Stars and Celebrities: A Network Creation Game

C. Àlvarez, M. Blesa, A. Duch, A. Messegué, and M. Serna

ALBCOM Research Group, Computer Science Department, UPC, Barcelona
{alvarez,mjblesa,duch,messegue,mjserna}@cs.upc.edu

Abstract. *Celebrity games*, a new model of network creation games is introduced. The specific features of this model are that players have different celebrity weights and that a critical distance is taken into consideration. The aim of any player is to be close (at distance less than critical) to the others, mainly to those with high celebrity weights. The cost of each player depends on the cost of establishing direct links to other players and on the sum of the weights of those players at a distance greater than the critical distance. We show that celebrity games always have pure Nash equilibria and we characterize the family of subgraphs having connected Nash equilibria, the so called *star celebrity games*. Exact bounds for the PoA of non star celebrity games and a bound of $O(n/\beta + \beta)$ for star celebrity games are provided. The upper bound on the PoA can be tightened when restricted to particular classes of Nash equilibria graphs. We show that the upper bound is $O(n/\beta)$ in the case of 2-edge-connected graphs and 2 in the case of trees.

1 Introduction

Nowadays social networks have become a huge interdisciplinary research area with important links to Sociology, Economics, Epidemiology, Computer Science, and Mathematics among others [16,13,4,15]. We propose to analyze network creation in the context of social networks by taking into consideration two of their fundamental aspects: the *small-world phenomenon* and a *celebrity weight*: a ranking of the participants on some particular feature as popularity, relevance, influence, etc. The well known empirical study of the small-world phenomenon was undertaken by Stanley Milgram [21,22], who asked to some randomly chosen individuals to forward a letter to a designated "target" person. A third of the letters eventually arrived to their target, in six steps on average. Currently in social networks, celebrities using Tumblr, Instagram, or Twitter communicate with millions of fans through posted messages which are read and resent by their followers reaching, in this way, a much wider audience than strictly their fans.

In this work we want to understand the impact of the features mentioned above in a process of network creation and, inspired in the model of network creation proposed by Fabrikant et al. [14], on one hand we assign a weight (celebrity rank) to each player (vertex in the network), on the other, we impose, motivated by the small-world phenomenon, the existence of a critical distance . This leads us to introduce the *celebrity games* model, defined by $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$.

In our interpretation, V is a set of nodes with weights $(w_u)_{u \in V}$ measuring the level of celebrity of each node. The parameter α is the cost of establishing a link, while β establishes the desirable critical distance. The cost of a node has two components. The first one corresponds (as usual) to the cost of the links bought by the player. The second one is a refinement of the usual distance metric in which the player pays the weight of those nodes who are farther away than the critical distance β.

To the best of our knowledge, celebrity games are new although there are several network creation games grounded on similar ideas. Indeed, a general framework for the study of network creation with selfish agents was proposed by Jackson and Wolinsky [18]. Different extensions to this model are numerous in Economics (see the survey [17]). Fabrikant et al. [14] introduce a novel game that models the creation of Internet-like networks by selfish node-agents without central coordination. Nodes pay for links that they establish unilaterally and benefit from short paths to all destinations. The authors assume that all pair of nodes have the same interest (all-to-all communication pattern with identical weights), the cost of being disconnected is infinite and the edges paid by one node can be used by others. Albers et al. in [1] continue the study of the model of [14] improving some of their bounds.

Corbo and Parkes [8] generalize the model of Fabrikant et al. [14] by having links formed bilaterally. In subsequent works many other different network formation games have been defined [9,20,7,11,10,3,19,6,12,2].

Regarding our model, we are interested in the following questions. Which aspects affect link establishment? Who pays for being connected? What is the computational complexity of optimal play? What are the networks that optimize social welfare? And how does the worst-case equilibrium performance of a network compare to its best-case performance?

In fact, our analysis finds that the properties of the game when $\beta > 1$ are quite different from the properties when $\beta = 1$. For the case $\beta > 1$, our results can be summarized as follows. Computing a best response is NP-hard. Nevertheless pure Nash equilibria always exist and Nash equilibria graphs are either connected or a a set of isolated nodes. Furthermore, there is a relationship between the cost of the establishment of a link and the weight of the nodes. This leads to a natural definition of a *celebrity* in terms of link cost. A *celebrity* is a node whose weight is strictly greater than the cost of establishing a link. Having at least one celebrity guarantee that all the Nash equilibria graphs are connected, although there are celebrity games without celebrities that still have connected equilibria graphs. In those games having a connected Nash equilibrium, a star tree is always a Nash equilibrium graph. We call this subfamily *star celebrity games*.

The Price of Anarchy (PoA) and of stability (PoS) is analyzed under the usual sum cost. We show that the PoS is 1 for star celebrity games and that, for games that are non star celebrity games, the PoS = PoA= $\max\{1, W/\alpha\}$, where W is the sum of all the weights of the vertices. For star celebrity games we obtain a general upper bound of $O(\beta + n/\beta)$. We conjecture that the PoA is $O(n/\beta)$. Towards proving this conjecture we show that it holds when the PoA is taken over 2-edge connected NE graphs. To complement the result we prove that the PoA on NE trees is constant. Finally, for the case $\beta = 1$, we show that the Best Response problem is polynomial time solvable and the PoA is 2.

The paper is organized as follows. In Section 2 we introduce the basic definitions, the celebrity games and we analyze the complexity of the best response problem. In Section 3 we set the fundamental properties of NE and optimal graphs, characterize star celebrity games and provide the first bounds for the PoA and the PoS. Section 4 is devoted to the study of the diameter of NE graphs and Section 5 to derive the bounds for the PoA. In Section 6 we give the upper bound of the PoA over NE trees and in Section 7 we study the case $\beta = 1$. Finally we state some conclusions and open problems in Section 8.

2 The Model

In this section we introduce the celebrity games model and we analyze the complexity of computing a best response. Let us start with some definitions. We use standard notation for graphs and strategic games. All the graphs in the paper are undirected unless explicitly said otherwise. Given a graph $G = (V, E)$ and $u, v \in V$, $d_G(u, v)$ denotes the *distance* between u and v in G, i.e. the length of the shortest path from u to v. The *diameter* of a vertex $u \in V$ is $diam(u) = \max_{v \in V} d_G(u, v)$ and the *diameter* of G is $diam(G) = \max_{v \in V} diam(v)$. An *orientation* of an undirected graph is an assignment of a direction to every edge of the graph, turning it into a directed graph. For a weighted set $(V, (w_u)_{u \in V})$ we extend the weight function to subsets in the usual way. For $U \subseteq V$, $w(U) = \sum_{u \in U} w_u$. Furthermore, we set $W = w(V)$, $w_{max} = \max_{u \in V} w_u$ and $w_{min} = \min_{u \in V} w_u$.

