Annals of Operations Research

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--Manuscript Draft--

Manuscript Number:	ANOR-D-14-00775
Full Title:	Power in voting rules with abstention: an axiomatization of a two components power index
Article Type:	Original Research
Keywords:	Decision making process; Voting systems in democratic organizations; Abstention; Power; Axioms
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Power in voting rules with abstention: an axiomatization of a two components power index

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November 17, 2014

Abstract: In order to study voting situations when voters can also abstain and the output can be only binary, i.e., either approval or rejection, a new extended model of voting rule was defined. Accordingly, indices of power, in particular Banzhaf's index, were considered. In this paper we argue that in this context a power index should be a pair of real numbers, since this better highlights the power of a voter in two different cases, i.e., her being crucial when switching from being in favor to abstain, and from abstain to be contrary. We also provide an axiomatization for both indices, and from this a characterization as well of the standard Banzhaf index (the sum of the former two) is obtained. Some examples are provided to show how the indices behave.

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Mathematics Subject Classification (2000) 91A12 · 90B50 · 91A35 · 05C65 · 94C10

*Departament de Matemàtica Aplicada 3 i Escola Politècnica Superior d'Enginyeria de Manresa, Universitat Politècnica de Catalunya. Research partially supported by Grant MTM2012-34426/FEDER "del Ministerio de Economía y Competitividad".

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1 Introduction

In this paper we want to analyze the situation in which several voters must decide whether to approve or not a given proposal. They actually have three options of vote, either to vote “yes”, or to vote “no”, or else to vote for “abstention”. The final outcome of the procedure is dichotomic – the proposal must be either approved or rejected. In this context winning *tripartitions*, i.e., partitions of the set of players N made by three elements suffice to describe the voting situation. Thus if (A, B, C) is such a partition, this quite naturally means that if the voters in A are in favor, in B abstain, and in C are contrary, then the proposal is approved. Analogously to simple voting games, in this more general context minimal winning tripartitions suffice to describe the voting rule; minimal winning in this context can be defined as a winning tripartition such that moving in the right sense just one player makes automatically the new tripartition loosing, i.e., non-winning.

The context of voting rules with abstention is considered in [Felsenthal et al. 1997] and extended to voting rules with several ordered levels of approval in input and output in [Freixas et al. 2003]. In this latter work, the notion of weighted voting rule is settled, while the notion of the desirability relation was recently established in [Freixas et al. 2014b]. The desirability relation entails the idea of power and tries to compare the strength of each pair of voters. As proven in [Freixas et al. 2014b] the only reasonable notion of desirability compatible with the notion of weighted voting rules with abstention has two main components which induce two independent notions of power.

A power index for voting rules with abstention that captures the essence of the Banzhaf power index for simple voting games has been considered in [Felsenthal et al. 1998] and extended to voting rules with multiple levels of approval in [Freixas 2005b], where an axiomatization for the index was provided following the spirit of first Dubey and Shapley’s axiomatization for the Banzhaf index in simple games. The idea of the extended index captures the probability for a player of being crucial in the game. However, as it was already anticipated in [Freixas 2005b] expressing the power of the players by means of a single real number does not explain in which way the voter is crucial. Indeed, some voters can be very decisive in passing proposals, while not so in the defeating of proposals. On the other hand, some players can be decisive in the opposite sense, or some other players can be decisive in a more balanced way.

All these considerations led us to mainly pay attention to the notion of *two components power* for voting rules with abstention rather than to the global notion of just power as a single numerical allocation for each player. To justify this concept with a pair of examples from other and somewhat esoteric comparisons we can for instance think of the usual blood test to check cholesterol. A single number, the total cholesterol present, can bring some information, but it is by far much more meaningful to know both levels of the LDL and HDL cholesteroses. So that the total is just the sum of the two. Or also, if you know the external temperature this helps in dressing you before going to the department, but perceived temperature and the chilling factor (that automatically provide you with the current external temperature), are more interesting information.

Abstention plays a key role in almost all real voting systems that have been modeled by simple voting games such as the United Nations Security Council (UNSC), the most important body in the international political system. The scholars of the period accorded, see e.g. Straffin [Straffin 1982], pp. 314-315, that the Banzhaf ratio of power between a permanent member and a non-permanent member is approximately 10 : 1. Felsenthal and Machover [Felsenthal et al. 1998], remarked that the simple voting game modeling for the UNSC was based on the incorrect assumption that abstention by a permanent member is tantamount to a 'no" vote, having the effect of a veto. Using the more realistic model with abstention, they get a Banzhaf ratio of power of approximately 2 : 1. In this paper we go a step further in this analysis, since we propose a 2-components decomposition of power for voting rules with abstention. Even with abstention it is clear that non-permanent members have no power in defeating proposals, but they play an apparent symmetrical role[1] with permanent members in making proposals to pass. However we will show that this apparent symmetry is not appropriate and even for passing resolutions, permanent members are more powerful than non-permanent members.

The organization of the paper is as follows. In the remaining of this section we formally introduce binary voting rules with abstention. Section 2 introduces the idea of 2-components power and its simple conceptual relation with the Banzhaf extended index for voting rules with abstention. Section 3 introduces some properties of power indices for voting rules with abstention, with the purpose to provide, in Section 4, an axiomatization of the three notions of power introduced in section 2. In Section 5 we show that for weighted voting rules with abstention some properties for power indices that naturally arise for weighted simple games are lost.The conclusion ends the paper.

1.1 The class of $(3, 2)$ voting rules

The material on this section is essentially taken from Freixas and Zwicker [Freixas et al. 2003], where (j, k) voting rules are introduced; here we consider only the case $j = 3$ and $k = 2$. Before introducing the main notions we need some preliminary definitions. An *ordered tripartition* of the finite set N is a triple $S = (S_1, S_2, S_3)$ of mutually disjoint sets whose union is N. Any S_i is allowed to be empty, and we think of S_i as the set of those voters of N who vote approval level i for the issue at hand (where approval level 1 is the highest level of approval, 2 is the intermediate level and 3 the lowest level). The most relevant situation that happens in voting is when S_1 correspond to the set of 'yes' voters, S_2 to the set of abstainers and S_3 to the set of 'no' voters. Thus, an ordered tripartition is the analogue of a coalition for a standard simple game. Let 3^N

[1] If we modelize this example as a weighted game the weights for yes are the same, the details are exposed in Example 1.3-(*iii*).

denote the set of all ordered tripartitions of N. For $S, T \in 3^N$, we write $S \subseteq^3 T$ to mean that either $S = T$ or S may be transformed into T by shifting 1 or more voters to higher levels of approval. This is the same as saying $S_1 \subseteq T_1$ and $S_2 \subseteq T_1 \cup T_2$; we write $S \subset^3 T$ if $S \subseteq^3 T$ and $S \neq T$. The $\subseteq^3$ order defined on 3^N has minimum: the tripartition $\mathcal{N}$ such that $\mathcal{N}_3 = N$, and maximum: the tripartition $\mathcal{M}$ such that $\mathcal{M}_1 = N$; i.e., for every tripartition S, $\mathcal{N} \subseteq^3 S \subseteq^3 \mathcal{M}$ holds.

