Modal Analysis of Coupling Problems in Optical Fibers

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Abstract—**A modal analysis of the problems of excitation of the dominant mode in an optical fiber by incident plane waves and Gaussian beams has been carried out, and the results applied to the effect on transmission of misalignment in fiber junctions due to offsets, tilts, and gaps. The results in cases of matched media confirm the accuracy of previous theoretical treatments using the Born approximation, which in turn show good agreement with experimental results. In addition, the modal analysis gives more precise solutions when there is a mismatch of media and makes possible the treatment of some problems to which the Born approximation is not applicable.**

I. INTRODUCTION

OVER THE PAST five years the advances made in optical signal processing techniques and in the development of low-loss glasses for optical fibers have made the utilization of the enormous bandwidth available in optical communications an attractive possibility. However, if such fibers are actually to be employed as transmission channels, in the manner of transmission lines and microwave waveguides, one must be able to couple, splice, bifurcate, etc., in a way analogous to that used with those older channels. This is not a trivial technical problem, because the core of the fiber, where the fields are concentrated, is typically only a few microns in diameter so that the difficulty, for example, of aligning two fibers when splicing is critical.

The excitation of propagating modes on a fiber by various types of source illumination has been studied by Snyder [1]–[3] and Marcuse [4], both using the Born approximation. Furthermore, splicing techniques have been developed [5]–[7] and some measurements made [7]–[9] of the effects of imperfect alignment at the interface between two uniform fibers. Most recently Cook *et al.* [9] have also given a theoretical analysis of the effects of misalignment in splicing on the transmission coefficient, again making use of the Born approximation. The importance of such a theoretical calculation lies in the fact that it would enable one to set meaningful standards of precision which must be adhered to in the practical means used for splicing.

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However, all of the theoretical calculations mentioned employ the Born approximation, i.e., the assumption that the fields at the illuminated cross section of the fiber consist entirely of those in the incident wave. That there must be a discrepancy [10] resulting from the use of this approximation is indicated by the fact that Marcuse [4] calculated two values of transmission coefficient corresponding to the two possible boundary conditions to be used (continuity of tangential $\boldsymbol{E}$ or of tangential $\boldsymbol{H}$) and arbitrarily took their geometric mean. Although this mean result was always quite reasonable, in some cases one of the two boundary conditions led to a transmission coefficient larger than unity, which is clearly impossible.

Consequently, this paper will reexamine the basic excitation problems, making use of a more rigorous modal expansion of the fields, which in turn is based on the general techniques for hybrid modes developed by Yaghjian [11], [12]. The solutions will show that, rather surprisingly, the result of taking the geometric mean of the two Born approximation coefficients was remarkably accurate. Finally, the results and techniques will be applied to the problems of the effect on efficiency of transmission at the junction of two similar optical fibers of three types of defect in their alignment: tilt of one axis with respect to the other, offset of their axes, and small gaps between the fiber cores. Numerical results will be presented for all of these problems and compared with the corresponding Born approximation results.

II. FORMULATION

The surface modes of the infinite circular dielectric rod have been extensively studied [2], [13], [14], and their orthogonality well established [15]. The fact that the modes are hybrid and the existence of a continuous spectrum make the exact solution of diffraction and scattering problems for a fiber with infinite outer diameter (od) a formidable task. In fact, there is no known exact solution even for the axially symmetric case, which reduces to a scalar problem [16], [17]. An exact solution would require solving an integral equation, and one possible means of overcoming the difficulty of its solution would be to approximate the integrals by infinite summations and solve the resulting linear system. A natural way of achieving this is to enclose the dielectric rod in a concentric perfectly conducting cylinder with radius large compared to the core radius and the wavelength. As the radius of the metallic pipe increases, the surface modes are unaffected and the nonsurface type modes become a continuum

(Appendix). This method will allow us to solve the excitation and fiber coupling problems by a normal-mode analysis, as in a standard waveguide discontinuity problem, and to take into account the reflected energy.

To find the scattered and transmitted fields for a wave incident at $z = 0$, Fig. 1, we will have to expand the fields on both sides of the interface as an infinite summation of the modes in each structure. Continuity of the transverse components of the fields at $z = 0$ gives

$$\sum_n a_n' \boldsymbol{e}_n' + \boldsymbol{E}_i = \sum_m a_m \boldsymbol{e}_m$$

$$-\sum_n a_n' \boldsymbol{h}_n' + \boldsymbol{H}_i = \sum_m a_m \boldsymbol{h}_m \tag{1}$$

where $\boldsymbol{E}_i$, $\boldsymbol{H}_i$ are the transverse incident fields, $\boldsymbol{e}_m$, $\boldsymbol{h}_m$ the transverse fields of the surface and nonsurface modes of the fiber, and $\boldsymbol{e}_n'$, $\boldsymbol{h}_n'$ are the forward traveling modes of the homogeneously filled circular waveguide [18] in the launching case or the hybrid modes of the fiber in the fiber junction case.

