DIRECT DETERMINATION OF THE LOWPASS COMPONENTS OF A BANDPASS SIGNAL FROM SAMPLES OF IT AND DELAYED VERSIONS OF IT

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ABSTRACT

Grace-Pitt-Brown's method {1} {2} {3} to sample a band-limited bandpass signal $x(t)$ at the minimum average sampling rate obtaining separate interpolations for its in-phase and quadrature components is based on sampling $x(t)$ and $x(t - \pi/2\omega_0)$ (where ω_0 is the center angular frequency of the signal) at a rate $\sigma/2\pi$ (where σ is the angular bandwidth of the signal) assuming that $\omega_0 = k\sigma/2$. This method is particularly useful because it is easy to obtain $x(t - \pi/2\omega_0)$.

In this paper we discuss the possibility of generalizing the above method obtaining a separate interpolation for the lowpass components of $x(t)$ by sampling $x(t)$ and $2N - 1$ dealayed versions of it at a rate $\sigma/2\pi N$.

INTRODUCTION

A deterministic, real, finite-energy, σ-band-limited bandpass signal x(t) can be written in the form

$$x(t) = i_x(t)\cos\omega_0 t - q_x(t)\sin\omega_0 t \quad (1)$$

where ω_0 is its center (angular) frequency and $i_x(t)$, $q_x(t)$ are its in-phase and quadrature components, respectively: two real, finite-energy, $\sigma/2$-band-limited lowpass signals, related one-to-one with x(t) when ω_0 is given. It is well known that we can interpolate $i_x(t)$ and $q_x(t)$ separately from samples of

* $i_x(t)$, $q_x(t)$, or
* $x(t)$, $\hat{x}(t)$ (the Hilbert transform of the signal) when $\omega_0 = k\sigma/2$ (k an integer), or
* $x(t)$, $x(t - \pi/2\omega_0)$ when $\omega_0 = k\sigma/2$ (Grace-Pitt-Brown's theorem),

taken at a rate $T = \sigma/2\pi$ (that corresponds to the minimum average sampling rate σ/π) {1} {2} {3} {4} {5} . The last possibility is advantageous because the obtention of $x(t - \pi/2\omega_0)$ is easier than the obtention of $i_x(t)$, $q_x(t)$, or $\hat{x}(t)$.

In {4} {5} we have demostrated that Grace-Pitt-Brown's theorem can be viewed as a particular case of a more general possibility that consists in reconstructing $i_x(t)$, $q_x(t)$, from the samples of the outputs of two linear time-invariant systems driven by x(t), under the following conditions: the impulse response of the first system must have a zero quadrature component, and that of the second system must have a zero in-phase component. This corresponds to hermitian and anti-hermitian spectra of the corresponding complex envelopes, $b_{h_i}(t)$, i = 1, 2; i.e.;

$$B_{h_1}(\omega) = 2H_1(\omega + \omega_0)\, u(\omega + \omega_0) = B^*_{h_1}(-\omega) \quad (2a)$$

$$B_{h_2}(\omega) = 2H_2(\omega + \omega_0)\, u(\omega + \omega_0) = -B^*_{h_2}(-\omega) \quad (2b)$$

where $H_i(\omega)$, 1 = 1, 2, are the transfer functions, and u is the unit step function.

These conditions are also verified by the sets of functions ($\omega_0 - \sigma/2 < |\omega| < \omega_0 + \sigma/2$)

$$H_{1i}(\omega) = \exp(-j\omega i\pi/\omega_0), \quad i = 0,\dots,N-1 \quad (3a)$$

$$H_{2i}(\omega) = \exp[-j\omega(2i+1)\pi/2\omega_0], \quad i = 0,\dots,N-1 \quad (3b)$$

that correspond to delays $i\,\pi/\omega_0$ and $(2i+1)\pi/2\omega_0$, $i = 0,\dots, N-1$, respectively, since their complex envelope spectra are $(-\sigma/2 < |\omega| < \sigma/2)$

$$B_{h_{1i}}(\omega) = 2\exp[-j(\omega+\omega_0)i\pi/\omega_0] = 2(-1)^i \exp(-j\omega i\pi/\omega_0) = B^*_{h_{1i}}(-\omega), \quad i = 0,\ldots N-1 \tag{4a}$$

$$B_{h_{2i}}(\omega) = 2\exp[-j(\omega+\omega_0)(2i+1)\pi/2\omega_0] = 2(-j)^{2i+1}\exp[-j\omega(2i+1)\pi/2\omega_0] = -B^*_{h_{2i}}(-\omega), \quad i = 0,\ldots N-1 \tag{4b}$$

