j$, $j \geq 2$.

The set $\mathcal{X}_{j+1}$ is constructed as follows. Let $z_1, \ldots, z_r$ denote the vertices of $CH(\mathcal{X}_j)$ in clockwise order around $CH(\mathcal{X}_j)$. For every $1 \leq i \leq r$, let ℓ_i denote the line through z_i orthogonal to the bisector of the angle of $CH(\mathcal{X}_j)$ at z_i. Let z_i' and z_i'' be the two points in ℓ_i at infinitesimal distance ε from z_i. Now move simultaneously z_i' and z_i'' away from $CH(\mathcal{X}_j)$ in the direction orthogonal to ℓ_i by another infinitesimal distance δ, with $\varepsilon \gg \delta$, and denote the resulting points u_i' and u_i'', respectively.

It is proved in [15] that ε and δ camn be chosen small enough such that $\mathcal{R}_{j+1} = \{u_i', u_i'' | i = 1, \ldots, r\}$ and $\mathcal{X}_{j+1} := \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_{j+1}$ satisfy conditions $1, 2, 3$ above. See Figure 4(b).

With the preceding construction we are ready to prove:

Lemma 3.4. *There is a set P of n points in convex position with $n = 2^m$ that has a 5-blocking set consisting of $\frac{n}{2} - 2$ elements.*

Proof. Let $P = \mathcal{R}_m$ and $B = \mathcal{X}_{m-2}$. Then $|P| = n$ and $|B| = \frac{n}{2} - 2$. We will show that B is a 5-hole blocking set for P. Suppose that B is not a 5-hole blocking set of P, then there is a 5-hole H of P such that no point of B lies in the interior of the convex hull of H. Take a triangulation of H; it will have three triangles of P. By construction, each of them contains exactly one element of $\mathcal{X}_{m-1}$, since $B = \mathcal{X}_{m-1} \setminus \mathcal{R}_{m-1}$. Then these three points have to be elements of $\mathcal{R}_{m-1}$ and they form a triangle contained in H. By construction, such a triangle contains precisely one element q of $\mathcal{X}_{m-2}$. Thus q blocks H, which is a contradiction. Our result follows. □

The proof of Theorem 3.1 follows now immediately from Lemmas 3.3 and 3.4. □

Theorem 3.1 is frankly surprising to us. We believed that a similar result to that obtained for blocking the 4-holes of point sets in convex position would also hold for 5-blocking sets, i.e., we thought that a 5-blocking set of any point set P in convex position would always have $n - o(n)$ elements. We have seen that that is not always the case yet, we still believe that for some point sets in convex position that may be the right answer. We pose explicitly a related open problem:

Problem 3.5. *Is it true that if P is the set of vertices of a* regular polygon *with n vertices, then any 5-blocking set of P has at least $n - o(n)$ elements?*

3.1.2 Blocking 5-holes of regular polygons

While a solution of Problem 3.5 remains elusive to us, we give in this section a proof for a special case, because the technique is used in Section 3.1.3, and we also hope that it may inspire a general solution.

Let $\mathcal{Q}_n = \{p_0, \ldots, p_{n-1}\}$ be the vertices of a regular polygon $\mathcal{R}_n$ with n vertices, given as they appear on the boundary in clockwise order. The arithmetic of their indices is done modulo n. A subset of $\mathcal{Q}_n$ is called a *lateral k-hole* if its elements are k consecutive elements of $\mathcal{Q}_n$. To be more precise, we use the notation $S_{i,k} = \{p_i, \ldots, p_{i+k-1}\}$ for the i-th lateral k-hole of $\mathcal{Q}_n$, with $0 \leq i \leq n-1$ and $3 \leq k \leq n$. The convex hull of $S_{i,k}$ is a convex k-gon, which we denote $R_{i,k}$. Abusing slightly the notation, we also say that $R_{i,k}$ is a *lateral k-hole* of $\mathcal{R}_n$.

Lemma 3.6. *Any 5 blocking set of $\mathcal{Q}_{19}$ has at least eight elements.*

Proof. First, recall that according to Lemmas 3.2 and 3.3, to block the 5-holes of any convex polygon with 5, 8, 13, 16, 17, and 19 vertices, we need at least 1, 2, 5, 6, 7, and 7 points, respectively.