Definition 1.1 *A* $(3, 2)$ simple game $G = (N, V)$ *(henceforth* $(3, 2)$ *game) consists of a finite set* N *of voters together with a value function* $V : 3^N \longrightarrow \{1, 0\}$*, with the identifications* $1 = win > lose = 0$*,* $V(\mathcal{N}) = 0$*,* $V(\mathcal{M}) = 1$ *and which is monotonic: for all ordered tripartitions* $S \subseteq^3 T$ *then* $V(S) \leq V(T)$*.*

A $(3, 2)$ game is also defined by the set of *winning tripartitions* W, and it satisfies the monotonicity requirement: if $S \subset^3 T$ and $S \in W$ then $T \in W$.
$\mathfrak{S}_N$ will denote the class of $(3, 2)$ games on the players set N.

Standard notions for coalitions in simple games naturally extend for tripartitions in (3,2) games: S is a *losing* tripartition whenever $V(S) = 0$, L denotes the set of losing tripartitions, S is a *minimal* winning tripartition provided that S is winning and that T is a losing for each $T \subset^3 S$. The set of maximal losing tripartitions is analogously defined. It is clear that W and L form a bipartition of 3^N, and that each of the sets: W, L, W^m, and L^m uniquely determine the $(3, 2)$ game, where W^m, and L^m denote the set of minimal winning and maximal loosing tripartitions.

Definition 1.2 *Let* $G = (N, V)$ *be a* $(3, 2)$ *game. A* representation of G as a weighted $(3, 2)$ game *consists of a triple* $w = (w_1, w_2, w_3)$ *of* 3 *weight functions, where* $w_i : N \to \mathbb{R}$ *for each* i *and the weight functions satisfy the additional weight-monotonicity requirement that for each* $p \in N$*,* $w_1(p) \geq w_2(p) \geq w_3(p)$*, together with a real number quota* Q *such that for every tripartition* S*,* $V(S) = 1$ if and only if $w(S) \geq Q$*, where* $w(S)$ *denotes*

$$\sum_{i=1}^{3} \sum_{p \in S_i} w_i(p).$$

We say that $G = (N, V)$ *is a weighted* $(3, 2)$ *game if it has such a representation.*

As was observed in [Freixas et al. 2003], each 'yes' voter p contributes the weight $w_1(p)$ to the total weight H; each abstainer p contributes $w_2(p)$ to H, and each 'no' voter p contributes $w_3(p)$ to H, with the issue passing exactly if H meets or exceeds some preset quota Q. That is, before any voting takes place each voter is pre-assigned three weights with $w_1(p) \geq w_2(p) \geq w_3(p)$ for each voter p. As occurs for simple games where two weights represent superfluous information, three weights represent superfluous information. If we renormalize by subtracting $w_2(p)$ from each of the weights $w_1(p)$, $w_2(p)$ and $w_3(p)$ then the new triple of weights $w^+(p) = w_1(p) - w_2(p)$, 0, and $w^-(p) = w_3(p) - w_2(p)$ describes the same voting system, and satisfies $w^+(p) \geq 0 \geq w^-(p)$. As a two components weight is enough for a $(3, 2)$ weighted representation of a weighted game, it is intuitive that a two components vector of power might be enough to explain power of voters in a $(3, 2)$ game. We now describe some motivating examples.

Example 1.3 (i) Let $N = \{a, b, c\}$ be the set of players, $Q = 1$ the quota for a weighted game with abstention, with respective weights $w(a) = (1, 0, 0)$, $w(b) = (1, 0, 0)$, and $w(c) = (0, 0, -2)$. If $Q = 1$, then (we omit brackets for the sake of getting simpler notation) $W^m = \{(a, c, b), (b, c, a)\}$. In this game players a and b play symmetrical roles and each one of them can make the proposal pass if c does not cast a negative vote.Note that players a and b nothing can do to make the proposal pass if c casts a negative vote. It seems obvious that players a and b are equally powerful, however who is more powerful between a and c?

(ii) $N = \{a, b, c\}$, $Q = 1$, $w(a) = (2, 0, 0)$, $w(b) = (1, 0, -1)$, and $w(c) = (0, 0, -2)$. If $Q = 1$, then $W^m = \{(ab, \emptyset, c), (a, c, b), (b, c, a)\}$. Player c does not have any influence to pass the proposal at hand, while player a cannot force the failure of the proposal if the other two make it to pass. Player b is a middle on the road player and seems to have influence in both sides. In this voting situation we wonder: who is the most powerful voter?

Suppose that in this example we rise the quota from $Q = 1$ to $Q = 2$, or to $Q = 3$. Who is the most powerful voter in all these contexts?

(iii) A resolution is carried in the Security Council if at least nine of its fifteen members support it and no permanent member of the five is explicitly opposed. The formal description of the UNSC as a $(3, 2)$ game is as follows: let P and R be respectively the set of permanent members and nonpermanent members, and

$$V(S) = V(S_1, S_2, S_3) = \begin{cases} 1 & \text{if } |S_1| \geq 9 \text{ and } S_3 \cap P = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

This voting situation can be represented (see [Freixas et al. 2003]) by $Q = 9$, $w(p) = (1, 0, -6)$ for a permanent member p of the Council and $w(r) = (1, 0, 0)$ for a non-permanent member r of the Council. It is obvious that the permanent members are more powerful than non-permanent members. However, is a permanent member more powerful than a non-permanent member in making a proposal to pass? This is questionable.

We observe that Examples 1.3 are weighted $(3, 2)$ games. Later on, in Section 5, we shall revisit the examples with the help of the index we introduce.

2 Towards a two components power index

To denote a $(3, 2)$ simple game, in what follows we shall use the notation (N, V) or simply V if the set of players is clearly specified. We now introduce some further notation for tripartitions. Given (S_1, S_2, S_3) a tripartition such that $p \notin S_3$, we define:

$$S_{\downarrow p} = \begin{cases} (S_1 \setminus \{p\}, S_2 \cup \{p\}, S_3) & \text{if} \quad p \in S_1 \\ (S_1, S_2 \setminus \{p\}, S_3 \cup \{p\}) & \text{if} \quad p \in S_2 \end{cases}$$

and if $p \in S_1$

$$S_{\downarrow\downarrow p} = (S_1 \setminus \{p\}, S_2, S_3 \cup \{p\}).$$

Analogous notation can be defined when a player p is moved in the left direction. Thus, given (S_1, S_2, S_3) with $p \notin S_1$, we define:

$$S_{\uparrow p} = \begin{cases} (S_1 \cup \{p\}, S_2 \setminus \{p\}, S_3) & \text{if} \quad p \in S_2 \\ (S_1, S_2 \cup \{p\}, S_3 \setminus \{p\}) & \text{if} \quad p \in S_3 \end{cases}$$

and if $p \in S_3$

$$S_{\uparrow\uparrow p} = (S_1 \cup \{p\}, S_2, S_3 \setminus \{p\}).$$

Definition 2.1 *Let $V \in \mathfrak{S}_N$, we say that for any tripartition $S \in 3^N$:*