Then, it is interesting to study the possibility of generalizing Grace-Pitt-Brown's theorem to these sets of transforms.

I. EXTENDING PAPOULIS' GENERALIZED SAMPLING EXPANSION

In {5} , we have extended Papoulis' Generalized Sampling Expansion {6} {7} {8} to prove the generalization of Grace-Pitt-Brown's theorem. We will maodify this extension to apply it to the case considered here.

Let us form the system of 2N equations in $-\omega_0 - \sigma/2 < \omega < -\omega_0 - \sigma/2 + c$

$$\sum_{i=0}^{N-1}\{\exp[-j(\omega+rc)i\pi/\omega_0]Y_{1i}(\omega,t)+\exp[-j(\omega+rc)(2i+1)\pi/2\omega_0]Y_{2i}(\omega,t)=\exp(jrct), \quad r=0,\ldots,N-1$$

$$\sum_{i=0}^{N-1}\{\exp[-j(\omega+2\omega_0+rc)i\pi/\omega_0]Y_{1i}(\omega,t)+\exp[-j(\omega+2\omega_0+rc)(2i+1)\pi/2\omega_0]Y_{2i}(\omega,t) = \exp[j(2\omega_0+rc)t], \quad r=0,\ldots,N-1 \tag{5}$$

where $c = \sigma/N$. Assuming $\omega_0 = k\sigma/2N = kc/2$, we have $\exp[j(2\omega_0+rc)t] = \exp[j(k+r)ct]$, and, since $\exp[jk(k+r)c\,2\pi/c] = 1$, it is obvious that $\{Y_{1i}(\omega,t)\}$, $\{Y_{2i}(\omega,t)\}$, $i = 0,\ldots,N-1$, are periodic in t with a period $T_N = 2\pi/c = 2\pi N/\sigma$. Then, we can prove easily that, in $-\omega_0 - \sigma/2 < \omega < -\omega_0 - \sigma/2 + c$:

$$Y_{mi}(\omega,t)\exp(j\omega t) = \sum_{n=-\infty}^{\infty} y_{mi,0}(t-nT_N)\exp(jnT_N\omega), \quad m = 1, 2;\ i = 0,\ldots N-1 \tag{6}$$

where

$$y_{mi,0}(t) = \frac{1}{c}\int_{-\omega_0-\sigma/2}^{-\omega_0-\sigma/2+c} Y_{mi}(\omega,t)\exp(j\omega t)d\omega, \quad m = 1, 2;\ i = 0,\ldots,N-1 \tag{7}$$

From (6) and the (r+1)th and (N + r + 1)th equations of (5) multiplied by exp(jωt), we obtain

$$\exp(j\omega t) = \sum_{i=0}^{N-1}\{\exp(-j\omega i\pi/\omega_0)\sum_{n=-\infty}^{\infty} y_{1i,0}(t-nT_N)\exp(jnT_N\omega) + \exp[-j\omega(2i+1)\pi/2\omega_0]\sum_{n=-\infty}^{\infty} y_{2i,0}(t-nT_N)\exp(jnT_N\omega)\} \tag{8}$$

valid in $-\omega_0-\sigma/2 + rc < \omega < \omega_0 - \sigma/2 + (r+1)c$ and $\omega_0-\sigma/2 + rc < \omega < \omega_0 - \sigma/2 + (r+1)c$, respectively; then (8) is valid in $\omega_0-\sigma/2 < \omega < \omega_0 - \sigma/2$, and we can write