Definition 1. *A* celebrity game Γ *is defined by a tuple* $\langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ *where:* $V = \{1, \ldots, n\}$ *is the set of players, for each player* $u \in V$, $w_u > 0$ *is the* celebrity weight *of player* u, $\alpha > 0$ *is the cost of establishing a link, and* β, $1 \leq \beta \leq n - 1$, *is the* critical distance.

A strategy *for player* u *in* Γ *is a subset* $S_u \subseteq V - \{u\}$, *the set of players for which player* u *pays for establishing a direct link. A* strategy profile *for* Γ *is a tuple* $S = (S_1, \ldots, S_n)$ *that assigns a strategy to each player. Every strategy profile* S *has associated an* outcome graph, *the undirected graph defined by* $G[S] = (V, \{\{u, v\} | u \in S_v \vee v \in S_u\})$.

We denote by $c_u(S) = \alpha |S_u| + \sum_{\{v | d_{G[S]}(u,v) > \beta\}} w_v$, *the* cost *of player* u *in the strategy profile* S. *And, as usual, the* social cost *of a strategy profile* S *in* Γ *is defined as* $C(S) = \sum_{u \in V} c_u(S)$.

Observe that, even though a link might be established by only one of the players, we assume that once a link is established it can be used in both directions by any player. In our definition players may have different celebrity weights. The player's cost function have two components: the cost of establishing links and the sum of the weights of those players who are farther away than the critical distance β.

In what follows we assume that, for a celebrity game $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$, the parameters verify the required conditions. We use the following notation $n = |V|$, $W = \sum_{u\in V} w_u$, $\mathcal{S}(u)$ is the set of strategies for player u and $\mathcal{S}(\Gamma)$ is the set of strategy profiles of Γ. For a strategy profile S and a strategy S'_u for player u, (S_{-u}, S'_u) represents the strategy profile in which S_u is replaced by S'_u while the strategies of the other players remain unchanged. The *cost difference* $\Delta(S_{-u}, S'_u)$ is defined as $\Delta(S_{-u}, S'_u) = c_u(S_{-u}, S'_u) - c_u(S)$. Observe that, if $\Delta(S_{-u}, S'_u) < 0$, then player u has an incentive to deviate from S_u.

Let us recall the definition of Nash equilibrium.

Definition 2. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$ be a celebrity game. A strategy profile $S \in \mathcal{S}(\Gamma)$ is a* Nash equilibria *of Γ if no player has an incentive to deviate from his strategy. Formally, for any player u and any strategy $S'_u \in \mathcal{S}(u)$, $\Delta(S_{-u}, S'_u) \geq 0$.*

We denote by $\mathrm{NE}(\Gamma)$ the set of Nash equilibria of a game Γ and we use the term NE to refer to a strategy profile $S \in \mathrm{NE}(\Gamma)$. We say that a graph G is a NE *graph* of Γ if there is $S \in \mathrm{NE}(\Gamma)$ so that $G = G[S]$. We will drop the explicit reference to Γ whenever Γ is clear from the context. It is worth observing that, for $S \in \mathrm{NE}(\Gamma)$, it never happens that $v \in S_u$ and $u \in S_v$, for any $u, v \in V$. Thus, if G is the outcome of a NE profile S, then S corresponds to an orientation of the edges in G. Furthermore, a NE graph G can be the outcome of several strategy profiles but not all the orientations of a NE graph G are NE.

Let $\mathsf{opt}(\Gamma) = \min_{S\in\mathcal{S}(\Gamma)} C(S)$ be the minimum value of the social cost. We use the term OPT strategy profile to refer to one strategy profile with with optimal social cost.

Observe that when in a strategy profile S two players u and v are such that $u \in S_v$ and $v \in S_u$, the social cost is higher than when only one of them is paying for the connection $\{u, v\}$ and therefore, as for NE, this does not happen in an OPT strategy profile. In the following, as we are interested in NE and OPT strategies, among all the possible strategy profiles having the same outcome graph, we only consider those that correspond to orientations of the outcome graph. In this sense the social cost depends only on the outcome graph, the weights and the parameters. Thus, we can express the social cost of a strategy profile as a function of the outcome graph G as follows

$$C(G) = \alpha|E(G)| + \sum_{u\in V} \sum_{\{v|d_G(u,v)>\beta\}} w_j = \alpha|E(G)| + \sum_{\{(u,v)|u<v \text{ and } d_G(u,v)>\beta\}} (w_i + w_j).$$

We make use of three particular outcome graphs on n vertices, K_n, the complete graph, I_n, the independent set and S_n a star graph, i.e., a tree in which one of the vertices, the *central* one, has a direct link to all the other $n-1$ vertices. For those graphs, we have the following values of the social cost. For $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, \beta\rangle$, with $|V| = n$, we have: $C(K_n) = \alpha n(n-1)/2$, $C(I_n) = W(n-1)$ and $C(S_n) = \alpha(n-1)$, for $1 < \beta \leq n-1$, and $C(S_n) = \alpha(n-1) + (n-2)(W - w_c)$ where c is the central vertex, for $\beta = 1$.

We define the PoA and the PoS as usual.

Definition 3. *Let Γ be a celebrity game. The* Price of anarchy *of Γ is defined as $PoA(\Gamma) = \max_{S \in NE(\Gamma)} C(S)/\mathit{opt}(\Gamma)$ and the* Price of stability *of Γ as $PoS(\Gamma) = \min_{S \in NE(\Gamma)} C(S)/\mathit{opt}(\Gamma)$.*

Whenever there is no possible confusion we drop the reference to Γ, by referring to $\mathsf{opt}(\Gamma)$, $PoA(\Gamma)$, and $PoS(\Gamma)$ by opt, PoA, and PoS, respectively.

Our first result shows that computing a best response in celebrity games is NP-hard. The hardness follows from a reduction from the minimum dominating set problem.

Proposition 1. *Computing a best response for a player to a strategy profile in a celebrity game is NP-hard, even restricted to the case in which all the weights are equal and $\beta = 2$.*

Proof. We provide a reduction from the problem of computing a dominating set of minimum size which is a classical NP-hard problem. Recall that a dominating set of a graph $G = (V, E)$ is a set $U \subset V$ such that any vertex $u \in V$ is in U or has a neighbor in U.

Let $G = (V, E)$ be a graph, we associate to G the following instance of the BestResponse problem (Γ, S, u), with $\Gamma = \langle V', (w_v)_{v \in V'}, \alpha, \beta \rangle$, where

- The set of players is $V' = V \cup \{u\}$, where u is a new player (i.e. $u \notin V$).
- $\beta = 2$, $\alpha = 1.5$,
- for every $v \in V \cup \{u\}$, $w_v = 2$.
- The strategy profile S is obtained from an orientation of the edges in G setting $S_u = \emptyset$. Observe that by construction $G[S]$ is the disjoint union of G with the isolated vertex u.
- Finally, u is the player for which we want to compute the best response.

