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SOME NEW RESULTS IN SAMPLING DETERMINISTIC SIGNALS

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Whittaker's {1} (or Shannon's {2}) Sampling Theorem is a well-known interpolation formula that has been extended in many directions. In this paper, we introduce two new formulations:

- The first follows Papoulis' {3}{4}{5} Generalized Sampling Expansion for reconstructing a signal from regular samples of N(linear, time-invariant) functionals of it, taking the samples at 1/N the Nyquist rate; but generalizing it for including linear T-periodically time-varying systems {6}{7}. This way is in close relation with works that extend sampling in other directions {8}{9} .

- The second generalizes Linden's {10} proof of Kohlenberg's {11} sampling for a bandpass signal, in order to maintain the minimum sampling rate (in the average) and to obtain a separate interpolation of the in-phase and quadrature components of the signal. This follows Grace-Pitt-Brown's {12}{13} theory of bandpass sampling.

1. LINEAR T-PERIODICALLY TIME-VARYING SYSTEMS

A linear time-varying system is a correspondence between two signals of the form

$$g(t) = L[f(t)] = \int_{-\infty}^{\infty} h(t,\tau)f(\tau)d\tau \qquad (1)$$

where $h(t,\tau) \triangleq L[\delta(t-\tau)]$ is the impulse response of the system. A linear T-periodically time-varying system shows the property

$$h(t,\tau) = h(t-T,\tau-T) \qquad (2)$$

The t-marginal transfer function of a linear time-varying system is

$$H^{(1)}(t,\omega) \triangleq \int_{-\infty}^{\infty} h(t,\tau)\exp(j\omega\tau)d\tau \qquad (3)$$

It is easy to show that

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^{(1)}(t,\omega)F(\omega)d\omega \qquad (4)$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

2. A NEW GENERALIZED SAMPLING EXPANSION

We will work with a deterministic, finite energy, σ-band limited function $f(t)$; then, $F(\omega) \equiv 0$ for $|\omega| \geqslant \sigma$.

Let us consider N linear T-periodically ($T=\pi N/\sigma$) time-varying systems having t-marginal transfer functions $\{H_i^{(1)}(\omega,t)\}$, $i = 1,\dots,N$, and let us form the system

$$\sum_{i=1}^{N} H_i^{(1)}(0,\omega+rc)Y_i(\omega,t) = \exp(jrct) \qquad (5)$$

($r=0,\dots,N-1$), where $c=2\sigma/N$ and ω varies between $-\sigma$ and $-\sigma+c$. We assume that $\{H_i^{(1)}(t,\omega)\}$ are selected verifying the classical restrictions in order that the system (5) defines N functions $\{Y_i(\omega,t)\}$, periodic in t with a period T. The Fourier series corresponding to $Y_i(\omega,t)\exp(j\omega t)$ (ω varying between $-\sigma$ and $-\sigma+c$) is

$$Y_i(\omega,t)\exp(j\omega t) = \sum_{n=-\infty}^{\infty} y_i(t-nT)\exp(jnT\omega) \qquad (6)$$

where

$$y_i(t) = \frac{1}{c}\int_{-\sigma}^{-\sigma+c} Y_i(\omega,t)\exp(j\omega t)d\omega \qquad (7)$$

then, multiplying both sides of the first equation of (5) by $\exp(j\omega t)$ and using the indicated Fourier series, we obtain

$$\sum_{i=1}^{N} H_i^{(1)}(0,\omega)\sum_{n=-\infty}^{\infty} y_i(t-nT)\exp(jnT\omega) = \exp(j\omega t) \qquad (8)$$

where ω varies between $-\sigma$ and $-\sigma+c$. By using: $\exp(jnTrc) = \exp(j2\pi nr) = 1$, and substituting ω for $\omega + rc$, we also obtain (8) from the rth equation of (5) with ω varying between $-\sigma+rc$ and $-\sigma+(r+1)c$; then, (8) is valid for $-\sigma < \omega < \sigma$. Inserting $\exp(j\omega t)$ from (8) in the inverse Fourier transform expression of $f(t)$, we have

