ASYMPTOTIC BEHAVIOR OF PALAIS-SMALE SEQUENCES ASSOCIATED WITH FRACTIONAL YAMABE TYPE EQUATIONS

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Abstract. In this paper, we analyze the asymptotic behavior of Palais-Smale sequences associated to fractional Yamabe type equations on an asymptotically hyperbolic Riemannian manifold. We prove that Palais-Smale sequences can be decomposed into the solution of the limit equation plus a finite number of bubbles, which are the rescaling of the fundamental solution for the fractional Yamabe equation on Euclidean space. We also verify the non-interfering fact for multi-bubbles.

Keywords. Palais-Smale sequence, asymptotically hyperbolic Riemannian manifolds, fractional Yamabe type equations

1. Introduction and statement of results

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. Fix a constant λ, and consider the Dirichlet boundary value problem of the elliptic PDE

$$\begin{cases} -\Delta u - \lambda u = u|u|^{\frac{4}{n-2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The associated variational functional of the equation (1.1) in the Sobolev space $W_0^{1,2}(\Omega)$ is

$$E(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx - \frac{n-2}{2n}\int_\Omega |u|^{\frac{2n}{n-2}}\,dx.$$

Suppose that the sequence $\{u_\alpha\}_{\alpha\in\mathbb{N}} \subset W_0^{1,2}(\Omega)$ satisfies the Palais-Smale condition, i.e.

$$\{E(u_\alpha)\}_{\alpha\in\mathbb{N}} \text{ is uniformly bounded and } DE(u_\alpha) \to 0, \text{ strongly in } (W_0^{1,2}(\Omega))',$$

as $\alpha \to +\infty$, where $(W_0^{1,2}(\Omega))'$ is the dual space of $W_0^{1,2}(\Omega)$. In an elegant paper [16], M. Struwe considered the asymptotic behavior of $\{u_\alpha\}_{\alpha\in\mathbb{N}}$. In fact, in the $W_0^{1,2}(\Omega)$ norm, u_α can be approximated by the solution to (1.1) plus a finite number of bubbles, which are the rescaling of the non-trivial entire solution of

$$-\Delta u = u|u|^{\frac{4}{n-2}} \quad \text{in } \mathbb{R}^n \quad \text{and} \quad u(x) \to 0 \quad \text{as } |x| \to +\infty.$$

One may pose the analogous problem on a manifold. Let (M^n, g) be a smooth compact Riemannian manifold without boundary. Consider a sequence of elliptic PDEs like

$$-\Delta_g u + h_\alpha u = u^{\frac{n+2}{n-2}}, \tag{E_α}$$

where $\alpha \in \mathbb{N}$ and Δ_g denotes the Laplace-Beltrami operator of the metric g. Assume that h_α satisfies that there exists $C > 0$ with $|h_\alpha(x)| \leq C$ for any α and any $x \in M$; also $h_\alpha \to h_\infty$ in $L^2(M)$ as $\alpha \to +\infty$. The limit equation is denoted by

$$-\Delta_g u + h_\infty u = u^{\frac{n+2}{n-2}}. \tag{E_∞}$$

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The related variational functional for (E_α) is

$$E_g^\alpha(u)=\frac{1}{2}\int_M|\nabla u|_g^2dv_g+\frac{1}{2}\int_M h_\alpha u^2dv_g-\frac{n-2}{2n}\int_M|u|^{\frac{2n}{n-2}}dv_g.$$

Suppose that $\{u_\alpha\geq 0\}_{\alpha\in\mathbb{N}}\subset W^{1,2}(M)$ also satisfies the Palais-Smale condition. O. Druet, E. Hebey and F. Robert [5] proved that, in the $W^{1,2}(M)$-sense, u_α can be decomposed into the solution of (E_∞) plus a finite number of bubbles, which are the rescaling of the non-trivial solution of

$$-\Delta u=u^{\frac{n+2}{n-2}}\ \text{ in }\ \mathbb{R}^n.$$

Next, let (M^n,g) be a compact Riemannian manifold with boundary ∂M. Recently, S. Almaraz [1] considered the following sequence of equations with nonlinear boundary value condition

