A practical test for assessing the reachability of discrete-time Takagi-Sugeno fuzzy systems

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Abstract

This paper provides a necessary and sufficient condition for the reachability of discrete-time Takagi-Sugeno fuzzy systems that is easy to apply, such that it constitutes a practical test. The proposed procedure is based on checking if all the principal minors associated to an appropriate matrix are positive. If this condition holds, then the rank of the reachability matrix associated to the Takagi-Sugeno fuzzy system is full for any possible sequence of premise variables, and thus the system is completely state reachable. On the other hand, if the principal minors are not positive, the property of the matrix being a block P one with respect to a particular partition of a set of integers is studied in order to conclude about the reachability of the Takagi-Sugeno system. Examples obtained using an inverted pendulum are used to show that it is easy to check this condition, such that the reachability analysis can be performed efficiently using the proposed approach.

Keywords: Reachability, controllability, fuzzy systems, Takagi-Sugeno models, block P-matrices.

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1. Introduction

Reachability and controllability are among the most important properties of dynamical systems. In simple words, reachability means that it is possible to steer a system from an arbitrary initial state to an arbitrary final state using the set of admissible controls [1]. On the other hand, controllability concerns the existence of an input sequence that transfers the state from an arbitrary initial condition to the origin [1]. Reachability always implies controllability, but the converse is true only when the state transition matrix is nonsingular [1].

The systematic study of these properties started at the beginning of the 1960s, when the corresponding theory began to be developed for time-invariant and time-varying linear control systems, leading to a big number of publications, the list of which can be found in the monographs, e.g. [2], or survey papers [3], [4].

The last decades have attracted a growing interest in the development of reachability and controllability theory for dynamical systems. A few papers have dealt with the case of nonlinear and semi-linear systems, for which linearization methods and generalization of open mapping theorem are extensively used [5, 6, 7, 8]. The controllability of infinite dimensional systems has been studied in [9, 10]. Further investigation has addressed the problem for stochastic [11, 12, 13, 14], delayed [15, 16], fractional [17, 18, 19] and switched [20, 21, 22, 23] systems.

In the last decades, gain-scheduling control techniques have consolidated as an efficient tool for analysis and synthesis problems for nonlinear systems [24]. The strength of these techniques consists in the fact that the properties of the nonlinear systems are checked on the basis of a collection of linear systems, that is also used for designing the controller. This is done in a divide and conquer fashion so that well established linear methods can be applied to nonlinear problems. Among the successful gain scheduling approaches, Takagi-Sugeno (TS) systems [25] provide an effective way of representing nonlinear systems with the aid of fuzzy sets, fuzzy rules and a set of local linear models (see [26, 27, 28, 29, 30, 31, 32, 33] and the references therein). The overall model of the system is obtained by merging the local models through fuzzy membership functions.

The a priori determination of reachability and controllability properties for TS fuzzy models is an open problem, and there exist very few works related with this issue. In [34], the authors introduced the idea of soft-controllability, i.e. the existence of a control sequence that forces the system to a reachable state close enough to the desired final state, and have analyzed this property using graph representations. A sufficient condition for assessing the robust controllability of TS fuzzy systems with parametric uncertainties was provided in [35]. An approach based on the linearization of the TS fuzzy model was introduced in [36]. However, when the resulting linearized system is non-controllable, the controllability property of the overall fuzzy system must be analyzed through a complex algorithm that may fail in some cases. Finally, [37] analyzes the controllability property for a class of TS fuzzy systems, independently of the form of the membership functions, but considering that at most two fuzzy rules are activated at the same time. From the literature review, it seems that at this moment there is a lack of a practical test capable of assessing the reachability property of TS fuzzy systems, and with the relevant feature of being not only a sufficient condition but a necessary one too.

The goal of this paper is to fill this gap, by providing a necessary and sufficient condition for the reachability of discrete-time TS fuzzy systems. This condition is derived using the results on the rank characterization of convex combination of matrices [38]. The provided condition involves checking whether or not a matrix is a block P one [39] with respect to some partition of an appropriate set of integers. Illustrative examples are used to show that it is easy to check the proposed condition, such that it constitutes a practical test for assessing the reachability of discrete-time TS fuzzy systems.

The paper is structured as follows: Section 2 recalls some notions that are used throughout the paper. In particular, the notions of real P-matrices, block P-matrices, TS systems, and complete state reachability are recalled. Section 3 introduces the main result that is illustrated through examples in Section 4. Finally, Section 5 draws the conclusions.

2. Preliminaries

2.1. Real and block P-matrices

Real P-matrices are well known in matrix theory because they play an important role in many applications [40]. In [41], it was shown that the P-property of a single matrix is equivalent to the nonsingularity of all matrices in a certain convex matrix set. This motivated generalizing this notion, introducing block P-matrices [39] that were later used to study the Schur [42] and Hurwitz [43] stability of convex combinations of matrices.

