# Models for the Displacement Calculus

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**Abstract.** The displacement calculus **D** is a conservative extension of the Lambek calculus $\mathbf{L}_*$ (with empty antecedent allowed in sequents). $\mathbf{L}_*$ can be said to be the logic of concatenation, while **D** can be said to be the logic of concatenation and intercalation. In many senses, it can be claimed that **D** mimics $\mathbf{L}_*$, namely that the proof theory, generative capacity and complexity of the former calculus are natural extensions of the latter calculus. In this paper, we strengthen this claim. We present the appropriate classes of models for **D** and prove its completeness results, and strikingly, we see that these results and proofs are natural extensions of the corresponding ones for $\mathbf{L}_*$.

## 1 Introduction

The displacement calculus **D** is a quite well-studied extension of the Lambek calculus (with empty antecedent allowed in sequents) $\mathbf{L}_*$. In many papers (see [6], [9] and [8]), **D** has proved to provide elegant accounts of a variety of linguistic phenomena of English, and of Dutch, namely a processing interpretation of the so-called Dutch cross-serial dependencies.

The hypersequent calculus **hD**[1] is a pure sequent calculus free of structural rules, which subsumes the sequent calculus for $\mathbf{L}_*$. The Cut elimination algorithm for **hD** provided in [9] mimics the one of Lambek's syntactic calculus (with some minor differences concerning the possibility of empty antecedents). Like $\mathbf{L}_*$, **D** enjoys some nice properties such as the subformula property, decidablity, the finite reading property and the focalisation property ([4]).

Like $\mathbf{L}_*$, **D** is known to be NP-complete [3]. Concerning the (weak) generative capacity, **D** recognises the class of well-nested multiple context-free languages ([10]). In this sense, this result on generative capacity generalises the result that states that $\mathbf{L}_*$ recognises the class of context-free languages. One point of divergence in terms of generative capacity is that **D** recognises the class of the permutation closure of context-free languages ([7]). Finally, it is important to note that a Pentus-like upper bound theorem for **D** is not known.

In this paper we present natural classes of models for **D**. Several strong completeness results are proved, in particular strong completeness w.r.t. the class of residuated displacement algebras (a natural extension of residuated monoids).

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[1] Not to be confused with the hypersequents of Avron ([1]).

Powerset frames for $\mathbf{L}_*$ are considered of interest from the linguistic point of view. Here, again powerset residuated displacement algebras over displacement algebras are given, which generalise the powerset residuated monoids over monoids as well as free monoids. Strong completeness result for the so-called implicative fragment of **D**, which is very relevant linguistically, is proved in the style of Buszkowski ([2]). Moreover, full completeness with respect powerset residuated displacement algebras over displacement algebras is given following again Buszkowski's methods.

The structure of the paper is as follows. In Section 2 we present the basic proof-theoretic tools (useful for the construction of canonical models) for the study of **D** from a semantic point of view. In Section 3 we provide the proof of strong completeness of the implicative fragment w.r.t. L-models. In Section 4, the proof of strong completeness of full **D** w.r.t. powerset residuated displacement algebras over displacement algebras is outlined. Finally, we conclude in the last section.

## 2 The Hypersequent Calculus hD and the Categorical calculus cD

**D** is model-theoretically motivated, and the key to its conception is the class of standard displacement algebras. Some definitions are needed. Let $\mathcal{M} = (|\mathcal{M}|, +, 0, 1)$ be a free monoid where 1 is a distinguished element of the set of generators $X$ of $\mathcal{M}$. We call such an algebra a *separated monoid*. Given an element $a \in |\mathcal{M}|$, we can associate to it a number, called its *sort* as follows:

$$(1)\quad \begin{aligned} s(1) &= 1 \\ s(a) &= 0 \text{ if } a \in X \text{ and } a \neq 1 \\ s(w_1 + w_2) &= s(w_1) + s(w_2) \end{aligned}$$

This induction is well-defined, for $\mathcal{M}$ is free, and 1 is a (distinguished) generator. The sort function $s(\cdot)$ in a separated monoid simply counts the number of separators an element contains.

**Definition 1.** *(***Sort Domains***)*
Where $\mathcal{M} = (|\mathcal{M}|, +, 0, 1)$ is a separated monoid, the *sort domains* $|\mathcal{M}|_i$ of sort $i$ are defined as follows:

$$|\mathcal{M}|_i = \{a \in |\mathcal{M}| : s(a) = i\}, i \geq 0$$

It is readily seen that for every $i, j \geq 0$, $|\mathcal{M}|_i \cap |\mathcal{M}|_j = \emptyset$ iff $i \neq j$.

**Definition 2.** *(***Standard Displacement Algebra***)*
The *standard displacement algebra* (or standard DA) defined by a separated monoid $(|\mathcal{M}|, +, 0, 1)$ is the $\mathbb{N}$-sorted algebra with the $\mathbb{N}$-sorted signature $\Sigma_D = (+, \{\times_i\}_{i>0}, 0, 1\})$ with sort functionality $((i, j \to i+j)_{i,j\geq 0}, (i, j \to i+j-1)_{i>0, j\geq 0}, 0, 1)$:

$$(\{|\mathcal{M}|_i\}_{i\geq 0}, +, \{\times_i\}_{i>0}, 0, 1)$$

where:

| operation | is such that |
|---|---|
| $+ : \|\mathcal{M}\|_i \times \|\mathcal{M}\|_j \to \|\mathcal{M}\|_{i+j}$ | as in the separated monoid |
| $\times_k : \|\mathcal{M}\|_i \times \|\mathcal{M}\|_j \to \|\mathcal{M}\|_{i+j-1}$ | $\times_k(s,t)$ is the result of replacing the $k$-th separator in $s$ by $t$ |

We will usually write standard DA instead of standard displacement algebra. The sorted types of $\mathbf{D}$, which we will define residuating w.r.t the sorted operations in definition 2, are defined by mutual recursion in Figure 1. We let $\mathbf{Tp} = \bigcup_{i \geq 0} \mathbf{Tp}_i$. A subset $B$ of $|\mathcal{M}|$ is called a *same-sort* subset iff there exists an $i \in \mathbb{N}$ such that for every $a \in B$, $s(a) = i$. $\mathbf{D}$ types are to be interpreted as same-sort subsets of $|\mathcal{M}|$. I.e. every inhabitant of $[\![A]\!]$ has the same sort. The semantic

| | |
|---|---|
| $\mathbf{Tp}_i ::= \mathbf{Pr}_i$ | where $\mathbf{Pr}_i$ is the set of atomic types of sort $i$ |
| $\mathbf{Tp}_0 ::= I$ | Continuous unit |
| $\mathbf{Tp}_1 ::= J$ | Discontinuous unit |
| $\mathbf{Tp}_{i+j} ::= \mathbf{Tp}_i \bullet \mathbf{Tp}_j$ | continuous product |
| $\mathbf{Tp}_j ::= \mathbf{Tp}_i \backslash \mathbf{Tp}_{i+j}$ | under |
| $\mathbf{Tp}_i ::= \mathbf{Tp}_{i+j} / \mathbf{Tp}_j$ | over |
| $\mathbf{Tp}_{i+j} ::= \mathbf{Tp}_{i+1} \odot_k \mathbf{Tp}_j$ | discontinuous product |
| $\mathbf{Tp}_j ::= \mathbf{Tp}_{i+1} \downarrow_k \mathbf{Tp}_{i+j}$ | extract |
| $\mathbf{Tp}_{i+1} ::= \mathbf{Tp}_{i+j} \uparrow_k \mathbf{Tp}_j$ | infix |

**Fig. 1.** The sorted types of **D**

interpretations of the connectives are shown in Figure 2. This interpretation is called the *standard interpretation*. We observe that for any type $A \in \mathbf{Tp}$, the interpretation of $A$, i.e. $[\![A]\!]$, is contained in $M_{s(A)}$, where the sort map $s(\cdot)$ for the set $\mathbf{Tp}$, is such that ():

$$
(2)\quad
\begin{array}{lll}
s(p) & = i & \text{for p} \in \mathbf{Pr}_i \\
s(I) & = 0 & \\
s(J) & = 1 & \\
s(A \bullet B) & = s(A) + s(B) & \\
s(B/A) = s(A \backslash B) & = s(B) - s(A) & \\
s(A \odot_k B) & = s(A) + s(B) - 1 & \\
s(A \downarrow_k B) = s(B \uparrow_k A) & = s(B) - s(A) + 1 &
\end{array}
$$

$$
\begin{array}{rll}
[\![p]\!] \subseteq |\mathcal{M}|_i \text{ for } i \geq 0 & & p \in \mathbf{Pr}_i \\
\\
[\![I]\!] = \{0\} & & \text{continuous unit} \\
[\![J]\!] = \{1\} & & \text{discontinuous unit} \\
\\
[\![A \bullet B]\!] = \{s_1 + s_2 |\ s_1 \in [\![A]\!] \ \&\ s_2 \in [\![B]\!]\} & & \text{product} \\
[\![A \backslash C]\!] = \{s_2 |\ \forall s_1 \in [\![A]\!], s_1 + s_2 \in [\![C]\!]\} & & \text{under} \\
[\![C / B]\!] = \{s_1 |\ \forall s_2 \in [\![B]\!], s_1 + s_2 \in [\![C]\!]\} & & \text{over} \\
\\
[\![A \odot_k B]\!] = \{\times_k(s_1, s_2) |\ s_1 \in [\![A]\!] \ \&\ s_2 \in [\![B]\!]\} & k > 0 & \text{discontinuous product} \\
[\![A \downarrow_k C]\!] = \{s_2 |\ \forall s_1 \in [\![A]\!], \times_k(s_1, s_2) \in [\![C]\!]\} & k > 0 & \text{infix} \\
[\![C \uparrow_k B]\!] = \{s_1 |\ \forall s_2 \in [\![B]\!], \times_k(s_1, s_2) \in [\![C]\!]\} & k > 0 & \text{extract}
\end{array}
$$