The set of modal coefficients a_m, which will give the efficiencies of excitation, can be obtained from (1) by cross multiplication by $\boldsymbol{h}_n'$ and $\boldsymbol{e}_n'$, integration over the cross-sectional area A at the left of the interface, and making use of the orthogonality of the modes, giving

$$a_n' N_{nn}' + A_n = \sum_m a_m M_m{}^n$$

$$-a_n' N_{nn}' + B_n = \sum_m a_m N_n{}^m \tag{2a}$$

these two equations can be added to obtain a linear system for the modal coefficients of the transmitted modes

$$\sum_{m=1}^{\infty} a_m \, | M_m{}^n + N_n{}^m | = A_n + B_n, \qquad n = 1,2,\cdots \tag{2b}$$

where $N_{nn}' = \int \boldsymbol{e}_n' \times \boldsymbol{h}_n' \cdot \hat{z} \, da$ are the normalization factors of the TE and TM modes and

$$A_n = \int_A (\boldsymbol{E}_i \times \boldsymbol{h}_n') \cdot \hat{z} \, da \tag{3}$$

$$B_n = \int_A (\boldsymbol{e}_n' \times \boldsymbol{H}_i) \cdot \hat{z} \, da \tag{4}$$

$$N_n{}^m = \int_A (\boldsymbol{e}_n' \times \boldsymbol{h}_m) \cdot \hat{z} \, da \tag{5}$$

$$M_m{}^n = \int_A (\boldsymbol{e}_m \times \boldsymbol{h}_n') \cdot \hat{z} \, da. \tag{6}$$

A_n and B_n can be calculated from the expressions of the incident fields, and the cross-normalization factors, $N_n{}^m$ and $M_m{}^n$, have been calculated by Yaghjian [12] by reducing the surface integrals to line integrals and are given in the Appendix.

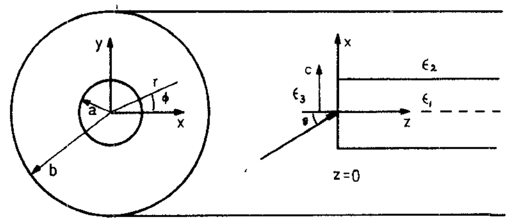

Fig. 1.

The summation over n includes for the launching case all TE and TM modes of the homogeneous circular metallic waveguide arranged in order of increasing eigenvalues, and in the fiber junction case all the propagating surface modes and all the nonsurface modes (Appendix).

To obtain a numerical solution of (2b), we shall have to truncate the system and solve the resulting finite system, adding more equations until the modal coefficients obtained become stable within the accuracy of the computations, and the addition of new equations does not produce further variations.

Practical fibers are made with the radius of the cladding large compared to the core radius and usually are externally coated. This in fact reduces the continuous spectrum to a discrete one but does not affect significantly the surface modes. The preceding model can be viewed not only as a solution for the transversally infinite fiber but as an exact study of the propagation and excitation of modes in a fiber with a metallic external coating.

III. EXCITATION COEFFICIENTS

A. Truncated Uniform Plane Wave

For this case, the incident fields at $z = 0$ are taken as those of a uniform plane wave in a medium of dielectric constant ϵ_3 illuminating a circle of radius c concentric with the fiber and propagating at an angle θ with respect to the axis of the fiber z. The plane of the fiber axis and the direction of incidence is taken to be the x, z plane, so that

$$\boldsymbol{E}_i = E_0 \exp\left(-jk_3 \, x \sin\theta\right) \hat{E} \tag{7}$$

$$\boldsymbol{H}_i = (\epsilon_3/\mu_0)^{1/2} \hat{n} \times \boldsymbol{E}_i \tag{8}$$

with $k_3{}^2 = \omega^2 \mu_0 \epsilon_3$, $\hat{n} = \hat{x} \sin\theta + \hat{z} \cos\theta$, and an $e^{j\omega t}$ time dependence is assumed.

We will consider only x-polarized plane waves, which excite only modes with $g_1(\phi)$, $g_2(\phi)$ given by (A6) with $n = 1$. Expressions for the other polarization could be similarly derived. Also we will assume the fields incident at $z = 0$, but if the source were at $z = z_0 (z_0 < 0)$ with ϵ_3 or a different medium in the region $z_0 < z < 0$, the problem could be treated in a similar way by writing the continuity equations at the $z = z_0$ and $z = 0$ interfaces and eliminating coefficients until we are left with a system relating incident and transmitted fields, as will be done in studying the effects of gaps in fiber joints.

For an x-polarized uniform plane wave, we obtain

$$E_z = -E_0 \exp(-j\Delta r \cos\phi) \sin\theta \quad (9a)$$

$$E_r = E_0 \exp(-j\Delta r \cos\phi) \cos\theta \cos\phi \quad (9b)$$

$$E_\phi = -E_0 \exp(-j\Delta r \cos\phi) \cos\theta \sin\phi \quad (9c)$$

with

$$\Delta = k_3 \sin\theta \quad (10)$$

the $\boldsymbol{H}_i$ field components are obtained from (8) and (9).

We shall have to solve the system (2). Substitution into (3), (4) of the fields of the homogeneously filled circular waveguide [18] gives for A_n and B_n the following expressions.

For n odd, TE_{1n} modes

$$A_n = -j \frac{\beta_n^2 E_0}{\omega\mu_0\gamma_n} \cos\theta \int_0^c \int_0^{2\pi} \exp(-j\Delta r \cos\phi) \cdot \left[\frac{J_1(\gamma_n r)}{\gamma_n r} \cos^2\phi + J_1'(\gamma_n r) \sin^2\phi\right] r\, dr\, d\phi \quad (11)$$

with $\gamma_n = P_{1m}'/b$ ($n = 2m - 1$, $m = 1,2,\cdots$), where P_{1m}' is the mth order zero of J_1', b is the radius of the guide, and β_n is the propagation constant of the nth mode. Use of the recurrence relations for the Bessel fuctions gives

$$A_n = -j \frac{\beta_n^2 E_0}{2\omega\mu_0\gamma_n} \cos\theta \int_0^c \int_0^{2\pi} \exp(-j\Delta r \cos\phi) [J_0(\gamma_n r) + J_2(\gamma_n r) \cos 2\phi] r\, dr\, d\phi. \quad (12)$$