A matrix $A \in \mathbb{R}^{n\times n}$ is called a P-matrix if all its principal minors are positive [40]. On the other hand, a matrix $A \in \mathbb{R}^{n\times n}$ is a block P-matrix with respect to a partition $N(\lambda)$ of $N = \{1,\dots,n\}$ into $\lambda \in [1,n]$ pair-wise disjoint nonvoid subsets N_i of cardinality n_i, $i = 1,\dots,\lambda$, if for any $T \in \mathscr{T}_n^{(\lambda)}$

$$\det\left(TA + (I - T)\right) \neq 0 \tag{1}$$

where $\mathscr{T}_n^\lambda$ is the set of all diagonal matrices $T \in \mathbb{R}^{n\times n}$ such that $T[N_i] = t_i I$, $t_i \in [0,1]$, $i = 1,\ldots,\lambda$, where $T[N_i]$ is the principal submatrix of T with row and column indices in N_i [40]. A P-matrix is also block P-matrix with respect to any partition [40].

Example. Consider the matrices:

$$A^{'} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \qquad A^{''} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

It is easy to verify that $A^{'}$ is a P-matrix, since all its principal minors are positive:

$$|A'| = 4 \quad a'_{11} = 1 \quad a'_{22} = 1 \quad a'_{33} = 1$$
$$\begin{vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{vmatrix} = 2 \quad \begin{vmatrix} a'_{11} & a'_{13} \\ a'_{31} & a'_{33} \end{vmatrix} = 1 \quad \begin{vmatrix} a'_{22} & a'_{23} \\ a'_{32} & a'_{33} \end{vmatrix} = 3$$

On the other hand, $A^{''}$ is not a P-matrix, since $a^{''}_{22} = 0$. Let us consider $T^{'} = diag\{t_1,t_2,t_2\}$, which corresponds to the partition of $N = \{1,2,3\}$ into the subsets $\{1\}$ and $\{2,3\}$. It follows that:

$$\det\left(T'A'' + I - T'\right) = \det\left(\begin{bmatrix} 1 & -t_1 & 0 \\ t_2 & 1-t_2 & -t_2 \\ 0 & 2t_2 & 1 \end{bmatrix}\right) = 1 - t_2 + 2t_2^2 + t_1 t_2 = 0$$

has the following solution:

$$t_1 = -\frac{2t_2^2 - t_2 + 1}{t_2}$$

where t_2 has an arbitrary value. Hence, since whenever $t_2 \in [0,1]$, $t_1 \notin [0,1]$, it can be concluded that $A^{''}$ is a block P-matrix with respect to the partition of $\{1,2,3\}$ into $\{1\}$ and $\{2,3\}$. If $T'' = diag\{t_1,t_2,t_1\}$ is considered instead, then:

$$\det\left(T''A'' + I - T''\right) = \det\left(\begin{bmatrix} 1 & -t_1 & 0 \\ t_2 & 1-t_2 & -t_2 \\ 0 & 2t_1 & 1 \end{bmatrix}\right) = 1 - t_2 + 3t_1 t_2 = 0$$

has the following solution:

$$t_1 = \frac{t_2 - 1}{3t_2}$$

for an arbitrary value of t_2. Since when $t_2 = 1$, then $t_1 = 0$, it can be concluded that A'' is not a block P-matrix with respect to the partition of $\{1,2,3\}$ into $\{1,3\}$ and $\{2\}$.

The following lemmas, taken from [38], play an important role in the development of a practical test for assessing the reachability of discrete-time Takagi-Sugeno fuzzy systems.

Lemma 1. *Let $\mathscr{M}^j \in \mathbb{C}^{n_r \times n_c}$, $j = 1,\dots,J$, and let us define:*

$$Q_{j,j} = \mathscr{M}^j \left(\mathscr{M}^j\right)^T \qquad j = 1,\dots,J \tag{2}$$

$$Q_{j,a} = \mathscr{M}^j \left(\mathscr{M}^a\right)^T + \mathscr{M}^a \left(\mathscr{M}^j\right)^T - \mathscr{M}^a \left(\mathscr{M}^a\right)^T - \mathscr{M}^j \left(\mathscr{M}^j\right)^T \quad j < a \tag{3}$$

and the matrices R_j, $j = 1,\dots,J$:

$$R_j = (R^j_{a,b})_{a,b \in [1,J]} = \begin{bmatrix} R^j_{1,1} & R^j_{1,2} & \dots & R^j_{1,J} \\ R^j_{2,1} & R^j_{2,2} & \dots & R^j_{2,J} \\ \vdots & \vdots & \ddots & \vdots \\ R^j_{J,1} & R^j_{J,2} & \dots & R^j_{J,J} \end{bmatrix} \tag{4}$$

with the generic block entry $R^j_{a,b}$ defined as:

$$R^j_{a,b} = \begin{cases} Q_{j,j} & \text{if } a = 1, b = 1 \\ Q_{b-1,j} & \text{if } a = 1, b = 2,\dots,j \\ I_{n_r} & \text{if } a = b, 1 < b \le J \\ -I_{n_r} & \text{if } b = 1, a = j+1 \\ O_{n_r} & \text{otherwise} \end{cases} \tag{5}$$

Then, the following statements are equivalent:

(a) *All convex combinations of matrices $\mathscr{M}^j$, $j = 1,\dots,J$ are full row-rank;*

