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A SIMPLE SPECTRUM ESTIMATION TECHNIQUE BASED ON THE ANALYTIC CEPSTRUM

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Several papers have been devoted to the alternative maximum entropy method (MEM2) which uses a finite-length cepstrum modelling to estimate the spectrum from a given set of autocorrelations. In this paper, a simple technique that avoids the computational burden of MEM2 by using the causal part of the autocorrelation instead of the complete two-sided sequence is presented. Its finite-length cepstrum model, that arises from the minimization of a root-mean-square measure of spectral distance, can readily include prior information without increasing the computational complexity of the algorithm. As illustrated with some numerical examples, the new method also demonstrates a better performance than MEM2.

1. INTRODUCTION

The hereafter so-called 2nd. maximum entropy method (MEM2) [1,2] analogous approach to the well known maximum entropy method or MEM1 [1]. Their common starting point is a constrained optimization problem where the objective is to maximize a measure of entropy subject to a given set of autocorrelation constraints $\{r_n, n=0,1,...,M\}$. Both MEM1 and MEM2 differ by the entropy measure used, resulting that MEM1 has a rational all-pole model whereas the MEM2 spectral model is

$$S(\omega) = e^{P(\omega)} \qquad (1)$$

where $P(\omega)$ is a trigonometric polynomial of order M. Thus, the spectral estimate obtained by MEM2 is the spectrum $S(\omega)$ that: 1) matches the first M+1 autocorrelations, and 2) agrees with the specific spectral estructure (1). The consequence is, therefore, an extrapolation of the autocorrelation beyond M according to (1). Since $P(\omega)$ is the logarithm of $S(\omega)$, the type of spectra described by (1) have a finite number of non-zero cepstral coefficients, i.e.

$$C(n)=0 , \quad \text{for } |n|>M \qquad (2)$$

where C(n) is the (real) cepstrum corresponding to $S(\omega)$ or, equivalently, the Fourier's series coefficients of $P(\omega)$.

If, along with the autocorrelation values r_n, an a-priori spectrum $S_p(\omega)$ is given, from the constrained maximization problem, it follows the spectral model [2,3]

$$S(\omega) = S_p(\omega)\, e^{P(\omega)} \qquad (3)$$

where the coefficients of $P(\omega)$ are such that the given autocorrelations are matched. Now the cepstrum has two components, i.e.

$$C(n) = C_p(n) + P(n) \qquad (4)$$

where $C_p(n)$ is the (non-finite) cepstrum corresponding to $S_p(\omega)$ and P(n) are the Fourier's series coefficients of $P(\omega)$, which are zero beyond M.

To find the spectral estimate $S(\omega)$, a system of non-linear of equations has to be solved by means , for instance, of an iterative algorithm of Newton-Raphson's type [1,2]. Therefore, MEM2 involves a high computational complexity.

Consequently, there has been some attempts to avoid this drawback [4,6]. First of all, Wu [4] proposed a non-iterative algorithm; however, his estimate really does not match a given set of autocorrelations. Analogously, Liefhebber-Boekee [5] base their various proposed techniques on the application of a rectangular window to the cepstrum so the given autocorrelations are not preserved (excepting the trivial case in which they already correspond to a MEM2 spectrum). Burr-Lytle [6] propose a computationally easy algorithm to solve the problem (unlike the other two papers, their approach considers a prior spectrum as additional information); however, their method actually uses an accurate set of cepstral values instead of autocorrelations.

Thus, all three papers start from the same constrained optimization problem using the MEM2 entropy measure and the autocorrelation constraints and arriving at the MEM2 spectral model (1). Then, they use this model to obtain spectral estimates with simple, non-iterative, algorithms which actually perform different types of cepstral smoothing. As a consequence, their autocorrelation estimates do not match the given values that were used as constraints of the starting approach shared by all three papers. Thus, although all those authors keep the name MEM2 (or other related names) for their techniques, they actually are not maximizing the MEM2 entropy measure but other (not explicit) functionals which give rise to the MEM2 spectral model when their real constraints (different from autocorrelations) are used in the optimization problem. In fact, the MEM2 spectrum is unique [1] and it demands

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solving the system of non-linear equations; any kind of simplification gives rise to a different spectrum.

The aim of this paper is to present a new technique that also avoids the computational drawback of MEM2 but still preserving the known autocorrelation values. It is based on using the cepstrum of the causal part of the autocorrelation instead of the cepstrum of the complete two-sided autocorrelation sequence R(n).