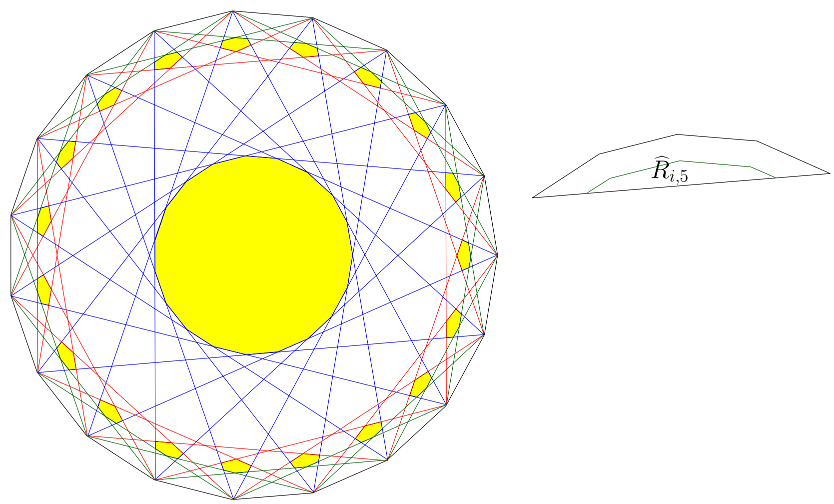

Figure 5: A regular 19-gon.

We prove now our claim by contradiction: Suppose that there is a 5-blocking set B of $\mathcal{Q}_{19}$ consisting of seven points. Observe first that if we remove a lateral 4-hole $R_{i,4}$ from $\mathcal{R}_{19}$, we obtain a convex 17-gon, namely $R_{i+3,17}$. As mentioned in the preceding paragraph, to block the 5 holes of $R_{i+3,17}$ we need at least seven points. It follows that all the elements of B lie in the interior of $R_{i+3,17}$ and therefore, that no lateral 4-hole $R_{i,4}$ contains any element of B. Let W_4 the union of these regions, i.e., $W_4 = \bigcup_{i=0,\ldots,n-1} R_{i,4}$, a polygonal annulus that contains no point from B.

Let $R_{i,5}$ be a lateral 5-hole of $\mathcal{R}_{19}$, and $\widehat{R}_{i,5}$ the subset of $R_{i,5}$ obtained by removing from $R_{i,5}$ all the points that belong to some lateral 4-hole of $\mathcal{R}_{19}$: Equivalently, $\widehat{R}_{i,5} = R_{i,5} \setminus W_4$ (see Figure 5, upper part). Since the elements of B block all the 5-holes of $\mathcal{Q}_{19}$, every lateral 5-hole $R_{i,5}$ of $\mathcal{R}_{19}$ contains at least one element of B, which must belong to $\widehat{R}_{i,5}$.

Observe that the polygonal region that complements $R_{i,5}$ in $\mathcal{R}_{19}$ is precisely $R_{i+4,16}$. As we know that we need at least six points to block the 5 holes of the vertices of any convex polygon with 16 vertices, each lateral 5-hole of $\mathcal{R}_{19}$ must contain exactly one blocking point.

In a similar way, if we remove a lateral 8-hole $R_{i,8}$ from $\mathcal{R}_{19}$, we are left with a convex polygon $R_{i+7,13}$ with 13 vertices, and thus at least five elements of B belong to the interior of $R_{i+7,13}$. It follows that each lateral 8-hole of Q contains exactly two elements of B.

Observe that for each lateral 8-hole $R_{i,8}$ of $\mathcal{R}_{19}$, there are exactly two lateral 5-holes of $\mathcal{R}_{19}$, namely $R_{i,5}$ and $R_{i+3,5}$, such that their corresponding regions $\widehat{R}_{i,5}$ and $\widehat{R}_{i+3,5}$ are disjoint and contained in $R_{i,8}$. Let $H_{i,8} = R_{i,8} \setminus (\widehat{R}_{i,5} \cup \widehat{R}_{i+3,5})$. The preceding discussion implies that the two blocking points of B in $R_{i,8}$ must be one in $\widehat{R}_{i,5}$ and the other one in $\widehat{R}_{i+3,5}$, and that $H_{i,8}$ is empty of points from B.