(b) *$\mathscr{M}^J$ is full row-rank and the $(J-1)Jn_r \times (J-1)Jn_r$ matrix:*

$$\begin{bmatrix} R_1 R_J^{-1} & (R_2 - R_1) R_J^{-1} & \cdots & (R_{J-2} - R_{J-3}) R_J^{-1} & (R_{J-1} - R_{J-2}) R_J^{-1} \\ -I_{Jn_r} & I_{Jn_r} & \cdots & O_{Jn_r} & O_{Jn_r} \\ O_{Jn_r} & -I_{Jn_r} & \cdots & O_{Jn_r} & O_{Jn_r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O_{Jn_r} & O_{Jn_r} & \cdots & I_{Jn_r} & O_{Jn_r} \\ O_{Jn_r} & O_{Jn_r} & \cdots & -I_{Jn_r} & I_{Jn_r} \end{bmatrix} \tag{6}$$

is a block P-matrix with respect to the partition $\{F_1,\dots,F_{J-1}\}$ of $\{1,\dots,(J-1)Jn_r\}$, with $F_i = \{(i-1)Jn_r + 1,\dots,iJn_r\}$, $i = 1,\dots,J-1$.

Proof*:* See Theorem 2 in [38]. □

In the case of square matrices, the full row-rank condition corresponds to nonsingularity, and the following lemma can be used instead.

Lemma 2. *Let $\mathscr{M}^j \in \mathbb{C}^{n\times n}$, $j = 1,\dots,J$. Then, the following statements are equivalent:*

(a) *All convex combinations of matrices $\mathscr{M}^j$, $j = 1,\dots,J$ are nonsingular;*

(b) *$\mathscr{M}^J$ is nonsingular and the $(J-1)n\times(J-1)n$ matrix:*

$$\mathscr{W} = \begin{bmatrix} \mathscr{M}^1(\mathscr{M}^J)^{-1} & \left(\mathscr{M}^2-\mathscr{M}^1\right)(\mathscr{M}^J)^{-1} & \cdots & \left(\mathscr{M}^{J-1}-\mathscr{M}^{J-2}\right)(\mathscr{M}^J)^{-1} \\ -I_n & I_n & \cdots & O_n \\ O_n & -I_n & \cdots & O_n \\ \vdots & \vdots & \ddots & \vdots \\ O_n & O_n & \cdots & O_n \\ O_n & O_n & \cdots & I_n \end{bmatrix} \tag{7}$$

is a block P-matrix with respect to the partition $\{F_1,\dots,F_{J-1}\}$ of $\{1,\dots,(J-1)n\}$, with $F_i = \{(i-1)n+1,\dots,in\}$, $i = 1,\dots,J-1$.

Proof*:* See Theorem 4 in [42]. □

2.2. Takagi-Sugeno systems

TS systems, as proposed by Takagi and Sugeno [25], are described by local models merged together using fuzzy IF-THEN rules [28], as follows:

$$\begin{array}{l} IF\ \vartheta_1(k)\ is\ M_{i1}\ AND\ \dots\ AND\ \vartheta_p(k)\ is\ M_{ip} \\ \qquad THEN \left\{ \begin{array}{l} x_i(k+1) = A_i x(k) + B_i u(k) \\ y_i(k) = C_i x(k) + D_i u(k) \end{array} \right. \qquad i = 1,\dots,N \end{array} \tag{8}$$

where $x_i \in \mathbb{R}^{n_x}$, $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $y_i \in \mathbb{R}^{n_y}$ are the local state, the global state, the input, and the local output vector, respectively, and $\vartheta_1(k),\dots,\vartheta_p(k)$ are the premise variables, that can be functions of the state variables, external disturbances and/or time. Each linear consequent equation represented by $A_i x(k) + B_i u(k)$ is called a *subsystem*.

Given a pair $(x(k), u(k))$, the state and output of the TS system can be easily inferred:

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{N} w_i\left(\vartheta(k)\right)\left(A_i x(k) + B_i u(k)\right) \Big/ \sum_{i=1}^{N} w_i\left(\vartheta(k)\right) \\ &= \sum_{i=1}^{N} \rho_i\left(\vartheta(k)\right)\left(A_i x(k) + B_i u(k)\right) = A_k x(k) + B_k u(k) \end{aligned} \tag{9}$$

$$
\begin{aligned}
y(k) &= \sum_{i=1}^{N} w_i(\vartheta(k))(C_i x(k) + D_i u(k)) \Big/ \sum_{i=1}^{N} w_i(\vartheta(k)) \\
&= \sum_{i=1}^{N} \rho_i(\vartheta(k))(C_i x(k) + D_i u(k)) = C_k x(k) + D_k u(k)
\end{aligned} \tag{10}
$$

where $\vartheta(k) = [\vartheta_1(k), \ldots, \vartheta_p(k)]$ is the vector containing the premise variables, and $w_i(\vartheta(k))$ and $\rho_i(\vartheta(k))$ are defined as follows:

$$
w_i(\vartheta(k)) = \prod_{j=1}^{p} M_{ij}\left(\vartheta_j(k)\right) \tag{11}
$$

$$
\rho_i(\vartheta(k)) = \frac{w_i(\vartheta(k))}{\sum\limits_{i=1}^{N} w_i(\vartheta(k))} \tag{12}
$$

where $M_{ij}\left(\vartheta_j(k)\right)$ is the grade of membership of $\vartheta_j(k)$ in M_{ij} and $\rho_i(\vartheta(k))$ is such that:

$$
\begin{cases}
\sum\limits_{i=1}^{N} \rho_i(\vartheta(k)) = 1 \\
\rho_i(\vartheta(k)) \geq 0, \quad i = 1, \ldots, N
\end{cases} \tag{13}
$$

2.3. Reachability of TS systems

Let us recall the definition of *complete state reachability* [1].