$$f(t) = \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H_i^{(1)}(0,\omega)\exp(jnT\omega)\cdot d\omega\right] y_i(t-nT) \qquad (9)$$

but $1/2\pi$ times the integral equals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)h_i(0,\tau)\exp[j(\tau+nT)\omega]\, d\omega\, d\tau = \int_{-\infty}^{\infty} h_i(0,\tau)\, f(\tau+nT)\, d\tau = \int_{-\infty}^{\infty} h_i(nT,\tau+nT)\cdot f(\tau+nT)d\tau = g_i(nT) \qquad (10)$$

where $g_i(t)$ is the output of the ith linear T-periodically time-varying system; then, we can write the interpolation formula

$$f(t) = \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} g_i(nT)y_i(t-nT) \qquad (11)$$

We note that the case of linear time-invariant systems is a particular situation of the theorem, because (2) holds for any T, since $h(t,\tau)$ depends only of $t-\tau$.

3. COMMENTS

We must remark a particular conclusion from (11), for which we need only N = 1:

$$f(t) = \sum_{n=-\infty}^{\infty} g(nT)y(t-nT) \qquad (12)$$

indicates that the (linear) distortion introduced on a σ-band limited signal by a linear

T-periodically ($T=\pi/\sigma$) time-varying system can be equalized by sampling and filtering with a linear time-invariant circuit having an impulse response y(t).

It is also possible to show the validity of the previous theorem in a different way, considering the τ-inverse Fourier transforms of Zadeh's "system functions", $z_i(t,\tau) = h_i(t, t-\tau)$, that are (in this case) periodic in t with period T, and expand them in Fourier series (maintaining τ as a parameter); then, we can write

$$h_i(t,\tau) = z_i(t,t-\tau) = \sum_{k=-\infty}^{\infty} z_{ik}(t-\tau)\exp(j2\pi kt/T) \tag{13}$$

(i - 1,...,N), having an obvious interpretation: the sum of an infinite number of systems composed by a linear time-invariant circuit with impulse response $z_{ik}(\cdot)$ followed by a complex exponential multiplier. The multipliers do not alter the values of the outputs taken at times nT; hence, it is enough to apply Papoulis' theorem with transfer functions $\{\sum_{k=-\infty}^{\infty} Z_{ik}(\omega)\}$, i = 1,...,N. But the drawback of this procedure is in manipulating functions that have infinite-sum expressions, while the previous analysis gives directly the interpolating functions.

4. SAMPLING BANDPASS SIGNALS

We will consider initially here a deterministic, real, finite energy, σ-band limited bandpass signal x(t). We must remark that band-limitation is a theoretical approximation for any physical signal (see {14} , for example).

Let ω_0 be the "central" (angular) frequency of x(t). We can write {15}

$$x(t) = i_x(t)\cos\omega_0 t - q_x(t)\,\mathrm{sen}\,\omega_0 t \tag{14}$$

where $i_x(t)$, $q_x(t)$, are the in-phase and quadrature components of x(t) (with respect to ω_0); these two functions will be one-to-one related with x(t) if ω_0 is fixed, and they are real, finite-energy, lowpass signals having an (angular) bandwidth $\sigma/2$.