Let R_B be the region obtained by removing from $\mathcal{R}_{19}$ all the empty regions $H_{i,8}$ defined the lateral 8-holes $R_{i,8}$ of $\mathcal{R}_{19}$, with $0 \leq i \leq n-1$. All the points of B must lie in R_B. It is easy to see that R_B consists of a 19-regular polygon $C_{19} = \bigcap_{i=0,\ldots,18} R_{i,13}$, with the same center than $\mathcal{R}_{19}$, and 19 hexagons, which we call A_i, for $0 \leq i \leq n-1$, where we denote by A_i the hexagon that is closer to p_i. To be precise, $A_i = R_{i-3,5} \cap R_{i-1,5} \cap R_{i,12} \cap R_{i+1,17} \cap R_{i+2,17} \cap R_{i+7,12}$. The twenty connected components of R_B are shaded in yellow in Figure 5.

No point in the central 19-gon C_{19} can block any lateral 5-hole. In addition, putting a blocking point in one of the hexagonal regions A_i, we only block 3 lateral 5-holes, $R_{i-3,5}$, $R_{i-2,5}$ and $R_{i-1,5}$.

Therefore, to block the 19 lateral 5-holes of $\mathcal{R}_{19}$, we need to put the seven blocking points from B in the hexagonal regions. As every lateral 5-hole contains three of these hexagons, one of the lateral 5-holes of $\mathcal{R}_{19}$ will contain two blocking points, contradicting the fact that each lateral 5-hole of $\mathcal{R}_{19}$ contains exactly one point in B. $\square$

3.1.3 Geometry matters

Lemmas 3.4 and 3.6 indicate that the geometry and distribution of the points has to be considered when finding 5-blocking sets for point sets, even in convex position. In this section we go deeper in that direction, and show two set of 11 points in convex position, for which their smallest 5-blocking point sets have different cardinalities.

Our first point set is $\mathcal{Q}_{11}$, the set of vertices of a regular polygon $\mathcal{R}_{11}$ with eleven vertices. With an approach along the lines of the proof of Lemma 3.6 it is easy to see that the 5-holes of $\mathcal{Q}_{11}$ can be blocked with exactly three points, see Figure 6.

Our second point set, $\mathcal{S}_{11} = \{p_0, \ldots, p_{10}\}$ is shown in Figure 7. First note that the four blue dots shown in Figure 7, block all the 5-holes of $\mathcal{S}_{11}$. We now prove that the 5-holes of $\mathcal{S}_{11}$ cannot be blocked with three points. Let $\mathcal{P}_{11}$ be the convex polygon with vertex set $\mathcal{S}_{11}$.

For any $0 \leq i \leq 10$ let T_i be the triangle bounded by the segments $p_i - p_{i+1}$, $p_i - p_{i+4}$, and $p_{i-3} - p_{i+1}$, addition taken *mod* 11. Observe that any

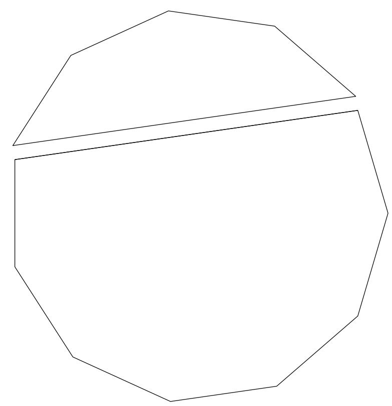

Figure 6: A regular 11-gon.

point of the plane can block at most four lateral 5-holes of $\mathcal{S}_{11}$, and that if it does, it must belong to some T_i, in which case it blocks the laterals 5-holes of $\mathcal{S}_{11}$ with vertex sets $\{p_{i-3}, \ldots, p_{i+1}\}$, $\{p_{i-2}, \ldots, p_{i+2}\}$, $\{p_{i-1}, \ldots, p_{i+3}\}$, and $\{p_i, \ldots, p_{i+4}\}$. Suppose now that the 5-holes of $\mathcal{S}_{11}$ can be blocked with a set of three points $\{x, y, z\}$. In particular $\{x, y, z\}$ also block the eleven lateral 5-holes of $\mathcal{S}_{11}$, and thus at least two points among x, y, and z cover four lateral 5-holes of $\mathcal{S}_{11}$, and the other point three or four. From this we can infer that two points among x, y, and z, say x and y, must belong to two triangles T_i and T_j such that $j = i + 4$ for some $0 \leq i \leq 10$, addition taken *mod* 11.