Definition 1. *The TS system* (9)-(10) *is completely state reachable if there exist $k_f > k_0$ and sequences of inputs $u(k_0), u(k_0+1), \ldots, u(k_f - 1)$ able to move the state $x(k) \in \mathbb{R}^{n_x}$ from any initial state $x(k_0) \in \mathbb{R}^{n_x}$ to any other final state $x(k_f) \in \mathbb{R}^{n_x}$.*

By taking into account that, considering iteratively (9) starting from the initial state $x(k_0)$, the following is obtained:

$$
x(k_0+1) = A_{k_0} x(k_0) + B_{k_0} u(k_0) \tag{14}
$$

$$
\begin{aligned}
x(k_0+2) &= A_{k_0+1} x(k_0+1) + B_{k_0+1} u(k_0+1) \\
&= A_{k_0+1} A_{k_0} x(k_0) + A_{k_0+1} B_{k_0} u(k_0) + B_{k_0+1} u(k_0+1)
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
x_0(k+3) &= A_{k_0+2} x(k_0+2) + B_{k_0+2} u(k_0+2) \\
&= A_{k_0+2} A_{k_0+1} A_{k_0} x(k_0) + A_{k_0+2} A_{k_0+1} B_{k_0} u(k_0) \\
&\quad + A_{k_0+2} B_{k_0+1} u(k_0+1) + B_{k_0+2} u(k_0+2)
\end{aligned} \tag{16}
$$

and so on up to $x(k_0+n_x)$, one obtains:

$$x(k_0+n_x)-\prod_{k=k_0}^{k_0+n_x-1}A_k x(k_0)=\mathscr{C}_{k_0}\begin{bmatrix} u(k_0+n_x-1)\\ \vdots \\ u(k_0+1)\\ u(k_0)\end{bmatrix} \tag{17}$$

where $\mathscr{C}_{k_0}$ is the reachability matrix[1], defined as:

$$\mathscr{C}_{k_0}=\begin{bmatrix} B_{k_0+n_x-1} & A_{k_0+n_x-1}B_{k_0+n_x-2} & \dots & \prod_{k=k_0+1}^{k_0+n_x-1}A_k \;\; B_{k_0}\end{bmatrix} \tag{18}$$

Then, it is straightforward to see that any state can be reached at sample k_0+n_x if and only if $\mathscr{C}_{k_0}$ is full row rank, that is:

$$rank\left(\mathscr{C}_{k_0}\right)=n_x \tag{19}$$

that corresponds to the definition of complete state reachability provided previously.

3. Main result

The main difficulty related with the direct application of (19) for assessing the complete state reachability lies in the fact that each matrix A_k and B_k, $k=k_0,\dots,k_0+n_x-1$, is a convex sum of matrices A_i and B_i, $i=1,\dots,N$, respectively.

For this reason, the following theorem, that constitutes the main result of this work, is proposed in order to provide a practical test for assessing the reachability of a Takagi-Sugeno fuzzy system.

Theorem 1. *Let us associate the indices $l_{k_0}\in\{1,\dots,N\}$, $l_{k_1}\in\{1,\dots,N\}$, …, $l_{k_{n_x-1}}\in\{1,\dots,N\}$ with $k_0,k_0+1,\dots,k_0+n_x-1$, such that l_{k_0} corresponds to k_0, l_{k_1} corresponds to k_0+1, etc., and let us consider a bijection between the first $J=N^{n_x}$ natural numbers and all the possible combinations of indices $l_{k_0},l_{k_1},\dots,l_{k_{n_x-1}}$. Then, taking into account the aforementioned bijection, for the j-th combination of indices $l_{k_0},l_{k_1},\dots,l_{k_{n_x-1}}$ ($j\in\{1,\dots,J\}$), let us define the matrix $\mathscr{M}^j$, as follows:*

[1] As stated by [1], the term *controllability matrix* is usually used for referring to the matrix $\mathscr{C}_{k_0}$. However, the terms *reachability matrix* or *controllability-from-the-origin matrix* are more appropriate, thus the former will be used throughout the paper.

$$\mathscr{M}^j = \begin{bmatrix} I_{n_x} & 0 & \cdots & 0 & 0 & \cdots & 0 & B_{lk_0} \\ -A_{l_{k_1}} & I_{n_x} & \cdots & 0 & 0 & \cdots & B_{l_{k_1}} & 0 \\ 0 & -A_{l_{k_2}} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n_x} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -A_{l_{k_{n_x-1}}} & B_{l_{k_{n_x-1}}} & \cdots & 0 & 0 \end{bmatrix} \tag{20}$$

The following two statements are equivalent:

(a) *The Takagi-Sugeno system* (8) *is completely state reachable;*

(b) $\mathscr{M}^J$ *is full row-rank and the* $(J-1)Jn_x^2 \times (J-1)Jn_x^2$ *matrix:*

$$\mathscr{V} = \begin{bmatrix} R_1 R_J^{-1} & (R_2 - R_1) R_J^{-1} & \cdots & (R_{J-2} - R_{J-3}) R_J^{-1} & (R_{J-1} - R_{J-2}) R_J^{-1} \\ -I_{Jn_x^2} & I_{Jn_x^2} & \cdots & O_{Jn_x^2} & O_{Jn_x^2} \\ O_{Jn_x^2} & -I_{Jn_x^2} & \cdots & O_{Jn_x^2} & O_{Jn_x^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O_{Jn_x^2} & O_{Jn_x^2} & \cdots & I_{Jn_x^2} & O_{Jn_x^2} \\ O_{Jn_x^2} & O_{Jn_x^2} & \cdots & -I_{Jn_x^2} & I_{Jn_x^2} \end{bmatrix} \tag{21}$$

with R_j*,* $j = 1, \ldots, J$*, defined as in* (2)-(5)*, is a block P-matrix with respect to the partition* $\{F_1, \ldots, F_{J-1}\}$ *of* $\{1, \ldots, (J-1)Jn_x^2\}$*, with* $F_i = \{(i-1)Jn_x^2 + 1, \ldots, iJn_x^2\}$*,* $i = 1, \ldots, J-1$.

Proof: The statement (a) is true if and only if the $n_x^2 \times n_x(n_x + n_u - 1)$ matrix:

$$\mathscr{S}_{k_0} = \begin{bmatrix} I_{n_x} & 0 & \cdots & 0 & 0 & \cdots & 0 & B_{k_0} \\ -A_{k_0+1} & I_{n_x} & \cdots & 0 & 0 & \cdots & B_{k_0+1} & 0 \\ 0 & -A_{k_0+2} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{n_x} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -A_{k_0+n_x-1} & B_{k_0+n_x-1} & \cdots & 0 & 0 \end{bmatrix} \tag{22}$$

has rank n_x^2. In fact, multiplying the first block row of (22) by A_{k_0+1} and adding the result to the second one, multiplying the second block row of the resulting matrix by A_{k_0+2} and adding the result to the third one, and proceeding in a similar way with the remaining rows (the procedure resembles the one provided by [44] for the LTI case), the matrix:

$$\mathscr{S}'_{k_0} = \begin{bmatrix} I_{n_x} & 0 & \cdots & 0 & 0 & \cdots & \cdots & B_{k_0} \\ 0 & I_{n_x} & \cdots & 0 & 0 & \cdots & \cdots & A_{k_0+1}B_{k_0} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & A_{k_0+2}A_{k_0+1}B_{k_0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_x} & 0 & \cdots & \cdots & \prod_{k=k_0+1}^{k_0+n_x-2} A_k B_{k_0} \\ 0 & 0 & \cdots & 0 & B_{k_0+n_x-1} & A_{k_0+n_x-1}B_{k_0+n_x-2} & \cdots & \prod_{k=k_0+1}^{k_0+n_x-1} A_k B_{k_0} \end{bmatrix} \tag{23}$$

is obtained. It is straightforward to see that (19) holds if and only if the matrix $\mathscr{S}'_{k_0}$ defined in (23) has rank n_x^2.

Associating the indices $l_{k_0} \in \{1,\ldots,N\}$, $l_{k_1} \in \{1,\ldots,N\}$, …, $l_{k_{n_x-1}} \in \{1,\ldots,N\}$ with $k_0, k_0+1, \ldots, k_0+n_x-1$, such that l_{k_0} corresponds to k_0, l_{k_1} corresponds to k_0+1, etc., the matrix $\mathscr{S}_{k_0}$ can be rewritten as:

$$\begin{aligned} \mathscr{S}_{k_0} = & \sum_{l_{k_0}=1}^{N} \rho_{l_{k_0}}(\vartheta(k_0)) \sum_{l_{k1}=1}^{N} \rho_{l_{k_1}}(\vartheta(k_0+1)) \cdots \\ & \cdots \sum_{l_{k_{n_x-1}}=1}^{N} \rho_{l_{k_{n_x-1}}}(\vartheta(k_0+n_x-1)) \mathscr{M}^{l_{k_0} l_{k_1} \cdots l_{k_{n_x-1}}} \end{aligned} \tag{24}$$

where the N^{n_x} matrices $\mathscr{M}^{l_{k_0} l_{k_1} \cdots l_{k_{n_x-1}}}$ are defined as in (20) (the notation $\mathscr{M}^j$ is recovered by considering the bijection between the first $J = N^{n_x}$ natural numbers and all the possible combinations of indices $l_{k_0}, l_{k_1}, \ldots, l_{k_{n_x-1}}$).