1. *$p \in S_1$ is a* YA-(down) swing *in S if $V(S) = 1$ but $V(S_{\downarrow p}) = 0$. Analogously, $p \in S_2$ is a* YA-(up) swing *in S if $V(S) = 0$ but $V(S_{\uparrow p}) = 1$.*
2. *$p \in S_2$ is a* AN-(down) swing *in S if $V(S) = 1$ but $V(S_{\downarrow p}) = 0$. Analogously, $p \in S_3$ is a* AN-(up) swing *in S if $V(S) = 0$ but $V(S_{\uparrow p}) = 1$.*
3. *$p \in S_1$ is a* YN-(down) swing *in S if $V(S) = 1$ but $V(S_{\downarrow\downarrow p}) = 0$. Analogously, $p \in S_3$ is a* YN-(up) swing *in S if $V(S) = 0$ but $V(S_{\uparrow\uparrow p}) = 1$.*

Since the number of X-down swings equals the number of the X-up swings, where X stands for either YA, AN or YN, we simply consider from now on down swings and call them simply *swings*.
Let us denote $\eta_p^{YA}[V]$, $\eta_p^{AN}[V]$, $\eta_p^{YN}[V]$ respectively the number of swings of each type for an arbitrary player $p \in N$.

Banzhaf's $(3, 2)$ extension Penrose [Penrose 1946] and Banzhaf [Banzhaf 1965] independently considered a very well recognized index for simple games. What is known as Banzhaf's 'raw' extended power index for a voter $p \in N$ in a $(3, 2)$ game $V \in \mathfrak{S}_N$ is defined in [Felsenthal et al. 1997] (see also [Felsenthal et al. 1998]) and extended to (j, k) games in [Freixas 2005b] as

$$\eta_p[V] = \eta_p^{YA}[V] + \eta_p^{AN}[V].$$

The raw extended Banzhaf index $\eta_p[V]$ counts the number of winning tripartitions in which p is a swing descending one single level of approval.
By observing that

$$|\{S : p \in S_2,\ V(S) = 1,\ V(S_{\downarrow p}) = 0\}| = |\{S : p \in S_1,\ V(S_{\downarrow p}) = 1,\ V(S_{\downarrow\downarrow p}) = 0\}|,$$

we see that $\eta_p[V]$ is nothing more than $\eta_p^{YN}[V]$ and thus

$$\eta_p^{YN}[V] = \eta_p^{YA}[V] + \eta_p^{AN}[V]. \tag{1}$$

The extended Banzhaf index to games with abstention, $I_p[V]$, can be directly interpreted as the probability of being decisive when the player is voting at the "yes"-level. In fact,

$$I_p[V] = \frac{\eta_p[V]}{\text{number of tripartitions with } p \in S_1} = \frac{\eta_p[V]}{3^{|N|-1}} \tag{2}$$

The next definition naturally introduces some differentiated indices, two of them I^{YA} and I^{AN} capture our idea of two components power on which we want to deepen in this paper. The third index is nothing else than index I.

Definition 2.2 *(X-component power). Consider the three power measures:*

$$I_p^X[V] = \frac{\eta_p^X[V]}{3^{|N|-1}} \tag{3}$$

where X stands for either YA, AY, and YN.

Note that I^{YN} is the extended Banzhaf index for voting rules with abstention. Due to (1) it is clear that only two of these indices really matter. Thus, we choose I^{YA} and I^{AN} and just regard to I^{YN} as the amalgamation of the two former indices. We do claim that the pair of numbers (I_p^{YA}, I_p^{AN}) better captures the idea of power for a player in a game with abstention. Due to (1), I_p^{YN} can be regarded as a 'total power' property for a player, that does not distinguish on the different types of being crucial in the game.

3 Definitions and preliminaries for games with abstention

In this section we introduce some further material that will be needed for the main result of the paper, i.e., an axiomatization of the two components power index. In particular we introduce the definitions necessary to establish our axioms.

Definition 3.1 *Let $S \neq \mathcal{N}$ be a tripartition, the S-*unanimity game *(N, U_S) is the game whose only minimal winning tripartition is S.*

Note that the unanimity games are the only games with a single minimal winning tripartition. Other games have at least two.

The next definition focuses on dummy players, i.e., those players whose marginal contribution play extreme roles.

Definition 3.2 *Let $V \in \mathfrak{S}_N$, voter $p \in N$ is called a:*

1. *YA-dummy if $V(S) - V(S_{\downarrow p}) = V(p, \emptyset, N \setminus p) - V(\emptyset, p, N \setminus p)$ for all $S \in 3^N$ with $p \in S_1$.*

2. *AN-dummy if $V(S) - V(S_{\downarrow p}) = V(\emptyset, p, N \setminus p)$ for all $S \in 3^N$ with $p \in S_2$.*

3. *YN-dummy if $V(S) - V(S_{\downarrow\downarrow p}) = V(p, \emptyset, N \setminus p)$ for all $S \in 3^N$ with $p \in S_1$.*

A remark and some immediate consequences.

- We distinguish between two different types for $p \in N$ of being X-dummy for each $X = YA$, AN, YN:
  1. p is an X-*null* player if the right-hand side expression in Definition 3.2 is 0.
  2. p is an X-*dictator* if the right-hand side expression in Definition 3.2 is 1.
- Player $p \in N$ in the $(3, 2)$ game is
  1. YN-null if and only if it is both YA-null and AN-null.
  2. YN-dictator if and only if p is either YA-dictator and AN-null or is YA-null and AN-dictator.
- If a game V has an X-dictator, the rest of players are X-nulls.

The next definition concerns operations on $(3, 2)$ games.

Definition 3.3 *Given $V_1, V_2 \in \mathfrak{S}_N$:*

1. *$V_1 \vee V_2$ is the game such that $(V_1 \vee V_2)(S) = \max\{V_1(S), V_2(S)\}$*

2. *$V_1 \wedge V_2$ is the game such that $(V_1 \wedge V_2)(S) = \min\{V_1(S), V_2(S)\}$*

Thus, in order to win in $V_1 \vee V_2$, a tripartition must win in either V_1 or in V_2, whereas to win in $V_1 \wedge V_2$, a tripartition must win in both V_1 and V_2.

Note that $\wedge$ is a closed operation inside the set of unanimity games. Indeed, $U_R \wedge U_S = U_T$ where $p \in T_i$ if $i = \min\{j, k\}$, $p \in R_j$, and $p \in S_k$ where $j, k \in \{1, 2, 3\}$.

The two next definitions give two different ways to reduce a game when two players decide to vote together as a single one. In the first version the merge is produced at the highest level of approval (of the two involved players) for each tripartition, while in the second version the merge is produced as close as possible to the abstention level for each tripartition.