Since blocking the 5-holes of nine points in convex position requires at least three blocking points, all the lateral 4-holes of $\mathcal{P}_S$ must be empty. Since T_1, T_2, T_4, T_7, T_9 and T_{10} are contained in lateral 4-holes of $\mathcal{P}_S$, they cannot contain any of the points x, y, or z. Then x and y are in T_0, T_3, T_5, T_6, or T_8.

But x and y must belong to some T_i and T_{i+4}, which is not possible: Therefore, to block the 5-holes of $\mathcal{S}_{11}$ we need at least four points, as claimed.

Thus, we have proved:

Theorem 3.7. *There are two different sets of eleven points in convex position such that their smallest* 5*-blocking sets have different cardinalities.*

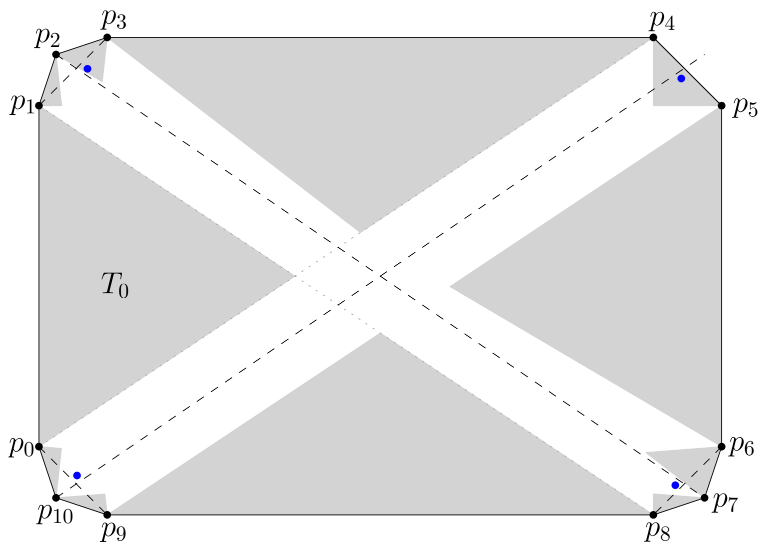

Figure 7: A set of 11 points in convex position that requires 4 points to block its 5-holes.

3.2 Point sets in general position

3.2.1 Geometry matters

As mentioned in the introduction, the number of points needed to block the set of triangles of a point set P, is exactly $2n - c_P - 2$, where $n = |P|$ and c_P is the number of elements from P that are vertices of $CH(P)$. A similar formula does not exist for blocking the 5-holes of a point set: We are next constructing point sets of the same cardinality, and having the same number of elements on their convex hulls, for which the number of points required to block their 5-holes are different.

In other words, we are giving here a result for points in general position, similar to Theorem 3.7, proving that the specific geometry and distribution of the points can change the size of the minimal 5-blocking stes.

We show first that there exist families of point sets with $4m$ elements, with $2m$ of them on the convex hull, such that all of their 5-holes can be blocked with $m - 2$ points.

Lemma 3.8. *For any m there is a point set P_{4m} in general position with $|P_{4m}| = n = 4m$ points and $c_P = 2m$, such that $m - 2$ points are sufficient and necessary to block all the 5-holes of P_{4m}.*