It is easy to show that, if we sample ideally x(t) at a rate σ/π, the obtained values will define uniquely x(t) only when $\omega_0 = m\sigma/2$, m an odd (positive) integer (m an odd or even (positive) integer if $q_x(t) \equiv 0$). This fact and the possibility of equivalent lowpass processing {15} with $i_x(t)$ and $q_x(t)$ have originated some research for sampling signals associated with x(t) which allow a separate interpolation for $i_x(t)$ and $q_x(t)$ maintaining simultaneously the minimum theoretical sampling rate (in the average), σ/π. A separate interpolation allows
- a reconstruction of x(t) by means of two lowpass interpolations and two (in quadrature) DSB modulations;
- to obtain the envelope and the instantaneous phase of x(t)

$$e_x(t) \triangleq \sqrt{x^2(t)+\hat{x}^2(t)} = \sqrt{i_x^2(t)+q_x^2(t)} \tag{15}$$

$$\phi_x(t) = \omega_0(t) + \Delta\phi_x(t) \triangleq \omega_0(t) + \tan^{-1}[q_x(t)/i_x(t)] \tag{16}$$

respectively {15} (^ indicates Hilbert transform) in a simple way.

A first possibility consists on sampling $i_x(t)$ and $q_x(t)$ at a rate $\sigma/2\pi = 1/T$ each; the (obvious) corresponding interpolation formula shows the separate interpolation property. Note that if we want, for example, as in Radar problems {16} , to determine a maximum of $e_x(t)$ by digital processing, we will use directly $i_x(nT)$ and $q_x(nT)$, an additional advantage of this possibility.

Another alternative is to sample at $\sigma/2\pi$ the signal and its Hilbert transform {17} ; the interpolation formula results

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\frac{\sin[\sigma(t-nT)/2]}{\sigma(t-nT)/2}\cos[\omega_0(t-nT)] - \sum_{n=-\infty}^{\infty} \hat{x}(nT)\,\frac{\sin[\sigma(t-nT)/2]}{\sigma(t-nT)/2}\sin[\omega_0(t-nT)] \tag{17}$$

Note that there is not a strictly separate interpolation (except when $\omega_0 = k\sigma/2$, k a positive integer; this equality will be indicated as Brown's condition {13}). Nevertheless, advantages of the previous case are basically maintained; for example, $e_x(t)$ can be obtained as easily as previously (see (15)).

The drawback of these alternatives is in obtaining the associated functions to be sampled: $i_x(t)$ and $q_x(t)$ (with the classical difficulties of the in quadrature synchronous demodulation) and $\hat{x}(t)$ (a constant phase shift can be only approximated for signals having a nonzero bandwidth). We will discuss in the following Sections the possibility of introducing different pairs of associated signals preserving the above basic advantages. We will restrict the possible associated signals to those resulting of x(t) through linear, time-invariant systems, which represent the simplest family of manipulations on x(t). A previously known result (Grace-Pitt-Brown's method {12} {13}) results included as a particular case in our formulation.

5. GENERALIZING LINDEN'S EQUATIONS

The first investigation in a line similar to that proposed here corresponds to Kohlenberg {11} ; his method can be reduced to consider x(t) and $x(t-t_0)$ as associated signals. Linden's {10} proof of Kohlenberg's interpolation formula can be easily generalized for a pair of linear time-invariant systems with transfer functions $H_1(\omega)$, $H_2(\omega)$. We will work with $\omega_0 - \sigma/2 < \omega < \omega_0 + \sigma/2$ because signals and filters have hermitic spectra and transfer functions.

From a graphical representation of the spectra resulting of sampling at $\sigma/2\pi$ the signals $g_i(t) = F^{-1}[X(\omega)H_i(\omega)]$ (i = 1,2), it is easy to determine as necessary for recuperating x(t) the equations

$$\begin{cases} H_1(\omega)F_1(\omega) + H_2(\omega)F_2(\omega) = 2\pi/\sigma \\ \begin{cases} H_1(\omega - m\sigma)F_1(\omega) + H_2(\omega - m\sigma)F_2(\omega) = 0 \\ H_1[\omega - (m+1)\sigma]F_1(\omega) + H_2[\omega - (m+1)\sigma]F_2(\omega) = 0 \end{cases} \end{cases} \quad (18)$$

