Letters to the Editor_____

On Generalized Sampling Expansions for Deterministic Signals

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Abstract—Papoulis [1]–[3] has introduced a theorem for determining a band-limited function from samples of the outputs of linear time-invariant systems having this function as input. We generalize his result to include linear T-periodic time-varying systems, in close relation with works that extend sampling in other directions [4], [5].

I. Linear Time-Varying Systems

A linear time-varying system is a correspondence between signals of the form

$$g(t) = L[f(t)] = \int_{-\infty}^{\infty} h(t,\tau)f(\tau)\,d\tau \tag{1}$$

where

$$h(t,\tau) \stackrel{\Delta}{=} L[\delta(t-\tau)] \tag{2}$$

is the impulse response of the system. The system is called T-periodic [6] if

$$h(t-T, \tau-T) = h(t,\tau). \tag{3}$$

For a linear time-varying system, we can introduce the t-marginal transfer function

$$H^{(1)}(t,\omega) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} h(t,\tau)\exp(j\omega\tau)\,d\tau \tag{4}$$

(an inverse Fourier transform), having the property

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^{(1)}(t,\omega)F(\omega)\,d\omega \tag{5}$$

where $F(\omega)$ is the Fourier transform of $f(t)$.

In the following, we will consider $f(t)$ as a deterministic finite-energy, σ-band-limited function; then,

$$F(\omega) \equiv 0, \quad |\omega| \geqslant \sigma. \tag{6}$$

II. A Generalized Sampling Theorem

We consider N linear T-periodic time-varying systems having t-marginal transfer functions $\{H_i^{(1)}(t,\omega)\}$, $i=1,\cdots,N$, and $T = \pi N/\sigma$, and we form the system

$$\sum_{i=1}^{N} H_i^{(1)}(0, \omega+rc)Y_i(\omega,t) = \exp(jrct), \quad r=0,\cdots,N-1 \tag{7}$$

where $c = 2\sigma/N$ and ω varies between $-\sigma$ and $-\sigma+c$. We assume the classical restrictions for the $\{H_i^{(1)}(0,\omega)\}$. Then, we have defined N functions $\{Y_i(\omega,t)\}$, $i=1,\cdots,N$, periodic in t with period $T = \pi N/\sigma$.

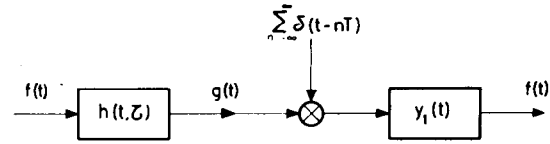Following the steps indicated in [3], we can conclude that, for

Fig. 1.

every ω in $(-\sigma, \sigma)$

$$\exp(j\omega t) = \sum_{i=1}^{N} H_i^{(1)}(0,\omega) \sum_{n=-\infty}^{\infty} y_i(t-nT)\exp(jnT\omega) \tag{8}$$

where

$$y_i(t) = \frac{1}{c}\int_{-\sigma}^{-\sigma+c} Y_i(\omega,t)\exp(j\omega t)\,d\omega, \quad i=1,\cdots,N. \tag{9}$$

Inserting (8) into the inverse Fourier transform of $F(\omega)$, and noting that, if $f(t)$ is σ-band limited

$$\frac{1}{2\pi}\int_{-\sigma}^{\sigma} F(\omega)\exp(j\omega t)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(j\omega t)\,d\omega \tag{10}$$

we obtain

$$f(t) = \sum_{i=1}^{N}\sum_{n=-\infty}^{\infty} y_i(t-nT)$$
$$\cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H_i^{(1)}(0,\omega)\exp(jnT\omega)\,d\omega \tag{11}$$

but

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H_i^{(1)}(0,\omega)\exp(jnT\omega)\,d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)h(0,\tau)\exp[j(\tau+nT)\omega]\,d\omega\,d\tau$$
$$= \int_{-\infty}^{\infty} h(0,\tau)f(\tau+nT)\,d\tau \tag{12}$$

and

$$h(0,\tau) = h(nT, \tau+nT) \tag{13}$$

then, following (1)

$$f(t) = \sum_{i=1}^{N}\sum_{n=-\infty}^{\infty} g_i(nT)y_i(t-nT) \tag{14}$$

where

$$g_i(t) = \int_{-\infty}^{\infty} h_i(t,\tau)f(\tau)\,d\tau \tag{15}$$

is the output of the ith linear T-periodic time-varying system.

We remark that the case of linear time-invariant systems is a particular situation of the theorem shown. We must remark also a particular conclusion from (14), for which we need only $N=1$: if we define

$$\hat{Y}_1(\omega) = \int_{-\infty}^{\infty} y_1(t)\exp(-j\omega t)\,dt \tag{16}$$

we can equalize the linear distortion introduced on a $(\sigma-)$ band-limited signal $f(t)$ by a linear T-periodic time-varying system $(T=\pi/\sigma)$ by resampling and filtering with a circuit having a transfer function $\hat{Y}_1(\omega)$. Fig. 1 shows the above discussed possibility.