**Fig. 2.** Standard semantic interpretation of **D** types

### 2.1 The Hypersequent Calculus hD

We will now consider the *string-based* hypersequent syntax from [5].[2] The reason for using the prefix *hyper* in the term *sequent* is that the data-structure used in hypersequent antecedents is quite nonstandard. *Type segments* are defined as follows

$$
(3) \quad \begin{array}{l}
\mathbf{TypSeg}_0 ::= A \text{ for } a \in \mathbf{Tp}_0 \\
\mathbf{TypSeg}_n ::= \lceil A \rceil_i \text{ for } 0 \leq i \leq n := s(A)
\end{array}
$$

Type segments of sort 0 are types. But, type segments of sort greater than 0 are no longer types. Strings of type segments can form meaningful logical material like the set of hyperconfigurations, which we now define. The *hyperconfigurations* **HConfig** are defined unambiguously by mutual recursion as follows, where $\Lambda$ is the empty string and 1 is the metalinguistic separator::

$$
(4) \quad \begin{array}{l}
\mathbf{HConfig} ::= \Lambda \\
\mathbf{HConfig} ::= A, \mathbf{HConfig} \text{ for } s(A) = 0 \\
\mathbf{HConfig} ::= 1, \mathbf{HConfig} \\
\mathbf{HConfig} ::= \lceil A \rceil_0, \mathbf{HConfig}, \lceil A \rceil_1, \cdots, \lceil A \rceil_{s(A)-1}, \mathbf{HConfig}, \lceil A \rceil_{s(A)}, \mathbf{HConfig} \\
\qquad\qquad\quad \text{for } s(A) > 0
\end{array}
$$

The semantic interpretation of the last clause from (4) consists of elements $\alpha_0 + \beta_1 + \alpha_1 + \cdots + \alpha_{n-1} + \beta_n + \alpha_n$ where $\alpha_0 + 1 + \alpha_1 + \cdots + \alpha_{n-1} + 1 + \alpha_n \in [\![A]\!]$ and $\beta_1, \cdots, \beta_n$ are the interpretations of the intercalated hyperconfigurations. The syntax in which **HConfig** has been defined is called *string-based hypersequent syntax*. An equivalent syntax for **HConfig** is called *tree-based hypersequent syntax* which was defined in [6], [9]. For proof-search and human readability, the tree-based notation is more convenient than the string-based notation, but for semantic purposes, the string-based notation turns out to be very useful, for

[2] Term which must not be confused with Avron's hypersequents ([1]).

the canonical model construction considered in Section 3 which relies on the set $\mathbf{TypSeg} = \bigcup_{n \geq 0} \mathbf{TypSeg}_n$.

In string-based notation the *figure* $\overrightarrow{A}$ of a type $A$ is defined as follows:

$$(5)\quad \overrightarrow{A} = \begin{cases} A & \text{if } s(A) = 0 \\ \lceil A \rceil_0, 1, \lceil A \rceil_1, \cdots, \lceil A \rceil_{s(A)-1}, 1, \lceil A \rceil_{s(A)} & \text{if } s(A) > 0 \end{cases}$$

The sort of a hyperconfiguration is the number of metalinguistic separators it contains. We have $\mathbf{HConfig} = \bigcup_{i \geq 0} \mathbf{HConfig}_i$, where $\mathbf{HConfig}_i$ is the set of hyperconfigurations of sort $i$. Where $\Gamma$ and $\Phi$ are hyperconfigurations and the sort of $\Gamma$ is at least 1, $\Gamma|_k\Phi$ $(k > 0)$ signifies the hyperconfiguration which is the result of replacing the $k$-th separator in $\Gamma$ by $\Phi$. Where $\Gamma$ is a hyperconfiguration of sort $i$ and $\Phi_1, \cdots, \Phi_i$ are hyperconfigurations, the *fold* $\Gamma \otimes \langle \Phi_1, \cdots, \Phi_i \rangle$ is the result of simultaneously replacing the successive separators in $\Gamma$ by $\Phi_1, \cdots, \Phi_i$ respectively. $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0(\Gamma \otimes \langle \Delta_1, \cdots, \Delta_i \rangle)$. When a type-occurrence $A$ in a hyperconfiguration is written without vectorial notation, that means that the sort of $A$ is 0. However, when one writes the metanotation for hyperconfigurations $\Delta\langle\overrightarrow{A}\rangle$, this does not mean that the sort of $A$ is necessarily greater than 0.

A *hypersequent* $\Gamma \Rightarrow A$ comprises an antecedent hyperconfiguration in string-based notation of sort $i$ and a succedent type $A$ of sort $i$. The hypersequent calculus for $\mathbf{D}$ is as shown in Figure 3. The following lemma is useful for the strong completeness results of section 3:

**Lemma 1.** *The set of* $\mathbf{HConfig}$ *is a subset of* $(\mathbf{TypSeg} \cup \{1\})^*$. *We have that:*

*i)* $\mathbf{HConfig}$ *is closed by concatenation and intercalation.*
*ii) If* $\Delta \in (\mathbf{TypSeg} \cup \{1\})^*$, $\Gamma \in \mathbf{HConfig}$, *and* $\Delta, \Gamma \in \mathbf{HConfig}$, *then* $\Delta \in \mathbf{HConfig}$. *Similarly, if we have* $\Gamma, \Delta \in \mathbf{HConfig}$ *instead of* $\Delta, \Gamma \in \mathbf{HConfig}$. *Finally, If* $\Delta \in (\mathbf{TypSeg} \cup \{1\})^*$, $\Gamma \in \mathbf{HConfig}$, *and* $\Delta|_i\Gamma \in \mathbf{HConfig}$, *then* $\Delta \in \mathbf{HConfig}$.

### 2.2 D and its Categorical Presentation cD

In [11] it is proved that the identities (or equations) true of standard DAs has as equational theory the so-called class of (general) displacement algebras (DA) (see Figure 4).

The categorical calculus **cD** for **D** is as follows:

(6)

$A \bullet B \to$ iff $A \to C/B$ iff $B \to A\backslash C$
$A \odot_i B \to$ iff $A \to C\uparrow_i B$ iff $B \to A\downarrow_i C$

We add as postulates the ones corresponding to the set of equations defining $\mathcal{DA}$

The set of postulates of **cD** would be in the case of the categorical calculus with unit for $\mathbf{L}_*$, the postulates corresponding to the equations defining the class of monoids. Let $\mathcal{RD}$ be the class of residuated DAs. Again, in [11], it is proved the following embedding translation:

$\overrightarrow{A} \Rightarrow A$ if $A$ is primitive

$$\frac{\Delta\langle\Lambda\rangle \Rightarrow A}{\Delta\langle I\rangle \Rightarrow A}\,IL \qquad \frac{}{\Lambda \Rightarrow I}\,IR$$

$$\frac{\Delta\langle 1\rangle \Rightarrow A}{\Delta\langle\overrightarrow{J}\rangle \Rightarrow A}\,JL \qquad \frac{}{1 \Rightarrow J}\,JR$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{B}\rangle \Rightarrow C}{\Delta\langle\overrightarrow{B/A}, \Gamma\rangle \Rightarrow C}\,/L \qquad \frac{\Delta, \overrightarrow{A} \Rightarrow B}{\Delta \Rightarrow B/A}\,/R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{B}\rangle \Rightarrow C}{\Delta\langle\Gamma, \overrightarrow{A\backslash B}\rangle \Rightarrow C}\,\backslash L \qquad \frac{\overrightarrow{A}, \Delta \Rightarrow B}{\Delta \Rightarrow A\backslash B}\,\backslash R$$

$$\frac{\Delta\langle\overrightarrow{A}, \overrightarrow{B}\rangle \Rightarrow C}{\Delta\langle\overrightarrow{A \bullet B}\rangle \Rightarrow C}\,\bullet L \qquad \frac{\Delta \Rightarrow A \qquad \Gamma \Rightarrow B}{\Delta, \Gamma \Rightarrow A \bullet B}\,\bullet R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{B}\rangle \Rightarrow C}{\Delta\langle\overrightarrow{B\uparrow_i A}|_i\Gamma\rangle \Rightarrow C}\,\uparrow_i L \qquad \frac{\Delta|_i\overrightarrow{A} \Rightarrow B}{\Delta \Rightarrow B\uparrow_i A}\,\uparrow_i R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{B}\rangle \Rightarrow C}{\Delta\langle\Gamma|_i\overrightarrow{A\downarrow_i B}\rangle \Rightarrow C}\,\downarrow_i L \qquad \frac{\overrightarrow{A}|_i\Delta \Rightarrow B}{\Delta \Rightarrow A\downarrow_i B}\,\downarrow_i R$$

$$\frac{\Delta\langle\overrightarrow{A}|_i\overrightarrow{B}\rangle \Rightarrow C}{\Delta\langle\overrightarrow{A \odot_i B}\rangle \Rightarrow C}\,\odot_i L \qquad \frac{\Delta \Rightarrow A \qquad \Gamma \Rightarrow B}{\Delta|_i\Gamma \Rightarrow A \odot_i B}\,\odot_i R$$