Let $D \subseteq V$ be a strategy for player u. Notice that, if D is a dominating set of G, then $c_u(S_{-u}, D) = \alpha|D| + \sum_{x \in V, d(u,x)>2} 2 = \alpha|D|$. If D is not a dominating set of G, $c_u(S_{-u}, D) = \alpha|D| + \sum_{x \in V, d(u,x)>2} 2 > \alpha(|D| + |\{x \in V | d(u,x) > 2\}|$. Then, $D \cup \{x \in V | d(u,x) > 2\}$ is a better response than D. Hence, the best response of player u is a dominating set D of G of minimum size.

To conclude the proof just notice that the described reduction is polynomial time computable. $\square$

The problem becomes tractable for $\beta = 1$ as we show in Section 7.

3 Social Optimum and NE

We analyze here the main properties of OPT and NE strategy profiles in celebrity games. For $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, we assume that $\beta > 1$, the case $\beta = 1$ is considered in Section 7. The optimal cost is characterized by the next proposition.

Proposition 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. We have that $\mathit{opt}(\Gamma) = \min\{\alpha, W\}(n-1)$.*

Proof. Let $S \in \textsc{opt}(\Gamma)$, and let $G = G[S]$ with connected components $G_1, ..., G_r$, $V_i = V(G_i)$, $k_i = |V_i|$, and $W_i = w(V_i)$, for $1 \leq i \leq r$. Observe that the social cost of a disconnected graph can be expressed as the sum of the social cost of its connected components. Each connected component must be a tree of diameter at most β, otherwise a strategy profile with smaller social cost could be obtained by replacing the connections on V_i by such a tree. W.l.o.g we can assume that, for $1 \leq i \leq r$, the i-th connected component is a star graph S_{k_i} of k_i vertices. Since $C(S_k) = \alpha(k-1)$ we have that

$$C(G) = \sum_{i=1}^{r} \alpha(k_i - 1) + \sum_{i=1}^{r} k_i(W - W_i) = \alpha(n-r) + nW - \sum_{i=1}^{r} k_i W_i.$$

As $1 \leq k_i \leq n - (r-1)$, we have

$$W \leq \sum_{i=1}^{r} k_i W_i \leq (n - r + 1)W.$$

Therefore, $\alpha(n-r) + (r-1)W \leq C(G)$. We consider two cases.

Case 1: $\alpha \geq W$.

We have $W(n-1) \leq C(G)$. Since $C(I_n) = W(n-1) \leq C(G)$ and G is an optimal graph, then $C(G) = W(n-1)$.

Case 2: $\alpha < W$.

Now $\alpha(n-1) \leq C(G)$. As $C(S_n) = \alpha(n-1) \leq C(G)$, the optimal graph G has a social cost $C(G) = \alpha(n-1)$. We conclude that $\textsc{opt} = \min\{\alpha, W\}(n-1)$. $\square$

Now we turn our attention to the study of the NE graph topologies.

Proposition 3. *Every* NE *graph of a celebrity game either is connected or is the graph I_n.*

Proof. If $n \leq 2$ the proposition follows immediately. Let $n > 2$. Let us suppose that there is a NE S such that the graph $G = G[S]$ is not connected and different from I_n. In this case G is composed of at least two different connected components G_1 and G_2. Furthermore, as $G \neq I_n$, we can assume that $|V(G_1)| > 1$ as at least one of the connected components must contain at least two vertices and one edge. Let $u \in V(G_1)$ be such that $S_u \neq \emptyset$. Let $x \in S_u$ and $v \in V(G_2)$. Let us consider the strategies $S'_u = S_u \setminus \{x\}$ and $S'_v = S_v \cup \{x\}$. As S is a NE we know that $\Delta(S_{-u}, S'_u) \geq 0$. Let $G' = G[S_{-v}, S'_v]$, observe that $d_{G'}(v,u) = 2 \leq \beta$, therefore $\Delta(S_{-v}, S'_v) \leq -\Delta(S_{-u}, S'_u) - w_u < 0$. This contradicts the hypothesis that G is a NE graph. $\square$

Next result establishes that celebrity games always have NE graphs.

Proposition 4. *Every celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ has a* NE*. Indeed, when $\alpha \geq w_{max}$, I_n is a* NE *graph, otherwise S_n is a* NE *graph but I_n is not.*

Proof. When $\alpha \geq w_{max}$ let us show that I_n is a NE graph. Observe that $G = I_n$ is the outcome of a unique strategy profile S, such that $S_u = \emptyset$, for any player $u \in V$. Let us consider a player u and a strategy $S'_u \neq \emptyset$. The cost difference of player i is then $\Delta(S_{-u}, S'_u) = \alpha|S_u| - \sum_{v \in S_u} w_v = \sum_{v \in S_u}(\alpha - w_v) \geq 0$. Therefore player u has no incentive to deviate from S_i and I_n is a NE graph.

When $\alpha < w_{max}$, let u be a vertex with $w_u = w_{max}$ and let S_n be a star graph on n vertices in which the center is u. Let us show that S_n is a NE graph.

Consider the strategy profile S in which $S_u = \emptyset$ and $S_v = \{u\}$, for any $v \in V$ different from u, in which the center is the vertex with maximum weight. As $\beta > 1$ no player will get a cost decrease by connecting to more players. Furthermore, for $u \neq v$, $w_v + \alpha < w_v + w_{max} < W$. Thus $\alpha < W - w_v$ and v will not get any benefit by deleting the actual connection. The only remaining possibility is to reconnect to another vertex, but in such a case the cost cannot decrease. Therefore S_n is a NE graph. Notice that in this case I_n can not be a NE, every player u has incentive to connect with any other player v such that $w_v = w_{max}$. □

In the following we analyze the conditions in which I_n is the unique NE graph.

Proposition 5. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game with $\alpha \geq w_{max}$. If there are more than one vertex $u \in V$ with $\alpha > W - w_u$, then I_n is the unique* NE *graph of Γ, otherwise S_n is a* NE *graph of Γ.*

Proof. Assume that there are at least two vertices u, v with $\alpha > W - w_u, W - w_v$ and that there exists a NE graph $G = G[S]$ different from I_n. By proposition 3, G is connected. Therefore, it has at least $n-1$ edges. Since, $\alpha > W - w_u, W - w_v$, we have that $S_u = S_v = \emptyset$, otherwise S will not be a NE. Therefore, there must be a vertex, $z \neq u, v$ such that $|S_z| \geq 2$. Let $x, y \in S_z$ and let $S'_z = S_z \setminus \{x, y\}$. Then, $\Delta(S_{-z}, S'_z) \leq -2\alpha + W - w_z$. Since G is a NE graph and $\alpha > W - w_u, W - w_v$ we have that $W - w_z \geq 2\alpha > W - w_u + W - w_v$. Hence, $W < w_u + w_v - w_z < w_u + w_v$, which is impossible.

