FURTHER RESULTS IN DESIGNING DIGITAL INTERPOLATORS

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ABSTRACT

One of the two approaches to the design of digital interpolators uses averaged second-order statistics to minimize the mean square interpolation error. This approach has been considered only when the number of samples employed to compute a final interpolated value is even. In the present paper, we generalize systematically this method to consider even or odd numbers of samples in that computation and values of sampling period ratio, with lineal phase non-recursive interpolating filters.

INTRODUCTION

To interpolate between samples of a sequence constitutes a useful technique in many signal processing applications: speech processing, modulation systems, filter banks, narrowband filters, etc. Basically, the interpolation process can be considered as a filtering {1} that obtains a sequence with a sampling period $T' = T/L$ from other sequence with a sampling period $T > T'$. L is the sampling frequency increase.

The design of interpolating filters has been fundamentally considered from two points of view. The first poses the problem as the design of a lowpass filter having specifications independent of the properties of the sequence to be interpolated {1} {2} {3} {4} . This method has its main goal in reducing the number of operations. The second point of view minimizes the interpolation error using the spectrum or the autocorrelation of the final sequence {5} {6} . The present work follows the last approach.

I. INTERPOLATOR DESIGN

Associating z^{-1} with the sampling period of the interpolated sequence ($T' = T/L$), the z-transform of the original sequence can be expressed as $X(z^L)$ and the transfer function of the interpolating filter can be written

$$H(z) = z^{-PL} + \sum_{j=1}^{L-1} z^{-j} H_j(z^L) \qquad (1)$$

where

$$H_j(z^L) = \sum_{i=0}^{Q-1} h_j(i)\, z^{-Li}, \quad j = 1,\dots,L-1 \qquad (2)$$

assuming that L is an integer and a non-recursive interpolating filter. The z-transform of the final sequence is

$$Y(z) = H(z)\ X(z^L) = X(z^L)\ z^{-PL} + \sum_{j=1}^{L-1} z^{-j} H_j(z^L) X(z^L) \tag{3}$$

Introducing

$$Y_0(z^L) = X(z^L) z^{-PL} \tag{4a}$$

$$Y_j(z^L) = H_j(z^L)\ X(z^L),\ j = 1, \ldots, L-1 \tag{4b}$$

formula (3) becomes

$$Y(z) = \sum_{j=0}^{L-1} z^{-j}\ Y_j(z^L) \tag{5}$$

This result admits the following intérpretation: the output sequence $y(n)$ is obtained as a time multiplexing of

$$y_0(rL) = x(rL - PL),\ r = 0, \pm 1, \pm 2, \ldots \tag{6a}$$

and

$$y_j(rL) = \sum_{i=0}^{Q-1} h_j(r - i)\ x(iL),\ j = 1, \ldots, L-1;\ r = 0,\ 1,\ 2 \ldots \tag{6b}$$

$y_0(rL)$ is the original sequence delayed P sampling periods (PT). The $\{y_j(rL)\}$ are obtained by filtering the original sequence through the $\{H_j(z^L)\}$; they represent the resulting values at distances jT' from the original samples. Fig. 1 illustrates this interpretation.

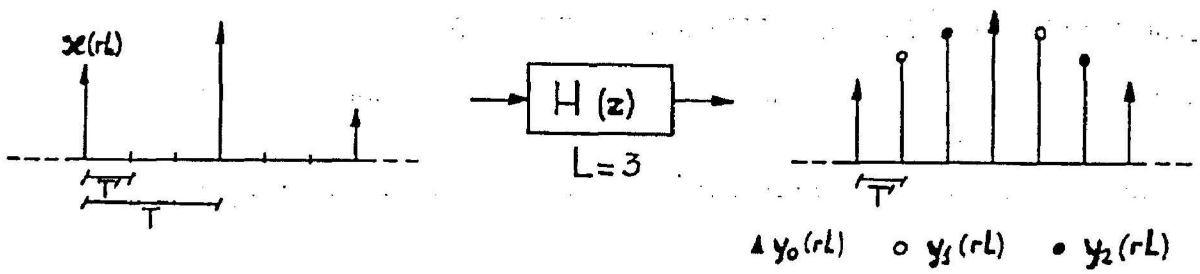

Fig. 1

Then, the final samples between $x(rL)$ and $x(rL + L)$ are given by

$$\hat{x}(rL + j) = y_j\ [(P + r)L] = \sum_{i=0}^{Q-1} x\ [(P + r - i)L]\ h_j(i),\ j = 1, \ldots, L-1 \tag{7}$$