Hence, if all the convex combinations of matrices $\mathscr{M}^j$, $j = 1,\ldots,J$, are full row-rank, the matrix $\mathscr{S}_{k_0}$ is full row-rank as well ($rank(\mathscr{S}_{k_0}) = n_x^2$). By applying Lemma 1, statement (b) is obtained, that completes the proof. □

Theorem 1 states a necessary and sufficient condition for the reachability of discrete-time TS fuzzy systems, that can be used to derive a practical algorithm for assessing the reachability of (8), summarized as follows:

1. Obtain matrices $\mathscr{M}^j$, $j = 1,\ldots,J$, as in (20).
2. Check if all matrices $\mathscr{M}^j$ are full row-rank. If not, the system (8) is non-reachable. Otherwise, continue the algorithm.

3. Calculate matrices R_j, $j = 1, \dots, J$.

4. Calculate matrix $\mathscr{V}$ according to (21).

5. Calculate the principal minors of $\mathscr{V}$.

6. If all the principal minors of $\mathscr{V}$ are positive, then the system (8) is reachable. Otherwise, continue the algorithm.

7. Calculate the symbolic matrix $T\mathscr{V} + (I - T)$, where T is a generic diagonal matrix such that $T\left\{(i-1)Jn_x^2 + 1, \dots, iJn_x^2\right\} = t_i$, $i = 1, \dots, J-1$.

8. Calculate the solutions of $det\left(T\mathscr{V} + (I - T)\right) = 0$. If any of the solutions is such that $t_i \in [0, 1]$ $\forall i = 1, \dots, J-1$, then the system (8) is non-reachable; otherwise, it is reachable.

The proposed algorithm can be employed for a general class of multiple input systems, but when a single input system is considered, then the algorithm explained above can be significantly simplified. Indeed, in this case, the matrices $\mathscr{M}^j$ are square ones, and this allows formulating an alternative procedure to tackle this special case, as summarized by the following corollary.

Corollary 1. *The following two statements are equivalent:*

(a) *The Takagi-Sugeno system* (8) *is completely state reachable;*

(b) *Given the matrices $\mathscr{M}^j$, $j = 1, \dots, J$, defined as in* (20)*, $\mathscr{M}^J$ is nonsingular and the $(J-1)n_x^2 \times (J-1)n_x^2$ matrix:*

$$\mathscr{W} = \begin{bmatrix} \mathscr{M}^1(\mathscr{M}^J)^{-1} & \left(\mathscr{M}^2 - \mathscr{M}^1\right)(\mathscr{M}^J)^{-1} & \cdots & \left(\mathscr{M}^{J-1} - \mathscr{M}^{J-2}\right)(\mathscr{M}^J)^{-1} \\ -I_{n_x^2} & I_{n_x^2} & \cdots & O_{n_x^2} \\ O_{n_x^2} & -I_{n_x^2} & \cdots & O_{n_x^2} \\ \vdots & \vdots & \ddots & \vdots \\ O_{n_x^2} & O_{n_x^2} & \cdots & O_{n_x^2} \\ O_{n_x^2} & O_{n_x^2} & \cdots & I_{n_x^2} \end{bmatrix} \tag{25}$$

is a block P-matrix with respect to the partition $\{F_1, \dots, F_{J-1}\}$ of $\left\{1, \dots, (J-1)n_x^2\right\}$, with $F_i = \left\{(i-1)n_x^2 + 1, \dots, in_x^2\right\}$, $i = 1, \dots, J-1$.

Proof: The proof follows the reasoning already provided for Theorem 1, and applies Lemma 2 instead of Lemma 1 in order to obtain statement (b). □

In this case, the algorithm for assessing the reachability of (8) is as follows:

1. Obtain matrices $\mathscr{M}^j$, $j = 1, \ldots, J$.
2. Check if all matrices $\mathscr{M}^j$ are non singular. If not, the system (8) is not completely state reachable. Otherwise, continue the algorithm.
3. Calculate matrix $\mathscr{W}$ according to (25).
4. Calculate the principal minors of $\mathscr{W}$.
5. If all the principal minors of $\mathscr{W}$ are positive, then the system (8) is reachable. Otherwise, continue the algorithm.
6. Calculate the symbolic matrix $T\mathscr{W} + (I - T)$, where T is a generic diagonal matrix such that $T\left\{(i-1)n_x^2 + 1, \ldots, in_x^2\right\} = t_i$, $i = 1, \ldots, J-1$.
7. Calculate the solutions of $det\left(T\mathscr{W} + (I - T)\right) = 0$. If any of the solutions is such that $t_i \in [0, 1]\ \forall i = 1, \ldots, J-1$, then the system (8) is non-reachable; otherwise, it is reachable.

Remark: The proposed test allows assessing the property of reachability of a given TS system. In many practical cases, the TS system is a representation of an underlying nonlinear system. When this representation is exact, e.g. when it has been obtained using the sector nonlinearity approach, the reachability of the TS system implies the reachability of the nonlinear system. In general, the converse is not true, i.e. a reachable nonlinear system can have non-reachable TS representations. It is also worth highlighting that TS representations of nonlinear systems are usually valid within a region of the state space, thus the conclusions obtained from the reachability test would be valid only for operating points belonging to the considered region.

4. Illustrative Examples

The objective of this section is to provide an example that will clearly and step-by-step illustrate the proposed algorithm. To this end, let us consider the two-rule fuzzy model of the inverted pendulum [45] obtained by local approximation in fuzzy partition spaces [28].