Using the associated series

$$\exp(-j\Delta r \cos\phi) = J_0(\Delta r) + 2 \sum_{k=1}^{\infty} (-j)^k J_k(\Delta r) \cos k\phi \quad (13)$$

and integrating over ϕ gives

$$A_n = -j \frac{\beta_n^2 E_0}{k_3\gamma_n} (\epsilon_3/\mu_0)^{1/2} \pi \cos\theta I_n^1 \quad (14)$$

with

$$I_n^1 = \int_0^c [J_0(\gamma_n r) J_0(\Delta r) - J_2(\gamma_n r) J_2(\Delta r)] r\, dr \quad (15)$$

which can be integrated analytically to give

$$I_n^1 = \begin{cases} -[c/(\gamma_n^2 - \Delta^2)]\{\Delta[J_0(\gamma_n c) + J_2(\gamma_n c)] J_1(\Delta c) \\ \quad - \gamma_n [J_0(\Delta c) + J_2(\Delta c)] J_1(\gamma_n c)\}, & \Delta \neq \gamma_n. \\ (2/\Delta^2) J_1^2(\Delta c), & \Delta = \gamma_n \end{cases} \quad (16)$$

B_n is obtained in the same way as

$$B_n = -j \frac{\beta_n E_0}{\gamma_n} (\epsilon_3/\mu_0)^{1/2} \pi I_n^1 \quad (17)$$

for normal incidence ($\theta = 0$), $\Delta = 0$, and

$$I_n^1 = \frac{c J_1(\gamma_n c)}{\gamma_n}, \qquad \Delta = 0. \quad (18)$$

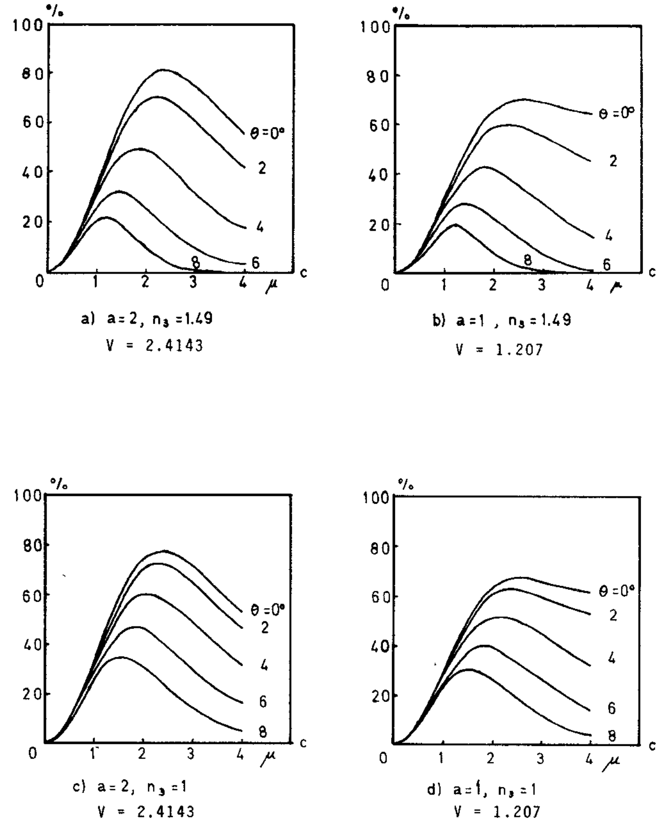

Fig. 2. HE_{11} mode launching efficiency in a fiber with $n_1 = 1.50$, $n_2 = 1.49$, $b = 25\ \mu$, $a = 1,2\ \mu$, $\lambda_0 = 0.9\ \mu$, excited by a truncated uniform plane wave at oblique incidence versus radius of illuminated area.

Similarly for n even, TM_{1n} modes, we have

$$A_n = -j \frac{k_3 E_0}{\gamma_n} (\epsilon_3/\mu_0)^{1/2} \pi \cos\theta I_n^2 \quad (19)$$

$$B_n = -j \frac{\beta_n E_0}{\gamma_n} (\epsilon_3/\mu_0)^{1/2} \pi I_n^2 \quad (20)$$

with $\gamma_n = P_{1m}/b$ ($n = 2m$, $m = 1,2,\cdots$), where P_{1m} is the mth order zero of J_1.

$$I_n^2 = \int_0^c [J_0(\gamma_n r) J_0(\Delta r) + J_2(\gamma_n r) J_2(\Delta r)] r\, dr \quad (21)$$

and

$$I_n^2 = \begin{cases} -[c/(\gamma_n^2 - \Delta^2)]\{\Delta[J_0(\gamma_n c) - J_2(\gamma_n c)] J_1(\Delta c) \\ \quad - \gamma_n [J_0(\Delta c) - J_2(\Delta c)] J_1(\gamma_n c)\}, & \Delta \neq \gamma_n \\ (c^2/2)\{J_0^2(\Delta c) + J_1^2(\Delta c) + J_2^2(\Delta c) \\ \quad - J_3(\Delta c) J_1(\Delta c)\}, & \Delta = \gamma_n \end{cases} \quad (22)$$

at normal incidence

$$I_n^2 = \frac{c J_1(\gamma_n c)}{\gamma_n}, \qquad \Delta = 0. \quad (23)$$

Substitution in (2) gives a system with the modal coefficients in the fiber as unknowns. Fig. 2 represents numerical