It is possible to show the validity of our theorem in a different

way, by using the function

$$z_i(t, \tau) = h_i(t, t - \tau) \qquad (17)$$

for each T-periodic system, and expanding $z_i(t, \tau)$ in a Fourier series maintaining τ as a parameter. But this procedure has the drawback of manipulating functions having an infinite-sum expression, while the previous analysis gives directly the interpolating functions.

III. CONCLUSION

We have introduced an interpolation formula for a σ-band limited deterministic signal $f(t)$ in function of the samples of the outputs of N linear T-periodically time-varying systems having $f(t)$ as input, the samples being taken at $1/N$ the Nyquist rate. The result is a generalization of Papoulis' theorem [1]–[3] and a partial derivation from Jerri's interpretation [5] of Kramer's sampling theorem [4].

We are now searching for equivalent versions for kernels different from those corresponding to the case of linear T-periodic time-varying systems.

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An Analysis of Errors in Wave Digital Filters

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Abstract—The roundoff noise properties of wave digital filters realized with fixed-point arithmetic are studied. The dynamic range of these filters is considered and the effects of scaling on the roundoff noise are determined. Claims have been made in the literature concerning the favorable coefficient sensitivity properties of wave digital filters. It is shown that specific configurations of wave digital filters do exhibit favorable coefficient sensitivity properties, and that filters can be realized which are insensitive to overall coefficient error at zero frequency (low pass).

I. INTRODUCTION

Wave digital filters exhibit low coefficient sensitivity [1], [2]. As a consequence of this, these filters can be realized with fewer bits. However, as the number of data bits used to realize a filter is decreased, the roundoff noise will increase. Therefore, it is important not only to consider a digital filter from the standpoint of coefficient sensitivity, but also in terms of roundoff

noise. A theoretical analysis of roundoff errors in wave digital filters using fixed-point arithmetic is presented in this paper. Filters are analyzed by computer programs from which the roundoff noise power spectral density has been determined both theoretically and experimentally. From these results it is possible to determine wave digital filter configurations which minimize roundoff errors. In addition, the dynamic range of these filters is considered. From this analysis, the wave digital filters are scaled and the subsequent effect on the roundoff noise is determined. Finally, the relationship of dynamic range and roundoff noise to coefficient sensitivity of individual multipliers is considered.

In Section II, we present the general analysis of the roundoff noise power spectral density contributions for multipliers in parallel and series adaptors used to build low-pass wave digital filters. Section III presents a comparative analysis of the roundoff noise power spectral densities of low-pass wave digital filters at zero frequency. In Section IV, the dynamic range of a wave digital filter is considered. Section V contains an analysis of coefficient sensitivity.

Many details, specifically derivations, can be found in [3]. In addition, numerous filters analyzed by computer program can be found in this reference.

II. ANALYSIS OF ROUNDOFF NOISE

The roundoff noise power spectral density for a digital filter can be computed if the usual assumptions are made that the rounding errors are uncorrelated and are independent of the input signal [4]. The general expression for the roundoff noise power spectral density is

$$N_y(\omega) = \sigma_o^2 \sum_{j=1}^{m} k_j |G_j(\omega)|^2 \qquad (1)$$

k_j is the number of multipliers whose outputs are summed at the jth summation node, and $G_j(\omega)$ is the transfer function from the noise injected into the jth summation node to the filters's output. σ_o^2 is the variance of the rounding error generated by each multiplication. Given that n bits are used to realize the multipliers of the filter, then

$$\sigma_o^2 = \frac{(2^{-n})^2}{12}. \qquad (2)$$

The analysis of the roundoff noise power spectral density in parallel and cascade realizations has been done by Jackson [4]–[6]. In the case of these digital filter realizations, the expressions for $G_j(\omega)$ involve quadratic factors. However, for a wave digital filter, the complexity of the expressions is much greater due to the fact that the wave digital filter has feedback throughout the whole structure. The approach taken in this paper has been to first determine the general expressions for $G_j(\omega)$ using wave chain matrix notation, and then resolve these general expressions at a specific frequency. From these specific values, from simulations done by computer, and from theoretical calculations, observations can be made and roundoff noise minimization procedures can be developed.

Wave Chain Matrix Relationships

The wave chain matrix [7] relates the input incident and reflected voltage waves of an adaptor to the output incident and reflected voltage waves of an adaptor as follows:

$$\begin{pmatrix} a_o \\ b_o \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} b_i \\ a_i \end{pmatrix}. \qquad (3)$$