$$x(t) = \frac{1}{2\pi}\int_{\omega_0-\sigma/2<|\omega|<\omega_0+\sigma/2} X(\omega)\exp(j\omega t)d\omega = \sum_{i=0}^{N-1}\Big\{\sum_{n=-\infty}^{\infty} y_{1i,0}(t-nT_N)\frac{1}{2\pi}\int_{\omega_0-\sigma/2<|\omega|<\omega_0+\sigma/2} X(\omega)\exp(-j\omega i\pi/\omega_0)$$

$$\exp(jnT_N\omega)d\omega + \sum_{n=-\infty}^{\infty} y_{2i,0}(t-nT_N)\frac{1}{2\pi}\int_{\omega_0-\sigma/2<|\omega|<\omega_0+\sigma/2} X(\omega)\exp[-j\omega(2i+1)\pi/2\omega_0]\exp(jnT_N\omega)d\omega\Big\} =$$

$$= \sum_{i=1}^{N-1}\Big\{\sum_{n=-\infty}^{\infty} x(nT_N - i\pi/\omega_0)y_{1i,0}(t-nT_N) + \sum_{n=-\infty}^{\infty} x\big[nT_N(2i+1)\pi/2\omega_0\big]\, y_{2i,0}(t-nT_N)\Big\} \tag{9}$$

that is a general interpolation formula. Obviously, (9) can be generalized to cover the case of arbitrary delays $\{t_{0i}\}$, $i = 0,\dots,2N-1$; but the particular case considered allows us to obtain a separate interpolation.

II. THE SEPARATE INTERPOLATION

Introducing the change of variable $\omega' = \omega+\omega_0$ in (5), we obtain in $-\sigma/2 < \omega' < -\sigma/2 + c$

$$\sum_{i=0}^{N-1}\{\exp[-j(\omega'-\omega_0+rc)i\pi/\omega_0]Y_{1i}(\omega'-\omega_0,t)+\exp[-j(\omega'-\omega_0+rc)(2i+1)\pi/2\omega_0]Y_{2i}(\omega'-\omega_0,t) =$$

$$= \exp(jrct), \quad r = 0,\dots, N-1$$

$$\sum_{i=0}^{N-1}\{\exp[-j(\omega'+\omega_0+rc)i\pi/\omega_0]Y_{1i}(\omega'-\omega_0,t)+\exp[-j(\omega'+\omega_0+rc)(2i+1)\pi/2\omega_0]Y_{2i}(\omega'-\omega_0,t) =$$

$$= \exp[j(2\omega_0+rc)t], \quad r = 0,\dots,N-1 \tag{10}$$

By adding the n-th and $(N + n)$-th equations and substacting the first from the last, $n = 1,\dots,N$, considering that

$$\exp(j\omega_0 i\pi/\omega_0) = \exp(-j\omega_0 i\pi/\omega_0) = (-1)^i \tag{11a}$$

$$-\exp[j\omega_0(2i+1)\pi/2\omega_0] = \exp[-j\omega_0(2i+1)\pi/2\omega_0] = (-j)^{2i+1} = -j(-1)^i \tag{11b}$$

we obtain the equivalent system in $-\sigma/2<\omega'<-\sigma/2 + c$

$$\sum_{i=0}^{N-1}(-1)^i\exp[-j(\omega'+rc)i\pi/\omega_0]Y_{1i}(\omega'-\omega_0,t) = \exp(jrct)[\exp(j2\omega_0 t)+1]/2, \quad r = 0,\dots,N-1 \tag{12a}$$

$$-j\sum_{i=0}^{N-1}(-1)^i\exp[-j(\omega'+rc)(2i+1)\pi/2\omega_0]Y_{2i}(\omega'-\omega_0,t) = \exp(jrct)[\exp(j2\omega_0 t)-1]/2, \quad r = 0,\dots,N-1 \tag{12b}$$

and since we can rewrite (7) in the form

$$y_{mi,0}(t) = \exp(-j\omega_0 t)\frac{1}{c}\int_{-\sigma/2}^{-\sigma/2+c} Y_{mi}(\omega'-\omega_0,t)\exp(j\omega' t)d\omega', \quad m = 1,2;\ i = 0,\dots,N-1 \tag{13}$$