2. THE ANALYTIC SPECTRUM AND CEPSTRUM

Let $R^+(n)$ denote the "causal part" of the autocorrelation R(n), namely

$$\begin{aligned} R^+(n) &= 2R(n) \quad \text{for} \quad n > 0 \\ &= R(n) \quad \text{for} \quad n = 0 \\ &= 0 \quad \text{for} \quad n < 0 \end{aligned} \tag{5}$$

Its Fourier transform $S^+(\omega)$ is

$$S^+(\omega) = S(\omega) + jH(\omega) = |S^+(\omega)|e^{j\theta(\omega)} \tag{6}$$

where $H(\omega)$ denotes the Hilbert transform of $S(\omega)$. $S^+(\omega)$ may be referred to as the "analytic spectrum" [7], in correspondence with the analogous definition used in amplitude modulation. Accordingly, we may define the spectral envelope

$$E(\omega)=|S^+(\omega)| \tag{7}$$

which has already been used for spectral estimation, and the spectral "instantaneous frequency"

$$\tau(\omega) = -d\theta(\omega)/\omega \tag{8}$$

Since $R^+(n)$ is a minimum phase function [8], either $E(\omega)$ or $\tau(\omega)$ specify $S(\omega)$ uniquely and viceversa. Moreover, the poles of $S(\omega)$ appear in both $\tau(\omega)$ and the square envelope $E^2(\omega)$.

The complex cepstrum $C^+(n)$ of $R^+(n)$, i.e. the Fourier's series coefficients of $\log S^+(\omega)$, will be hereafter referred to as the analytic cepstrum. It directly characterizes both functions $\log E(\omega)$ and $\tau(\omega)$, since

$$\log E(\omega) = C^+(0) + \sum_{n=1}^{\infty} C^+(n) \cos n\omega \tag{9}$$

and

$$\tau(\omega) = \sum_{n=1}^{\infty} n\, C^+(n) \cos n\omega \tag{10}$$

On the other hand, due to the causality and minimum phase properties of $R^+(n)$, the analytic cepstrum $C^+(n)$ can be obtained by means of the following recursion [9].

$$\begin{aligned} C^+(n) &= \frac{2}{R(0)}\left[R(n) - \sum_{k=1}^{n-1} \frac{k}{n} C^+(k)R(n-k) \right], \; n > 0 \\ &= \log R(n), \; n = 0 \\ &= 0, \quad n < 0 \end{aligned} \tag{11}$$

where the relationship (5) between $R^+(n)$ and R(n) has been used.

Observe that to compute $C^+(n_1)$, apart from its previous values, only the autocorrelations from lag 0 to lag n_1 are needed. Hence, the first N values of $C^+(n)$ are completely determined by the first N values of R(n) and vice versa.

3. MODELLING OF THE ANALYTIC CEPSTRUM

The problem that will be considered here is the extrapolation of the given set of autocorrelation values r_n, or, equivalently, the estimation of a spectral function real or complex $T(\omega)$ that agrees with them. This function can be the spectrum $S(\omega)$ or another function related with it, provided that this function fully characterizes the autocorrelation R(n).

On the other hand, some additional information about the spectral features of the underlying random process can be, in general, available. Assuming that this prior information is contained in the function $T_p(\omega)$, a reasonable criterion for extrapolating r_n may be to minimize a measure of spectral distance $D(T,T_p)$ between $T(\omega)$ and $T_p(\omega)$. Thus the problem is

$$\begin{aligned} &\text{to minimize} \quad D(T,T_p) \\ &\text{subject to} \quad R(n) = r_n, \quad n=-M,., 0,., M \end{aligned}$$

In our approach, the starting data are a set of autocorrelations r_n. However, according to (11), there exists an one to one correspondence between these first M+1 autocorrelation values and the corresponding M+1 first coefficients of the analytic cepstrum. Therefore, we can compute these coefficients C_n^+ with (11) and use them as the constraints of the optimization problem.

Although different kinds of spectral functions and distances may be considered, we will just concentrate in this paper on one of them. We select the spectral function

$$T(\omega) = \log S^+(\omega) \tag{12}$$

and the root-mean-square distance

$$D_2(T,T_p) = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |T(\omega)-T_p(\omega)|^2 \, d\omega \right]^{1/2}$$

Thus, we aim at

$$\begin{aligned} &\text{minimizing} \quad D_2(\log S^+, \log S_p^+) \\ &\text{subject to} \quad C^+(n) = c_n^+, \; n = 0, \ldots, M \end{aligned}$$

To find the solution of this optimization problem we can equal to zero the derivatives of D_2 (log S^+, log S_p^+) with respect to the unknown cepstral coefficients, resulting

$$C^+(n) = C_p^+(n), \qquad n > M \qquad (13)$$

where $C_p^+(n)$ are the cepstral coefficients of the prior analytic spectrum.