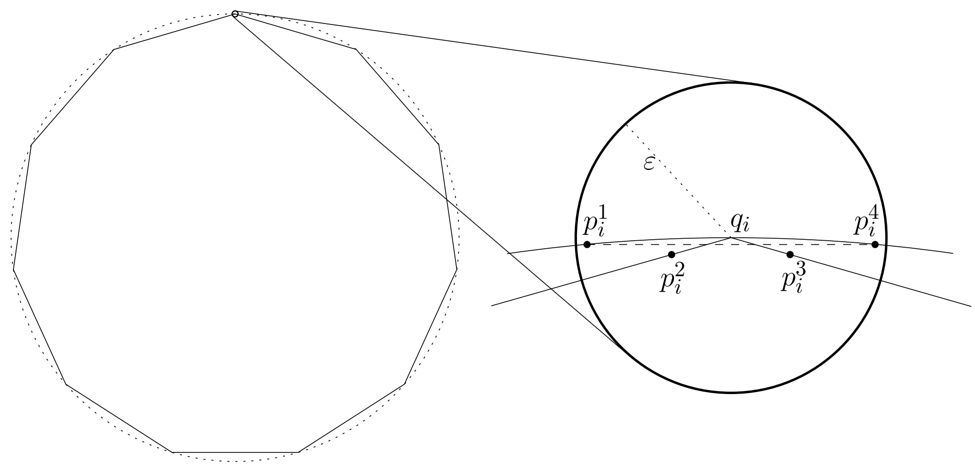

Figure 8: A point set in general position in which $\frac{n}{4} - 2$ points are sufficient and necessary to block all of its convex 5-holes. The image on the right is a close up look at each fat point of the regular m-gon at the left.

Proof. Let $\mathcal{R}_m = \{q_1, \ldots, q_m\}$ be a regular m-gon. From the results in [5, 16], we can choose $m-2$ points $B = \{b_1, \ldots, b_{m-2}\}$ such that any triangle with vertices in $\mathcal{R}_m$ contains exactly an element of B in its interior. It is not hard to see that given such B, we can move the vertices of $\mathcal{R}_m$ around some sufficiently small $\varepsilon > 0$, such that any triangle in the perturbed set contains exactly one element of B.

We construct a set P_{4m} with $4m$ points as follows. We substitute each vertex q_i of $\mathcal{R}_m$, $i = 1, 2, \ldots, m$, by a set of 4 points $S_i = \{p_i^1, p_i^2, p_i^3, p_i^4\}$, each of them at distance no more than ε from q_i, and consider the set $P_{4m} = S_1 \cup \ldots \cup S_m$. The replacement is as follows: Consider the bisector b_i of the internal angle of $\mathcal{R}_m$ at q_i. Let ℓ_i be a line orthogonal to b_i that intersects the edges of $\mathcal{R}_m$, incident to q_i, infinitesimally enough to q_i. Let p_i^1 and p_i^4 be the points of intersection of ℓ_i with the circumcircle C of $\mathcal{R}_m$. Let p_i^2 and p_i^3 be two points equidistant to q_i, below ℓ_i, one on each of the edges of $\mathcal{R}_m$ incident to q_i, and such that the angles $\angle p_i^1 p_i^2 p_i^3$ and $\angle p_i^4 p_i^3 p_i^2$ are close to π, see Figure 8. With this replacement, the convex hull of P_{4m} has $2m$ vertices.

Observe that one can choose p_i^1 and p_i^4 such that one of the open half-planes bounded by the line passing trough p_i^1 and p_i^3 (resp. p_i^4 and p_i^2) contains p_i^4, (resp. p_i^1,) and no other point of P_{4m}. See Figure 8.

Observe next, that no 5-hole can use more than two elements of S_i. It follows now that any 5-hole has vertices in at least three different sets S_i, S_j, S_k.

Moreover, since the elements of S_i are at distance no more than ε from q_i, any triangle containing a point in any three sets S_i, S_j, and S_k contains a point of B in its interior. Therefore the elements of B block all of the 5-holes of P_{4m}.

Observe now that any 5-blocking set for P_{4m} can not have fewer points than $m-2$. First, suppose that B' is a 5-blocking set for P_{4m} with at most $m-3$ elements, then at least one triangle with vertices in $\mathcal{R}_m$ that is not blocked (since the number of triangles in any triangulation of $\mathcal{R}_m$ is $m-2$). Assume that the vertices of one such triangle are q_i, q_j, q_k. Then, by taking two elements in S_i and S_j and one in S_k, we obtain a 5-hole of P_{4m} that is not blocked by any element of B'. Thus, P_{4m} requires $m-2$ points in order to block all of its 5-holes. $\square$

We construct now point sets P'_{4m} with $4m$ elements, $2m$ on its convex hull, such that to block all of its 5 holes we need more than $2m$ points, roughly twice as many as for P_{4m}.