Starting from a given set N of players and a given $(3, 2)$ game (N, V), the idea is to construct a new set of players N' and two new $(3, 2)$ games (N', V') and (N', V'') in the following way. We imagine that there are two players, say p, r in N and we imagine that player r merges in p to obtain the new set of players $N' = (N \setminus \{p, r\}) \cup \{pr\}$. The two next definitions capture two possible ways for the merge of players p and r.

Definition 3.4 *Let $V \in \mathfrak{S}_N$. The* voting game at the highest level of approval *is the pair (N', V') such that a winning tripartition S' in game V' is constructed by a winning tripartition S in game V in the following way:*

1. *if either p or r belong to S_1, then*
$$S_1' = (S_1 \setminus \{p,r\}) \cup \{pr\}, \quad S_2' = S_2 \setminus \{p,r\}, \quad S_3' = S_3 \setminus \{p,r\},$$
2. *if both p and r belong to S_3, then*
$$S_1' = S_1, \quad S_2' = S_2, \quad S_3' = (S_3 \setminus \{p,r\}) \cup \{pr\},$$
3. *if either p or r belong to S_2 but neither p nor r belong to S_1, then*
$$S_1' = S_1, \quad S_2' = (S_2 \setminus \{p,r\}) \cup \{pr\}, \quad S_3' = S_3 \setminus \{p,r\}.$$

Definition 3.5 *Let $V \in \mathfrak{S}_N$. The* voting game at the intermediate level of approval *is the pair (N', V'') such that a winning tripartition S'' in game V'' is constructed by a winning tripartition S in game V in the following way:*

1. *if both p and r belong to S_1, then*
$$S_1'' = (S_1 \setminus \{p,r\}) \cup \{pr\}, \quad S_2'' = S_2, \quad S_3'' = S_3,$$
2. *if both p and r belong to S_3, then*
$$S_1'' = S_1, \quad S_2'' = S_2, \quad S_3'' = (S_3 \setminus \{p,r\}) \cup \{pr\},$$
3. *if either p or r belong to S_2 then*
$$S_1'' = S_1 \setminus \{p,r\}, \quad S_2'' = (S_2 \setminus \{p,r\}) \cup \{pr\}, \quad S_3'' = S_3 \setminus \{p,r\}.$$

The two reduced games are well-defined, in the sense that they satisfy the requirements for a $(3,2)$ game in Definition 1.1.

Note that a relationship for the $(3,2)$ games (N', V') and (N', V'') obtained from (N, V) is: $V'(S) \leq V''(S)$ for all $S \in 3^{N'}$. Thus, (N', V') is more stringent than (N', V'') in making collective proposals to win. For instance, assume $N = \{p, r\}$ and $V = U_{(p,r,\emptyset)}$ then $N' = \{pr\}$ and $V' = U_{(pr,\emptyset,\emptyset)}$ while $V'' = U_{(\emptyset,pr,\emptyset)}$.

3.1 Axioms

We are now in a position to present sets of three axioms suitable to single out I_p^{YA} and I_p^{AN} appearing in (1) as the power indices $P_p : \mathfrak{S}_N \to \mathbf{R}_+$ for the family $(3,2)$ games (with N as set of players).[2] In what follows we shall use X for a choice between YA

[2]In the Conclusion section we shall say something about axiomatization of I_p^{YN}.

and AN, whenever nothing changes when using one of them rather the other one. The first axiom shows that the sum of powers of voter p in the games $V_1 \vee V_2$ and $V_1 \wedge V_2$ is equal to the sum of powers in V_1 and V_2, so that power is transferred in these games. This is an adaptation, to our context, of the same well known axiom in the context of classical simple games. This axiom (and the second one to be introduced later) was already considered for $(3,2)$ games in [Freixas 2005a] and in [Freixas 2005b].

Axiom 1 *(Transfer) For $V_1, V_2 \in \mathfrak{S}_N$,*

$$P_p[V_1 \vee V_2] + P_p[V_1 \wedge V_2] = P_p[V_1] + P_p[V_2]. \tag{4}$$

The idea of the next axiom, which makes a specification about maximal and minimal power specification in different situations X, i.e., a X-dictator should posses maximum power in the game as she can be characterized as the only non-X-null voter. On the other hand X-null voters should have minimum power with respect to non-X-null voters. These two extreme measures of power are quantified here by 1 and 0 respectively, but arbitrary choices could be considered instead.

Axiom 2 *(Extreme power specification): Let $V \in \mathfrak{S}_N$. If the voter p is:*

(i) either a YA-null player or a YA-dictator in V, then

$$P_p[V] = V(p, \emptyset, N \setminus p) - V(\emptyset, p, N \setminus p), \tag{5}$$

(ii) either an AN-null player or an AN-dictator in V, then

$$P_p[V] = V(\emptyset, p, N \setminus p), \tag{6}$$

(iii) either a YN-null player or a YN-dictator in V, then

$$P_p[V] = V(p, \emptyset, N \setminus p). \tag{7}$$

Axiom 2 can be regarded as a normalization axiom in which the two bounds for extreme power are specified.

The third axiom shows the relationship between the power of a member and the power of that member after forming a block in each of the two reduced games considered in definitions 3.4 and 3.5.

Axiom 3 *(Individual block effect) Let $V = U_S \in \mathfrak{S}_N$.*

(i) Let $V' \in \mathfrak{S}_{N'}$ be the $(3,2)$ game obtained from V when the voters $p, r \in N$ form a block pr at the highest level of approval. Suppose moreover $p \in S_1$. Then

1. *if $r \in S_1$, then $P_{pr}[V'] = 3P_p[V]$,*
2. *if $r \in S_2$, then $P_{pr}[V'] = (3/2)P_p[V]$,*

3. *if* $r \in S_3$*, then* $P_{pr}[V'] = P_p[V]$.

(*ii*) *Let* $V'' \in \mathfrak{S}_{N'}$ *be the* $(3,2)$ *game obtained from game* $V \in \mathfrak{S}_N$ *above, when the voters* $p, r \in N$ *form a block* pr *at the intermediate level of approval, where* $V = U_S$ *and* $p \in S_2$*. Then*

1. $r \in S_1$*, then* $P_{pr}[V''] = 3P_p[V]$,
2. $r \in S_2$*, then* $P_{pr}[V''] = (3/2)P_p[V]$,
3. $r \in S_3$*, then* $P_{pr}[V''] = P_p[V]$.

(*iii*) *Let* $V''' \in \mathfrak{S}_{N'}$ *be the* $(3,2)$ *game obtained from game* $V \in \mathfrak{S}_N$ *above, when the voters* $p, r \in N$ *form a block* pr*: at the highest level of approval if* $p \in S_1$*, and at the intermediate level of approval if* $p \in S_2$*. Then*

1. $r \in S_1$*, then* $P_{pr}[V'''] = 3P_p[V]$,
2. $r \in S_2$*, then* $P_{pr}[V'''] = (3/2)P_p[V]$,
3. $r \in S_3$*, then* $P_{pr}[V'''] = P_p[V]$.