$$\begin{cases}-\Delta_g u=0 & \text{in } M,\\ -\frac{\partial}{\partial\eta_g}u+h_\alpha u=u^{\frac{n}{n-2}} & \text{on } \partial M,\end{cases}\tag{1.2}$$

where $\alpha\in\mathbb{N}$ and η_g is the inward unit normal vector to ∂M. The associated energy functional for equation (1.2) is

$$\bar{E}_g^\alpha(u)=\frac{1}{2}\int_M|\nabla u|_g^2dv_g+\frac{1}{2}\int_{\partial M}h_\alpha u^2d\sigma_g-\frac{n-2}{2(n-1)}\int_{\partial M}|u|^{\frac{2(n-1)}{n-2}}d\sigma_g,$$

for $u\in H^1(M):=\{u|\nabla u\in L^2(M),u\in L^2(\partial M)\}$. Here dv_g and $d\sigma_g$ are the volume forms of M and ∂M, respectively. He also showed that a nonnegative Palais-Smale sequence $\{u_\alpha\}_{\alpha\in\mathbb{N}}$ of $\{\bar{E}_g^\alpha\}_{\alpha\in\mathbb{N}}$ converges, in the $H^1(M)$-sense, to a solution of the limit equation (the equation replacing h_α by h_∞ in (1.2)) plus a finite number of bubbles.

Motivated by these facts and the original study of the fractional Yamabe problem by M.d.M. González and J. Qing [8], in this paper we shall be interested in the asymptotic behavior of nonnegative Palais-Smale sequences associated with the fractional Yamabe equation on an asymptotically hyperbolic Riemannian manifold.

Let (X^{n+1},g^+), $n\geq 3$, be a smooth Riemannian manifold with smooth boundary $\partial X^{n+1}=M^n$. A function ρ_* is called a defining function of the boundary M^n in X^{n+1} if it satisfies

$$\rho_*>0\ \text{ in }\ X^{n+1},\quad \rho_*=0\ \text{ on }\ M^n,\quad d\rho_*\neq 0\ \text{ on }\ M^n.$$

We say that a metric g^+ is conformally compact if there exists a defining function ρ_* such that $(X^{n+1},\overline{g}_*)$ is compact for $\overline{g}_*=\rho_*^2g^+$. This induces a conformal class of metrics $\hat{h}=\overline{g}_*|_{M^n}$ when defining functions vary. The conformal manifold $(M^n,[\hat{h}])$ is called the conformal infinity of (X^{n+1},g^+). A metric g^+ is said to be asymptotically hyperbolic if it is conformally compact and the sectional curvature approaches -1 at infinity. It is easy to check then that $|d\rho_*|^2_{\overline{g}_*}=1$ on M^n.

Using the meromorphic family of scattering operators $S(s)$ introduced by C.R. Graham and M. Zworski [10], we will define the so-called fractional order scalar curvature. Given an asymptotically hyperbolic Riemannian manifold (X^{n+1},g^+) and a representative $\hat{h}$ of the conformal infinity $(M^n,[\hat{h}])$, there is a unique geodesic defining function ρ_* such that, in $M^n\times(0,\delta)$ in X^{n+1}, for small δ, g^+ has the normal form

$$g^+=\rho_*^{-2}(d\rho_*^2+h_{\rho_*})$$

where h_{ρ_*} is a one parameter family of metric on M^n such that

$$h_{\rho_*}=\hat{h}+h^{(1)}\rho_*+O(\rho_*^2).$$

It is well-known [10] that, given $f \in \mathbb{C}^\infty(M^n)$, and $s \in \mathbb{C}$, $Re(s) > n/2$ and $s(n-s)$ is not an L^2 eigenvalue for $-\Delta_{g^+}$, then the generalized eigenvalue problem

$$-\Delta_{g^+}\tilde{u} - s(n-s)\tilde{u} = 0 \quad \text{in } X^{n+1} \tag{1.3}$$

has a solution of the form

$$\tilde{u} = F(\rho_*)^{n-s} + G(\rho_*)^s, \quad F, G \in \mathcal{C}^\infty(\overline{X^{n+1}}), \quad F|_{\rho_*=0} = f.$$