In the case that there is at most one vertex u with $\alpha > W - w_u$. The star S_n with center u, strategy set $S_u = \emptyset$, and such that $S_v = \{u\}$ for all $v \neq u$ is, clearly, a NE graph. □

Corollary 1. *Let $\langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. I_n is the unique* NE *graph of Γ if and only if $\alpha \geq w_{max}$ and there are more than one vertex $u \in V$ such that $\alpha > W - w_u$.*

This characterization allows us to identify the subfamily of celebrity games that have always a connected NE graph. Observe that those games have S_n as a NE graph.

Definition 4. *$\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ is a star celebrity game if Γ has a* NE *graph that is connected.*

Corollary 2. *For a celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, the following are equivalent:*

(1) Γ is a star celebrity game,
(2) either $\alpha < w_{max}$ or $\alpha \geq w_{max}$ and there is at most one $u \in V$ for which $\alpha > W - w_u$, and
(3) S_n is a NE *graph of Γ.*

Those results allow us to characterize the PoS and, in some cases, the PoA.

Theorem 1. *Let Γ be a celebrity game. Then:*
- If Γ is a star celebrity game, $PoS(\Gamma) = 1$.
- If Γ is not a star celebrity game and $\alpha \geq W$, $PoS(\Gamma) = PoA(\Gamma) = 1$.
- If Γ is not a star celebrity game and $\alpha < W$, $PoS(\Gamma) = PoA(\Gamma) = W/\alpha > 1$.

Proof. From Proposition 2, we have that $\mathsf{opt}(\Gamma) = W(n-1)$ if $\alpha \geq W$ and $\mathsf{opt}(\Gamma) = \alpha(n-1)$, otherwise.

If Γ is a star celebrity game, by Corollary 2 we know that S_n is a NE graph. Let us see that in star celebrity games it can only occur that $\alpha < W$. If $\alpha < w_{max}$, clearly $\alpha < W$. If $\alpha \geq w_{max}$, by Corollary 2 we have that there is at most one $u \in V$ for which $\alpha > W - w_u$. Assuming that $w_{u_1} \leq \ldots \leq w_{u_{n-1}} \leq w_{u_n}$, we have that $W > W - w_{u_1} \geq \ldots \geq W - w_{u_{n-1}} \geq W - w_{u_n}$, and then $W - w_{u_{n-1}} \geq \alpha$. Hence, $PoS(\Gamma) = 1$.

When Γ is not a star celebrity game, I_n is the unique NE graph. Thus, when $\alpha \geq W$ we have, $PoS(\Gamma) = PoA(\Gamma) = 1$ and, when $\alpha < W$ we have, $PoS(\Gamma) = PoA(\Gamma) = W/\alpha > 1$. □

4 Critical Distance and Diameter in NE graphs

In this section we analyze the diameter of NE graphs and its relationship with the parameters defining the game. We are interested only in games in which NE graphs with finite diameter exist, thus we only consider star celebrity games. In stating the characterization, vertices with a high celebrity weight with respect to the link cost play a fundamental role.

Definition 5. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a celebrity game. We say that a vertex $u \in V$ is a* celebrity *if $\alpha < w_u$.*

Given a celebrity u, any other node v with $d(u, v) > \beta$ has an incentive to pay for connecting to u. Thus, in any NE graph G, every celebrity node u satisfies that $diam(u) \leq \beta$.

Proposition 6. *For a* NE *graph G of a star celebrity game, $diam(G) \leq 2\beta + 1$.*

Proof. Let S be a NE of Γ such that $G = G[S]$ and suppose that $diam(G) \geq 2\beta + 2$. Then, there exist two nodes $u, v \in V$ such that $d(u, v) = 2\beta + 2$. Let $u = u_0, u_1, \ldots, u_{2\beta+1}, u_{2\beta+2} = v$ be a shortest path from u to v. Let $A_u = \{x \in V | d(u, x) \leq \beta\}$ and let $A_{u_1} = \{x \in V | d(u_1, x) \leq \beta\}$. Let us show that if a node $x \in A_u \cup A_{u_1}$, then $d(x, v) > \beta$. If $x \in A_u$ then $d(x, v) > \beta$, otherwise $d(u, v) \leq d(u, x) + d(x, v) \leq 2\beta$ contradicting the fact that $d(u, v) = 2\beta + 2$. Moreover, if

$x \in A_{u_1}$ then $d(x,v) > \beta$, otherwise $d(u,v) \leq 1 + d(u_1,x) + d(x,v) \leq 2\beta + 1$ which also contradicts the fact that $d(u,v) = 2\beta + 2$.

Consider the edge $\{u, u_1\}$. Then, either $u_1 \in S_u$ or $u \in S_{u_1}$. In the case that $u_1 \in S_u$, let $S'_u = S_u \setminus \{u_1\}$ and $S'_v = S_v \cup \{u_1\}$. Observe that,

$$\Delta(S_{-u}, S'_u) \leq -\alpha + w(A_{\{u,u_1\}}(u) \cap A_u)$$

By the previous remark about distances, we know that all the vertices $r \in A_{\{u,u_1\}}(u) \cap A_u$ verify $d(r,v) > \beta$, but after adding $\{v, u_1\}$ all of them and u become at distance less than β from v, therefore

$$\Delta(S_{-v}, S'_v) \leq \alpha - w_u - w(A_{\{u,u_1\}}(u) \cap A_u)$$

Hence, $\Delta(S_{-u}, S'_u) + \Delta(S_{-v}, S'_v) \leq -w_u < 0$. Therefore, either $\Delta(S_{-u}, S'_u) < 0$ or $\Delta(S_{-v}, S'_v) < 0$ and then S can not be a NE.

The case $u \in S_{u_1}$, follows in a similar way by interchanging the roles of u and u_1. □

The previous result can be refined to get better bounds on the diameter when all the nodes are celebrities or when at least one of the nodes is a celebrity. In particular, it is easy to see the following cases. Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let G be a NE graph of Γ, then (i) If $\alpha < w_{min}$, $diam(G) \leq \beta$ and, (ii) If $w_{min} \leq \alpha < w_{max}$, $diam(G) \leq 2\beta$.

5 Bounding the Price of Anarchy

We derive some upper bounds for the PoA for star celebrity games. Our first result establishes a upperbound on the PoA in terms of W and α.