For instance, forming a block at the highest level (the same for the intermediate level) between $p \in S_1$ and $r \in N$ in the S-unanimity game U_S means for p a collective gain of the triple[3] if r is also a yes-voter in S, a collective gain of just 1.5 times[4] if r is an abstainer in S, while no gain is obtained if r is a no-voter in S.

Note that the action of power P_p over the S-unanimity game U_S as defined in A3-(i) coincides with the action of power as defined in A3-(iii) whenever $p \in S_1$. Similarly, the action of power P_p over the S-unanimity game U_S as defined in A3-(ii) coincides with the action of power as defined in A3-(iii) whenever $p \in S_2$.

4 A characterization theorem

In this section we state the main results of the paper. They are axiomatic characterizations of the indices.

Theorem 4.1 *A power index* P_p *satisfies axioms* A1*,* A2-(*i*)*, and* A3-(*i*) *if and only if* P_p *is the index* I_p^{YA} *in* (3).

Theorem 4.2 *A power index* P_p *satisfies axioms* A1*,* A2-(*ii*)*, and* A3-(*ii*) *if and only if* P_p *is the index* I_p^{AN} *in* (3).

[3]Note that for the Banzhaf index for simple games the gain is just the double. The multiplier effect is 2 for simple games which coincides with the levels of approval 2, while for games with abstention the multiplier effect is 3 which coincides with number of levels of approval. For $(j,2)$ games the multiplier effect would be j.

[4]i.e., the half of the number of input levels.

We are going to prove only the first theorem because, essentially, the same proof can be given to the second one.

Proof. We will first show that I_p^{YA} satisfies axioms A1, A2-(i), and A3-(i). Let us start by seeing A1. We only need to prove

$$\eta^{YA}[V_1 \vee V_2] + \eta^{YA}[V_1 \wedge V_2] = \eta^{YA}[V_1] + \eta^{YA}[V_2]. \tag{8}$$

Consider the following sets of tripartitions:

$$\begin{array}{rcl} X & = & \{S \in 3^N \,:\, p \in S_1\} \cap \{S \in W_1 \setminus W_2,\ S_{p\downarrow} \notin W_1\}, \\ Y & = & \{S \in 3^N \,:\, p \in S_1\} \cap \{S \in W_2 \setminus W_1,\ S_{p\downarrow} \notin W_2\}, \\ Z_1 & = & \{S \in 3^N \,:\, p \in S_1\} \cap \{S \in W_1 \cap W_2,\ S_{p\downarrow} \notin W_1,\ S_{p\downarrow} \in W_2\}, \\ Z_2 & = & \{S \in 3^N \,:\, p \in S_1\} \cap \{S \in W_1 \cap W_2,\ S_{p\downarrow} \in W_1,\ S_{p\downarrow} \notin W_2\}, \\ Z_3 & = & \{S \in 3^N \,:\, p \in S_1\} \cap \{S \in W_1 \cap W_2,\ S_{p\downarrow} \notin W_1,\ S_{p\downarrow} \notin W_2\}. \end{array}$$

Clearly,

1. the sets X, Z_1, and Z_3 form a partition of the set of YA-swings of p in V_1,
2. the sets Y, Z_2, and Z_3 form a partition of the set of YA-swings of p in V_2,
3. the sets X, Y, Z_1, Z_2, and Z_3 form a partition of the set of YA-swings of p in $V_1 \vee V_2$,
4. the set Z_3 coincides with the set of YA-swings of p in $V_1 \wedge V_2$.

To check that I_p^{YA} satisfies A2-(i), note that if p is YA-dictator it means (see Definition 3.2-1) that $V = U_{(p,\emptyset,N\setminus p)}$ which implies $\eta_p^{YA}[V] = 3^{n-1}$ and from (3) $I_p^{YA}[V] = 1$, while r is an AY-null player if $r \neq p$ and therefore $\eta_r^{YA}[V] = I_r^{YA}[V] = 0$.

Note that if p is a YA-null player it means (see Definition 3.2-1) that $p \notin S_1$ for all tripartitions $S \in W^m$ which implies $\eta_p^{YA}[V] = I_p^{YA}[V] = 0$.

Finally, to verify A3-(i) for I_p^{YA}, let $G = (N, V)$ be such that $V = U_S$ and $p \in S_1$. Let $G' = (N', V')$ be the $(3, 2)$ game introduced in Definition 3.4-1.

The set of tripartitions in which p is a YA-swing for V is $\mathcal{P} = \{T \in 3^N \,:\, S \subseteq^3 T\}$, i.e., tripartitions in which all players vote either in the same level of approval or in a higher level. Let $|S_i| = s_i$ for $i = 2, 3$ and $|N| = n$, then $\eta_p^{YA}[V] = |\mathcal{P}|$ and as $|\mathcal{P}| = 2^{s_2} \cdot 3^{s_3}$, it follows $I_p^{YA}[V] = 2^{s_2} \cdot 3^{s_3+1-n}$.

Consider now $G' = (N', V')$ which is the S'-unanimity game $V' = U_{S'}$ with S' given as in Definition 3.4-1 with $pr \in S'_1$. We can perform similar computations to obtain:

$$\eta_{pr}^{YA}[V'] = \begin{cases} 2^{s_2} \cdot 3^{s_3} & \text{if } r \in S_1 \\ 2^{s_2-1} \cdot 3^{s_3} & \text{if } r \in S_2 \\ 2^{s_2} \cdot 3^{s_3-1} & \text{if } r \in S_3 \end{cases}$$

Hence,

$$I_{pr}^{YA}[V'] = \begin{cases} 2^{s_2} \cdot 3^{s_3+2-n} & \text{if } r \in S_1 \\ 2^{s_2-1} \cdot 3^{s_3+2-n} & \text{if } r \in S_2 \\ 2^{s_2} \cdot 3^{s_3+1-n} & \text{if } r \in S_3 \end{cases}$$

and A3-(i) is verified.

Let us now prove that if a power index P_p that satisfies properties $A1$, $A2$-(i), and $A3$-(i), then it must be I_p^{YA}. First observe that any P_p is uniquely determined by its action on unanimity games. Indeed, suppose the set of minimal winning tripartitions in V is $W^m = \{S_1, \dots, S_r\}$. Then we can write

$$V = U_{S_1} \vee U_{S_2} \vee \cdots \vee U_{S_r},$$

and by A1 the conclusion follows. Now, let us proceed by induction on the number r of winning tripartitions. If $k = 1$ then V is itself a unanimity game and so the conclusion is obvious. If $k > 1$, then V can be written as

$$V = U \vee U_{S_k},$$

where $U = U_{S_1} \vee U_{S_2} \vee \cdots \vee U_{S_{k-1}}$. Therefore, by the distributive law:

$$U \wedge U_{S_k} \quad = \quad (U_{S_1} \wedge U_{S_k}) \vee \cdots \vee (U_{S_{k-1}} \wedge U_{S_k}) \quad = \quad U_{T_1} \vee \cdots \vee U_{T_{k-1}}$$

where T_i $(i = 1, \dots, k-1)$ is the tripartition obtained from S_i and S_k as described in Definition 3.3. According to the transfer axiom, it follows that