The scattering operator on M^n is then defined as

$$S(s)f = G|_{M^n}.$$

Now we consider the normalized scattering operators

$$P_\gamma[g^+, \hat{h}] = d_\gamma S\left(\frac{n}{2} + \gamma\right), \qquad d_\gamma = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)}.$$

Note $P_\gamma[g^+, \hat{h}]$ is a pseudo-differential operator whose principal symbol is equal to the one of $(-\Delta_{\hat{h}})^\gamma$. Moreover, $P_\gamma[g^+, \hat{h}]$ is conformally covariant, i.e. for any $\varphi, w \in \mathcal{C}^\infty(\overline{X^{n+1}})$ and $w > 0$, it holds

$$P_\gamma[g^+, w^{\frac{4}{n-2\gamma}}\hat{h}](\varphi) = w^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma[g^+, \hat{h}](w\varphi). \tag{1.4}$$

Thus we shall call $P_\gamma[g^+, \hat{h}]$ the conformal fractional Laplacian for any $\gamma \in (0, n/2)$ such that $n^2/4 - \gamma^2$ is not an L^2 eigenvalue for $-\Delta_{g^+}$.

The fractional scalar curvature associated to the operator $P_\gamma[g^+, \hat{h}]$ is defined as

$$Q_\gamma^{\hat{h}} = P_\gamma[g^+, \hat{h}](1).$$

The scattering operator has a pole at the integer values γ. However, in such cases the residue may be calculated and, in particular, when g^+ is Poincaré-Einstein metric, for $\gamma = 1$ we have

$$P_1[g^+, \hat{h}] = -\Delta_{\hat{h}} + \frac{n-2}{4(n-1)}R_{\hat{h}}$$

is exactly the so-called conformal Laplacian, and

$$Q_1^{\hat{h}} = \frac{n-2}{4(n-1)}R_{\hat{h}}.$$

Here $R_{\hat{h}}$ is the scalar curvature of the metric $\hat{h}$.

For $\gamma = 2$, $P_2[g^+, \hat{h}]$ is precisely the Paneitz operator and its associated curvature is known as Q-curvature [15]. In general, $P_k[g^+, \hat{h}]$ for $k \in \mathbb{N}$ are precisely the conformal powers of the Laplacian studied in [9].

We consider the conformal change $\hat{h}_w = w^{\frac{4}{n-2\gamma}}\hat{h}$ for some $w > 0$, then by (1.4), we have

$$P_\gamma[g^+, \hat{h}](w) = Q_\gamma^{\hat{h}_w} w^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{in } (M^n, \hat{h}).$$

If for this conformal change $Q_\gamma^{\hat{h}_w}$ is a constant C_γ on M^n, this problem reduces to

$$P_\gamma[g^+, \hat{h}](w) = C_\gamma w^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{in } (M^n, \hat{h}), \tag{1.5}$$

which is the so-called the fractional Yamabe equation or the γ-Yamabe equation studied in [8].

From now on, we always suppose that $\gamma \in (0, 1)$ throughout the paper, and such that $n^2/4 - \gamma^2$ is not an L^2 eigenvalue for $-\Delta_{g^+}$.

It is well known that the above fractional Yamabe equation may be rewritten as a degenerate elliptic Dirichlet-to-Neumann boundary problem. For that, we first recall some results obtained by S.A. Chang and M.d.M. González in [3]. Suppose that u^* solves

$$\begin{cases} -\Delta_{g^+}u^* - s(n-s)u^* = 0 \quad \text{in} \quad X^{n+1}, \\ \lim_{\rho_*\to 0} \rho_*^{s-n}u^* = 1 \quad \text{on } M^n. \end{cases} \tag{1.6}$$