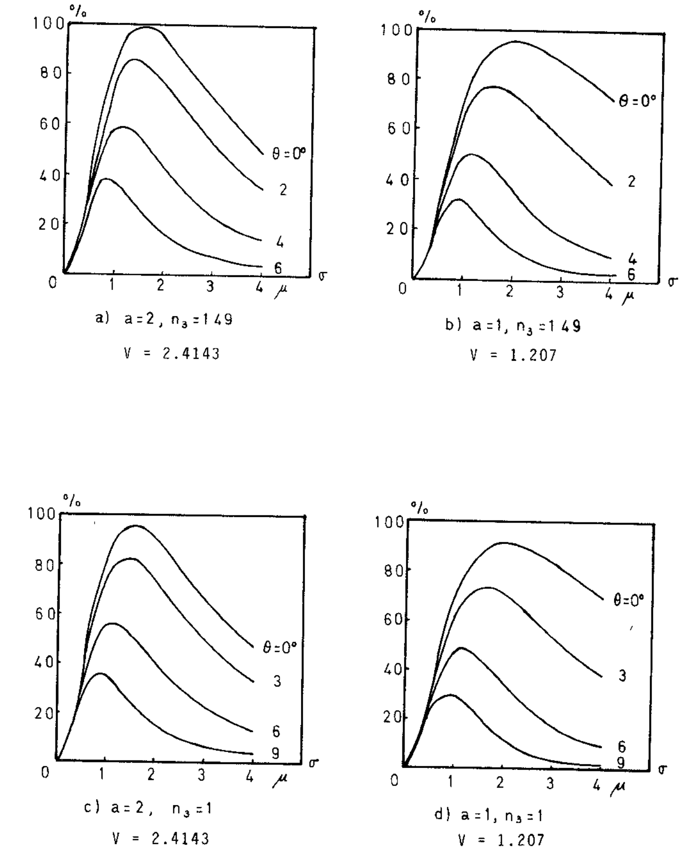

Fig. 3. HE_{11} mode launching efficiency in a fiber with $n_1 = 1.50$, $n_2 = 1.49$, $b = 25\ \mu$, $a = 1{,}2\ \mu$, $\lambda_0 = 0.9\ \mu$, excited by a Gaussian beam at oblique incidence versus Gaussian width σ.

TABLE I

$\sigma(\mu)$	Modal Solution for $n_3 = 1$	Modal Solution for $n_3 = 1.49$	Born Approximation (Geometric mean)
0.5	33.83 percent	35.24 percent	34.16 percent
1.	80 72	84.08	83.81
1.5	95.78	99.75	99.60
2.	90.12	93.84	93.69
2.5	78.10	81.32	81.19
3.	65.81	68.52	68.40
3.5	55.09	57.36	57.26
4.	46.25	48.15	48.07

Note: HE_{11} mode launching efficiency in a fiber with $n_1 = 1.50$, $n_2 = 1.49$, $b = 25\ \mu$, $a = 2\ \mu$, $\lambda_0 = 0.9\ \mu$ excited by a Gaussian beam of width σ at normal incidence from a medium with refraction index n_3.

solution of that system and gives the efficiency of excitation of the HE_{11} mode versus radius of the illuminated area for various angles of incidence. A more extensive set of curves for different fibers, outer radius and medium of incidence together with the corresponding Born approximations can be found in [19].

The transmitted HE_{11} mode was found to be in phase with the incident fields; values of Bessel functions were calculated to six digits accuracy and the ratio of imaginary to real part of the modal coefficient is of the order of 10^{-5}.

Maximum launching efficiencies of 80 percent with matched media and 77 percent for vacumn were obtained for the $a = 2\mu$ fiber which is just below cutoff of the TM_{01} mode [13], and at 4° deviation from normal incidence it drops to about half that value for the matched case. As the fiber radius decreases, so does the maximum of the curves. The effect of tilts increases with the refractive index of the medium of incidence. Fibers of small radius need a much greater illuminating area, as expected, i.e., most of the power is in the clading. The outer pipe does not have a noticeable effect in the fiber well above cutoff.

B. Gaussian Beam

We shall now consider an incident Gaussian beam propagating at an angle θ with respect to the axis of the fiber z. Again, the plane of the axis and the direction of incidence is taken to be the x, z plane, and the center of the beam is displaced a distance d on the positive x-axis.

For small angles of incidence we have

$$\boldsymbol{E}_i = \frac{E_0}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{d^2}{2\sigma^2}\right) \exp\left(-\frac{r^2}{2\sigma^2}\right) \exp(\zeta r \cos\phi)\hat{E}. \tag{24}$$

$\boldsymbol{H}_i$ is given by (8) with

$$\zeta = D - j\Delta \tag{25}$$

$$D = d/\sigma^2 \tag{26}$$

and Δ given by (10).

We shall consider only x-polarized beams and treat oblique incidence and offsets separately. If both were considered, it would be necessary to perform numerical integrations with Bessel functions of the complex argument ζr. If the center of the beam were not on the x-axis but at a point (d,α), then the argument of $e^{Dr\cos\phi}$ in (24) would be replaced by $Dr\cos(\phi - \alpha)$ and both polarizations of the HE_{11} mode would be excited. The formalism to treat those cases is the same as that for the cases $\Delta = 0$ or $D = 0$, but the computations become much more cumbersome.

For oblique incidence ($D = 0$) we obtain the expressions for A_n and B_n given by (14), (17), (19), (20) but with

$$I_n{}^1 = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_0^b \exp\left(-\frac{r^2}{2\sigma^2}\right) [J_0(\gamma_n r)J_0(\Delta r) - J_2(\gamma_n r)J_2(\Delta r)]r\, dr \tag{27}$$

$$I_n{}^2 = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_0^b \exp\left(-\frac{r^2}{2\sigma^2}\right) [J_0(\gamma_n r)J_0(\Delta r) + J_2(\gamma_n r)J_2(\Delta r)]r\, dr. \tag{28}$$

Both of these expressions require numerical integration, and if both tilts and offsets were to be considered simultaneously, a multiplicative factor $\exp(-d^2/2\sigma^2)$ would appear and ζ would replace Δ.

Fig. 3 shows two of the curves obtained [19]. Maximum values of efficiency are 99.7 percent for $a = 2\mu$ and 98.7 percent for $a = 1.5\mu$. As the radius of the core a decreases, the maximum decreases and the effect of tilts increases.