$$\begin{aligned} P_p[V] &= P_p[U] + P_p[U_{S_k}] - P_p[U \wedge U_{S_k}] \\ &= P_p[U_{S_1} \vee \cdots \vee U_{S_{k-1}}] + P_p[U_{S_k}] - P_p[U_{T_1} \vee \cdots \vee U_{T_{k-1}}] \end{aligned}$$

so that the claim readily follows from the inductive hypothesis. Thus I_p^{YA} is uniquely determined by its values on unanimity games. Let us obtain these $3^n - 1$ values by induction on the number of players $|N| = n$. Assume $|N| = 1$ and $p \in N$, then there are just two (unanimity) games: $U_{(p,\emptyset,\emptyset)}$ and $U_{(\emptyset,p,\emptyset)}$; in the former case p is a YA-dictator, while in the second one p is a YA-null. Thus by A2-(i)

$$P_p^{YA}(U_{(p,\emptyset,\emptyset)}) = 1 = 1/3^{|N|-1} \tag{9}$$

and

$$P_p^{YA}(U_{(\emptyset,p,\emptyset)}) = 0. \tag{10}$$

Hence, P^{YA} is completely determined when $|N| = 1$ and it coincides with I^{YA}.

So assume $|N| > 1$. Let $p \neq r$ be two voters in N, consider $V = U_S$ an arbitrary S-unanimity game and the game (N', V') as defined in Definition 3.4. Assume:

1. $\{p, r\} \cap S_1 = \emptyset$. In this case, both p and r are YA-null voters and therefore by A2-(i), $P_p^{YA}[V] = P_r^{YA}[V] = 0 \; \forall p, r \notin S_1$.

2. $\{p,r\}\cap S_1 \neq \emptyset$. Then assume, w.l.o.g., that $p\in S_1$. From A3-(i) it follows that:

   (a) if $r\in S_1$, then $P_p^{YA}[V]=(1/3)P_{pr}^{YA}[V']$,

   (b) if $r\in S_2$, then $P_p^{YA}[V]=(2/3)P_{pr}^{YA}[V']$,

   (c) if $r\in S_3$, then $P_p^{YA}[V]=P_{pr}^{YA}[V']$.

   Thus by the inductive assumption the right-hand side expressions are determined, and also $P_r^{YA}[V]=(1/3)P_{pr}^{YA}[V']$ when $r\in S_1$. Thus,

$$P_p^{YA}[V]=P_r^{YA}[V] \quad \forall\, p,r\in S_1$$

   and

$$P_p^{YA}[V]=P_r^{YA}[V]=0 \quad \forall\, p,r\in S_2\cup S_3.$$[5]

This ends the proof. □

Remark 4.3 *1. Note that equations* (9) *and* (10) *for one-player* $(3,2)$ *games derived from A2-(i) have analogous versions if, instead, we consider A2-(ii) or A2-(iii) respectively. By A2-(ii),*

$$P_p^{AN}(U_{(p,\emptyset,\emptyset)})=0, \tag{11}$$

and

$$P_p^{AN}(U_{(\emptyset,p,\emptyset)})=1=1/3^{|N|-1}. \tag{12}$$

Hence, P^{AN} *is completely determined when* $|N|=1$ *and it coincides with* I^{AN}*.*
By A2-(iii)

$$P_p^{YN}(U_{(p,\emptyset,\emptyset)})=1=1/3^{|N|-1}, \tag{13}$$

and

$$P_p^{YN}(U_{(\emptyset,p,\emptyset)})=1=1/3^{|N|-1}. \tag{14}$$

Hence, P^{YN} *is completely determined when* $|N|=1$ *and it coincides with* I^{YN}*.*

It follows that I_p^{YN} over unanimity games is given by

$$I_p^{YN}[U_S]=\begin{cases} 2^{s_2}\cdot 3^{s_3+1-n} & \text{if } \ p\in S_1\\ 2^{s_2-1}\cdot 3^{s_3+1-n} & \text{if } \ p\in S_2\\ 0 & \text{otherwise}\end{cases}$$

We conclude by showing that all axioms are needed to identify the indices. We only prove the result for the YA-version since the result for the AN-version is, *mutatis mutandis*, the same.

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[5]This shows that the null-axiom and the reduced axiom at the highest level imply YA-equal treatment on unanimity games, which by transfer can be extended to all games. YA-equal treatment for the $(3,2)$ game V and players $p,r\in N$ means that $P_p^{YA}[V]=P_r^{YA}[V]$ whenever $V(S)-V(S_{p\downarrow})=V(S)-V(S_{r\downarrow})$ for all tripartitions $S\in 3^N$ with $p,r\in S_1$. Analogously, one may consider YN- and AN-equal treatment with the corresponding implications.

Theorem 4.4 *Axioms A1, A2-(i), and A3-(i) are independent.*

Theorem 4.5 *Axioms A1, A2-(ii), and A3-(ii) are independent.*

Proof. The index given by

$$I_p[V] = \begin{cases} \dfrac{\eta_p^{YA}[V]}{3^{|N|-1}} & \text{if } V = U_S \text{ for some tripartition } S \\ \left(\dfrac{\eta_p^{YA}[V]}{3^{|N|-1}}\right)^2 & \text{if } V \neq U_S \end{cases}$$

where $|N| > 1$, is nonlinear in η_p^{YA}, and thus it fails to satisfy the transfer axiom A1, but it satisfies A2-(i) and A3-(i).

Since the index

$$J_p[V] = k \cdot I_p^{YA}[V],$$

for some $k \geq 0$, $k \neq 1$, is not appropriately normalized, it violates A2-(i) whenever p is a YA-dictator, while it satisfies A1 and A3-(i) and A2-(i) for null players.

The index

$$K_p[V] = I_p^{YN}[V]$$

satisfies A1 and A3-(i). Moreover, if p is a YA-dictator then p is a YN-dictator as well. Hence, K satisfies A2-(i) for dictators, whereas if p is a YA-null player, p is not necessarily a YN-null player and therefore A2-(i) fails for null players.

Finally, let V be a $(3, 2)$ game, consider the power index B for $(3, 2)$ games that applies to V in the following way. Consider for V an associated simple game v with the same set of players N which is uniquely determined by the winning coalitions as follows. Coalition $S_1 \neq \emptyset$ is winning in v if and only if tripartition $S = (S_1, N \setminus S_1, \emptyset)$ is winning in V, and $v(\emptyset) = 0$. The simple game (N, v) is well-defined since: $v(N) = 1$, $v(\emptyset) = 0$, and v is monotonic. If Bz denotes the Banzhaf index for simple games, we take

$$B_p[V] = Bz_p[v].$$

B satisfies the transfer axiom since Bz for simple games does so. If p is a YA-null player in V it means that $p \notin S_1$ for all $S \in W^m$, hence p does not belong to any minimal winning coalition in v as well. Thus, $Bz_p[v] = 0$ and therefore $B_p[V] = 0$.