Proposition 1.1. [3, 8] *Let $f \in \mathcal{C}^\infty(M)$. Assume that $\tilde{u}, u^*$ are solutions to (1.3) and (1.6), respectively. Then $\rho = (u^*)^{1/(n-s)}$ is a geodesic defining function. Moreover, $u = \tilde{u}/u^* = \rho^{s-n}\tilde{u}$ solves*

$$\begin{cases} -\operatorname{div}(\rho^{1-2\gamma}\nabla u) = 0 \quad \textit{in} \quad X^{n+1}, \\ u = f \quad \textit{on } M^n, \end{cases} \tag{1.7}$$

with respect to the metric $g = \rho^2 g^+$ and u is the unique minimizer of the energy functional

$$I(v) = \int_{X^{n+1}} \rho^{1-2\gamma}|\nabla v|_g^2 dv_g$$

among all the extensions $v \in W^{1,2}(X^{n+1}, \rho^{1-2\gamma})$ (see Definition 2.1) satisfying $v|_{M^n} = f$. Moreover,

$$\rho = \rho_* \left(1 + \frac{Q_\gamma^{\hat{h}}}{(n-s)d_\gamma}\rho_*^{2\gamma} + O(\rho_*^2) \right)$$

near the conformal infinity and

$$P_\gamma[g^+, \hat{h}](f) = -d_\gamma^* \lim_{\rho\to 0} \rho^{1-2\gamma}\partial_\rho u + Q_\gamma^{\hat{h}} f, \quad d_\gamma^* = -\frac{d_\gamma}{2\gamma} > 0,$$

provided that $\operatorname{Tr}_{\hat{h}} h^{(1)} = 0$ when $\gamma \in (1/2, 1)$. Here $g|_{M^n} = \hat{h}$, and has asymptotic expansion

$$g = d\rho^2[1 + O(\rho^{2\gamma})] + \hat{h}[1 + O(\rho^{2\gamma})].$$

We fix $\gamma \in (0, 1)$. By Proposition 1.1, one can rewrite the Yamabe equation (1.5) into the following problem:

$$\begin{cases} -\operatorname{div}(\rho^{1-2\gamma}\nabla u) = 0 \quad \text{in } (X^{n+1}, g), \\ u = w \quad \text{on } (M^n, \hat{h}), \\ -d_\gamma^* \lim_{\rho\to 0} \rho^{1-2\gamma}\partial_\rho u + Q_\gamma^{\hat{h}} w = C_\gamma w^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on } (M^n, \hat{h}). \end{cases} \tag{1.8}$$

In this paper we consider the positive curvature case $C_\gamma > 0$. Without loss of generality, we assume $C_\gamma = d_\gamma^*$.

In the particular case $\gamma = 1/2$, one may check that (1.8) reduces to (1.2), which was considered in [1]. The main difficulty we encounter here is the presence of the weight that makes the extension equation only degenerate elliptic.

Next, we introduce the so-called γ-Yamabe constant (c.f. [8]). For the defining function ρ mentioned above, we set

$$I_\gamma[u, g] = \frac{d_\gamma^* \int_X \rho^{1-2\gamma}|\nabla u|_g^2 \, dv_g + \int_M Q_\gamma^{\hat{h}} u^2 \, d\sigma_{\hat{h}}}{\left(\int_M |u|^{2^*} \, d\sigma_{\hat{h}}\right)^{\frac{2}{2^*}}},$$

then the γ-Yamabe constant is defined as

$$\Lambda_\gamma(M,[\hat{h}])=\inf\{I_\gamma[u,g]: u\in W^{1,2}(X,\rho^{1-2\gamma})\}. \tag{1.9}$$

It was shown in [8] that in the positive curvature case $C_\gamma>0$ we must have $\Lambda_\gamma(M,[\hat{h}])>0$.

Now we take a perturbation of the linear term $Q_\gamma^{\hat{h}}w$ to a general $-d_\gamma^* Q_\alpha^\gamma w$, where $Q_\alpha^\gamma\in\mathcal{C}^\infty(M^n)$, $\alpha\in\mathbb{N}$. Suppose that for any $\alpha\in\mathbb{N}$ and any $x\in M^n$, there exists a constant $C>0$ such that $|Q_\alpha^\gamma(x)|\le C$. And we also assume that $Q_\alpha^\gamma\to Q_\infty^\gamma$ in $L^2(M^n,\hat{h})$ as $\alpha\to+\infty$. We will consider a family of equations

$$\begin{cases} -\operatorname{div}(\rho^{1-2\gamma}\nabla u)=0 & \text{in } (X^{n+1},g),\\ u=w & \text{on } (M^n,\hat{h}),\\ -\lim\limits_{\rho\to0}\rho^{1-2\gamma}\partial_\rho u+Q_\alpha^\gamma w=w^{\frac{n+2\gamma}{n-2\gamma}} & \text{on } (M^n,\hat{h}).\end{cases} \tag{1.10}$$