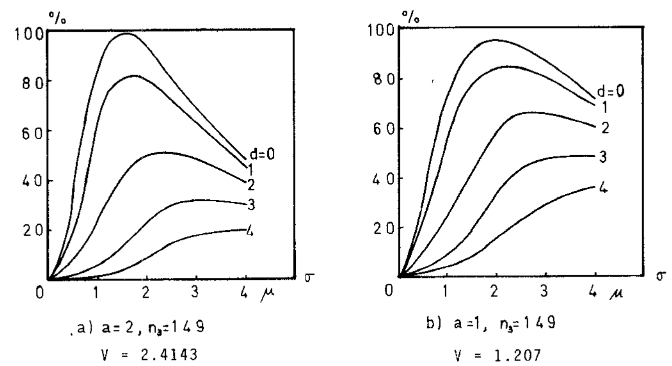

Fig. 4. HE_{11} mode launching efficiency in a fiber with $n_1 = 1.50$, $n_2 = 1.49$, $b = 25\ \mu$, $a = 1,2\ \mu$, $\lambda_0 = 0.9\ \mu$, excited by a Gaussian beam with the axis displaced d versus Gaussian width σ.

Again, the efficiency for the vacuum was about 4 percent less than for $n_3 = n_2$. The results are insensitive to changes in the pipe radius except for fibers with very small core radius. A Born approximation was carried out for the same cases, and the geometric mean of the two values obtained for the modal coefficient was in very good agreement for matched media, but for incidence from the vacuum was about 4 percent higher than the corresponding modal solution (Table I). Since the theoretical calculations of Cook *et al.* [9] are also based on the Born approximation, the same degree of precision presumably also applies to them.

For beam offsets ($\Delta = 0$), using the associated series

$$\exp(Dr\cos\phi) = I_0(Dr) + 2\sum_{k=1}^{\infty} I_k(Dr)\cos k\phi \qquad (29)$$

we again obtain A_n and B_n given by (14), (17), (19), (20) but with

$$I_n{}^1 = \frac{\exp(-d^2/2\sigma^2)}{(2\pi\sigma^2)^{1/2}} \int_0^b \exp\left(-\frac{r^2}{2\sigma^2}\right)[J_0(\gamma_n r)I_0(Dr) + J_2(\gamma_n r)I_2(Dr)]r\,dr \qquad (30)$$

$$I_n{}^2 = \frac{\exp(-d^2/2\sigma^2)}{(2\pi\sigma^2)^{1/2}} \int_0^b \exp\left(-\frac{r^2}{2\sigma^2}\right)[J_0(\gamma_n r)I_0(Dr) - J_2(\gamma_n r)I_2(Dr)]r\,dr. \qquad (31)$$

Fig. 4 shows the effect of offset misalignments and indicates that they are more critical than tilts. Alignments of cores of 1 or 2 microns in fiber joints is not an easy task, especially in the field; also in the launching system there is the possibility of some misalignment. In fibers close to cutoff of the TM_{01} mode an offset equal to the core radius reduces the efficiency by more than half. As the radius of the core is decreased the effect decreases, as expected. Also as σ increases the effect becomes less important but then the efficiency drops and the effect of tilts increases. Born approximations for these cases can be found in [19].

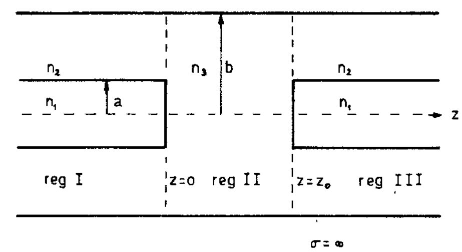

Fig. 5.

IV. THE FIBER BUTT JOINT WITH A GAP

The problem depicted in Fig. 5 is that of two identical metallic coated fibers aligned on the same axis, but whose ends are separated by the gap region $0 < z < z_0$ which has a refractive index n_3. In region I, $z < 0$, there is a set of forward-moving incident modes with amplitudes $a_k{}^0$, $k = 1, 2, \cdots, k$, and backward-moving reflected modes with coefficients $a_m{}^{(1)}$. Region II, $0 < z < z_0$, is characterized by a set of TE and TM circular waveguide modes with forward-directed modal coefficients $a_n{}^{(2)}$ and backward-directed ones $b_n{}^{(2)}$; and in region III, $z > z_0$, a set of transmitted hybrid modes $a_l{}^{(3)}$ is excited.

If one takes the cross product of the modal fields $\boldsymbol{h}_m$ and $\boldsymbol{e}_m$ with the equations for continuity at $z = 0$ of tangential $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively, and then integrates over the cross section, the result is

$$\sum_{k=1}^{K} \delta_{km} a_k{}^0 N_{kk} - N_{mm} a_m{}^{(1)} = \sum_{n=1}^{\infty} a_n{}^{(2)} N_n{}^m - \sum_{n=1}^{\infty} b_n{}^{(2)} N_n{}^m \qquad (32)$$

$$\sum_{k=1}^{K} \delta_{km} a_k{}^0 N_{kk} + N_{mm} a_m{}^{(1)} = \sum_{n=1}^{\infty} a_n{}^{(2)} M_m{}^n + \sum_{n=1}^{\infty} b_n{}'^{(2)} M_m{}^n, \qquad m = 1,2,\cdots \qquad (33)$$

where the cross-normalization factors $N_n{}^m$ and $M_m{}^n$ are defined as in (5), (6), and where

$$N_{mm} = \int_A (\boldsymbol{e}_m \times \boldsymbol{h}_m)\cdot\hat{z}\,da. \qquad (34)$$

An exactly similar procedure applied at $z = z_0$ yields

$$a_n{}^{(2)} N_{nn}{}' \exp(-j\beta_n{}' z_0) - b_n{}^{(2)} N_{nn}{}' \exp(j\beta_n{}' z_0) = \sum_{l=1}^{\infty} a_l{}^{(3)} M_l{}^n \exp(-j\beta_l z_0) \qquad (35)$$

$$a_n{}^{(2)} N_{nn}{}' \exp(-j\beta_n{}' z_0) + b_n{}^{(2)} N_{nn}{}' \exp(j\beta_n{}' z_0) = \sum_{l=1}^{\infty} a_l{}^{(3)} N_n{}^l \exp(-j\beta_l z_0) \qquad (36)$$

where β_n is the propagation constant of the nth mode and the primes refer to region II.