If p is a YA-dictator it means that $V = U_{(p,\emptyset,N\setminus p)}$, hence p is a dictator for v as well. Thus, $Bz_p[v] = 1$ and therefore $B_p[V] = 1$.

However, B does not satisfy A3-(i) since for example $B_p[U_{\mathcal{M}}] = Bz_p[u_N] = 2^{1-n}$, while $B_{pr}[U_{\mathcal{M}'}] = Bz_{pr}[u_{N'}] = 2^{2-n}$ where $p, r \in S_1 \subseteq N$ and u_S denotes the S-unanimity game of coalition S for simple games. Thus, $2^{2-n} = B_{pr}[U_{\mathcal{M}'}] = 2B_p[U_{\mathcal{M}}] \neq 3B_p[U_{\mathcal{M}}]$. □

	η^{YA}	η^{AN}	η^{YN}
$U_{(a,c,b)}$	(6,0,0)	(0,0,3)	(6,0,3)
$U_{(b,c,a)}$	(0,6,0)	(0,0,3)	(0,6,3)
$U_{(ab,c,\emptyset)}$	(2,2,0)	(0,0,1)	(2,2,1)
$U_{(a,c,b)} \vee U_{(b,c,a)}$	(4,4,0)	(0,0,5)	(4,4,5)

Table 1: Raw indices for Example 1.3-(i).

We conclude this section by revisiting the first example considered.

Example 1.3-(i) (revisited). Since $I^X[U_{(a,c,b)} \vee U_{(b,c,a)}] = I^X[U_{(a,c,b)}] + I^X[U_{(b,c,a)}] - I^X[U_{(a,c,b)} \wedge U_{(b,c,a)}]$ and $U_{(a,c,b)} \wedge U_{(b,c,a)} = U_{(ab,c,\emptyset)}$ by the transfer axiom we obtain the value of I^X for the game by its action on unanimity games. As the denominator is constant we just consider η^X instead of I^X for the three possible choices of X. The next table summarizes the results. Thus, clearly c is null from the viewpoint of YA-power, while a and b are nulls for YN-power. When the amalgamation of the two versions of power into YN-power is produced, player c becomes one more time crucial than players a or b, and thus a bit more powerful, e.g., the difference of YN-power between c and a is 1/9. Table 1 contains the raw indices.

5 A remark on the rankings of power in comparing two different weighted (3,2) games

In weighted simple games it is well-known that if two players have in a weighted representation the same weights then they are equally powerful for all symmetric power indices, including the Banzhaf power index. In weighted games with abstention and for the three power indices considered in this paper this property is still (trivially) true. For instance, in the previous example players a and b have the same X-power.

If $(q; w) \equiv [q; w_1, w_2, \ldots, w_n]$ is a weighted representation of a weighted simple game and $(q'; w)$ is another representation of another weighted simple game with equal weights but different quota, then the rankings of the Banzhaf index for players cannot be opposite, i.e. it is not possible to have

$$Bz_p(q; w) > Bz_r(q; w) \qquad \text{and} \qquad Bz_p(q'; w) < Bz_r(q'; w).$$

This is true because having more weight than another player in a weighted representation means being crucial at least as many times as the player with less weight. This property is satisfied by the Banzhaf, the Shapley-Shubik (see [Diffo Lambo 2002]) and Johnston (see [Freixas et al. 2012]) indices, but not by some other indices, like the Holler or Deegan and Packel indices (see e.g. [Holler 2001] or [Holler et al. 2004]).

	η^{YA}	η^{AN}	η^{YN}
$U_{(ab,\emptyset,c)}$	(3,3,0)	(0,0,0)	(3,3,0)
$U_{(a,c,b)}$	(3,0,0)	(0,0,3)	(3,0,3)
$U_{(b,c,a)}$	(0,3,0)	(0,0,3)	(0,3,3)
$U_{(ab,c,\emptyset)}$	(2,2,0)	(0,0,1)	(2,2,1)
$U_{(ab,\emptyset,c)} \vee U_{(a,c,b)} \vee U_{(b,c,a)}$	(2,2,0)	(0,0,4)	(2,2,4)

Table 2: The raw indices for Example 1.3-(ii) and $Q = 1$.

Thus, if in a weighted representation of a simple game the quota is increased (or decreased) the Banzhaf power of two arbitrary players cannot be reversed.

The next example, which is also useful to revise the properties considered in the previous sections, illustrates that the natural extension of the Banzhaf index for games with abstention, I_p^{YN} fails to fulfill this property.

Example 1.3-(ii) (revisited): Let $N = \{a, b, c\}$, $w(a) = (2, 0, 0)$, $w(b) = (1, 0, -1)$, and $w(c) = (0, 0, -2)$.

- Consider $Q = 1$. Then $W^m = \{(ab, \emptyset, c), (a, c, b), (b, c, a)\}$. It holds that

$$\begin{aligned} I^X[U_{(ab,\emptyset,c)} \vee U_{(a,c,b)} \vee U_{(b,c,a)}] &= I^X[U_{(ab,\emptyset,c)}] + I^X[U_{(a,c,b)}] + I^X[U_{(b,c,a)}] \\ &\quad -2I^X[U_{(ab,c,\emptyset)}] \end{aligned}$$

since

$$U_{(ab,\emptyset,c)} \wedge U_{(a,c,b)} = U_{(ab,c,\emptyset)}$$

$$U_{(ab,\emptyset,c)} \wedge U_{(b,c,a)} = U_{(ab,c,\emptyset)}$$

$$U_{(a,c,b)} \wedge U_{(b,c,a)} = U_{(ab,c,\emptyset)}$$

and

$$U_{(ab,\emptyset,c)} \wedge U_{(a,c,b)} \wedge U_{(b,c,a)} = U_{(ab,c,\emptyset)}.$$

Thus we can obtain by the transfer property the values of I^X for the game by its action on unanimity games. Table 2 contains the raw indices.

Concerning YN-power c is doubly powerful than a and b which are equally powerful. Note that a does not benefit with respect to b of having a greater "yes"-weight.

- Consider $Q = 2$. Then $W^m = \{(a, bc, \emptyset)\}$.

As illustrated in Table 3 voter a is by far the most powerful while b cannot benefit of her positive "yes"-weight.

- Consider $Q = 3$. Then $W^m = \{(ab, c, \emptyset)\}$.

	η^{YA}	η^{AN}	η^{YN}
$U_{(a,bc,\emptyset)}$	(4,0,0)	(0,2,2)	(4,2,2)

Table 3: The raw indices for Example 1.3-(iii) and $Q = 2$.

	η^{YA}	η^{AN}	η^{YN}
$U_{(ab,c,\emptyset)}$	(2,2,0)	(0,0,1)	(2,2,1)

Table 4: The raw indices for Example 1.3-(iii) and $Q = 3$.

As illustrated in Table 4 the "yes"-weights and "no"-weights for a and b produce the same effect in the game. An effect on decisiveness which is not compensated by the "YN"-power of c.

Thus, when simply considering these three quotas the rankings obtained for the YN-power are all different (see Table 5).