Lemma 1. *For a star celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$, $PoA(\Gamma) = O(\frac{W}{\alpha})$.*

Proof. Let S be a NE of Γ and let $G = G[S] = (V, E)$. As S is a NE, no player has an incentive to deviate from S. In particular, for any $u \in V$

$$0 \leq \Delta(S_{-u}, \emptyset) \leq -\alpha |S_u| + \sum_{\{v | d(u,v) \leq \beta\}} w_v - w_u.$$

Summing up, for all $u \in V$, we have

$$0 \leq \sum_{u \in V} \left(-\alpha |S_u| + \sum_{\{v | d(u,v) \leq \beta\}} w_v - w_u \right) = -\alpha |E| + \sum_{u \in V} \sum_{\{v | d(u,v) \leq \beta\}} w_v - W.$$

Therefore $\alpha |E| \leq \sum_{u \in V} \sum_{\{v | d(u,v) \leq \beta\}} w_v - W$ and $C(G)$ can be upper bounded as follows:

$$\begin{aligned} C(G) &= \alpha |E| + \sum_{u \in V} \sum_{\{v | d(u,v) > \beta\}} w_v \\ &\leq \sum_{u \in V} \left(\sum_{\{v | d(u,v) \leq \beta\}} w_v + \sum_{\{v | d(u,v) > \beta\}} w_v \right) - W \\ &= (n-1)W \end{aligned}$$

Hence, $PoA(\Gamma) \leq \frac{(n-1)W}{\alpha(n-1)} = \frac{W}{\alpha}$. □

Using the previous lemma we get an $O(n)$ upper bound on the PoA of star celebrity games.

Theorem 2. *For a star celebrity game Γ, $PoA(\Gamma) = O(n)$.*

Proof. When $\alpha > w_{max}$, $W < n\alpha$ and, from Lemma 1, the claim follows. We consider now the case $\alpha \leq w_{max}$. Let S be any NE of Γ and $G = G[S] = (V, E)$. If $\alpha < w_v$ and $d(u, v) > \beta$ then player u has an incentive to buy a link to v, contradicting the fact that S is a NE. Therefore, the social cost of S can be expressed as:

$$\begin{aligned} C(S) &= \alpha|E| + \sum_{u \in V} \sum_{\{v | d(u,v) > \beta\}} w_v \leq \alpha|E| + \sum_{u \in V} \sum_{\{v \neq u, \alpha \geq w_v\}} w_v \\ &\leq \alpha|E| + \alpha n(n-1)/2 \leq \alpha n(n-1). \end{aligned}$$

Hence, $PoA(\Gamma) \leq \frac{\alpha n(n-1)}{\alpha(n-1)} = n$. □

From Theorem 2 we get a tight bound for some particular cases.

Corollary 3. *For a star celebrity game $\Gamma = \langle V, (w_i)_{i \in V}, \alpha, \beta \rangle$ with $\alpha \leq w_{min}$, $\beta = \Theta(1)$, and $\beta > 2$, $PoA(\Gamma) = \Theta(n)$.*

We can also derive a partial upper bound on the cost component corresponding to the vertices' weights. Define the *weight component* of the social cost, for a critical distance β, $W(G, \beta)$, as

$$W(G, \beta) = \sum_{u \in V(G)} \sum_{\{v | d(u,v) > \beta\}} w_v = \sum_{\{\{u,v\} | d(u,v) > \beta\}} (w_u + w_v).$$

Proposition 7. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. In a NE graph G, $W(G, \beta) = O(\alpha n^2/\beta)$.*

Proof. Let S be a NE and $G = G[S]$ be a NE graph. Let $u \in V$ and let $b = diam(u)$. Recall that, by Proposition 6, $b \leq 2\beta + 1$. We have three cases.
Case 1: $b < \beta$. For any node $v \in V \setminus \{u\}$ consider the strategy $S'_v = S_v \cup \{u\}$, and let $G' = G[S_{-v}, S'_v]$. By connecting to u we have $diam_{G'}(v) \leq \beta$ and, as S is a NE, we have $\Delta(S_{-v}, S'_v) = \alpha - \sum_{\{x | d_G(x,v) > \beta\}} w_x \geq 0$. Therefore we have $\sum_{\{x | d_G(x,v) > \beta\}} w_x \leq \alpha$. As $b < \beta$ we conclude that $W(G, \beta) \leq n\alpha$. Since $1 < \beta \leq n - 1$, we get $n/\beta \leq \alpha n^2/\beta$.
Case 2: $b \geq \beta$ and $b \geq 6$. For $1 \leq i \leq b$, consider the set $A_i(u) = \{v \mid d(u, v) = i\}$ and the sets

$$\begin{aligned} C_1 &= \{v \in V \mid 1 \leq d(u, v) \leq b/3\} = \cup_{1 \leq i \leq b/3} A_i(u), \\ C_2 &= \{v \in V \mid b/3 < d(u, v) \leq 2b/3\} = \cup_{b/3 < j \leq 2b/3} A_j(u), \\ C_3 &= \{v \in V \mid 2b/3 < d(u, v) \leq b\} = \cup_{2b/3 < k \leq b} A_k(u). \end{aligned}$$

As $b = diam(u)$, $A_\ell(u) \neq \emptyset$, $1 \leq \ell \leq b$, and all those sets constitute a partition of $V \setminus \{u\}$. As $b \geq 6$, for each ℓ, $1 \leq \ell \leq 3$, C_ℓ contains vertices at a $b/3 \geq 2$ different distances. Therefore, for $1 \leq \ell \leq 3$, it must exist i_ℓ such that $A_{i_\ell}(u) \subseteq C_\ell$ and $|A_{i_\ell}(u)| \leq 3n/b$, otherwise the total number of elements in C_ℓ would be bigger than n.

For any $v \in V$, let $S'_v = (S_v \cup A_{i_1}(u) \cup A_{i_2}(u) \cup A_{i_3}(u)) \setminus \{v\}$ and let $G' = G[S_{-v}, S'_v]$. Since $b \leq 2\beta+1$, we have that $b/3 < \beta$. Hence, by construction, $diam_{G'}(v) \leq \beta$. Therefore, as S is a NE, we have

$$0 \leq \Delta(S_{-v}, S'_v) \leq \frac{9n\alpha}{\beta} - \sum_{\{x \mid d_G(x,v) > \beta\}} w_x.$$

Thus, $\displaystyle\sum_{\{x \mid d_G(x,v) > \beta\}} w_x \leq \frac{9n\alpha}{\beta}$ and $W(G, \beta) \leq \dfrac{9n^2\alpha}{\beta}$.

Case 3: $b \geq \beta$ and $b \leq 6$. Consider the sets $A_i(u) = \{v \mid d(u,v) = i\}$, $0 \leq i \leq b$, and the sets $C_0 = \{v \in V \mid d(u,v) \text{ is even}\}$ and $C_1 = V \setminus C_0$. Both sets are non-empty and one of them must have $\leq n/2$ vertices. By connecting to all the vertices in the smaller of those sets the diameter of the resulting graph is 2. Therefore, using a similar argument as in case 2, we get $W(G, \beta) \leq \frac{n^2\alpha}{2}$, which is $O(n^2/\beta)$ as $\beta < 6$. □

Observe that, from the previous result it would be enough to show that in a NE graph G, $|E| = O(n/\beta)$ to prove that the $PoA(\Gamma) = O(n/\beta)$. We could not prove this bound for the general case so we analyze the PoA for some particular cases. To do so, in the following technical Lemma, that allows us to derive a a general upper bound in terms of n and β, we establish some topological properties of NE graphs.