The reflected mode coefficients and those in region II may be eliminated from the four sets of equations obtained, leaving

$$\sum_{l=1}^{\infty} a_l{}^{(3)} \left\{ \sum_{n=1}^{\infty} \frac{U_{m,n}U_{l,n}\exp[j(\beta_n{}' - \beta_l)z_0] - V_{m,n}V_{l,n}\exp[-j(\beta_n{}' + \beta_l)z_0]}{2N_{nn}{}'} \right\} = \sum_{k=1}^{K} \delta_{km} 2 a_k{}^0 N_{kk}, \qquad m = 1,2,3,\cdots \qquad (37)$$

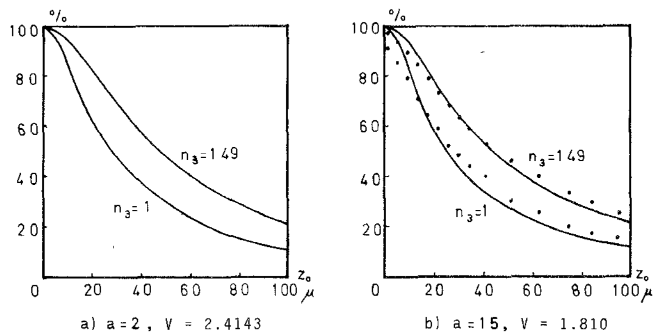

Fig. 6. —: Coupling efficiency for a gap of length z_0 in a fiber with $n_1 = 1.50$, $n_2 = 1.49$, $b = 25\ \mu$, $a = 1.5\ \mu$, $\lambda_0 = 0.9\ \mu$. °: Experimental results [8] for a fiber of 3.7-μ diameter core, $n_1 = 1.6171$, $n_2 = 1.6038$, at $\lambda = 0.6328$ presented on the same normalized separation-core radii scale.

where $U_{m,n} = N_n{}^m + M_m{}^n$ and $V_{m,n} = M_m{}^n - N_n{}^m$.

When there is only a single incident mode, the HE_{11}, all terms except the first one on the right hand side are zero, and that one is $2a_1{}^\circ N_{11}$. Numerical solutions for this case have been found by truncating the system of equations to $m \leq 40$ for four different sizes of fiber with $n_3 = n_2 = 1.49$, corresponding to a gap filled with matching oil, and $n_3 = 1$ corresponding to an air gap. The results are shown in Fig. 6, which also includes a comparison with the experimental results of Bisbee [8]. Even though his ratio a/λ is somewhat different, the agreement is good, as it was for his data on offsets as well; however, it should be emphasized that, unlike the other problems previously considered, no Born approximation solution can be obtained for this one, and the theoretical values are the only ones available. It might also be noted that the reason the curves are relatively insensitive to the value of a/λ is that the absolute transverse distribution of the fields remains nearly the same even though more of it is contained in the cladding at smaller a/λ.

The same technique could be applied to the case of lateral displacement as well, but the lack of rotational symmetry would make the calculations of the cross-normalization factors much more complicated, requiring numerical integration.

V. SUMMARY AND CONCLUSIONS

The effect of including a conducting boundary at $r = b$ was found to be insignificant as long as $\chi = b/a > 4$; there was less than 0.3 percent difference between the calculated launching efficiencies for pipes of 8-μ and 25-μ radius. The type of coating used on the fiber seems to be irrelevant.

The maximum transmission coefficient for excitation by a normally incident uniform plane wave was 81 percent, as compared with a value of 80 percent obtained by Snyder [3] with a Born approximation. The corresponding values for excitation by a Gaussian beam were 99.6 percent, as compared with a value of 99.7 percent reported by Marcuse [4] when the beam is incident from a medium matched to the cladding. In fact, if one takes the geometric mean of the two possible Born approximation solutions, the results are surprisingly close (within 0.2 percent) to the modal solution in all cases where the media were matched; but if the medium of incidence is vacuum, the Born approximation solution is about 4 percent higher (Table I).

The practical effect of tilts or offsets of the incident beam is quite pronounced; an offset of one core radius or a tilt of 4° reduces the coupling efficiency to less than 50 percent for the cases treated. However, a gap of 25 radii is required to produce a comparable effect when matching oil is used or about 15 radii for an air gap, so that in practice the gap is apt to be a much less serious problem. The effect of a gap of given length is insensitive to the core diameter when it is of the order of a wavelength because the lateral extent of the fields does not change very much. However, as Cook *et al.* [9] have pointed out, a smaller core diameter makes the effect of offsets, as measured in core diameters, less critical and of angular displacements more critical when one finally approaches the situation where the fields begin to spread laterally.

Although the gap problem was solved by rigorous application of the boundary conditions to the modal expansions, in the tilt and offset problems in fiber junctions computational symplicity prompted an approximate treatment, replacing the incident HE_{11} mode in the fiber by a Gaussian beam. This should, however, give excellent results for the case of incidence from an actual fiber for that value of σ corresponding to maximum transmission at normal incidence and no offset, since at this point the two types of incident fields are very nearly identical. That choice of σ is appropriate because the two fibers were assumed identical, and coupling is maximum when the incident wave is very nearly matched to the transmitted mode. This simplification is also borne out by the experimental results of Cook *et al.* [9], who got good agreement with their Born approximation calculations.