An appropriate criterion to determine the $\{h_j(i)\}$ is to select those values that minimize the mean square error

$$e^2 = \frac{1}{L-1} \sum_{j=1}^{L-1} E\{[x(rL + j) - \hat{x}(rL + j)]^2\} \tag{8}$$

where $\{x(rL + j)\}$ are the samples of $x(n)$ to be estimated. Since each $h_j(i)$ appears only in a term of the sum in (8), e^2 can be minimized by minimizing each term separately. Using (7), it can be easily obtained

$$e_j^2 = E\{[x(rL+j) - \hat{x}(rL+j)]^2\} = R_{xx}(0) - 2\sum_{i=0}^{Q-1} R_{xx}[(i-P)L+j]\, h_j(i) + \sum_{i=0}^{Q-1}\sum_{k=0}^{Q-1} R_{xx}[(i-k)L]\, h_j(i)\, h_j(k) \tag{9}$$

where $R_{xx}(m)$ is the autocorrelation of $x(n)$. The minimum of e_j^2 appears when {7}

$$\bar{h}_j = R^{-1}\,\bar{b}_j$$

where

$$\bar{h}_j = [h_j(0), h_j(1), \ldots, h_j(Q-1)]^T \tag{11a}$$

$$\bar{b}_j = [R_{xx}(-PL+j), \ldots, R_{xx}((Q-1-P)L+j)]^T \tag{11b}$$

and

$$R = \| R_{xx}[(i-k)L] \| \tag{11c}$$

is the autocorrelation matrix.

This criterion to determine the interpolating filter coefficients has been proposed independently in {5} and {6} , but only in {5} the representation (1), that allows an easy design, is used, and only for an even Q.

Although (1) and (2) are valid in any case, the resulting design satisfies the usual restrictions for non-recursive interpolators only when Q is an even number. This seems to be the reason to consider only this case in {5} and {6} .

In the following, we will indicate how to use (1) to verify the above mentioned restrictions for even or odd Q and L.

II. USUAL RESTRICTIONS ON A NON-RECURSIVE INTERPOLATOR

Although not all restrictions have been explicitly expressed, they are usually the following:

1.- The same number Q of original samples has to be used to obtain each final sample.

2.- If d_A and d_P are the distances from a final sample to the furthest previous and later original samples needed to compute it, respectively, (12) will be verified

$$|d_P - d_A| < L \tag{12}$$

Fig. 2 illustrates the meaning of this condition.

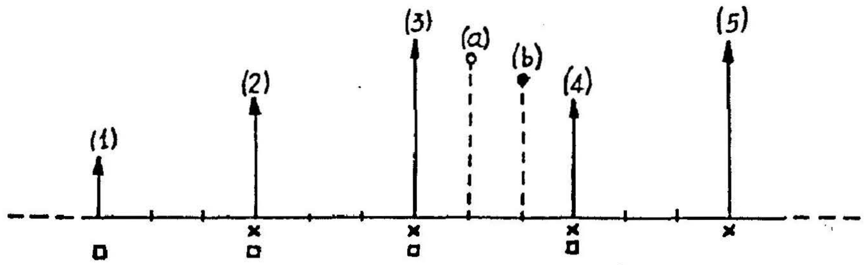

Fig. 2: × indicates a correct selection of original samples
□ indicates an incorrect selection
$Q = 4$

3.- The same coefficient has to correspond to equal distances between final and original samples. For example, in Fig. 2, to compute the sample (a), the coefficients for the samples (2), (3), (4) and (5) have to be equal to the corresponding to the samples (5), (4), (3) and (2), respectively, in computing (b).