The model has two states: $x_1(t)$ and $x_2(t)$, denoting the angle (in radians) of the pendulum from the vertical and the angular velocity, respectively.

When $x_1(t)$ is near zero, the inverted pendulum is described by:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{gx_1(t) - au(t)}{4l/3 - aml} \end{cases} \tag{26}$$

where g is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum and u is the force applied to the cart; $a = 1/(m+M)$.

The continuous-time model (26) can be discretized through an Euler method with sampling time T_s, obtaining:

$$\begin{cases} x_1(k+1) = x_1(k) + T_s x_2(k) \\ x_2(k+1) = x_2(k) + T_s \frac{gx_1(k) - au(k)}{4l/3 - aml} \end{cases} \tag{27}$$

When $x_1(t)$ is near $\pm\pi/2$, the inverted pendulum is described by:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{2gx_1(t)/\pi - a\cos\beta u(t)}{4l/3 - aml\cos^2\beta} \end{cases} \tag{28}$$

whose discretized version is:

$$\begin{cases} x_1(k+1) = x_1(k) + T_s x_2(k) \\ x_2(k+1) = x_2(k) + T_s \frac{2gx_1(k)/\pi - a\cos\beta u(k)}{4l/3 - aml\cos^2\beta} \end{cases} \tag{29}$$

where $\beta \approxeq \pi/2$. The following values will be used for the inverted pendulum parameters: $g = 9.8\,m/s^2$, $m = 0.1\,kg$, $M = 0.9\,kg$, $l = 0.1\,m$, $a = 1\,kg^{-1}$. The sampling time is chosen as $T_s = 0.1\,s$.

The proposed example concerns a single input system, for which the simplified algorithm based on Corollary 1 can be applied. The examples show the reachable and the non-reachable case, respectively.

4.1. Example 1 - reachable system

By considering $\beta = 88^\circ$, a TS representation as in (8) with $N = 2$ can be obtained from (27)-(29), with:

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 7.95 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0.1 \\ 4.68 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -0.81 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -0.03 \end{bmatrix}$$

Following the algorithm given at the end of Section 3, the matrices $\mathscr{M}^j$, $j = 1,2,3,4$, are calculated:

$$\mathscr{M}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.81 \\ -1 & -0.1 & 0 & 0 \\ -7.95 & -1 & -0.81 & 0 \end{bmatrix} \quad \mathscr{M}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.81 \\ -1 & -0.1 & 0 & 0 \\ -4.68 & -1 & -0.03 & 0 \end{bmatrix}$$

$$\mathscr{M}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.03 \\ -1 & -0.1 & 0 & 0 \\ -7.95 & -1 & -0.81 & 0 \end{bmatrix} \quad \mathscr{M}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.03 \\ -1 & -0.1 & 0 & 0 \\ -4.68 & -1 & -0.03 & 0 \end{bmatrix}$$

It can be easily verified that each of the matrices $\mathscr{M}^i, i = 1,\dots,4$ has rank $n_x^2 = 4$, which means that the considered system is locally reachable. The resulting matrix $\mathscr{W}$ is:

$$\mathscr{W} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 260 & 27 & 260 & 0 & 0 & 0 & 0 & 0 & -260 & -26 & -260 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -141.59 & 0 & -260 & 27 & 141.59 & 0 & 260 & -26 & -141.59 & 0 & -260 & 26 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and it can be verified that all its principal minors are positive [46], which means that the considered system is reachable.

For further illustration, let us consider the problem of designing a deadbeat state-feedback controller:

$$K(\rho_1) = \begin{bmatrix} K_1(\rho_1) & K_2(\rho_1) \end{bmatrix}$$

where ρ_1 corresponds to the level of activation of the subsystem described by A_1 and B_1 (see (9)). Taking into account that $\rho_2 = 1 - \rho_1$ due to (13), the reachable TS representation obtained with $\beta = 88^\circ$ can be equivalently rewritten as:

$$A(\rho_1) = \rho_1 \begin{pmatrix} 1 & 0.1 \\ 7.95 & 1 \end{pmatrix} + (1-\rho_1) \begin{pmatrix} 1 & 0.1 \\ 4.68 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.1 \\ 3.27\rho_1 + 4.68 & 1 \end{pmatrix}$$

$$B(\rho_1) = \rho_1 \begin{pmatrix} 0 \\ -0.81 \end{pmatrix} + (1-\rho_1) \begin{pmatrix} 0 \\ 0.03 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.78\rho_1 - 0.03 \end{pmatrix}$$

which leads to:

$$A(\rho_1) + B(\rho_1)K(\rho_1) = \begin{pmatrix} 1 & 0.1 \\ a_{21}(\rho_1) & a_{22}(\rho_1) \end{pmatrix}$$

with:

$$a_{21}(\rho_1) = 3.27\rho_1 + 4.68 - 0.78\rho_1 K_1(\rho_1) - 0.03K_1(\rho_1)$$
$$a_{22}(\rho_1) = 1 - 0.78\rho_1 K_2(\rho_1) - 0.03K_2(\rho_1)$$

It is easy to check by means of symbolic calculations that, in order to place the closed-loop eigenvalues in $\{0,0\}$, the controller coefficients should be calculated as:

$$K_1(\rho_1) = \frac{3.27\rho_1 + 14.68}{0.78\rho_1 + 0.03}$$
$$K_2(\rho_1) = \frac{2}{0.78\rho_1 + 0.03}$$

which are defined $\forall \rho_1 \in [0,1]$.