**Fig. 3.** Hypersequent Calculus for D

(7) For any type $A, B \in$, $\mathbf{cD} \vdash A \to B$ iff $\overrightarrow{A} \Rightarrow B$

We can define the well-known Lindenbaum-Tarski construction to see that **cD** is strongly complete w.r.t. $\mathcal{RD}$. The canonical model $(\mathcal{L}, v)$ where $\mathcal{L}$ is $(\mathbf{Tp}/\theta, \circ, (\circ_i)_{i>0}, \backslash\backslash, //, (\downdownarrows_i)_{i>0}(\upuparrows_i)_{i>0}; \leq)$. $\theta$ is the equivalence relation on $\mathbf{Tp}$ defined as follows: $A\,\theta B$ iff $R \vdash_{\mathbf{cD}} A \to B$ and $R \vdash_{\mathbf{cD}} B \to A$, where $R$ is a set of non-logical axioms. Using the classical tonicity properties for the connectives of $\mathbf{Tp}$, one proves that $\theta$ is a congruence. Where $A$ is a type, $\overline{A}$ is an element of $\mathbf{Tp}/\theta$ modulo $\theta$. We define $\overline{A} \leq \overline{B}$ iff $R \vdash_{\mathbf{cD}} A \to B$. We define the valuation $v$ as $v(p) = \overline{p}$ ($p$ is a primitive type). We have that for every type $A$, $[\![A]\!]_v^{\mathcal{L}} = \overline{A}$. Finally, one has that $(\mathcal{L}, v) \models A \to B$ iff $R \vdash_{\mathbf{cD}} A \to B$. From this, we infer the following theorem:

**Theorem 1.** **cD** *is strongly complete w.r.t.* $\mathcal{RD}$.

Using the embedding translation (7), we see that **hD** is strongly complete w.r.t. $\mathcal{RD}$. Since $\mathcal{DA}$ is an equational class (see Figure 4), it is closed by subalgebras,

**Continuous associativity**

$x + (y + z) \approx (x + y) + z$

**Discontinuous associativity**

$x \times_i (y \times_j z) \approx (x \times_i y) \times_{i+j-1} z$
$(x \times_i y) \times_j z \approx x \times_i (y \times_{j-i+1} z)$ if $i \leq j \leq 1 + s(y) - 1$

**Mixed permutation**

$(x \times_i y) \times_j z \approx (x \times_{j-\,S(y)\,+1} z) \times_i y$ if $j > i + s(y) - 1$
$(x \times_i z) \times_j y \approx (x \times_j y) \times_{i+S(y)-1} z$ if $j < i$

**Mixed associativity**

$(x + y) \times_i z \approx (x \times_i y) + z$ if $1 \leq i \leq s(x)$
$(x + y) \times_i z \approx x + (y \times_{i-s(x)} z)$ if $x + 1 \leq i \leq s(x) + s(y)$

**Continuous unit and discontinuous unit**

$0 + x \approx x \approx x + 0$ and $1 \times_1 x \approx x \approx x \times_i 1$

**Fig. 4.** Equational theory for $\mathcal{DA}$

direct products and homomorphic images. We have other interesting examples of DAs, namely the *powerset DA over* $\mathcal{A} = (|\mathcal{A}|, +, \{\times_i\}_{i>0}, 0, 1)$, which we denote $\mathcal{P}(A)$. We have:

(8) $\mathcal{P}(A) = (|\mathcal{P}(A)|, \circ, \{\circ_i\}_{i>0}, \mathbb{I}, \mathbb{J})$

A subset $B$ of $|\mathcal{A}|$ is called a *same-sort* subset iff:

(9) There exists an $i \in \mathbb{N}$ such that for every $a \in B$, $s(a) = i$

The notation of the carrier set of $\mathcal{P}(A)$ presupposes that its members are same-sort subsets. Where $A$, $B$ and $C$ denote same-sort subsets of $|\mathcal{A}|$, the operations $\mathbb{I}$, $\mathbb{J}$, $\circ$ and $\circ_i$ are defined as follows:

(10)
$$\begin{array}{ll} \mathbb{I} & \triangleq \{0\} \\ \mathbb{J} & \triangleq \{1\} \\ A \circ B & \triangleq \{a + b : a \in A \text{ and } b \in B\} \\ A \circ_i B & \triangleq \{a \times_i b : a \in A \text{ and } b \in B\} \end{array}$$

It is readily seen that for every $\mathcal{A}$, $\mathcal{P}(A)$ is in fact a DA. A residuated powerset displacement algebra over a displacement algebra $\mathcal{P}(A)$ is the following:

(11) $\mathcal{P}(A) = (|\mathcal{P}(A)|, \circ, \backslash\backslash, //, \{\circ_i\}_{i>0}, , \{\Uparrow_i\}_{i>0}, \{\Downarrow_i\}_{i>0}, \mathbb{I}, \mathbb{J}; \subseteq)$

$\backslash\backslash$, $//$, $\upuparrows_i$ and $\downdownarrows_i$ are defined as follows:

$$(12)\quad \begin{array}{l} A\backslash\backslash B \triangleq \{d : \text{ for every } a \in A,\ a + d \in B\} \\ A//B \triangleq \{d : \text{ for every } a \in A,\ a + d \in B\} \\ B\upuparrows_i B \triangleq \{d : \text{ for every } a \in A,\ d \times_i b \in B\} \\ A\downdownarrows_i B \triangleq \{d : \text{ for every } a \in A,\ a \times_i d \in B\} \end{array}$$

The class of powerset residuated DAs over a DA is denoted $\mathcal{PRDD}$. The class of powerset residuated DAs over a standard DA is denoted $\mathcal{PRSD}$. Finally, the subclass of powerset residuated algebras over finitely-generated standard DA, is denoted $\mathcal{PRSD}_{fg}$. Models over residuated DAs of this subclass, are called L-models. Every standard DA $\mathcal{A}$ has two remarkable properties, namely the property that sort domains $|\mathcal{A}|_i$ (for $i > 0$) can be defined in terms of $|\mathcal{A}|_0$, and the property that every element $a$ of a sort domain $|\mathcal{A}|_i$ is decomposed uniquely around the separator 1:

$$(13)\quad \begin{array}{l} \text{a) For } i > 0,\ |\mathcal{A}|_i = \underbrace{|\mathcal{A}|_0 \circ \{1\} \cdots \{1\} \circ |\mathcal{A}|_0}_{(i-1)\ 1's} \qquad (1) \\ \text{b) For } i > 0, \text{ If } a_0 + 1 + \cdots + 1 + a_i = b_0 + 1 + \cdots + 1 + b_i \text{ then} \\ a_k = b_k \text{ for } 0 \leq k \leq i \end{array}$$

We say that the sort domains of $|\mathcal{A}|$ are *separated* by 1. The single Property (13 a) is called *weakly separtion*, and both properties of (13) constitute what we call *strong separation*.

Standard DAs, as suggests its denomination, are effectively general DAs:

**Lemma 2.** $\mathcal{SD} \subseteq \mathcal{DA}$.[3]

*Proof.* We define a useful notation which will help us to prove the lemma. Where $\mathcal{A} = (|\mathcal{A}|, +, (\times_i)_{i>0}, 0, 1)$ is a standard DA, let $a$ be an arbitrary element of sort $s(a)$. We associate to every $a \in |\mathcal{A}|$ a sequence of elements $a_0, \cdots, a_{s(A)}$. We have the following vectorial notation:

$$(14)\quad \overrightarrow{a}_i^j = \begin{cases} a_i, \text{ if } i = j \\ \overrightarrow{a}_i^{j-1} + 1 + a_j, \text{ if } j - i > 0 \end{cases}$$

Since $\mathcal{A}$ is standard DA, the $a_i$ associated to a given $\overrightarrow{a}$ are unique (by freeness of the underlying monoid). We have that $a = \overrightarrow{a}_0^{s(A)}$, and we write $\overrightarrow{a}$ in place of $\overrightarrow{a}_0^{s(A)}$. Consider arbitrary elements of $|\mathcal{A}|$, $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$:

- Continuous associativity is obvious.