($\omega_0 - \sigma/2 < \omega < \omega_0 + \sigma/2$, $\omega_0 - \sigma/2 < \omega < \alpha$ and $\alpha < \omega < \omega_0 + \sigma/2$, respectively), where $m = E[2\omega_0/\sigma]$, $E[\cdot]$ indicates the integer part function, $\alpha = m\sigma - \omega_0 + \sigma/2$, and $F_1(\omega)$, $F_2(\omega)$, are the spectra of the interpolating functions; i.e., the functions $f_1(t)$, $f_2(t)$, necessary for writting

$$x(t) = \sum_{n=-\infty}^{\infty} [g_1(nT)\, f_1(t-nT) + g_2(nT) f_2(t-nT)] \quad (19)$$

(We will assume an unique solution pair for the system; this imposes classical indirect restrictions on $H_1(\omega)$, $H_2(\omega)$). $f_1(t)$, $f_2(t)$ will be, clearly, real bandpass signals confined to $\omega_0 - \sigma/2 < |\omega| < \omega_0 + \sigma/2$; thus, we can write them in the in-phase/quadrature form, and, developping the cosinus and sinus functions, we will obtain a direct interpolation formula, but showing four lowpass interpolations for obtaining $i_x(t)$ and another four (though basically indentical) for $q_x(t)$.

Are there cases in which a simplified result can appear? A first cause of the complexity of this interpolation is the double expression for the second equation of (18): this indicates a frequency "jump" that, in a general case, originates a basic difference between $i_{f_i}(t)$ and $q_{f_i}(t)$, doing not allow more simplicity. But

- if $H_1(\omega)$, $H_2(\omega)$, are constant for $\omega > 0$ (and $\omega < 0$), or
- if $\alpha = \omega_0 - \sigma/2$ or $\alpha = \omega_0 + \sigma/2$,

the jump dissapears. The second case corresponds to Brown's condition, $\omega_0 = k\sigma/2$.

It is easy to show that the first possibility (constant H_1 and H_2) leads us (except for irrelevant constant factors) to signal/Hilbert transform pair as the only possible pair of linear time-invariant modifications of x(t) that allows a "separate" (with "carriers" $\cos[\omega_0(t-nT)]$ and $\sin[\omega_0(t-nT)]$) interpolation without forcing any relation between ω_0 and σ. We will examine in more detail the second possibility.

6. WHEN BROWN'S CONDITION IS VERIFIED

For an easier discussion, it is convenient to introduce the spectra of the complex envelopes ($b_{f_i}(t)$ and $b_{h_i}(t)$) {15} of the interpolating functions and of the impulse responses of the filters. These spectra are

$$B_{f_i}(\omega) = 2F_i(\omega + \omega_0)\, u(\omega + \omega_0) \quad (20)$$

$$B_{h_i}(\omega) = 2H_i(\omega + \omega_0)\, u(\omega + \omega_0) \quad (21)$$

($i = 1, 2$); then, and considering that the system functions are hermitic, we can rewrite (18) in the form:

$$\begin{cases} B_{h_1}(\omega)\, B_{f_1}(\omega) + B_{h_2}(\omega) B_{f_2}(\omega) = 8\pi/\sigma \\ B^*_{h_1}(-\omega) B_{f_1}(\omega) + B^*_{h_2}(-\omega)\, B_{f_2}(\omega) = 0 \end{cases} \quad (22)$$

$|\omega| < \sigma/2$; from which

$$B_{f_1}(\omega) = 8\pi B^*_{h_2}(-\omega)/\sigma\, [B_{h_1}(\omega) B^*_{h_2}(-\omega) + B^*_{h_1}(-\omega) B_{h_2}(\omega)] \quad (23a)$$

$$B_{f_2}(\omega) = 8\pi B^*_{h_1}(-\omega)/\sigma\, [B_{h_2}(\omega) B^*_{h_1}(-\omega) + B^*_{h_2}(-\omega) B_{h_1}(\omega)] \quad (23b)$$