4.2. Example 2 - non-reachable system

By considering $\beta = 92^\circ$, a TS representation as in (8) with $N = 2$ can be obtained from (27)-(29), with:

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 7.95 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0.1 \\ 4.68 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -0.81 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0.03 \end{bmatrix}$$

Following the algorithm given at the end of Section 3, the matrices $\mathscr{M}^j$, $j = 1,2,3,4$, are calculated:

$$\mathscr{M}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.81 \\ -1 & -0.1 & 0 & 0 \\ -7.95 & -1 & -0.81 & 0 \end{bmatrix} \quad \mathscr{M}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.81 \\ -1 & -0.1 & 0 & 0 \\ -4.68 & -1 & 0.03 & 0 \end{bmatrix}$$

$$\mathscr{M}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.03 \\ -1 & -0.1 & 0 & 0 \\ -7.95 & -1 & -0.81 & 0 \end{bmatrix} \quad \mathscr{M}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.03 \\ -1 & -0.1 & 0 & 0 \\ -4.68 & -1 & 0.03 & 0 \end{bmatrix}$$

It can be easily verified that each of the matrices $\mathscr{M}^i, i = 1, \ldots, 4$ has rank $n_x^2 = 4$, which means that the considered system is locally reachable. The resulting matrix $\mathscr{W}$ is:

$$\mathscr{W} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -280 & -27 & -280 & 0 & 0 & 0 & 0 & 0 & 280 & 28 & 280 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 145.69 & 0 & 280 & -27 & -145.69 & 0 & -280 & 28 & 145.69 & 0 & 280 & -28 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{30}$$

and it can be verified that some of its principal minors are not positive [46]. In this case, the next step of the proposed algorithm involves calculating the matrix $T\mathscr{W} + (I - T)$, with:

$$T = diag\,(t_1, t_1, t_1, t_1, t_2, t_2, t_2, t_2, t_3, t_3, t_3, t_3) \tag{31}$$

which leads to:

$$\det(T\mathscr{W} + (I - T)) = -784t_1^2t_2^2t_3^2 + 784t_1^2t_2^2t_3 - 784t_1^2t_2 + 784t_1^2 + 28t_1t_2 - 56t_1 + 1 = 0 \tag{32}$$

which has the following solutions, calculated using Maple [47]:

$$\left\{ t_1 = t_1, t_2 = \frac{1}{28}\frac{28t_1 - 1}{t_3t_1}, t_3 = t_3 \right\}$$

$$\left\{ t_1 = t_1, t_2 = t_2, t_3 = \frac{1}{28}\frac{28t_1t_2 - 28t_1 + 1}{t_1t_2} \right\}$$

where $t_i = t_i$, $i = 1, \ldots, 4$ means that t_i has an arbitrary value.

In particular, by considering the second set of solutions, there exist solutions such that $t_1, t_2, t_3 \in [0, 1]$, e.g. $t_1 = 0.1, t_2 = 0.9, t_3 = 0.2857$, which proves that the considered system is non-reachable. As a matter of fact, the convex combination of matrices A_1, A_2, B_1 and B_2 with $\rho_1 = 0.0357$:

$$0.0357A_1 + 0.9643A_2 = \begin{bmatrix} 1 & 0.1 \\ 4.7967 & 1 \end{bmatrix}$$

$$0.0357B_1 + 0.9643B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

corresponds to an operating condition for which the system (9) is non-reachable.

Let us consider the problem of designing a deadbeat state-feedback controller for the non-reachable TS representation obtained with $\beta = 92°$. By performing symbolic calculations similar to the previous example, it can be shown that, in order to place the closed-loop eigenvalues in $\{0,0\}$, the controller coefficients should be calculated as:

$$K_1(\rho_1) = \frac{3.27\rho_1 + 14.68}{0.84\rho_1 - 0.03}$$

$$K_2(\rho_1) = \frac{2}{0.84\rho_1 - 0.03}$$

In this case, it is easy to check that the controller coefficients are not defined for $\rho_1 = 0.0357$, due to the loss of reachability that occurs for this value, as shown previously. Notice that, as expected, the loss of reachability corresponds to the impossibility of performing closed-loop pole placement.

5. Conclusions

In this paper, the problem of developing a practical reachability test for Takagi-Sugeno fuzzy systems has been tackled. The proposed solution is based on checking if all the principal minors associated to an appropriate matrix are positive. If this condition holds, then the rank of the reachability matrix associated to the Takagi-Sugeno fuzzy system is full for any possible sequence of premise variables, and thus the system is completely state reachable. On the other hand, if the principal minors are not positive, the property of the matrix being a block P one with respect to a particular partition of a set of integers is studied in order to conclude about the reachability of the TS system. Two illustrative examples, obtained by local approximation in fuzzy partition spaces of an inverted pendulum nonlinear model, have shown the application and effectiveness of the proposed reachability test.

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