- Discontinuous associativity. Let $i, j$ be such that $i \leq j \leq i + s(\overrightarrow{a}) - 1$:

$$(15)\quad \begin{array}{l} \overrightarrow{b} \times_j \overrightarrow{c} = \overrightarrow{b}_0^{i-1} + \overrightarrow{c} + \overrightarrow{b}_i^{s(b)}, \text{ therefore:} \\ \overrightarrow{a} \times_i (\overrightarrow{b} \times_j \overrightarrow{c}) = \boxed{\overrightarrow{a}_0^{i-1} + \overrightarrow{b}_0^{j-1} + \overrightarrow{c} + \overrightarrow{b}_j^{s(b)} + \overrightarrow{a}_i^{s(a)}} \end{array}$$

[3] Later we see that the inclusion is proper, i.e. $\mathcal{SD} \subsetneq \mathcal{DA}$

On the other hand, we have that:

$$\overrightarrow{a} \times_i \overrightarrow{b} = \overrightarrow{a}_0^{i-1} + \overrightarrow{b} + \overrightarrow{a}_i^{s(\overrightarrow{a})} = \overrightarrow{a}_0^{i-1} + \overrightarrow{b}_0^{j-1} + \underbrace{1}_{(i+j-1)\text{-th } separator} + \overrightarrow{b}_j^{s(\overrightarrow{b})} + \overrightarrow{a}_i^{s(\overrightarrow{a})}$$

It follows that:

(16) $(\overrightarrow{a} \times_i \overrightarrow{b}) \times_{i+j-1} \overrightarrow{c} = \boxed{\overrightarrow{a}_0^{i-1} + \overrightarrow{b}_0^{j-1} + \overrightarrow{c} + \overrightarrow{b}_j^{s(\overrightarrow{b})} + \overrightarrow{a}_i^{s(\overrightarrow{a})}}$

By comparing the rhs of (15) and (16), we have therefore:

$$\overrightarrow{a} \times_i (\overrightarrow{b} \times_j \overrightarrow{c}) = (\overrightarrow{a} \times_i \overrightarrow{b}) \times_{i+j-1} \overrightarrow{c}$$

- Mixed Permutation. Consider $(\overrightarrow{a} \times_i \overrightarrow{b}) \times_j \overrightarrow{c}$ and suppose that $i + s(\overrightarrow{b}) - 1 < j$:

$$\overrightarrow{a} \times_i \overrightarrow{b} = \underbrace{\overrightarrow{a}_0^{i-1} + \overrightarrow{b} + \overrightarrow{a}_i^{j-s(\overrightarrow{b})}}_{j-s(\overrightarrow{b})+s(\overrightarrow{b})-1=j-1 \text{ separators}} + 1 + \overrightarrow{a}_{j-s(\overrightarrow{b}+1)}^{s(\overrightarrow{a})}$$

It follows that:

(17) $(\overrightarrow{a} \times_i \overrightarrow{b}) \times_j \overrightarrow{c} = \boxed{\overrightarrow{a}_0^{i-1} + \overrightarrow{b} + \overrightarrow{a}_i^{j-s(\overrightarrow{b})} + \overrightarrow{c} + \overrightarrow{a}_{j-s(\overrightarrow{b})+1}^{s(\overrightarrow{a})}}$

Since $i + s(\overrightarrow{b}) - 1 < j$, then $i < j - s(\overrightarrow{b}) + 1$. Then we have that:

$$\overrightarrow{a} \times_{j-s(\overrightarrow{b})+1} \overrightarrow{c} = \overrightarrow{a}_0^{i-1} + 1 + \overrightarrow{a}_i^{j-s(\overrightarrow{b})} + \overrightarrow{c} + \overrightarrow{a}_{j-s(\overrightarrow{b})+1}^{s(\overrightarrow{a})}$$

It follows that:

(18) $(\overrightarrow{a} \times_{j-s(\overrightarrow{b})+1} \overrightarrow{c}) \times_i \overrightarrow{b} = \boxed{\overrightarrow{a}_0^{i-1} + \overrightarrow{b} + \overrightarrow{a}_i^{j-s(\overrightarrow{b})} + \overrightarrow{c} + \overrightarrow{a}_{j-s(\overrightarrow{b})+1}^{s(\overrightarrow{a})}}$

By comparing the rhs of (17) and (18), we have therefore:

$$(\overrightarrow{a} \times_i \overrightarrow{b}) \times_j \overrightarrow{c} = (\overrightarrow{a} \times_{j-s(\overrightarrow{b})+1} \overrightarrow{c}) \times_i \overrightarrow{b}$$

- Mixed associativity. There are two cases: $i \leq s(\overrightarrow{a})$ or $i > s(\overrightarrow{a})$. Considering the first one, this is true for:

$$(\overrightarrow{a} + \overrightarrow{b}) \times \overrightarrow{c} = (\overrightarrow{a}_0^{i-1} + 1 + \overrightarrow{a}_i^{s(\overrightarrow{a})}) \times_i \overrightarrow{c} = \overrightarrow{a}_0^{i-1} + \overrightarrow{c} + \overrightarrow{a}_i^{s(\overrightarrow{a})} + \overrightarrow{b} = (\overrightarrow{a} \times_i \overrightarrow{c}) + \overrightarrow{b}$$

The other case corresponding to $i > s(\overrightarrow{a})$ is completely similar.

- The case corresponding to the units is completely trivial.

$\square$

### 2.3 Some special DAs

The standard DA $\mathcal{S}$, induced by the separated monoid with generator set $\mathbf{TypSeg}_* \cup \{1\}$, plays an inportant role. The interpretation of the signature $\Sigma_D$ in $|\mathcal{S}|$ is:

(19) $\mathcal{S} = ((\mathbf{TypSeg}_* \cup \{1\})^*, (,), \{|_i\}_{i>0}, \Lambda, 1)$

We have seen in section 2 that **HConfig** is closed by concatenation $(,)$ and intercalation $|_i$, $i > 0$, i.e. $\mathcal{C} = (\mathbf{HConfig}, (,), (|_i)_{i>0}, \Lambda, 1)$ is a $\Sigma_D$-algebra.[4] Since $\mathcal{DA}$ is an equational class, it is closed by $(\Sigma_D)$-subalgebras. Since $\mathcal{C}$ is a subalgebra of $\mathcal{S}$, hence $\mathcal{C}$ is a (general) DA, concretely a nonstandard DA. To see that $\mathcal{C}$ cannot be standard we notice that the sort domains of $\mathcal{C}$ are not separated by $\{1\}$. Recall that $|\mathcal{C}| = \bigcup_{i\geq 0} \mathbf{HConfig}_i$ ($|\mathcal{C}|_i = \mathbf{HConfig}_i$, for every $i \geq 0$). We have that:

(20) For $i > 0, |\mathcal{C}|_i \neq \underbrace{\mathbf{HConfig}_0 \circ \cdots \circ \mathbf{HConfig}_0}_{i \text{ times}}$

For, for example let us take $\overrightarrow{p\uparrow_1 p} = \lceil p\uparrow_1 p\rceil_0, 1, \lceil p\uparrow_1 p\rceil_1$, where $p \in \mathbf{Pr}$. Type $p\uparrow_1 p$ has sort 1, but clearly neither $\lceil p\uparrow_1 p\rceil_0$ nor $\lceil p\uparrow_1 p\rceil_1$ are members of $\mathbf{HConfig}_0$. In fact, we have the proper inclusion:

(21) For $i > 0, \underbrace{\mathbf{HConfig}_0 \circ \cdots \circ \mathbf{HConfig}_0}_{i \text{ times}} \subsetneq |\mathcal{C}|_i$

It follows that the class of standard DAs is a proper subclass of the class of general DAs:

(22) $\mathcal{SD} \subsetneq \mathcal{DA}$

The Lindenbaum algebra $\mathcal{L}$ defined in the previous Subsection is a nonstandard DA, because again, the sort domains are not separated by $\{1\}$. The initial $\Sigma_D$-algebra $\mathcal{N}$ in $\mathcal{DA}$ is the standard displacement algebra induced by the singleton generator set $\{1\}$, where $\mathcal{N} = (\mathbb{N}, +^{\mathcal{N}}, \{\times_i^{\mathcal{N}}\}_{i>0}, 0^{\mathcal{N}}, 1^{\mathcal{N}})$. The elements of the signature are interpreted as follows:

(23)
$$\begin{array}{ll}
0^{\mathcal{N}} & \triangleq 0 \\
1^{\mathcal{N}} & \triangleq 1 \\
+^{\mathcal{N}}(n,m) & \triangleq n+m, \text{ where } n,m \in \mathbb{N} \\
\text{Where } i>0,\ \times_i^{\mathcal{N}}(n,m) & \triangleq n+m-1, \text{ where } n>0 \text{ and } m \geq 0
\end{array}$$

[4] Observe that the sort functionalities of $(,)$ and $|_i$ $(i > 0)$ are respectively $(i, j \to i + j)_{i,j\geq 0}$, and $(i, j \to i + j - 1)_{i>0,j\geq 0}$, where $s(0) = 0$, and $, s(1) = 1$.

### Synthetic Connectives and the Implicative fragment

From a logical point of view, synthetic connectives abbreviate derivations mainly in sequent systems. They form new connectives with left and right sequent rules. Using a linear logic slogan, synthetic connectives help to eliminate some *bureaucracy* in cut-free proofs and in the (syntactic) Cut elimination algorithms. We consider a set of synthetic connectives which are of linguistic interest (Figure 5 corresponds to their semantic interpretation, Figure 6 and Figure 7 correspond to their hypersequent rules). By these definition it is readily seen that the unary synthetic connectives form residuated pairs, and the binary synthetic connectives form residuated triples. We define what we call *the implicative fragment* of **D** which contains all the continuous and discontinous implicative rules, as well as the synthetic connectives defined here, which are considered implicative in the sense that their semantic interpretations in Figure 5 are defined in terms of implicational subset connectives such as $\backslash\backslash, //, \downarrow\downarrow_i$ and $\uparrow\uparrow_i$. We have the following notations: where $\mathbf{Tp}_*$ is the set of unit-free types, $\mathbf{Tp}_*[\rightarrow]$ is the set of $\mathbf{Tp}_*$-types restricted to the implicative fragment. Similarly, we have $\mathbf{hD}_*[\rightarrow]$, $\mathbf{TypSeg}_*[\rightarrow]$, and $\mathbf{HSeq}_*[\rightarrow]$ (where $\mathbf{HSeq}_*$ is the set of hypersequents over unit-free types).[5]

| | | |
|---|---|---|
| $[\![\triangleleft^{-1}A]\!] \triangleq [\![A]\!]//\mathbb{J}$ | | left projection |
| $[\![\triangleright^{-1}A]\!] \triangleq \mathbb{J}\backslash\backslash[\![A]\!]$ | | right projection |
| $[\![\check{}^{\,i}A]\!] \triangleq [\![A]\!]\uparrow_i\mathbb{I}$ | | $i$-th split |
| $B\Uparrow A$ | $\triangleq B\uparrow_1 A \,\&\, \cdots \,\&\, B\uparrow_{s(B)-s(A)+1}A$ | nondeterministic extract |
| $A\Downarrow B$ | $\triangleq A\downarrow_1 B \,\&\, \cdots \,\&\, A\downarrow_{s(B)-s(A)+1}B$ | nondeterministic infix |