A related problem, that of the radiation into a uniform medium from the termination of a fiber, has been treated, yielding a transmission coefficient of 96 percent for radiation into a vacuum and 100.0 percent for a matched medium.

In conclusion, the more rigorous modal analysis has shown that the Born approximation does give accurate results in all cases treated using matched media, and that, even though it requires somewhat greater computational effort, the modal approach would be advantageous in problems where serious mismatches of refractive index occur or where the Born approximation is inapplicable, as in the case of the gap.

APPENDIX

The set of functions $\boldsymbol{e}_m$, $\boldsymbol{h}_m$, are the fields of the modes of a dielectric rod of permittivity ϵ_1 and radius a in a medium of lower permittivity ϵ_2 surrounded by a perfectly conducting pipe of radius b and can be obtained from the usual longitudinal formulation [18].

For the range of the propagation constant $k_2{}^2 < \beta^2 < k_1{}^2 (k_{1,2}{}^2 = \omega^2\mu_0\epsilon_{1,2})$, we obtain the surface-type modes, whose existance is independent of the surrounding pipe and which have fields localized to the vicinity of the core with cutoff condition $\beta = k_2$. Their longitudinal components are given by

$$E_z = g_1(\phi)\begin{cases} J_n(\alpha_1 r), & r < a \\ J_n(u)\dfrac{K_n(\chi w)I_n(sr) - I_n(\chi w)K_n(sr)}{K_n(\chi w)I_n(w) - I_n(\chi w)K_n(w)}, & a < r < b \end{cases} \quad \text{(A1a)}$$

$$H_z = (\epsilon_1/\mu_0)^{1/2}(\beta/k_1)P_1 g_2(\phi) \cdot \begin{cases} J_n(\alpha_1 r), & r < a \\ J_n(u)\dfrac{K_n'(\chi w)I_n(sr) - I_n'(\chi w)K_n(sr)}{K_n'(\chi w)I_n(w) - I_n'(\chi w)K_n(w)}, & a < r < b \end{cases} \quad \text{(A1b)}$$

where

$$\chi = b/a \quad \text{(A2)}$$

$$u = \alpha_1 a, \qquad w = sa \qquad u^2 + w^2 = V^2 = a^2\omega^2\mu_0\epsilon_1\delta \quad \text{(A3)}$$

$$\delta = 1 - \epsilon_2/\epsilon_1 = 1 - n_2^2/n_1^2 \quad \text{(A4)}$$

$$\alpha_1^2 = k_1^2 - \beta^2 \qquad s^2 = \beta^2 - k_2^2 \quad \text{(A5)}$$

$$g_1(\phi) = \begin{Bmatrix} \sin n\phi \\ \cos n\phi \end{Bmatrix} \quad g_2(\phi) = \begin{Bmatrix} \cos n\phi \\ -\sin n\phi \end{Bmatrix} \quad \text{(A6)}$$

with n a nonnegative integer.

The eigenvalue equation is

$$F_1 = \frac{\beta^2}{k_1^2}P_1 \quad \text{(A7)}$$

with

$$P_1 = \frac{V^2}{u^2w^2}\frac{n}{\eta_1 + \eta_2\xi_2} \quad \text{(A8)}$$

$$F_1 = \frac{u^2w^2}{nV^2}(\eta_1 + (1-\delta)\eta_2\xi_1) \quad \text{(A9)}$$

$$\eta_1 = \frac{J_n'(u)}{uJ_n(u)} \qquad \eta_2 = \frac{K_n'(w)}{wK_n(w)} \quad \text{(A10)}$$

$$\xi_1 = \frac{I_n'(w)/K_n'(w) - I_n(\chi w)/K_n(\chi w)}{I_n(w)/K_n(w) - I_n(\chi w)/K_n(\chi w)} \quad \text{(A11a)}$$

$$\xi_2 = \frac{I_n'(w)/K_n'(w) - I_n'(\chi w)/K_n'(\chi w)}{I_n(w)/K_n(w) - I_n'(\chi w)/K_n'(\chi w)}. \quad \text{(A11b)}$$

As b increases, these modes become the surface modes of the open rod. In the limiting case $\chi \to \infty : \xi_1 \to 1$, $\xi_2 \to 1$, and (A1), (A7) become the expressions for the open fiber [2]. Use of the asymptotic expressions for the Bessel functions yields for $\chi \gg 1$

$$\xi_1 \sim \frac{-1 - e^\chi}{1 - e^\chi} \qquad \xi_2 \sim \frac{-1 + e^\chi}{1 + e^\chi}, \qquad \text{for} \quad w \gg 1$$

$$\xi_1 \sim 1 + 2/\chi^{2n} \qquad \xi_2 \sim 1 - 2/\chi^{2n}, \qquad \text{for} \quad \chi w \ll 1 \quad \text{and} \quad n \geq 1.$$