Indeed, this example illustrates a different behavior of the index I^{YN} for $(3, 2)$ games and the Banzhaf index for simple games. If we compare players b and c in the three previous games for $Q = 1$, $Q = 2$, and $Q = 3$ we observe that the three rankings of power for I^{YN} and players b and c are: $b < c$, $b = c$, and $b > c$ respectively.

One might expect that if two players have the same weight to vote affirmatively (or negatively) then they should have the same power to pass (veto) resolutions. For instance, in the weighted representation of the UNSC voting system with abstention giving in Example 1.3-(iii) all players, permanent and non-permanent, have the same "yes" which is 1. Should be expected that they have the same YA-power, I^{YA}?

	$Q = 1$	$Q = 2$	$Q = 3$
Ranking for YN-power	$c > a = b$	$a > b = c$	$a = b > c$

Table 5: Rankings of power for different quotas in Example 1.3-(ii).

Example 1.3-(iii) (revisited): Some computations to get the swings lead us to

$$\eta_p^{YA}[V] = \sum_{j=0}^{4} \binom{4}{j} \cdot \binom{10}{8-j} \cdot \left[\sum_{k=0}^{2+j} \binom{2+j}{k} \right] = 69876$$

$$\eta_r^{YA}[V] = \sum_{j=0}^{5} \binom{5}{j} \cdot \binom{9}{8-j} \cdot \left[\sum_{k=0}^{1+j} \binom{1+j}{k} \right] = 53154$$

$$\eta_p^{AN}[V] = \sum_{j=0}^{4} \sum_{k=0}^{j} \binom{4}{j-k} \cdot \binom{10}{10-k} 2^k + \sum_{k=0}^{4} \binom{4}{4-k} \binom{10}{10-k-1} 2^{1+k} = 38460$$

and as players $r \in R$ are AN-nulls it holds $\eta_r^{AN}[V] = 0$. From this we obtain:

$$\begin{array}{lll} I_p^{YA}[V] = 0.01460833575, & I_p^{AN}[V] = 0.0080410306, & I_p^{YN}[V] = 0.02265036633, \\ I_r^{YA}[V] = 0.01111318096, & I_r^{AN}[V] = 0, & I_r^{YN}[V] = 0.01111318096. \end{array}$$

so that the answer to the question posed is negative. Even for passing resolutions, permanent members are more powerful than non-permanent members.

Clearly, in weighted games with abstention I^{YA} power depends not only on the affirmative weights of players, but also on the weights against.

We finally conclude by just recalling that some authors (see [Parker 2012] and [Tchantcho et al. 2008]) have proved that the I^{YN} power index respects the I-influence, an extension of the desirability relation for simple games. However, not all weighted games are I-complete, i.e., games for which the I-influence is a total pre-ordering. Thus, their studies for I^{YN} power cannot be extended to all weighted $(3, 2)$ games.

Furthermore I^{YA} respects the D^{YA}-desirability relation, I^{AN} respects the D^{AN}-desirability relation, and I^{YN} respects the D^{YN}-desirability relation, the three desirability relations are introduced in [Freixas et al. 2014a] and are of fundamental importance for the consistency of the notion of weighted voting rule with abstention. The D^{YA}-desirability relation formalizes the intuitive notion that is the basis of the expression: "p has at least as YA-power as r" and it is formalized in terms of the formation of winning tripartition when swamping the voter from the abstention level to the yes level. Analogous intuition underlies under the other two relations: D^{AN} and D^{YN}. However, the equation 1 leaves only two degrees of freedom and two components power measure, we have chosen I^{YA} and I^{AN}, are necessary instead of only one, let's say I^{YN}.

6 Conclusion

The notion of two components power for voting rules with abstention, or simply $(3, 2)$ games, is introduced in this paper. It is compatible with the two components desirability relation considered in [Freixas et al. 2014b] which is also compatible with the notion of weighted voting rule with abstention. These two components of power have an independent but complementary meaning to explain the real ex-ante power that a player has in a $(3,2)$ game. We think we have shown evidence that a classical power measure assigning a single number to each player in a $(3,2)$ game is not enough to understand how influent a player is in the situations described by $(3,2)$ games, and in which way is influent in such games. The corresponding generalization of the Banzhaf index in the context of $(3,2)$ games is just the sum of the components of the two components vector considered in this paper.

To add theoretical robustness to this new notion in the paper we provided also an axiomatization of the most natural two components power index. We conclude by observing that our axiomatization of the two components can be easily extended to the generalized Banzhaf index just observing that all properties extend, *mutatis mutandis*, to sums.[6] The only point to put in evidence is now that in Axiom 2 the required normalization conditions must be written as in (7) instead of its similar versions in (5) or (6), and similarly occurs with Axiom 3. Thus it holds:

Theorem 6.1 *A power index P_p satisfies axioms A1, A2-(iii), and A3-(iii) if and only if P_p is the index I_p^{YN} in (3).*

Several axiomatizations for the Banzhaf index in simple games exist (see among others, [Albizuri et al. 2001, Barua 2005, Feltkamp 1995, Haller 1994, Lehrer 1988, Owen 1978]). All of them have been very useful to better understand this power index and to highlight different features for it. Our notion of two components power index needs the support of similar characterizations. We point out a certain similarity between our block axiom and the corresponding axiom considered in [Barua 2005]. Apparently some quite close connections might exist between some of these axiomatizations for simple games and new axiomatizations for our idea of 2-components power.

It is also remarkable that other notions of games with alternatives exist in the literature trying to extend values for cooperative games to these larger classes of games, we refer among others to [Amer 1998, Carreras 1998, Bolger 1986, Bolger 1993, Bolger 2000, Ono 2001]. All these models do not assume an ordering for the input and output levels as we do. However, they admit restrictions to some voting rules with ordered levels. In this context, our approach is still valid in these models.

While a power index for a simple game or a value for cooperative game gives a total ranking for players in the game, the idea introduced here of 2-components power for games with abstention or $(3,2)$-games (or more generally $(j-1)$-components power for $(j,2)$ games) loses this property since the 2-components power of two players need not to be Pareto comparable. This enforces the analysis of importance rankings in this more complex framework, which is also a significant issue in operational research. The approach in our paper is useful in ranking voters in voting institutions where abstention is allowed as a third input. Examples of application of our results naturally apply to political institutions, but also in management enterprisers and even in reliability systems where voters are replaced by device components with three input levels.

[6]Also a direct verification is quite straightforward, following the lines of our previous proofs.

Examples of the treatment of the importance of rankings in these different contexts can be found in: [Alonso-Meijide 2009, Bishnu 2012, Cook 2006, Freixas et al. 2014a, Jones et al. 2010, Levitin 2003, Obata et al. 2003].

Acknowledgments

The ideas of the paper were discussed, and the paper itself was prepared mostly during some exchange visits of the two authors. Both are grateful to the hosting departments for their warm hospitality. They also acknowledge a grant from GNAMPA, CNR, supporting the visit in Italy of the first author. The research of the second author was partially supported by Ministero dell'Istruzione, dell'Universitá e della Ricerca Scientifica (COFIN 2009).

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