We will see that these conditions force a linear phase for $H(z)$.

III. COMPUTING THE IMPULSE RESPONSE OF A NON-RECURSIVE INTERPOLATOR

Let us consider the fulfilment of the indicated three restrictions.

To write $H(z)$ in the form of (1) with

$$H_j(z^L) = \sum_{i=m}^{Q-1+m} h_j(i) z^{-Li} \quad , \quad j = 1,\dots,L-1 \tag{13}$$

obviously fulfils the first restriction. Though the use of (1) and (2) verifies this condition, (13) allows to verify the second restriction for any Q and L.

According with (13), expression (7) has to be rewritten in the form

$$\hat{x}(rL + j) = y_j\left[(P+r)L\right] = \sum_{i=m}^{Q-1+m} x\left[(P+r-i)L\right] h_j(i), \quad j=1,\dots,L-1 \tag{14}$$

Applying the definitions of d_A and d_P and using (14), we obtain

$$d_A = rL + j - (P+r-Q+1-m)L = j - (P-Q+1-m)L \tag{15a}$$

$$d_P = (P+r-m)L - (rL+j) = (P-m)L - j \tag{15b}$$

and

$$|d_P - d_A| = |(2P - 2m - Q + 1)L - 2j| \tag{16}$$

To consider the verification of the second restriction, expressed by (12), we will distinguish two cases.

<u>First case</u>: even Q.

(12) imposes

$$P = Q/2 + m \tag{17}$$

then, selecting $m = 0$, we arrive to (2). Since

$$H(z) = \sum_{i=0}^{N-1} h(i) z^{-i} \tag{18}$$

it is easy to see that (2) forces $h(0) = 0$. To avoid this, we introduce a translation to the left, obtaining finally

$$N = QL - 1 \tag{19a}$$

$$h\left[(N+1)/2 - 1\right] = h\left[(N-1)/2\right] = 1 \tag{19b}$$

$$h\,(rL - 1) = 0, \quad r = 1, 2, \ldots \tag{19c}$$

$$h\,(iL + j - 1) = h_j(i), \quad i = 0, 1, \ldots, Q-1; \; j = 1, 2, \ldots, L-1 \tag{19d}$$

Thus, we can find the impulse response of the interpolating filter from the solution of (10) with $P = Q/2$.

<u>Second case</u>: odd Q.

(12) requires

$$P = (Q + 1)/2 \tag{20}$$

being

$$m = 1 \quad \text{when} \quad 1 \leqslant j \leqslant \lfloor L/2 \rfloor \tag{21a}$$

$$m = 0 \quad \text{when} \quad \lfloor L/2 \rfloor < j \leqslant L - 1 \tag{21b}$$

where $\lfloor\;\rfloor$ indicates truncated greatest integer function. We can observe that, for even L, the interpolated sample cannot verify (12), because $|d_P - d_A| = L$. This case presents other anomalies, as we will see later.

At first, expression (13) and conditions (21a, b) implicate that the first $\lfloor L/2 \rfloor + 1$ samples of the impulse response of the interpolating filter are zero. Translating the same number of samples to the left, we obtain

$$N = LQ \tag{22a}$$

$$h\left[\lfloor (N - 1)/2 \rfloor\right] = 1 \tag{22b}$$

$$h\left[\lfloor (L - 1)/2 \rfloor + rL\right] = 0, \quad r = 1, 2, \ldots \tag{22c}$$

$$h\left[iL + j - \lfloor (L+1)/2 \rfloor\right] = h_j(i), \quad \begin{cases} i = 1, \ldots, Q; \; j = 1, \ldots, \; L/2 \\ i = 0, \ldots, Q-1; \; j = \lfloor (L+1)/2 \rfloor, \ldots, L-1 \end{cases} \tag{22d}$$

Then, we can find the impulse response of the interpolating filter by means of (22 a, b, c, d) from the solution of (10), where, in this case,

$$\bar{h}_j = \left[h_j(m), \ldots, h_j(Q - 1 + m)\right]^T \tag{23a}$$

$$\bar{b}_j = \left[R_{xx}\left(\left(-\frac{Q+1}{2} + m\right) L + j\right), \ldots, R_{xx}\left(\left(\frac{Q-3}{2} + m\right) L + j\right)\right]^T \tag{23b}$$

and m is given by (21 a, b).