**Fig. 5.** Semantic interpretation in standard DAs for the set of synthetic connectives

## 3 Strong Completeness of the implicative fragment w.r.t. L-models

In this section we prove theorem 3, which states that $\mathbf{D}_*[\rightarrow]$ is strongly complete w.r.t. language models. More concretely, the theorem establishes the strong completeness w.r.t. the hypersequent calculus $\mathbf{HSeq}_*[\rightarrow]$. In order to prove it, we demonstrate first strong completeness of $\mathbf{HSeq}_*[\rightarrow]$ w.r.t. powerset residuated DAs over standard DAs with a countable set of generators. Let $V = \mathbf{TypSeg}_*[\rightarrow] \cup \{1\}$. Clearly, $V$ is countably infinite since $\mathbf{TypSeg}_*[\rightarrow]$ is the countable union $\bigcup_i \mathbf{TypSeg}_*[\rightarrow]_i$ (see (3)), where each $\mathbf{TypSeg}_*[\rightarrow]_i$ is also countably infinite.

[5] The unary connectives are definable in terms of (full) **Tp** (using the units, e.g. $\check{}^{\,k}A \triangleq A\uparrow_k I$), but the nondeterministic $\Downarrow$ and $\Uparrow$ are no longer definable only in terms of **Tp**, (see [11]).

$$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow B}{\Gamma\langle\overrightarrow{\triangleleft^{-1}A, 1}\rangle \Rightarrow B}\triangleleft^{-1}L \qquad \frac{\Gamma, 1 \Rightarrow A}{\Gamma \Rightarrow \triangleleft^{-1}A}\triangleleft^{-1}R$$

$$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow B}{\Gamma\langle\overrightarrow{1, \triangleright^{-1}A}\rangle \Rightarrow B}\triangleright^{-1}L \qquad \frac{1, \Gamma \Rightarrow A}{\Gamma \Rightarrow \triangleright^{-1}A}\triangleright^{-1}R$$

$$\frac{\Delta\langle\overrightarrow{B}\rangle \Rightarrow C}{\Delta\langle\overrightarrow{\check{}^{i}B}|_i\Lambda\rangle \Rightarrow C}\check{}^{i}L \qquad \frac{\Delta|_i\Lambda \Rightarrow BB}{\Delta \Rightarrow \check{}^{i}B}\check{}R$$

**Fig. 6.** Hypersequent rules for synthetic unary implicative connectives

$$\frac{\Delta \Rightarrow A \qquad \Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{B\Uparrow A}|_i\Gamma\rangle \Rightarrow C}\Uparrow L \qquad \frac{\Delta|_1\overrightarrow{A} \Rightarrow B \qquad \cdots \qquad \Delta|_d\overrightarrow{A} \Rightarrow B}{\Delta \Rightarrow B\Uparrow A}\Uparrow R$$

$$\frac{\Delta \Rightarrow A \qquad \Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\Gamma|_i\overrightarrow{A\Downarrow B}\rangle \Rightarrow C}\Downarrow L \qquad \frac{\overrightarrow{A}|_1\Delta \Rightarrow B \qquad \cdots \qquad \overrightarrow{A}|_a\Delta \Rightarrow B}{\Delta \Rightarrow A\Downarrow B}\Downarrow R$$

**Fig. 7.** Hypersequent calculus rules for nondeterministic synthetic connectives

Let us consider the standard DA $\mathcal{S}$ (from (19)), induced by the (countably) infinite set of generators $V$:

$$\mathcal{S} = (V^*, (,), \{|_i\}_{i>0}, \Lambda, 1)$$

In order to prove completeness of $\mathbf{HSeq}^*[\rightarrow]$ w.r.t. the class of powerset DAs over (countable) standard DAs ($\mathcal{PRSD}$), we define some useful notation:

**Definition 3.** *For any type $C \in \mathbf{Tp}_*[\rightarrow]$, and a set of sequents $R$ of* **HSeq**:

$$[C]_R \triangleq \{\Delta : \Delta \in \mathbf{HConfig} \text{ and } R \vdash \Delta \Rightarrow C\}$$

In practice, when the set $R$ of hypersequents is clear by the context, we simply write $[C]$ instead of $[C]_R$. Let us fix some set of hypersequents $R$:

**Lemma 3.** (**Truth Lemma**)
Let $\mathcal{P}(S)$ be the powerset residuated DA over the standard DA $\mathcal{S}$ from (19) . Let $v$ be the following valuation on the powerset $\mathcal{P}(S)$:

$$\text{For every } p \in \mathbf{Pr},\ v(p) \triangleq [p]_R$$

Let $\mathcal{M} = (\mathcal{P}(S), v)$ be called as usual the canonical model. The following equality holds:

$$\text{For every } C \in \mathbf{Tp}_*[\rightarrow],\ [\![C]\!]_v^{\mathcal{P}(\mathcal{A})} = [C]$$

*Proof.* We proceed by induction on the structure of type $C$. Let us write $[\![\cdot]\!]$ instead of $[\![C]\!]_v^{\mathcal{P}(\mathcal{A})}$, and $[\cdot]$ instead of $[\cdot]_R$. We will say that an element $\Delta \in [\![A]\!]$ is *correct* iff $\Delta \in \mathbf{HConfig}$.

- $C$ is primitive. True by definition.
- $C = B{\uparrow_i}A$. Let us see:

$$[B{\uparrow_i}A] \subseteq [\![B{\uparrow_i}A]\!]$$

Let $\Delta$ be such that $R \vdash \Delta \Rightarrow B{\uparrow_i}A$. Let $\Gamma_A \in [\![A]\!]$. By induction hypothesis (i.h.), $[\![A]\!] = [A]$. Hence, $R \vdash \Gamma_A \Rightarrow A$ We have:

$$\frac{\Delta \Rightarrow B{\uparrow_i}A \qquad \overrightarrow{B{\uparrow_i}A}|_i\Gamma_A \Rightarrow B}{\Delta|_i\Gamma_A \Rightarrow B}\,Cut$$

By (i.h.), $[\![B]\!] = [B]$. It follows that $\Delta|_i\Gamma_A \in [\![B]\!]$, hence $\Delta \in [\![B{\uparrow_i}A]\!]$. Whence, $[B{\uparrow_i}A] \subseteq [\![B{\uparrow_i}A]\!]$.
Conversely, let us see:

$$[\![B{\uparrow_i}A]\!] \subseteq [B{\uparrow_i}A]$$

Let $\Delta \in [\![B{\uparrow_i}A]\!]$. By i.h. $[\![A]\!] = [A]$. For any type $A$, we have eta-expansion, i.e. $\overrightarrow{A} \Rightarrow A$. Hence, $\overrightarrow{A} \in [\![A]\!]$. We have that $\Delta|_i\overrightarrow{A} \in [\![B]\!]$. By i.h., $\Delta|_i\overrightarrow{A} \Rightarrow B$. Since $\overrightarrow{A}$ is correct, and by i.h. $\Delta|_i\overrightarrow{A}$ is correct, by lemma 1, $\Delta$ is correct. By applying the $\uparrow_i$ right rule to the provable hypersequent $\Delta|_i\overrightarrow{A} \Rightarrow B$ we get:

$$\Delta \Rightarrow B{\uparrow_i}A$$

This ends the case of $B{\uparrow_i}A$.

- $C = A{\downarrow_i}B$. Completely similar to case $B{\uparrow_i}A$.
- $C = B/A$. Let us see:

$$[B/A] \subseteq [\![B/A]\!]$$

Let $\Delta$ be such that $R \vdash \Delta \Rightarrow B/A$. Let $\Gamma_A \in [\![A]\!]$. By induction hypothesis (i.h.), $[\![A]\!] = [A]$. Hence, $R \vdash \Gamma_A \Rightarrow A$ We have:

$$\frac{\Delta \Rightarrow B{\uparrow_i}A \qquad \overrightarrow{B/A}, \Gamma_A \Rightarrow B}{\Delta, \Gamma_A \Rightarrow B}\,Cut$$

By (i.h.), $[\![B]\!] = [B]$. It follows that $\Delta, \Gamma_A \in [\![B]\!]$. Whence, $[B/A] \subseteq [\![B/A]\!]$.
Conversely, let us see:

$$[\![B/A]\!] \subseteq [B/A]$$

Let $\Delta \in [\![B/A]\!]$. By i.h. $[\![A]\!] = [A]$. For any type $A$, we have eta-expansion, i.e. $\overrightarrow{A} \Rightarrow A$. Hence, $\overrightarrow{A} \in [\![A]\!]$. We have that $\Delta, \overrightarrow{A} \in [\![B]\!]$. By i.h., $\Delta, \overrightarrow{A} \Rightarrow B$.