Thus, this very simple algorithm allows both the extrapolation of $C^+(n)$ according to (13) and the estimation of the spectral functions log $E(\omega)$ and $\tau(\omega)$, which can be obtained from $C^+(n)$ by means of the expressions (9) and (10).

On the other hand, the extrapolation of $C^+(n)$ also yields, by means of the inverse relations of (11), an extrapolation of $R(n)$ and therefore an estimation of both $R(n)$ and $S(\omega)$. Note that $E(\omega)$ is always positive since it is obtained from log $E(\omega)$; however the positivity of $S(\omega)$ is not guaranteed.

When a prior log analytic spectrum log $S_p^+(\omega)$ is not available, we can reasonably consider that it is flat, i.e. $\log S_p^+(\omega) = c_0^+$ (the average value of log $S^+(\omega)$). The solution of the minimization problem is then a zero extrapolation, namely,

$$C^+(n) = 0 \ , \ n > M \qquad (14)$$

Hence, the part of the analytic cepstrum corresponding to the given autocorrelations is always of finite length M+1 and it is obtained applying a rectangular window to the analytic cepstrum sequence. If a prior analytic cepstrum exists, it is just appended to the finite length sequence so that the first M+1 values of that prior analytic cepstrum are not used. Thus, if $S_e(w)$ is the exact or true spectrum and $W_M(n)$ is a rectangular window of length M+1, the analytic cepstrum estimated by this method when the given autocorrelation are exact can be written in the form

$$C^+(n) = W_M(n)\, C_e^+(n) + [1-W_M(n)]\, C_p^+(n) \qquad (15)$$

It should be noted the analogy between equations (14) and (2). Consequently, the version in the frequency domain of the spectral model arising from (14) is analogous to (1), namely

$$S^+(\omega) = e^{\,P^+(\omega)} \qquad (16)$$

where $P^+(\omega)$ is a complex polynomial of order M. Thus, the proposed method performs a correct matching of the given autocorrelations and uses a spectral model analogous to the MEM2 model. Additionally, it allows a readily inclusion of a-priori information in form of values of the analytic cepstrum. In this case, there also exist model expressions that are analogous to (3) and (4).

The algorithm is much simpler than the iterative algorithm used to find the MEM2 spectrum and even is simpler than the MEM1 algorithm (using the Levinson-Durbin recursion). Moreover, if a prior spectrum is available, the algorithm is also much simpler than those used for MEM2 or MEM1 [10], which are of iterative type.

4. NUMERICAL EXAMPLES

First of all, let us consider a spectrum of unit power consisting of a peak and a valley with Gaussian shapes and the same logarithmic amplitude, as shown in Fig.1. The spectra (in dB) obtained with the proposed method and MEM2, using 20 exact autocorrelations, are plotted in Fig.2. By comparing these spectra, we observe that the new technique is superior to the MEM2 spectrum in both peak and valley.

A second example will help us to illustrate the resolution capability of the new method and the use of prior information. For this purpose, a different spectrum having two close Gaussian peaks of the same amplitude was selected. The exact spectrum, which also has unit power, is given in Fig.3.

Fig.4 shows the MEM1 spectrum along with the new estimate which was obtained, according to the proposed method, by keeping only the first M+1 coefficients of the analytic cepstrum. In this case, the number of given autocorrelations is M+1=13. Clearly, both methods are capable of resolving the two peaks. However, when the number of autocorrelations is 12, both of them fail to distinguish the peaks.

However, even though the information about the existence of two separate peaks is lost in the spectrum of the proposed technique, it still remains in the spectral function $\tau(\omega)$ which, as observed in section 2, has the same poles than $S(\omega)$. In fact, Fig.5 shows that the peaks are resolved for 12 autocorrelations and even for 11. Also, observe that $\tau(\omega)$ sligthly shifts the peaks, separating one from the other. It is worth noticing that only positive values of $\tau(\omega)$ are considered, since they give the desired information about spectral poles.

Continuing with the same example, let suppose now that the highest frequency peak is previously known. As shown in Fig.6, the proposed technique only needs 8 autocorrelations to resolve the peaks. The corresponding spectral envelope has also been plotted in the same figure to show that it is also able to display the desired information.