Lemma 2. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let S be a* NE *of Γ and $G = G[S]$. If there is a vertex $v \in V$ such that $|S_v| > 6n/\beta$, then there exists $X \subseteq S_v$ with $|X| \geq 3n/\beta$ such that for each $x \in X$, $\{v, x\}$ is a bridge.*

Proof. Let $x \in S_v$ and let F_x be the set of nodes z for which every shortest path from v to z passes through x (those vertices are forced to use x in any shortest path). Notice that, for each pair of different vertices $x, x' \in S_v$, $F_x \cap F_{x'} = \emptyset$ and $|F_x| + |F_{x'}| < n$. This together with the fact that $|S_v| > 6n/\beta$ implies that there exist at least $3n/\beta$ nodes $x \in S_v$ such that $|F_x| < \beta/3$. Consider the set $X = \{x \in V \mid |F_x| < \beta/3\}$.

Let $x \in X$ and assume that $\{v, x\} \in E$ is not a bridge. In this case, there exist at least one more edge between F_x and $V \setminus F_x$. Let $\{y, y'\} \in E$ be an edge distinct from $\{v, x\}$ with $y \in F_x$ and $y' \in V(G) \setminus F_x$.

As $|F_x| < \beta/3$, $\max_{z \in F_x} d(v, z) < \beta/3$. Consider the strategy $S'_v = S_v \setminus \{x\}$ and let $G' = G[S_{-v}, S'_v]$, the above conditions imply that, for any $z \in F_x$,

$$d_{G'}(v, z) \leq d(v, y') + 1 + d(y, x) + d(x, z) \leq \beta/3 + 1 + 2(\beta/3 - 1) < \beta$$

Therefore, $\Delta(S_{-v}, S'_v) = -\alpha < 0$, which is a contradiction with the fact that G is a NE graph. Thus $\{v, x\}$ is a bridge for every $x \in X$ and therefore there are at least $|X| \geq 3n/\beta$ bridges having v as endpoint, as we wanted to see. $\square$

This leads us to a upper bound of the PoA in terms of n and β.

Theorem 3. *For a star celebrity game Γ, $PoA(\Gamma) = O(n/\beta + \beta)$.*

Proof. Let G be a NE of a star celebrity game $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$. Since we have that $W(G) = O(n^2\alpha/\beta)$ it is enough to see that $|E| = O(n^2/\beta + n\beta)$. Let $X = \{u \in V \mid |S_u| > 6n/\beta\}$. By Lemma 2, every $u \in X$ defines $3n/\beta$ bridges of the form $\{u, x\}$, so there are at most $n/(3n/\beta) = \beta/3$ nodes in X. We have

$$|E| = \sum_{u \in V} |S_u| = \sum_{u \in X} |S_u| + \sum_{u \notin X} |S_u| \leq n|X| + (n - |X|)(6n/\beta) < n\beta/3 + 6n^2/\beta.$$

Thus $|E| = O(n^2/\beta + n\beta)$. $\square$

Hence, when $\beta = O(\sqrt{n})$, we get an upper bound of $O(n/\beta)$.

Corollary 4. *For a star celebrity game Γ with $\beta = O(\sqrt{n})$, $PoA(\Gamma) = O(n/\beta)$.*

For the case of 2-edge connected graphs, as a consequence of Lemma 2, we have the following.

Corollary 5. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let S be a* NE *of Γ and $G = G[S]$. If there exists a vertex $v \in V$ such that $|S_v| > 6n/\beta$, then G cannot be 2-edge connected.*

Using this property we can tighten the upper bound of the PoA when considering only 2-edge connected NE graphs.

Proposition 8. *For celebrity games having a 2-edge connected* NE*, the PoA on 2-edge-connected graphs is $O(n/\beta)$.*

Proof. Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game and let S be a NE of Γ. Assume that $G = G[S]$ is 2-edge connected. From Lemma 2, if there exists a vertex $v \in V$ such that $|S_v| > 6n/\beta$, then G cannot be 2-edge connected. Therefore, for each $u \in V$, $|S_u| < 3n/\beta + 1$ and $|E(G)| \leq n(3n/\beta + 1)$. From Proposition 7 we have $W(G, \beta) = O(\alpha n^2/\beta)$. So, $C(G) = O(\alpha n^2/\beta)$. As $\alpha < W$, from Proposition 2, $\mathsf{opt}(\Gamma) = \alpha(n-1)$. We conclude that PoA on 2-edge-connected graphs is $O(n/\beta)$. $\square$

6 Price of Anarchy on NE trees

Now we complement the results of the previous section by providing a constant upper bound when we restrict the NE graphs to be trees. In order to get a tighter upper bound for the PoA on NE trees, we first improve the bound on the diameter of NE trees to $\beta + 1$.

Proposition 9. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ be a star celebrity game. If T is a* NE *tree of Γ, $diam(T) \leq \beta + 1$.*

Proof. Let T be a tree such that $T = G[S]$ where S is a NE of Γ. Let $d = diam(T)$ and let $P = u, u_1, \ldots, u_d$ be a diametral path of T. Assume that $d > \beta + 1$. For $1 \leq i < d$, let T_i be the connected subtree containing u_i after removing edges (u_{i-1}, u_i) and $(u_i, u_i + 1)$. As P is a diametral path, both u and u_d are leaves in T. Furthermore, T_1 and T_2 are star trees. In general, the distance from the leaves of any T_i to both u and u_d is at most d.

We consider two cases depending on whom is paying for the connections to the end points of P.

Case 1: $u \in S_{u_1}$ or $u_d \in S_{u_{d-1}}$.

W.l.o.g. assume that $u_d \in S_{u_{d-1}}$. As S is a NE we have $w_d \geq \alpha$. Consider the strategy $S'_{u_1} = S_{u_1} \cup \{u_{d-1}\}$, then $\Delta(S_{-u_1}, S'_{u_1}) \leq \alpha - w_{u_d} - w_{u_{d-1}} < 0$ and T can not be a NE graph.

Case 2: $u_1 \in S_u$ and $u_{d-1} \in S_{u_d}$.

When $\beta \geq 3$. Set $S'_u = S_u - \{u_1\} \cup \{u_2\}$ and $T' = G[(S_{-u}, S'_u)]$. Observe that, for $x \in T_1$, $d_{T'}(u, x) \leq 3 \leq \beta$ and, for $x \notin T_1 \cup \{u\}$, $d_{T'}(u, x) = d_T(u, x) - 1$. Therefore, $\Delta(S_{-u}, S'_u) \leq -w_{u_{\beta+1}} < 0$. Therefore, T is not a NE graph.