Since $\overrightarrow{A}$ is correct, and by i.h. $\Delta, \overrightarrow{A}$ is correct, by lemma 1 $\Delta$ is correct. By applying the / right rule to the provable hypersequent $\Delta, \overrightarrow{A} \Rightarrow B$ we get:

$$\Delta \Rightarrow B/A$$

This ends the case of $B/A$.

- $C = A\backslash B$. Completely similar to the case $C = B/A$.
- Nondeterministic connectives. Consider the case $C = B \Uparrow_i A$.

$$[B \Uparrow A] \subseteq [\![B \Uparrow A]\!]$$

Let $\Gamma_A \in [\![A]\!]$. By i.h, $\Gamma_A \Rightarrow A$. Let $\Delta \Rightarrow B \Uparrow A$. By $s(B) - s(A) + 1$ applications of $\Uparrow$ left rule, we have

$$\frac{\Gamma_A \Rightarrow A \qquad \overrightarrow{B} \Rightarrow B, \text{ by eta-expansion}}{\overrightarrow{B \Uparrow A}|_i \Gamma_A \Rightarrow B, \text{ for } i = 1, \cdots, s(B) - s(A) + 1} \Uparrow L$$

By $s(B) - s(A) + 1$ Cut applications with $\Delta \Rightarrow B \Uparrow A$, we get:

$$\Delta|_i \Gamma_A \Rightarrow B$$

Hence, for $i = 1, \cdots s(B) - s(A) + 1$, by i.h. $\Delta, \Gamma_A \in [\![B]\!]$. Hence, $\Delta \in [\![B \Uparrow A]\!]$. Conversely, let us see:

$$[\![B \Uparrow A]\!] \subseteq [B \Uparrow A]$$

By i.h, we see that $\overrightarrow{A} \in [\![A]\!]$. Let $\Delta \in [\![B \Uparrow A]\!]$. This means that for every $i = 1, \cdots, s(B) - s(A) + 1$ $\Delta|_i \overrightarrow{A} \in [\![B]\!]$. By i.h., $\Delta|_i \overrightarrow{A} \Rightarrow B$. By a similar reasoning to the deterministic case $C = B\uparrow_i A$, we see that $\Delta$ is correct. We have that:

$$\frac{\Delta|_1 \overrightarrow{A} \Rightarrow B \qquad \cdots \qquad \Delta|_{s(B)-s(A)+1} \overrightarrow{A} \Rightarrow B}{\Delta \Rightarrow B \Uparrow A} \Uparrow R$$

- The case $C = A \Downarrow B$ is completely similar to the previous one.
  Let us see the cases corresponding to the unary (implicative) connectives.
- Left projection case: $C = \triangleleft^{-1} A$. Let us see:

$$[\triangleleft^{-1} A] \subseteq [\![\triangleleft^{-1} A]\!]$$

Let $\Delta \in [\triangleleft^{-1} A]$. Hence, $\Delta \Rightarrow \triangleleft^{-1} A$. We have that:

$$\frac{\Delta \Rightarrow \triangleleft^{-1} A \qquad \dfrac{\overrightarrow{A} \Rightarrow A}{\overrightarrow{\triangleleft^{-1} A}, 1 \Rightarrow A} \triangleleft^{-1} L}{\Delta, 1 \Rightarrow A} Cut$$

By i.h., $\Delta, 1 \in [\![A]\!]$. Hence, $\Delta \in [\![\triangleleft^{-1} A]\!]$.
Conversely, let us see:

$$[\![\triangleleft^{-1} A]\!] \subseteq [\triangleleft^{-1} A]$$

Let $\Delta \in [\![\triangleleft^{-1} A]\!]$. By definition, $\Delta, 1 \in [\![A]\!]$. By i.h., $\Delta, 1 \Rightarrow A$, and by lemma 1, $\Delta$ is correct. By application of $\triangleleft^{-1}$ right rule, we get:

$$\Delta \Rightarrow \triangleleft^{-1} A$$

This proves the converse.

- Case $C = \triangleright^{-1} A$ is completely similar to the previous one.

- Case $C = \check{}_k A$. Let us see:

$$[\check{}_k A] \subseteq [\![\check{}_k A]\!]$$

Let $\Delta \Rightarrow \check{}_k A$. We have that:

$$\dfrac{\Delta \Rightarrow \check{}_i A \qquad \dfrac{\overrightarrow{A} \Rightarrow A}{\overrightarrow{\check{}_i A}|_k \Lambda \Rightarrow A}\,\check{}_k L}{\Delta|_k \Lambda \Rightarrow A}\,Cut$$

By i.h., $\Delta \in [\![\check{}_k A]\!]$.
Conversely, let us see that:

$$[\![\check{}_k A]\!] \subseteq [\check{}_k A]$$

Let $\Delta \in [\![\check{}_k A]\!]$. By definition, $\Delta|_k \Lambda \in [\![A]\!]$. By i.h. and lemma 1, $\Delta$ is correct and $\Delta|_k \Lambda \Rightarrow A$. By application of the $\check{}_k$ right rule:

$$\Delta \Rightarrow \check{}_k A$$

Hence, $\Delta \in [\check{}_k A]$.

We have seen all the cases of the so-called implicative fragment. We are done. $\square$

By induction on the structure of **HConfig** (see (4)) one proves the following lemma:

**Lemma 4.** (**Identity lemma**)
For any $\Delta \in \mathbf{HConfig}$, $\Delta \in [\![\Delta]\!]^{\mathcal{M}}$.

Let $(A_i)_{i=1,\cdots,n}$ be the sequence of type-occurrences in a hyperconfiguration $\Delta$. Let $\Delta\begin{pmatrix} \Gamma_1 \cdots \Gamma_n \\ A_1 \cdots A_n \end{pmatrix}$ be the result of replacing every type-occurrence $A_i$ with $\Gamma_i$. Recall that we have fixed a set of hypersequents $R$. We have the lemma:

**Lemma 5.** $\mathcal{M} = (\mathcal{P}(S), v) \models R$

*Proof.* Let $(\Delta \Rightarrow A) \in R$. For every type-occurrence $A_i$ in $\Delta$ (we suppose that the sequence of type occurrences in $\Delta$ is $(A_i)_{i=1,\cdots,n}$), we have that $[\![A_i]\!]^{\mathcal{M}}_v = [A_i]_R$. For any $\Gamma_i \in [\![A_i]\!]^{\mathcal{M}}_v$, we have by the truth lemma 3 that $R \vdash \Gamma_i \Rightarrow A_i$. Since $(\Delta \Rightarrow A) \in R$, we have then that $R \vdash \Delta \Rightarrow A$. By $n$ applications of the Cut rule with the premises $\Gamma_i$ we get from $R \vdash \Delta \Rightarrow A$ that $R \vdash \Delta\begin{pmatrix}\Gamma_1 \cdots \Gamma_n \\ A_1 \cdots A_n\end{pmatrix} \Rightarrow A$. We have that $[\![\Delta]\!]^{\mathcal{M}}_v = \{\Delta\begin{pmatrix}\Gamma_1 \cdots \Gamma_n \\ A_1 \cdots A_n\end{pmatrix} : \Gamma_i \in [\![A_i]\!]^{\mathcal{M}}_v\}$. Since, we have $R \vdash \Delta\begin{pmatrix}\Gamma_1 \cdots \Gamma_n \\ A_1 \cdots A_n\end{pmatrix} \Rightarrow A$, again, by the truth lemma, $\Delta\begin{pmatrix}\Gamma_1 \cdots \Gamma_n \\ A_1 \cdots A_n\end{pmatrix} \in [\![A]\!]^{\mathcal{M}}_v$. We have then that $[\![\Delta]\!]^{\mathcal{M}}_v \subseteq [\![A]\!]^{\mathcal{M}}_v$. We are done. □

**Theorem 2.** $\mathbf{D}_*[\to]$ *is strongly complete w.r.t. the class* $\mathcal{PRSD}$.

*Proof.* Suppose $\mathcal{PRSD}(R) \models \Delta \Rightarrow A$. Hence, in particular this is true of the canonical model $\mathcal{M}$. Since $\Delta \in [\![\Delta]\!]^{\mathcal{M}}$, it follows that $\Delta \in [\![A]\!]^{\mathcal{M}}$. By the truth lemma, $[\![A]\!]^{\mathcal{M}} = [A]_R$. Hence, $R \vdash \Delta \Rightarrow A$. We are done. □

Let $\mathcal{A}$ and $\mathcal{B}$ be respectively standard DAs over separated monoids with a countable set of generators $V_1 = (a_i)_{i>0} \cup \{1\}$, and a finitely generated standard DA, concretely a standard DA with a set of three generators $V_2 = \{a, b, 1\}$. We have that $|\mathcal{A}| = V_1^*$, and $|\mathcal{B}| = V_2^*$. Let $\bar{\rho}$ be the following injective mapping from $V_1$ into $V_2^*$:

(24) $$\begin{aligned}\bar{\rho}(1) &= 1\\ \bar{\rho}(a_i) &= a + b^i + a\end{aligned}$$

$\rho$ extends recursively to the morphism of standard DAs $\bar{\rho}$:

(25)
$$\begin{array}{lcl}\bar{\rho} : \mathcal{A} & \longrightarrow & \mathcal{B}\\ \quad 0 & \mapsto & 0\\ \quad 1 & \mapsto & 1\\ \quad \bar{\rho}(w_1 + w_2) & \mapsto & \bar{\rho}(w_1) + \bar{\rho}(w_2)\\ \quad \bar{\rho}(w_1 \times_i w_2) & \mapsto & \bar{\rho}(w_1) \times_i \bar{\rho}(w_2)\end{array}$$

Since for every $i > 0$, $\bar{\rho}(a_i)$ is prefix-free (and hence $\bar{\rho}$),[6] $\bar{\rho}$ is a monomorphism. $\bar{\rho}$ induces then a monomorphism of standard DAs. Let $A$, $B$ and $C$ range over subsets of $|\mathcal{A}|$. Since $\bar{\rho}$ is a monomorphism of DAs and the underlying monoids of $\mathcal{A}$ and $\mathcal{B}$ are free, one proves:

(26) $$\begin{array}{llll}\bar{\rho}(A\circ B) &= \bar{\rho}(A)\circ\bar{\rho}(B) & \bar{\rho}(A\circ_i B) &= \bar{\rho}(A)\circ_i\bar{\rho}(B)\\ \bar{\rho}(A\backslash\backslash B) &= \bar{\rho}(A)\backslash\backslash\bar{\rho}(B) & \bar{\rho}(B/\!/A) &= \bar{\rho}(B)/\!/\bar{\rho}(A)\\ \bar{\rho}(A\downdownarrows_i B) &= \bar{\rho}(A)\downdownarrows_i\bar{\rho}(B) & \bar{\rho}(B\upuparrows_i A) &= \bar{\rho}(B)\upuparrows_i\bar{\rho}(A)\\ \bar{\rho}(B\upuparrows_i \mathbb{I}) &= \bar{\rho}(B)\upuparrows_i \mathbb{I} & \bar{\rho}(A/\!/\mathbb{J}) &= \bar{\rho}(A)/\!/\bar{\rho}(\mathbb{J})\\ \bar{\rho}(\mathbb{J}\backslash\backslash A) &= \mathbb{J}\backslash\backslash\bar{\rho}(A) & & \end{array}$$

[6] If $w$ is a non-empty proper prefix of $\bar{\rho}(a_i)$, then $w \notin Im(\bar{\rho})$.

Moreover, one proves also

$$(27)\quad \begin{array}{l} \bar{\rho}(\bigcap_{i=1}^{s(B)-s(A)+1} B{\uparrow\uparrow}_i A) = \bigcap_{i=1}^{s(B)-s(A)+1} \bar{\rho}(B){\uparrow\uparrow}_i\bar{\rho}(A) \\ \bar{\rho}(\bigcap_{i=1}^{s(B)-s(A)+1} A{\downarrow\downarrow}_i B) = \bigcap_{i=1}^{s(B)-s(A)+1} \bar{\rho}(A){\downarrow\downarrow}_i\bar{\rho}(A) \end{array}$$

Now, let $(\mathcal{P}(\mathcal{A}), v)$ be the powerset residuated displacement model over the standard DA $\mathcal{A}$. For every $A \in \mathbf{Tp}_*[\rightarrow]$, and $\Delta \in \mathbf{HConfig}$, one has the following facts:

$$(28)\quad [\![A]\!]_v^{\mathcal{P}(\mathcal{A})} = [\![A]\!]_{\bar{\rho}\circ v}^{\mathcal{P}(\mathcal{B})} \text{ and } [\![A]\!]_v^{\mathcal{P}(\mathcal{A})} = [\![A]\!]_{\bar{\rho}\circ v}^{\mathcal{P}(\mathcal{B})}$$

By properties (28), (26), and (27) and the fact that $\bar{\rho}$ is a monomorphism, we have therefore the following equivalence:

$$(29)\quad [\![\Delta]\!]_v^{\mathcal{P}(\mathcal{A})} \subseteq [\![A]\!]_v^{\mathcal{P}(\mathcal{A})} \text{ iff } [\![\Delta]\!]_{\bar{\rho}\circ v}^{\mathcal{P}(\mathcal{B})} \subseteq [\![A]\!]_{\bar{\rho}\circ v}^{\mathcal{P}(\mathcal{B})}$$

Where $\mathcal{K}$ is a subclass of $\mathcal{RD}$, the notation $\mathcal{K}(R) \models \Delta \Rightarrow A$ (where $R$ is a set of hypersequents) means that $\mathcal{K} \models R$ and $\mathcal{K} \models \Delta \Rightarrow A$.

**Theorem 3.** $\mathbf{D}_*[\rightarrow]$ *is strongly complete w.r.t. L-models.*

*Proof.* Let $R$ be a set of sequents. By way of contradiction, consider a hypersequent $\Delta \Rightarrow A$ such that $\mathbf{D}_*[\rightarrow] + R \not\vdash \Delta \Rightarrow A$ but $\mathcal{PRSD}_{fg}(R) \models \Delta \Rightarrow A$. Since $\mathbf{D}_*[\rightarrow]$ is strongly complete w.r.t. $\mathcal{PRSD}$ (theorem 2), there exists a model $(\mathcal{A}, v) \models R$ but $(\mathcal{A}, v) \not\models \Delta \Rightarrow A$. Let $\bar{\rho}$ the coding morphism from (25). Let $\mathcal{B}$ be the finitely generated standard displacement algebra with 3 generators $a, b$ and 1. Since $\mathcal{PRSD}_{fg}(R) \models \Delta \Rightarrow A$, we have that for every valuation $v'$, $[\![\Delta]\!]_{v'}^{\mathcal{B}} \subseteq [\![A]\!]_{v'}^{\mathcal{B}}$, in particular for the valuation $\bar{\rho} \circ v$. By (29) we have that:

$$[\![\Delta]\!]_v^{\mathcal{A}} \subseteq [\![A]\!]_v^{\mathcal{A}} \text{ iff } [\![\Delta]\!]_{\bar{\rho}\circ v}^{\mathcal{B}} \subseteq [\![A]\!]_{\bar{\rho}\circ v}^{\mathcal{B}}$$

But, by assumption, $[\![\Delta]\!]_v^{\mathcal{A}} \not\subseteq [\![A]\!]_v^{\mathcal{A}}$. Contradiction. □

**Corollary 1.** $\mathbf{HSeq}^*[\rightarrow]$ *is strongly complete w.r.t. powerset residuated DAs overs standard DAs with* 3 *generators.*

## 4 Towards Strong Completeness of full D w.r.t. $\mathcal{PRDD}$

We sketch[7] in this section strong completeness of full $\mathbf{D}$ w.r.t. $\mathcal{PRDD}$. We get this result by proving a representation theorem between $\mathcal{RD}$ and $\mathcal{PRDD}$. In order to get this representation theorem we need to consider $\mathbf{D}_*$ (unit-free $\mathbf{D}$), and consequently, $\mathbf{Tp}_*$ (unit-free $\mathbf{Tp}$), and $\mathbf{HConfig}_*$ (unit-free $\mathbf{HConfig}$). This is a step to prove strong completeness for full $\mathbf{D}$ (without restrictions on the units). We give the mutually recursive definition of the set $\mathcal{T}$ of hypertrees, and the set of atomic terms:

[7] We do not have enough space to justify the main claims. But, we believe that this sketch is quite illuminating.

(30)
$$\begin{array}{l}
\Lambda, 1 \in \mathcal{T} \\
\text{If } A \in \mathbf{Tp}_*, \text{ then } A \text{ is an atomic term} \\
\text{If } T \in \mathcal{T}, A, B \in \mathbf{Tp}_*, \text{ then } (T; A{\bullet}B; \mathbf{f}) \text{ is an atomic term} \\
\text{If } T \in \mathcal{T}, A, B \in \mathbf{Tp}_*, \text{ then } (T; A{\bullet}B; \mathbf{s}) \text{ is an atomic term} \\
\text{If } T \in \mathcal{T}, A, B \in \mathbf{Tp}_*, \text{ then } (T; A{\odot_i}B; \mathbf{f}) \text{ is an atomic term} \\
\text{If } T \in \mathcal{T}, A, B \in \mathbf{Tp}_*, \text{ then } (T; A{\odot_i}B; \mathbf{s}) \text{ is an atomic term} \\
s((T; A{\bullet}B; \mathbf{f})) = s(A) \text{ and } s((T; A{\bullet}B; \mathbf{s})) = s(B) \\
s((T; A{\odot_i}B; \mathbf{f})) = s(A) \text{ and } s((T; A{\odot_i}B; \mathbf{s})) = s(B) \\
\Lambda \in \mathcal{T}, \text{ and } 1 \in \mathcal{T} \\
\text{If } L \text{ atomic, then } \overrightarrow{L} \triangleq L\{\underbrace{1 : \cdots : 1}_{s(L)\ 1\text{'s}}\} \in \mathcal{T} \\
\text{If } T, S \in \mathcal{T}, \text{ then } T, S \in \mathcal{T} \\
\text{If } T, S \in \mathcal{T}, \text{ then } T|_i S \in \mathcal{T}
\end{array}$$

Like **HConfig**, $\mathcal{T}$ is sorted, and for every $T \in \mathcal{T}$, $s(T)$ is simply the number of separators $T$ contains. We put $\mathcal{T}_i = \{T : T \in \mathcal{T} \text{ and } s(T) = i\}$ for $i \geq 0$. We have then that $\mathcal{T} = \bigcup_{i \geq 0} \mathcal{T}_i$. Notice that $\mathcal{T}$ includes the set **HConfig**. We define now a notion of reduction $\triangleright$ in $\mathcal{T}$. Where $A, B \in \mathbf{Tp}_*$, and $T_i, S_j$, $(i = 1, \cdots, s(A),$ and $, j = 1, \cdots, s(B))$ are hypertreees, we have:

(31)
$$\begin{cases} \overrightarrow{(T; A{\bullet}B; \mathbf{f})}\{T_1 : \cdots : T_{s(A)}\}, \overrightarrow{(T; A{\bullet}B; \mathbf{s})}\{R_1 : \cdots : R_{s(B)}\} \\ \triangleright T \otimes \langle T_1 : \cdots : T_{s(A)} : R_1 : \cdots : R_{s(B)} \rangle \end{cases}$$
$$\begin{cases} \overrightarrow{(T; A{\odot_i}B; \mathbf{f})}\{T_1 : \cdots : \overrightarrow{(T; A{\odot_i}B; \mathbf{f})}\{R_1 : \cdots : R_{s(B)}\} : \cdots : T_{s(A)}\} \\ \triangleright T \otimes \langle T_1 : \cdots : R_1 : \cdots : R_{s(B)} : \cdots : T_{s(A)} \rangle \end{cases}$$

By a simple primitive type counting argument, one sees that the transitive closure of $\triangleright$ $\triangleright^*$, is always terminating, i.e. $\triangleright^*$ is strongly normalising. Again, by primitive type counting arguments, one sees that $\triangleright^*$ is weakly Church-Rosser, and hence, by Newman's lemma, $\triangleright^*$ is Church-Rosser. This allows for every element of $T \in \mathcal{T}$ to define its normal form $irr(T)$. We put $\mathbf{Irr} = irr(\mathcal{T})$. Since $\mathcal{T}$ is sorted, **Irr** is also sorted. We have that $\mathbf{Irr} = \bigcup_{i \geq 0} \mathbf{Irr}_i$, where $\mathbf{Irr}_i = \{T : T \in \mathbf{Irr} \text{ and } s(T) = i\}$. Let us consider the $\Sigma_D$-algebra:

(32) $\mathbf{Irr} = (\mathbf{Irr}, \tilde{+}, (\tilde{\times}_i)_{i \geq 0}, \Lambda, 1)$

Where $\tilde{+}$, and $(\tilde{\times}_i)_{i \geq 0}$, are defined as follows::

(33)
$$\begin{array}{l} T \tilde{+} S \triangleq irr(T, S) \\ T \tilde{\times}_i S \triangleq irr(T, S) \end{array}$$

By Church-Rosser, **Irr** is easily seen to be a (nonstandard) DA. For, given the arbitrary hypertrees $T_1$, $T_2$ and $T_3$, for example discontinuous associativity is proved as follows:

(34)
$$\begin{aligned} T_1 \tilde{\times}_i (T_2 \tilde{\times}_j T_3) &= irr(T_1|_i irr(T_2|_j T_3)) \\ &= Irr(T_1|_i (T_2|_j T_3)) = Irr((T_1|_i T_2)|_{i+j-1} T_3) \\ &= Irr(Irr(T_1|_i T_2)|_{i+j-1} T_3) \\ &= (T_1 \tilde{\times}_i T_2) \tilde{\times}_{i+j-1} T_3 \end{aligned}$$

**Irr** induces the powerset residuated DA over the DA **Irr**, which we denote $\mathcal{P}(\mathbf{Irr})$.

Following Buszkowski's technics on labelled deductive systems ([2]), we can now introduce in Figure 8 a natural deduction system $\mathbf{nD}_*$ for a conservative extension of $\mathbf{D}_*$. $R$ is a given set of $\mathbf{D}_*$-hypersequents. The axiom rule has as

$$\overrightarrow{A}\twoheadrightarrow A \text{ for every } A \in \mathbf{Tp}_*$$

$$\frac{T\twoheadrightarrow B/A \qquad S\twoheadrightarrow A}{T, S\twoheadrightarrow B}\,/E \qquad \frac{T, \overrightarrow{A}\twoheadrightarrow B}{T\twoheadrightarrow B/A}\,/I$$

$$\frac{S\twoheadrightarrow A \qquad T\twoheadrightarrow A\backslash B}{T, S\twoheadrightarrow B}\,\backslash E \qquad \frac{\overrightarrow{A}, T\twoheadrightarrow B}{T\twoheadrightarrow A\backslash B}\,\backslash I$$

$$\frac{T\twoheadrightarrow A\bullet B}{\overrightarrow{(T; A\bullet B; \mathbf{f})}\twoheadrightarrow A}\,\bullet E1 \qquad \frac{T\twoheadrightarrow A\bullet B}{\overrightarrow{(T; A\bullet B; \mathbf{s})}\twoheadrightarrow B}\,\bullet E2$$

$$\frac{T\twoheadrightarrow A \qquad S\twoheadrightarrow B}{T, S\twoheadrightarrow A\bullet B}\,\bullet I$$

$$\frac{T\twoheadrightarrow B\uparrow_i A \qquad S\twoheadrightarrow A}{T|_i S\twoheadrightarrow B}\,\uparrow_i E \qquad \frac{T|_i \overrightarrow{A}\twoheadrightarrow B}{T\twoheadrightarrow B\uparrow_i A}\,\uparrow_i I$$

$$\frac{S\twoheadrightarrow A\downarrow_i T\twoheadrightarrow A\backslash B}{S|_i T\twoheadrightarrow B}\,\downarrow_i E \qquad \frac{\overrightarrow{A}|_i T\twoheadrightarrow B}{T\twoheadrightarrow A\downarrow_i B}\,\backslash I$$

$$\frac{T\twoheadrightarrow A\odot iB}{\overrightarrow{(T; A\odot iB; \mathbf{f})}\twoheadrightarrow A}\,\odot iE1 \qquad \frac{T\twoheadrightarrow A\odot iB}{\overrightarrow{(T; A\odot iB; \mathbf{s})}\twoheadrightarrow B}\,\odot iE2$$

$$\frac{T\twoheadrightarrow A \qquad S\twoheadrightarrow B}{T|_i S\twoheadrightarrow A\odot iB}\,\odot iI$$

$$\frac{T\twoheadrightarrow A}{S\twoheadrightarrow A}\,\mathit{Red} \text{ If T} \rhd^* \text{S}$$

$$\frac{T_i\twoheadrightarrow A_i \text{ where } i = 1, \cdots, n}{\Delta\begin{pmatrix} T_1 \cdots T_n \\ A_1 \cdots A_n \end{pmatrix}\twoheadrightarrow A}\,\mathit{Axiom}_R,\ (\Delta \Rightarrow A) \in R$$

**Fig. 8.** $\mathbf{nD}_*$ rules

premises $T_i\twoheadrightarrow A_i$ where $(A_i)_{i=1,\cdots,n}$ is the sequence of type-occurrences of $\Delta$. We can prove that for $R \vdash \Delta \Rightarrow A$ iff $R \vdash \Delta\twoheadrightarrow A$.

One considers the following canonical model $\mathcal{M} = (\mathcal{P}(\mathbf{Irr}), \alpha_R)$, where $\alpha_R(p) = [p]_R \triangleq \{T : R \vdash T \twoheadrightarrow p\}$. Writing $[\![ \cdot ]\!]$ instead of $[\![ \cdot ]\!]^{\mathcal{M}}_{\alpha_R}$, one proves that for every type $A \in \mathbf{Tp}_*$, $[\![A]\!] = [A]_R$. By rule $Axiom_R$ of $\mathbf{nD}_*$, it is readily seen that $\mathcal{M} \models R$. The product rules of elimination help to straightforwardly prove that $[\![A \star B]\!] = [A \star B]_R$, where $\star \in \{\bullet, \odot_i : i > 0\}$. To prove that for every residuated DA algebra $\mathcal{A}$ is isomorphically embeddable into a powerset rediduated DA over a DA, one defines a bijection between the carrier set of $\mathcal{A}$ and the set of primitive types $Pr = (p_a)_{a \in |\mathcal{A}|}$. We define the valuation $\mu(p_a) = a$, and the set of hypersequents which hold in $(\mathcal{A}, \mu)$, i.e. $R = \{\Delta \Rightarrow A : \mu(\Delta) \leq \mu(A)\}$. We consider the canonical model $\mathcal{M} = (\mathcal{P}(\mathbf{Irr}), \alpha_R)$, and we define the faithful monomorphism $h : |\mathcal{A}| \to |\mathcal{P}(\mathbf{Irr})|$ such that $h(a) := \alpha_R(p_a)$, and $h(0) = \Lambda$, and $h(1) = 1$. Finally, in order to obtain full strong completeness w.r.t. $\mathcal{PRDD}$, one uses the representation theorem and the fact that $\mathbf{hD}$ is strongly complete with respect residuated DAs (see Subsection 2.2).

# 5 Conclusions

The strong completeness theorems we have proved are quite analogous to the ones of $\mathbf{L}_*$. The semantics are quite natural as in the case of $\mathbf{L}_*$. We think that these results constitute a big step towards the study of the model theory of $\mathbf{hD}$.

It is known that $\mathbf{L}_*$ is not weakly complete w.r.t. free monoids (see [11]). Hence, weak completeness of full $\mathbf{D}$ (with units) w.r.t. L-models is not possible. It remains open whether $\mathbf{D}_*$ is weakly complete w.r.t. L-models.

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