The nonsurface type modes are obtained in the range $\beta^2 \leq k_2^2$ and the longitudinal components are given by

$$E_z = g_1(\phi)\begin{cases} J_n(\alpha_1 r), & r < a \\ J_n(u)\dfrac{J_n(\alpha_2 r)N_n(\chi v) - N_n(\alpha_2 r)J_n(\chi v)}{J_n(v)N_n(\chi v) - N_n(v)J_n(\chi v)}, & a < r < b \end{cases} \quad \text{(A12a)}$$

$$H_z = (\epsilon_1/\mu_0)^{1/2}(\beta/k_1)P_2 g_2(\phi) \cdot \begin{cases} J_n(\alpha_1 r), & r < a \\ J_n(u)\dfrac{J_n(\alpha_2 r)N_n'(\chi v) - N_n(\alpha_2 r)J_n'(\chi v)}{J_n(v)N_n'(\chi v) - N_n(v)J_n'(\chi v)}, & a < r < b \end{cases} \quad \text{(A12b)}$$

with

$$\alpha_1^2 = k_1^2 - \beta^2 \qquad \alpha_2^2 = k_2^2 - \beta^2 \quad \text{(A13)}$$

$$u = \alpha_1 a \qquad v = \alpha_2 a \qquad u^2 - v^2 = V^2 \quad \text{(A14)}$$

and the eigenvalue equation

$$F^2 = (\beta^2/k_1^2)P_2 \quad \text{(A15)}$$

$$P_2 = \frac{V^2}{u^2v^2}\frac{n}{-\eta_1 + \eta_3\gamma_2} \quad \text{(A16)}$$

$$F_2 = \frac{u^2v^2}{nV^2}(-\eta_1 + (1-\delta)\eta_3\gamma_1) \quad \text{(A17)}$$

$$\eta_3 = \frac{N_n'(v)}{vN_n(v)} \quad \text{(A18)}$$

$$\gamma_1 = \frac{J_n'(v)/N_n'(v) - J_n(\chi v)/N_n(\chi v)}{J_n(v)/N_n(v) - J_n(\chi v)/N_n(\chi v)} \quad \text{(A19a)}$$

$$\gamma_2 = \frac{J_n'(v)/N_n'(v) - J_n'(\chi v)/N_n'(\chi v)}{J_n(v)/N_n(v) - J_n'(\chi v)/N_n'(\chi v)}. \quad \text{(A19b)}$$

The following surface integrals have been reduced by Yaghjian [11] to simple line integrals. The index m is used for the modes of the fiber, and n refers to the modes of the homogeneous cylindrical waveguide.

$$M_m^n = \int_A (\boldsymbol{e}_m \times \boldsymbol{h}_n')\cdot\hat{z}\,da$$

$$= \frac{j\omega}{p_n'^2 - p_m^2}\oint_c (\mu_m h_{mz}\boldsymbol{h}_n' - \epsilon_n e_{nz}'\boldsymbol{e}_m)\cdot\hat{n}\,dl + \frac{j}{p_n'^2 - p_m^2}\oint_c (\beta_n' h_{nz}'\boldsymbol{e}_m + \beta_m e_{mz}\boldsymbol{h}_n')\cdot\hat{r}\,dl. \quad \text{(A20)}$$

The cross-normalization coefficient

$$N_n^m = \int_A (\boldsymbol{e}_n' \times \boldsymbol{h}_m)\cdot\hat{z}\,da$$

is obtained from (A20) with n and m interchanged:

$$N_{mm} = \int_A (\boldsymbol{e}_m \times \boldsymbol{h}_m) \cdot \hat{z}\, da$$

$$= \frac{-j}{2\beta_m} \left\{ \omega \oint_c \left(\mu_m h_{mz} \frac{d\boldsymbol{h}_m}{d\beta_m} - \epsilon_m \frac{de_{mz}}{d\beta_m} \boldsymbol{e}_m \right) \cdot \hat{n}\, dl + \beta_m \oint_c \left(\frac{dh_{mz}}{d\beta_m} \boldsymbol{e}_m + e_{mz} \frac{d\boldsymbol{h}_m}{d\beta_m} \right) \cdot \hat{r}\, dl \right\} \quad \text{(A21)}$$

where the contour c includes both sides of all the interfaces, and

$$p_n'^2 = \omega^2 \mu_0 \epsilon_n - \beta_n'^2 \quad \text{(A22)}$$

$$p_m^2 = \omega^2 \mu_0 \epsilon_m - \beta_m^2. \quad \text{(A23)}$$

For a fiber with ϵ_1 in the core and ϵ_2 in the cladding, we have

$$N_n^m = -j\pi a \left(\frac{1}{\gamma_n^2 - \alpha_{1m}^2} - \frac{1}{\gamma_n^2 - \alpha_{2m}^2} \right) (\omega\mu_0 h_{nz}' h_{mr} + \beta_m h_{mz} e_{n\phi}' + \beta_n' e_{nz}' h_{m\phi}) + j\pi a \omega \epsilon_0 \left(\frac{\epsilon_1'}{\gamma_n^2 - \alpha_{1m}^2} - \frac{\epsilon_2'}{\gamma_n^2 - \alpha_{2m}^2} \right) e_{mz} e_{nr}' \quad \text{(A24)}$$

$$M_m^n = j\pi a \left(\frac{1}{\gamma_n^2 - \alpha_{1m}^2} - \frac{1}{\gamma_n^2 - \alpha_{2m}^2} \right) (\omega\mu_0 h_{mz} h_{nr}' + \beta_n' h_{nz}' e_{m\phi} + \beta_m e_{mz} h_{n\phi}') - j\pi a \omega \epsilon_3 \left(\frac{1}{\gamma_n^2 - \alpha_{1m}^2} - \frac{\epsilon_1'/\epsilon_2'}{\gamma_n^2 - \alpha_{2m}^2} \right) e_{nz}' e_{mr}(a^-) \quad \text{(A25)}$$

where

$$\gamma_n^2 = \omega^2 \mu_0 \epsilon_3 - \beta_n'^2 \quad \text{(A26)}$$

$$\alpha_{1m}^2 = \omega^2 \mu_0 \epsilon_1 - \beta_m^2 \quad \text{(A27)}$$

$$\alpha_{2m}^2 = \omega^2 \mu_0 \epsilon_2 - \beta_m^2. \quad \text{(A28)}$$

All field components have the angular dependence removed and are evaluated at $r = a$, except $e_{mr}(a^-)$, which denotes the radial electric field of the core at $r = a$.

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