$|\omega| < \sigma/2$. We return to the general interpolation formula obtained from (19) by writting the interpolating functions in the in-phase/quadrature form. It is clear that, if $\omega_0 = k\sigma/2$,

$$\cos[\omega_0(t-nT)] = (-1)^{kn} \cos\omega_0 t \quad (24a)$$

$$\sin[\omega_0(t-nT)] = (-1)^{kn} \sin\omega_0 t \quad (24b)$$

then, for a separate interpolation, we need only $q_{f_1}(t) \equiv 0$, $i_{f_2}(t) \equiv 0$ (indexes are unimportant). Since these are the imaginary and real parts of the corresponding complex envelopes, we need a real $b_{f_1}(t)$ and an imaginary $b_{f_2}(t)$; i.e., a hermitic $B_{f_1}(\omega)$ and an antihermitic $B_{f_2}(\omega)$:

$$B_{f_1}(\omega) = B^*_{f_1}(-\omega) \quad ; \quad B_{f_2}(\omega) = -B^*_{f_2}(-\omega) \quad (25a,b)$$

With the help of (23a) and (23b), we arrive to the equivalent conditions for the systems

$$B_{h_2}(\omega) = -B^*_{h_2}(-\omega) \text{ (antihermitic)} \quad (26a)$$

$$B_{h_1}(\omega) = B^*_{h_1}(-\omega) \text{ (hermitic)} \quad (26b)$$

i.e., $Re[B_{h_1}(\omega)]$ and $Im[B_{h_2}(\omega)]$ need to be even functions, and $Im[B_{h_1}(\omega)]$ and $Re[B_{h_2}(\omega)]$, odd functions. These conditions apply to the parts of $H_1(\omega)\, u(\omega)$, $H_2(\omega)\, u(\omega)$, with respect to ω_0 (strictly speaking, these conditions are necessary only on the signal band). The resulting interpolation formula is

$$x(t) = \{\sum_{n=-\infty}^{\infty} g_1(nT)(-1)^{nk} i_{f_1}(t-nT)\}\cos\omega_0 t - \{\sum_{n=-\infty}^{\infty} g_2(nT)(-1)^{nk} q_{f_2}(t-nT)\}\operatorname{sen}\omega_0 t \quad (27)$$

having the basic advantages of sampling $i_x(t)$, $q_x(t)$, though requiring a previous interpolation for obtaining $i_x(nT)$, $q_x(nT)$.

A particular case in which the above conditions are verified is the well-known Grace-Pitt-Brown's method, in which $H_1(\omega) = 1$, $H_2(\omega) = \exp(-j\omega\pi/2\omega_0)$, in the signal band; the interpolating formula results

$$x(t) = \left\{\sum_{n=-\infty}^{\infty} x(nT)(-1)^{nk} \frac{\sin[\sigma(t-nT)/2]}{\sigma(t-nT)/2}\right\}\cos\omega_0 t - \left\{\sum_{n=-\infty}^{\infty} x(nT - \pi/2\omega_0)(-1)^{nk} \frac{\sin[\sigma(t+\pi/2\omega_0 - nT)/2]}{\sigma(t+\pi/2\omega_0 - nT)/2}\right\}\sin\omega_0 t \quad (28)$$

Clearly, this is not the only possible solution: $x(t)$ and $\hat{x}(t)$ provide another known example; and modifying Grace-Pitt-Brown's $H_1(\omega)$ and $H_2(\omega)$ to $H_1(\omega) = |H_1(\omega)| \exp\{j\, Arg[H_1(\omega)]\}$, $H_2(\omega) = |H_2(\omega)| \exp(j\omega\pi/2\omega_0)$, $H_i(\omega)\; u(\omega)$ ($i = 1, 2$) being symmetric and $Arg[H_1(\omega)]\, u(\omega)$ being antisymmetric with respecto to ω_0, a separate interpolation is obtained. This is the most general version of Grace-Pitt-Brown's method.