Finally, let us consider the third restriction. According with (14), the distance between the interpolated sample and the sample that is being used is given by

$$|(P + r - i)L - rL - j| = |(P - i)L - j| \tag{24}$$

The same distance will be obtained for i' and j' such that

$$j' + (i' - P)L = (P - i)L - j \tag{25}$$

i.e.,

$$i' = 2P - i - (j + j')/L \quad (26)$$

Then, since i' is an integer less than Q,

$$j' = L - j \quad (27a)$$

$$i' = 2P - 1 - i \quad (27b)$$

Thus, the third restriction imposes

$$h_j(i) = h_{L-j}(2P - 1 - i) \quad (28)$$

We remark that restriction (28) is compatible with (13) when $P = Q/2$ and $m = 0$ (first case) and when $P = (Q + 1)/2$ and m is given by (21a, b) (sencod case).

On the other hand, it can be easily checked that the solution of design equation (10) verifies (28), since R^{-1} is a symmetric matrix, and, if we express by $b_j(i)$ the components of $\overline{b}_j$, we will have

$$b_j(i) = b_{L-j}(2P - 1 - i) \quad (29)$$

since $R_{xx}(n)$ is an even function. Then, equation (10) has to be solved for $j = 1, 2, \ldots \lfloor (L + 1)/2 \rfloor$, only; obtaining the remaining components using (28).

IV. THE LINEAR PHASE PROPERTY

Expressions (19) and (22) and restriction (28) allow to establish that the obtained designs verify the linear phase property, except in a case that we will see later. Let us consider the two previously studied cases separately.

First case: even Q.

According with (19a, d) and (28), we can write

$$h(iL + j - 1) = h_j(i) = h_{L-j}(2P - 1 - i) = h\left[(2P - 1 - i)L + L - j - 1\right] = h\left[N - 1 - (iL + j - 1)\right] \quad (30)$$

(30) proves the linear phase property.

Second case: odd Q.

From (22a, d) and (28), we obtain

$$h(iL + j - \lfloor (L+1)/2 \rfloor) = h_j(i) = h_{L-j}(2P - 1 - i) = h\left[(2P - 1 - i)L + L - j - \lfloor (L+1)/2 \rfloor\right] = h\left[N - 1 - (iL + j - \lfloor (L + 2)/2 \rfloor)\right] \quad (31)$$

According with the equality

$$\lfloor (L + 2)/2 \rfloor = \lfloor (L + 1)/2 \rfloor, \text{ odd } L \quad (32a)$$

relation (31) establish the linear phase property when L is odd.

When L is even, we have

$$\lfloor (L+2)/2 \rfloor = \lfloor (L+1)/2 \rfloor + 1, \text{ even } L \qquad (32b)$$

and in (31) we can verify that there is not linear phase. Nevertheless, taking

$$h(N) = h(QL) = h_{L/2}(Q) = 0 \qquad (33)$$

that implicates a total number of samples

$$N' = N - 1 = QL - 1 \qquad (34)$$

it can be checked the verification of the linear phase property in (31) with the help of (32b). Condition (33) is compatible with (28), and it leads to the fulfilment of (12) when Q is odd and L is even.

CONCLUSION

A design method to obtain non-recursive interpolators using the autocorrelation or the spectrum of the original sequence has been introduced. This method allows to obtain the impulse response of the interpolating filter by means of solving systems of linear equations having an immediate formulation for even or odd values of the number of samples Q used to compute a final sample and of the ratio L between the original and final sampling periods. The obtained filters verify the linear phase property.

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