The previous argument fails for the case $\beta = 2$ as there might be $x \in T_1$ with $d_{T'}(u, x) = 3$. From Proposition 6, we know that $d \leq 2\beta + 1 \leq 5$. Let us see that it can not be the case that $d = 4$ or $d = 5$. Let $S'_u = S_u - \{u_1\} \cup \{u_{d-1}\}$ and $S'_{u_d} = S_{u_d} - \{u_{d-1}\} \cup \{u_1\}$. Let $T^1 = G[(S_{-u}, {S'}_u)]$ and $T^2 = G[(S_{-u_d}, {S'}_{u_d})]$.

When $d = 4$, for any $x \in T_2$, $d_{T^1}(u, x) = d_T(u, x)$ and $d_{T^2}(u_4, x) = d_T(u_4, x)$. Therefore, we have

$$\Delta(S_{-u}, {S'}_u) = w(T_1) - w(T_3) - w_{u_4} \text{ and } \Delta(S_{-u_4}, {S'}_{u_4}) = w(T_3) - w(T_1) - w_u.$$

Thus $\Delta(S_{-u}, {S'}_u) + \Delta(S_{-u_4}, {S'}_{u_4}) = -w_u - w_{u_4} < 0$ and one of the two players has an incentive to deviate.

When $d = 5$, we have $\Delta(S_{-u}, {S'}_u) = w(T_1) + w_{u_2} - w_{u_3} - w(T_4) - w_{u_5}$ and $\Delta(S_{-u_5}, {S'}_{u_5}) = w(T_4) + w_{u_3} - w_u - w(T_1) - w_{u_2}$. Therefore we have that $\Delta(S_{-u}, {S'}_u) + \Delta(S_{-u_5}, {S'}_{u_5}) = -w_u - w_{u_5} < 0$ and one of the two players has an incentive to deviate. ◻

Our next result shows a constant upper bound for the PoA on NE trees. In order to prove the upper bound on the PoA on NE trees we provide first an auxiliary result.

Lemma 3. *Let $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta \rangle$ a star celebrity game and let G be a* NE *graph of Γ. If there is $v \in V$ with $diam_G(v) \leq \beta - 1$, then $W(G, \beta) \leq \alpha(n-1)$.*

Proof. Let $S \in$ NE(Γ) and let $G = G[S]$. Let $u \in V$, $u \neq v$. If $v \notin S_u$, $\Delta(S_{-u}, S_u \cup \{v\}) \geq \alpha - \sum_{\{x | d_G(u,x) > \beta\}} w_x \geq 0$. But, if $v \in S_u$, $diam(u) \leq \beta$.

Hence, $\alpha \geq \sum_{\{x | d_G(u,x) > \beta\}} w_x$ and summing over all $u \neq v$ we have that $\alpha(n-1) \geq W(G, \beta)$.

The proof of the upperbound for the PoA on NE trees uses the previous statements and examines the particular cases $\beta = 2, 3$.

Theorem 4. *The PoA on* NE *trees of a star celebrity game is at most* 2*.*

Proof. Let T be a NE tree of Γ. From Proposition 9 we have a bound on the diameter, so we know that $diam(T) \leq \beta + 1$. Since T is a tree, we have that there exists $u \in V$ such that $diam(u) \leq (diam(T) + 1)/2 \leq \beta/2 - 1$ If $\beta \geq 4$, then $diam(u) \leq \beta - 1$. By lemma 3, $C(T) \leq 2\alpha(n - 1)$. Hence, the PoA of NE trees of Γ is at most 2 for $\beta \geq 4$.

In the case of $\beta = 3$, either $diam(T) \leq 3$ or $diam(T) = 4$. In the first case $C(T) = \alpha(n - 1)$ and in the second there is u with $diam_T(u) = 2 = \beta - 1$ and we can use Lemma 3.

Finally, we consider the case $\beta = 2$. Notice that the unique tree T with diameter 3 is a double star, a graph that is formed by connecting the centers of two stars. Assume that a NE tree T is formed by S_k, a star with k vertices and centered at u, and S_{n-k}, a star graph with $n - k$ vertices centered at v, joined by the edge (u, v). Let L_u (L_v) be the set of leaves in S_k (S_{n-k}). As T is a NE graph we have that $w(L_u), w(L_v) \leq \alpha$. Furthermore

$$\begin{aligned} C(T) =& \alpha(n-1) + \sum_{w \in L_u} w(L_v) + \sum_{w \in L_v} w(L_u) \leq \alpha(n-1) + \sum_{w \in L_u} \alpha + \sum_{w \in L_v} \alpha \\ &\leq \alpha(n-1) + \alpha(n-2) \leq 2\alpha(n-1). \end{aligned}$$

□

Note that in a NE tree T, if $\alpha > w_{max}$, for an edge (u, v) connecting a leaf u it must be the case that $v \in S_u$. Then, in the proof of Proposition 9, we only have the case $u_1 \in S_u$. In such case $diam(T) \leq \beta$. Hence, if $\alpha > w_{max}$, the PoA on NE trees is 1.

Corollary 6. *Let* $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, \beta\rangle$ *be a star celebrity game such that* $\alpha > w_{max}$*. For any* NE *tree of* Γ*,* $diam(T) \leq \beta$ *and therefore the PoA on* NE *trees is* 1*.*

7 Celebrity games for $\beta = 1$

We analyze now the case of celebrity games when $\beta = 1$. Our first result is that the problem of computing a best response becomes tractable when $\beta = 1$

Proposition 10. *The problem of computing a best response of a player to a strategy profile in celebrity games is polynomial time solvable when* $\beta = 1$*.*

Proof. Let S be a strategy profile of $\Gamma = \langle V, (w_u)_{u \in V}, \alpha, 1\rangle$ and let $u \in V$ and $D \subseteq V \setminus \{u\}$. As $\beta = 1$ we have

$$c_u(S_{-u}, D) = \alpha|D| + \sum_{v \notin D} w_v.$$

Notice that for sets with $|D| = k$ the first component of the cost is the same. Thus a best response on strategies with k players can be obtained by considering the set D_k formed by the players in $V \setminus \{u\}$ having the k-th highest weights. Let $W_k = W - w(D_k)$. Thus $c_u(S_{-u}, D_k) = \alpha k + W_k$. To obtain a best response it is enough to compute the value k for which $c_u(S_{-u}, D_k)$ is minimum and output D_k. Observe that the overall computation can be performed in polynomial time. □

When $\beta = 1$ the particular structure of NE and OPT graphs is different from the case of $\beta > 1$, as pairs of vertices at distance bigger than one are not directly connected. Such a property does not hold for higher distances.