Lemma 3.9. *For every positive integer m divisible by 15 there is a point set P'_{4m} in general position with $|P'_{4m}| = n = 4m$ elements and $c_P = 2m$, such that more than $2m$ are points necessary to block all the 5-holes of P'_{4m}.*

Proof. Let P'_{4m} be a set with $4m = 30k$ points, with $15k$ on its convex hull forming the set of vertices of a regular $15k$-gon. We consider on the boundary of $CH(P'_{4m})$ alternated subsets consisting of 10 and 5 vertices, yielding therefore k subsets of each class. For each of the subsets of 5 vertices, we form a *block* conecting with a chord the first and last element and adding 15 points to the interior of the region, in such a way that the region can be decomposed into 11 convex 5-gons (the pattern corresponds to the classical plane drawing of the dodecahedron graph). See figure 9, where each block is labelled "a".

The part of the convex hull of P'_{4m} that is not in the blocks is an empty convex polygon H with $12k$ vertices: $10k$ come from the subsets not used for the blocks and $2k$ come from gathering the first and last points of all the blocks.

By Lemma 3.3, H requires at least $12k/2-2$ points to block all of its 5-holes, and for the pentagonized blocks we need at least $11k$ points. Thus, any 5-blocking set for P'_{4m} contains at least $(6k-2)+11k = 17k-2$ points, which is larger than $2m = 15k$. $\square$

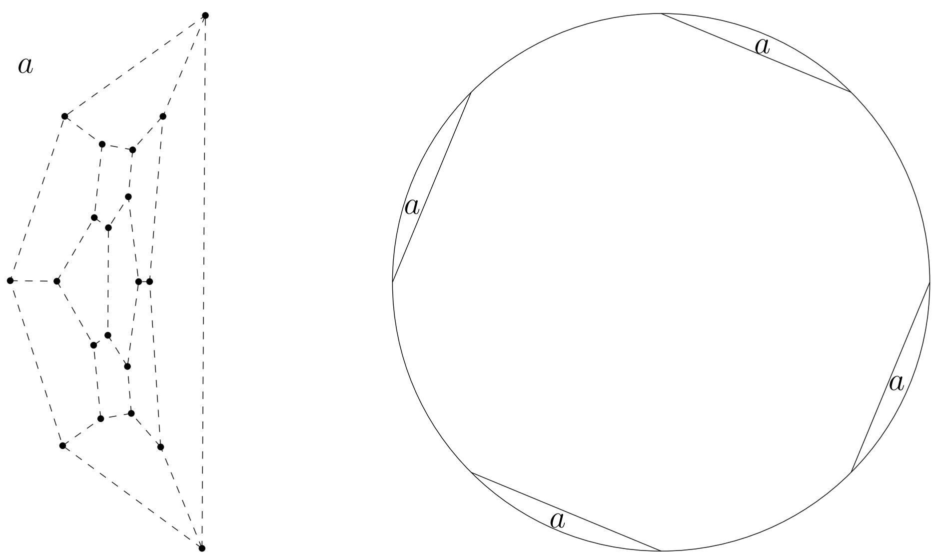

Figure 9:

Thus, combining Lemmas 3.8 and 3.9 we have proved:

Theorem 3.10. *There are two different sets of $n = 4m$ points in non-convex position, such that the number of vertices in the convex hull of each set $2m$, and such that their smallest 5-blocking sets have different cardinalities.*

3.2.2 Piercing the 5-holes of general point sets

We conclude this section with a general a lower bound on the number of points needed to block the 5-holes of any point set. We prove:

Theorem 3.11. *Let P be any set of n points in general position. Then any 5-blocking set of P has at least $2\lceil \frac{n}{9} \rceil - 3$ points.*