5. CONCLUSIONS

The initial goal of the work presented in this paper was to develop a spectral estimation technique based on the same type of model than MEM2 but avoiding its computational complexity. To accomplish this objective, we resorted to an already reported concept: the analytic spectrum. By modeling the analytic spectrum (or, equivalently, its Fourier's series coefficients, namely, the causal part of the autocorrelation sequence) instead of the spectrum itself, we were able of arriving at a very simple algorithm for spectral estimation that, as MEM2, uses a finite-length (analytic) cepstrum modelling.

Due to the one-to-one correspondence between the first M+1 autocorrelation values and the corresponding analytic cepstral coefficients, the proposed method actually consists of applying a rectangular window to the latter and allows a readily inclusion of prior spectral information. Nevertheless,

unlike the other published papers that avoid the iterative algorithm required by MEM2, the spectrum obtained with the presented method actually matches a given set of autocorrelations.

As shown with numerical examples, the new technique gives clear evidence, with both functions $S(\omega)$ and $E(\omega)$, of a performance greater than its homologous MEM2. Moreover, when the interest lies in the spectral poles, the spectral instantaneous frequency $\tau(\omega)$ can be used in lieu of the spectrum, so even achieving more resolution. This fact is a consequence of the high-pass liftering procedure used to obtain $\tau(\omega)$ with (10).

Finally, the proposed algorithm is simpler than the Levinson-Durbin recursion used in MEM1, and additionally the inclusion of prior information does not substantially affect it, unlike MEM1 which requires, in this case, an iterative algorithm.

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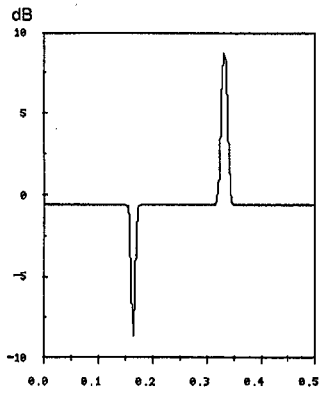

Figure 1

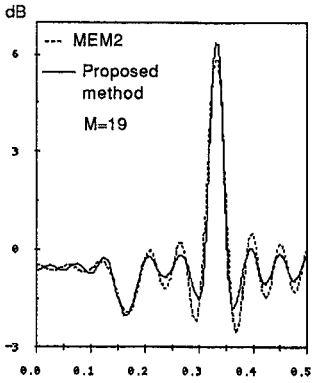

Figure 2

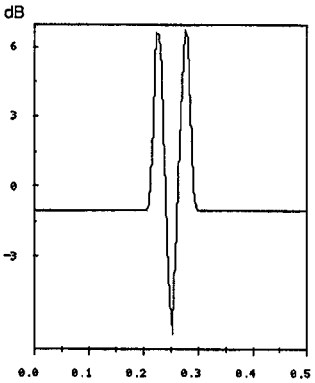

Figure 3

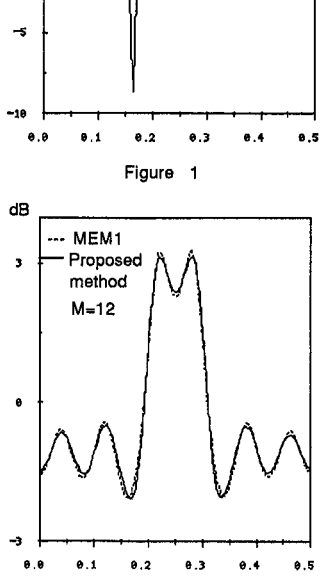

Figure 4

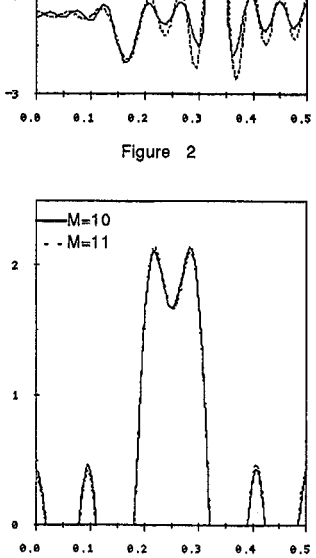

Figure 5

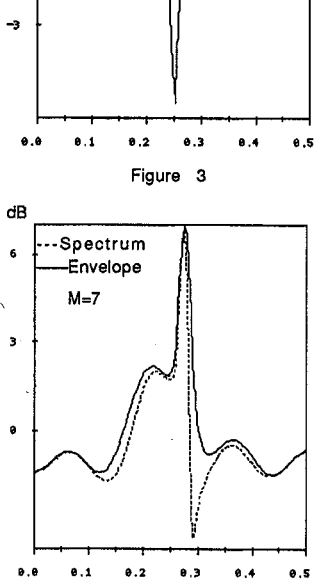

Figure 6