Proposition 11. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, 1\rangle$ be a celebrity game and $G = (V, E)$ a* NE *graph of Γ. Therefore, for any $u, v \in V$*

- *if either $w_u > \alpha$ or $w_v > \alpha$ then $\{u, v\} \in E$,*
- *if both $w_u < \alpha$ and $w_u < \alpha$ then $\{u, v\} \notin E$,*
- *otherwise the edge $\{u, v\}$ might or might not belong to E.*

Proof. Let S be a NE and let $G = G[S] = (V, E)$. Observe that due to the fact that $\beta = 1$, for any player u,

$$c_u(S) = \alpha|S_u| + \sum_{\{v|v\neq u, \{u,v\}\notin E\}} w_v.$$

The cost is thus expressed in terms of the existence or non existence of a connection between pairs of nodes and thus the strategy can be analyzed considering only deviations in which a single edge is added or removed. We analyze the different cases for players u and v.

Case 1: $w_u > \alpha$. For any player $v \neq u$, if the edge $\{u, v\}$ is not present in G the graph cannot be a NE graph as v improves its cost by connecting to u. For the same reason, if the edge is present either $u \in S_v$ or $v \in S_v$. The later case$v \in S_v$, can happend only when $w_v > \alpha$. Therefore, the player that is paying for the connection will not obtain any benefit by deviating.

Case 2: $w_u, w_v < \alpha$. If the edge $\{u, v\}$ is present in G the graph cannot be a NE graph as the player establishing the connection improves its cost by removing the connection to the other player. For the same reason, if the edge is not present none of the players will obtain any benefit by deviating and paying for the connection.

Case 3: $w_u, w_v = \alpha$. The cost, for any of the players, of establishing the connection or not is the same. In consequence the edge can or cannot be in a NE graph.

Case 4: $w_u = \alpha$ and $w_v < \alpha$. Player v is indifferent to be or not to be connected to u, but player u in a NE will never include v in its strategy. Observe that again the edge can or cannot exists in a NE graph but, if it exists, it can only be the case that $u \in S_v$. □

Let us analyze now the structure of the OPT graphs.

Proposition 12. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, 1\rangle$ be a celebrity game and $G = (V, E)$ a* OPT *graph of Γ. For any $u, v \in V$, we have*

- *if $w_u + w_v < \alpha$ then $\{u, v\} \notin E$,*
- *if $w_u + w_v > \alpha$ then $\{u, v\} \in E$,*
- *if $w_i + w_j = \alpha$ then $\{u, v\}$ might or not be an edge in G.*

Proof. Let S be a strategy profile and let $G = G[S] = (V, E)$ be an OPT graph. As we have seen before as $\beta = 1$, for any player u,

$$c_u(S) = \alpha|S_u| + \sum_{\{v|v\neq u, \{u,v\}\notin E\}} w_v,$$

and we get et the following expression for the social cost

$$C(G) = \alpha|E| + \sum_{\{u,v|u<v,\{u,v\}\notin E\}} (w_i + w_j).$$

The above expression shows that to minimize the contribution to the cost, an edge $\{u, v\}$ can be present in the graph only if $w_u + w_v \geq \alpha$ and will appear for sure only when $w_i + w_j > \alpha$. Thus the claim follows. □

From the previous characterizations we can derive a constant upper bound for the price of anarchy when $\beta = 1$.

Theorem 5. *Let $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, 1\rangle$ be a celebrity game. $PoA(\Gamma) \leq 2$. Furthermore the, ratio among the social cost of the best and the worst* NE *graphs of Γ is bounded by 2.*

Proof. $\Gamma = \langle V, (w_u)_{u\in V}, \alpha, 1\rangle$. Observe that due to the conditions given in Propositions 11 and 12 the social cost of an OPT graph is at most

$$\sum_{\{\{u,v\}|w_u+w_v\geq\alpha\}} \alpha + \sum_{\{\{u,v\}|w_u+w_v<\alpha\}} (w_u + w_v),$$

and the social cost of a NE graph with minimum number of edges, i.e. one in which all the optional are not present, is

$$\begin{aligned}\sum_{\{\{u,v\}|w_u>\alpha \text{ or } w_v>\alpha\}} \alpha + \sum_{\{\{u,v\}|w_u,w_v\leq\alpha\}} (w_u + w_v) = \\ = \sum_{\{\{u,v\}|w_u>\alpha \text{ or } w_v>\alpha\}} \alpha \\ + \sum_{\{\{u,v\}|w_u,w_v\leq\alpha \text{ and } w_u+w_v=\alpha\}} \alpha \\ + \sum_{\{\{u,v\}|w_u,w_v\leq\alpha \text{ and } w_u+w_v<\alpha\}} (w_u + w_v) \\ + \sum_{\{\{u,v\}|w_u,w_v\leq\alpha \text{ and } w_u+w_v>\alpha\}} (w_u + w_v) =\end{aligned}$$

Observe that the difference with the cost of an OPT graph is in the last term

$$D = \{\{u, v\} \mid w_u, w_v \leq \alpha \text{ and } w_u + w_v > \alpha\}.$$

Notice that $\{u, v\} \in D$ contributes to the cost of an OPT graph with α and to the cost of a NE graph with $w_u + w_v$. By taking Γ with $w_u = \alpha$, for any $u \in V$, we can maximize the size of D and this leads to the worst possible NE graph. For such a Γ, I_n is a NE graph and we have that $C(I_n) = \sum_{u,v \in V, u<v}(w_u + w_v) = \alpha n(n-1)$. Furthermore, in any OPT graph of Γ, all the edges will be present, thus we have $\text{OPT} = \alpha n(n-1)/2$. Thus

$$\text{PoA}(\Gamma) \leq \frac{n(n-1)\alpha}{\alpha n(n-1)/2} = 2$$

Observe that when $w_u = \alpha$, for any u, the complete graph is also a NE graph and thus we have that the ratio between the social cost of the worst and the best NE graph is bounded by 2. □

8 Conclusions and Open problems

We have introduced celebrity games: a new model of network creation whose cost function considers two features. The first one is that every player has a celebrity weight and the second is that the network has a critical distance. The relation between the weight of the players and the link cost establishes the conditions for the connectivity of NE graphs. For connected NE graphs we show that:

- the networks created by celebrity games have diameter $\leq 2\beta + 1$ (that depends on the given critical distance β) and that,
- the value of this critical distance has implications on the quality of the NE strategies.

Indeed, we obtain NE graphs in which all the nodes are as close as desired among them. In fact, since the PoA of star celebrity games is $O(\beta + n/\beta)$, we can observe that enlarging (below some reasonable bounds) the value of β improves the quality of the equilibrium and the other way round. In other words, there is a trade-off between the closeness of the nodes in the graph and the quality of the NE.

To go further, we propose two different lines of research. On one hand, to tighten the gap between the provided upper bound of $O(\beta + n/\beta)$ for the PoA. We conjecture that the upper bound can be improved to $O(n/\beta)$ and we have proved so for 2-edge connected NE graphs. On the other hand, we propose to study natural variations of our framework. Among the many possibilities to extend this model we think that the following three are interesting: (i) to analyze celebrity games under the max social cost measure (work in progress), (ii) to consider other definitions of the social cost measure and, (iii) to analyze the non uniform model in which the critical distance might be different for each pair of participants.

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