it is possible to write

$$y_{1i,0}(t) = \left[\frac{1}{c}\int_{-\sigma/2}^{-\sigma/2+c} Z_{1i,0}(\omega,t)\,\exp(j\omega t)d\omega\right]\cos\omega_0 t,\quad i = 0,\ldots,N-1 \tag{14a}$$

$$y_{2i,0}(t) = -\left[\frac{1}{c}\int_{-\sigma/2}^{-\sigma/2+c} Z_{2i,0}(\omega,t)\,\exp(j\omega t)d\omega\right]\sin\omega_0 t,\quad i = 0,\ldots,N-1 \tag{14b}$$

where $\{Z_{mi,0}(\omega,t)\}$, m = 1, 2, i = 0,...,N - 1, are the solutions of

$$\sum_{i=0}^{N-1} (-1)^i \exp[-j(\omega+rc)i\pi/\omega_0]\, Z_{1i,0}(\omega,t) = \exp(jrct),\quad r = 0,\ldots,N-1 \tag{15a}$$

$$\sum_{i=0}^{N-1} (-1)^i \exp[-j(\omega+rc)(2i+1)\pi/2\omega_0]\, Z_{2i,0}(\omega,t) = \exp(jrct),\quad r = 0,\ldots,N-1 \tag{15b}$$

in $-\sigma/2 < \omega < -\sigma/2 + c$. Note that, since

$$\begin{matrix}\cos\\ \sin\end{matrix}\,[\omega_0(t - n\,T_N)] = (-1)^{nk}\,\begin{matrix}\cos\\ \sin\end{matrix}\,\omega_0 t \tag{16}$$

we obtain the separate interpolation formulas

$$i_x(t) = \sum_{i=0}^{N-1}\sum_{n=-\infty}^{\infty} (-1)^{nk}\, x(n\,T_N - i\pi/\omega_0)\, z_{1i,0}(t - n\,T_N) \tag{17a}$$

$$q_x(t) = \sum_{i=0}^{N-1}\sum_{n=-\infty}^{\infty} (-1)^{nk}\, x[n\,T_N - (2i+1)\pi/2\omega_0]\, z_{2i,0}(t - nT_N) \tag{17b}$$

where

$$z_{mi,0}(t) = \frac{1}{c}\int_{-\sigma/2}^{-\sigma/2+c} Z_{mi,0}(\omega,t)\,\exp(j\omega t)d\omega\ ,\quad m = 1,2;\ n = 0,\ldots,N-1 \tag{18}$$

It is possible to simplify the obtention fo $\{z_{mi,0}(t)\}$, m = 1, 2, i = 0,...,N - 1, introducing the functions

$$A_{1i,0}(t) = (-1)^i \exp(-j\omega i\pi/\omega_0)\, Z_{1i,0}(\omega,t) \tag{19a}$$

$$A_{2i,0}(t) = (-1)^i \exp[-j\omega(2i+1)\pi/2\omega_0]\, Z_{2i,0}(\omega,t) \tag{19b}$$

the systems (15a, b) become

$$\sum_{i=0}^{N-1} w_k^{ri} A_{1i,0}(t) = \exp(jrct),\quad r = 0,\ldots,N-1 \tag{20a}$$

$$\sum_{i=0}^{N-1} w_k^{r(i+1/2)} A_{2i,0}(t) = \exp(jrct),\quad r = 0,\ldots,N-1 \tag{20b}$$

where $w_k = \exp(-j2\pi/k)$ is the k-th root of the unity. In accordance to this, we will have

$$z_{1i,0}(t) = (-1)^i A_{1i,0}(t) \frac{1}{c} \int_{-\sigma/2}^{-\sigma/2+c} \exp[j\omega(t + i\pi/\omega_0)] \, d\omega =$$

$$= (-1)^i A_{1i,0}(t) w_k^{i(N-1)/2} \frac{\sin[\sigma(t + i\pi/\omega_0)/2N]}{\sigma(t + i\pi/\omega_0)/2N} \exp(-j \frac{N-1}{2N} \sigma t), \quad i = 0,\ldots,N-1 \tag{21a}$$