7. WHEN BROWN'S CONDITION IS NOT VERIFIED

If $\omega_0 \neq k\sigma/2$ and we want to maintain ω_0 present in the final representation of x(t), the alternative will be that indicated by Brown{13}: to introduce a (fictitious) bandwidth $\sigma' > \sigma$ as low as possible and such that $2\omega_0/\sigma'$ be an integer. We obtain

$$\sigma' = 2\omega_0/E\left[2\omega_0/\sigma\right] \qquad (29)$$

and, consequently, $T' = 2\pi/\sigma'$ will be the new sampling period.

To maintain ω_0 can be necessary in many situations; for example, when x(t) has the forms $i_x(t)\cos\omega_0 t$ or $-q_x(t)\sin\omega_0 t$, or when ω_0 is the (angular) frequency of a certain "carrier" (i.e., the "important" information is the pair $i_x(t)$, $q_x(t)$, defined just respect to ω_0). But when x(t) is an arbitrary general bandpass signal, ω_0 has not a strict sense of "central" frequency, but only a sense of average of "extremes" ($\omega_0 - \sigma/2$ and $\omega_0 + \sigma/2$): a concept of secondary importance, since bandlimitation is not a physical possibility. The argument that the in-phase and quadrature components will have a bandwidth $\sigma/2$ only in ω_0 is used has the same practical weakness. We can remark also that their new expressions will be written in terms of σ' and T', and neitheir σ nor T (the interpolations show "apparent" bandwidths $\sigma'/2$). Thus, in some cases we can think that our main concern is with obtaining a sampling rate as low as possible, and that to maintain ω_0 is of secondary importance. Then, we will look for the minimum (average) sampling rate, $1/T'' = \sigma''/2\pi$, without maintaining ω_0. (We suppose that a "sufficient" part of the signal is in $\omega_0 - \sigma/2 < |\omega| < \omega_0 + \sigma/2$). If we divide $2\omega_0 + \sigma$ in a maximum number of equal segments (the extremes of which have to cover $(-\omega_0 - \sigma/2, -\omega_0 + \sigma/2)$ and $(\omega_0 - \sigma/2, \omega_0 + \sigma/2)$), these segments will have a width

$$\sigma'' = \frac{2\omega_0 + \sigma}{E\left[(2\omega_0 + \sigma)/\sigma\right]} < \sigma' = \frac{2\omega_0 + \sigma}{E\left[(2\omega_0 + \sigma)/\sigma\right]} \qquad (30)$$

and the situation is equivalent to that of (27), introducing σ'', $T'' = 2\pi/\sigma''$, and $\omega_0'' = \omega_0 - (\sigma'' - \sigma)/2$. The spectrum of x(t) is considered as confined to $\omega_0'' - \sigma''/2 < |\omega| < \omega_0'' + \sigma''/2$, and has the (theoretical) property of being in $\omega_0 - \sigma/2 < |\omega| < \omega_0'' + \sigma''/2 = \omega_0 - \sigma/2$ (note: $\omega_0 - \sigma/2 > \omega_0'' - \sigma''/2$). Then, an average sampling rate $\sigma''/2$ such that $\sigma/2\pi < \sigma''/2\pi < \sigma'/2\pi$ results.

8. CONCLUSIONS

We have introduced an interpolation formula in function of the samples of the outputs of N linear T-periodically time-varying systems having the signal as input, the samples being taken at 1/N the Nyquist rate. We are now searching for equivalent versions with kernels different to those corresponding to linear T-periodically time-varying systems.

We have also examinated the separate interpolation for a bandpass signals, determining general formulas and conditions under Brown's restriction, considering samples of (linear time-invariant) functionals of the signal, and discussing possibilities when the restriction is not verified. With some flexibility, it is possible to include in our formulation cases apparently not considered; but these remain as a part of an extension for including here some kinds of linear time-varying systems; we are working in this direction at the present time.

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