Proof. Harborth [13] proved that any set of ten points in general position in the plane always contains a 5-hole. Let B be a 5-blocking set of P. Take a geometric planar matching of the elements of B, and decompose the plane into convex regions by extending the segments in the matching as in Lemma 3.2. Then any convex region in our decomposition cannot contain more than nine points, otherwise there would be a 5-hole of P not blocked by any element of B. It now follows, as in the proof of Lemma 3.2, that $B \geq 2\lceil \frac{n}{9} \rceil - 3$. □

In view of the preceding results we conjecture:

Conjecture 3.12. *The number of points needed to block all the* 5*-holes of any point set with* n *elements is greater than or equal to* $\frac{n}{4} \pm c$*, where* c *is a constant.*

4 Blocking k-holes for points in convex position

In this last section before the concluding remarks, we consider the problem of blocking k-holes for larger values of k. As mentioned in the introduction, Horton [14] proved that for $k \geq 7$, there exist point sets that don't have any k-hole. Thus the question of finding *the minimum* number of blocking points is properly interesting only for some specific families of point sets always having k-holes; here we focus on point sets in convex position.

Let P be a set of n points in convex position. Using a similar argument as in the proof of Lemma 3.2, it can be verified that any k-blocking set for P has at least $2\left\lceil \frac{n}{k-1} \right\rceil - 3$ elements. This bound is essentially tight for odd values of k, as we show next.

We construct a point set P as follows: Let $\mathcal{R}_m$, C, B, and ϵ as in the proof of Lemma 3.8, i.e., $\mathcal{R}_m$ is a regular m-gon, C its circumcircle, B a set of $m-2$ points blocking all the triangles of $\mathcal{R}_m$, and ϵ is the radius of infinitesimal disks centered at the vertices of $\mathcal{R}_m$ in such a way that if these vertices are perturbed each to any position inside their associated disks, the set B keeps blocking all the triangles the perturbed vertices determine. Let $k = 2s+1$. Replace each vertex p_i of $\mathcal{R}_m$ with a set $S_i = \{p_1^i, \ldots, p_s^i\}$ of s points on C within the circle of radius ϵ centered at p_i, see Figure 10. Let $P = S_1 \cup \cdots \cup S_m$. Then P has $n = m \times s$ elements,

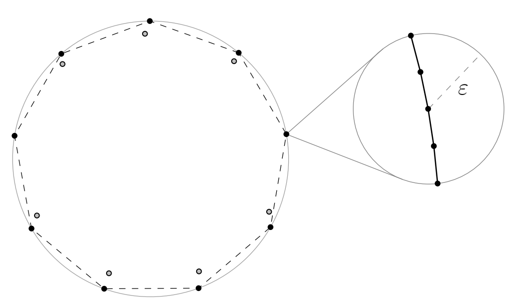

Figure 10: The general construction when $k = 11$.

Then any k-hole with vertices in P has vertices in at least three of the sets S_i, and thus the set B blocks all of the k-holes of P. But B has $m-2$ elements, and $2\left\lceil \frac{n}{k-1} \right\rceil - 2 = 2\left\lceil \frac{ms}{2s} \right\rceil - 2 = m-2$.

Therefore, we have proved:

Theorem 4.1. *$2\left\lceil \frac{n}{k-1} \right\rceil - 3$ points are always necessary, and $2\left\lceil \frac{n}{k-1} \right\rceil - 2$ are sometimes sufficient to block the k holes of a point set with n elements in convex position.*

Next we show that when k is even we can give a better lower bound.

Proposition 4.2. *Let $P = \{p_1, \ldots, p_n\}$ be a set of $n = mh$ points in convex position, with $h \geq 2$. Then, $\frac{n}{h} - O(\sqrt{(n)})$ points are necessary to block all the $2(h+1)$-holes of P.*

Proof. Let us denote by P' the set of points $p_{i \cdot h}$, for $i = 1, 2, \ldots, m$. Then, the number of points in P' is $n/h = m$. Figure 11 shows the case with $h = 2$ (6-holes), in which P' is the set of points with even indices.