$$z_{2i,0}(t) = (-1)^i A_{2i,0}(t) \frac{1}{c} \int_{-\sigma/2}^{-\sigma/2+c} \exp\{j\omega[t + (2i+1)\pi/2\omega_0]\} d\omega =$$

$$= (-1)^i A_{2i,0}(t) w_k^{(i+\frac{1}{2})(N-1)/2} \frac{\sin\{\sigma[t + (2i+1)\pi/2\omega_0]/2N\}}{\sigma[t + (2i+1)\sigma/2\omega_0]/2N} \exp(-j \frac{N-1}{2N} \sigma t), \quad i=0,\ldots,N-1 \tag{21b}$$

(17a, b) (with (20a, b) and (21a, b)) constitute the generalization of Grace-Pitt-Brown's bandpass sampling technique.

III. AN ILLUSTRATIVE EXAMPLE

If N = 2, systems (20a, b) will become

$$\begin{cases} A_{10,0}(t) + A_{11,0}(t) = 1 \\ A_{10,0}(t) + w_k A_{11,0}(t) = \exp(j\sigma t/2) \end{cases} \tag{22a}$$

$$\begin{cases} A_{20,0}(t) + A_{21,0}(t) = 1 \\ w_k^{1/2} A_{20,0}(t) + w_k^{3/2} A_{21,0}(t) = \exp(j\sigma t/2) \end{cases} \tag{22b}$$

from which

$$A_{10,0}(t) = [\exp(j\sigma t/2) - w_k] / (1 - w_k) \tag{23a}$$

$$A_{11,0}(t) = [1 - \exp(j\sigma t/2)] / (1 - w_k) \tag{23b}$$

$$A_{20,0}(t) = [\exp(j\sigma t/2) - w_k^{3/2}] / w_k^{1/2}(1 - w_k) \tag{23c}$$

$$A_{21,0}(t) = [w_k^{1/2} - \exp(j\sigma t/2)] / w_k^{1/2}(1 - w_k) \tag{23d}$$

introducing these values in (21a, b) and operating, the interpolation functions result

$$z_{10,0}(t) = \frac{\sin(\sigma t/4 + \pi/k)}{\sin(\pi/k)} \frac{\sin(\sigma t/4)}{\sigma t/4} \tag{24a}$$

$$z_{11,0}(t) = \frac{\sin(\sigma t/4)}{\sin(\pi/k)} \frac{\sin[\sigma(t + \pi/\omega_0)/4]}{\sigma(t + \pi/\omega_0)/4} \tag{24b}$$

$$z_{20,0}(t) = \frac{\sin(\sigma t/4 + 3\pi/2k)}{\sin(\pi/k)} \frac{\sin[\sigma(t + \pi/2\omega_0)/4]}{\sigma(t + \pi/2\omega_0)/4} \quad (24c)$$

$$z_{21,0}(t) = \frac{\sin(\sigma t/4 + \pi/2k)}{\sin(\pi/k)} \frac{\sin[\sigma(t + 3\pi/2\omega_0)/4]}{\sigma(t + 3\pi/2\omega_0)/4} \quad (24d)$$

and inserting them in (17a, b) with $N = 2$, $T_N = T_2 = \pi/2\sigma$, we obtain the corresponding separate interpolation formulas.

CONCLUSIONS

We have generalized Grace-Pitt-Brown's quadrature bandpass sampling method, introducing the following procedure: it is possible to recover separately the lowpass components of a band-limited bandpass signal $x(t)$ from the samples of $\{x(t - i\pi/\omega_0)\}$, $\{x[t -(2i + 1)\pi/2\omega_0]\}$, $i = 0,\ldots,N - 1$, (where ω_0 is the center angular frequency of the signal) taken at a rate $1/T_N = \sigma/2\pi N$ (where σ is the angular bandwidth of the signal) when $\omega_0 = k\sigma/2N$.

Generalizations of this result to transforms that correspond to zero quadrature and zero in-phase component impulse response systems are also possible.

ACKNOWLEDGEMENT

The authors wish to thank Prof. Athanasios Papoulis for introducing them to his Generalized Sampling Expansion.

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