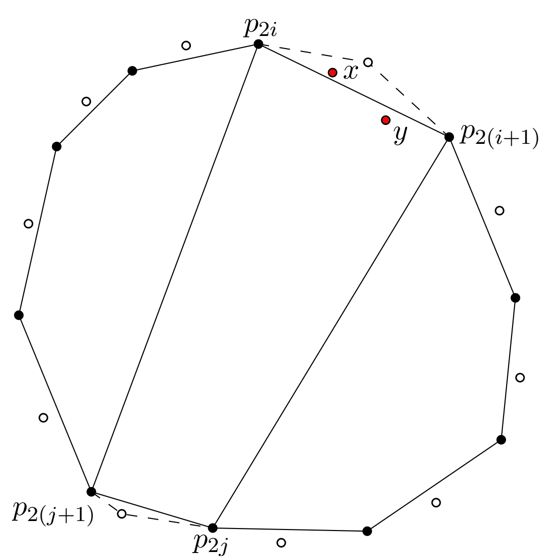

Figure 11: Illustration for Proposition 4.2.

Take two arbitrary edges $p_{i \cdot h} - p_{(i+1) \cdot h}$ and $p_{j \cdot h} - p_{(j+1) \cdot h}$, with $i+1 < j$, and let $H_{i,j}$ be the $2(h+1)$-hole of P determined by the set of points $\{p_{i \cdot h}, p_{i \cdot h+1}, \ldots, p_{(i+1) \cdot h}, p_{j \cdot h}, p_{j \cdot h+1}, \ldots, p_{(j+1) \cdot h}\}$. Consider now the 4-hole $H'_{i,j}$ of P' determined by $p_{i \cdot h}$, $p_{(i+1) \cdot h}$, $p_{j \cdot h}$ and $p_{(j+1) \cdot h}$. Observe that two edges of this 4-hole are on the convex hull of P' and that the other two edges are diagonals (see Figure 11).

Therefore, we can define a bijection between the set Q' of 4-holes in P' defined by pairs of edges $p_{i\cdot h}p_{(i+1)\cdot h}$ and $p_{j\cdot h}p_{(j+1)\cdot h}$, and the set Q of $2(h+1)$-holes $H_{i,j}$ defined above.

Now, take a set B of points blocking the $2h$-holes of Q. Suppose that one of the blocking points x is inside the polygon with vertices $p_{i\cdot h}, p_{i\cdot h+1}, \ldots, p_{(i+1)\cdot h}$ (a triangle in the case $h = 2$). Let R be the set of $2(h+1)$-holes of Q blocked only by x. Note that this point can only block the $2(h+1)$-holes of Q formed using edge $p_{i\cdot h}p_{(i+1)\cdot h}$.

Then, we can remove x and we can add a point y very close to the midpoint of the edge $p_{i\cdot h}p_{(i+1)\cdot h}$, inside the convex hull of P', such that y blocks at least the $2(h+1)$-holes in R (see Figure 11).

Then we can assume that, for any set B blocking the $2(h+1)$-holes of Q, all the blocking points are inside the convex hull of P'. In this case, note that, if a point z blocks a $2(h+1)$-hole of Q, then its corresponding 4-hole in Q' is also blocked by z and vice versa.

Since there is a bijection between Q and Q' and since we need $\frac{n}{h} - O(\sqrt(n))$ points to block all the 4-holes in Q' (as proved in Section 2), it is impossible that the size of a $2(h+1)$-blocking set for Q is smaller than $\frac{n}{h} - O(\sqrt(n))$, for otherwise we could block the 4-holes of Q' with less than $\frac{n}{h} - O(\sqrt(n))$ points. $\square$

5 Final remarks

Closing the gaps between the lower and upper bounds for this family of problems is obviously a main open problem for future research. Yet to be more specific, we would like to end this paper emphasizing the interest of bringing more light into two specific bounds.

As repeatedly mentioned in this paper, it is known that any point set S that blocks the set of triangles of any n-point set P in convex position, has at least $n-2$ points; moreover, if $|S| = n-2$, which is achievable, then any triangle with vertices in P has exactly one element of S in its interior. This gives a trivial upper bound on the number of elements sufficient to block the k-holes of P: Simply remove $k-3$ elements from S. However, we do not know a better upper bound than that! In fact, we conclude with an apparently simpler question:

Question 5.1. *Is it true that to block all the k holes of the set of vertices of a regular n-gon, we need $n - c(k)$ points?*

We believe that the answer to the preceding question should be positive, but a proof is still elusive to us.

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