The associated variational functional to (1.10) is

$$I_g^{\gamma,\alpha}(u)=\frac12\int_{X^{n+1}}\rho^{1-2\gamma}|\nabla u|_g^2\,dv_g+\frac12\int_{M^n}Q_\alpha^\gamma u^2\,d\sigma_{\hat{h}}-\frac{n-2\gamma}{2n}\int_{M^n}|u|^{\frac{2n}{n-2\gamma}}\,d\sigma_{\hat{h}}. \tag{1.11}$$

Hyperbolic space $(\mathbb{H}^{n+1},g_{\mathbb{H}})$ is the first example of a conformally compact Einstein manifold. As $(\mathbb{H}^{n+1},g_{\mathbb{H}})$ can be characterized as the upper half-space $\mathbb{R}_+^{n+1}$ endowed with metric $g^+=y^{-2}(|dx|^2+dy^2)$, where $x\in\mathbb{R}^n$, $y\in\mathbb{R}_+$, then the Dirichlet-to-Neumann problem (1.8) reduces to

$$\begin{cases} -\operatorname{div}(y^{1-2\gamma}\nabla u)=0 & \text{in } (\mathbb{R}_+^{n+1},|dx|^2+dy^2),\\ u=w & \text{on } (\mathbb{R}^n,|dx|^2),\\ -\lim\limits_{y\to0}y^{1-2\gamma}\partial_y u=w^{\frac{n+2\gamma}{n-2\gamma}} & \text{on } (\mathbb{R}^n,|dx|^2).\end{cases} \tag{1.12}$$

And the variational functional to (1.12) is defined as

$$\tilde{E}(u)=\frac12\int_{\mathbb{R}_+^{n+1}}y^{1-2\gamma}|\nabla u(x,y)|^2dxdy-\frac{n-2\gamma}{2n}\int_{\mathbb{R}^n}|u(x,0)|^{\frac{2n}{n-2\gamma}}\,dx.$$

Up to multiplicative constants, the only solution to problem (1.12) is given by the standard

$$w(x)=w_a^\lambda(x)=\left(\frac{\lambda}{|x-a|^2+\lambda^2}\right)^{\frac{n-2\gamma}{2}}$$

for some $a\in\mathbb{R}^n$ and $\lambda>0$ (c.f. [8],[11]). By L. Caffarelli and L. Silvestre's Poisson formula [2], the corresponding extension can be expressed as

$$U_a^\lambda(x,y)=\int_{\mathbb{R}^n}\frac{y^{2\gamma}}{(|x-\xi|^2+y^2)^{(n+2\gamma)/2}}\,w_a^\lambda(\xi)\,d\xi. \tag{1.13}$$

Here U_a^λ is called a "bubble". Note that all of them have constant energy. Indeed:

Remark 1.2. *For any $a\in\mathbb{R}^n$ and $\lambda>0$, we have*

$$\tilde{E}(U_a^\lambda)=\tilde{E}(U_0^1)=\frac{\gamma}{n}\int_{\mathbb{R}^n}|U_0^1(x,0)|^{\frac{2n}{n-\gamma}}\,dx.$$

Now we give some notations which will be used in the following. In the half space $\mathbb{R}^{n+1}_+ = \{(x,y) = (x^1, \cdots, x^n, y) \in \mathbb{R}^{n+1} \, : \, y > 0\}$ we define, for $r > 0$,

$$B_r^+(z_0) = \{z \in \mathbb{R}^{n+1}_+ : |z - z_0| < r, z_0 \in \mathbb{R}^{n+1}_+\},$$
$$D_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r, x_0 \in \mathbb{R}^n\},$$
$$\partial' B_r^+(z_0) = B_r^+(z_0) \cap \mathbb{R}^n, \quad \partial^+ B_r^+(z_0) = \partial B_r^+(z_0) \cap \mathbb{R}^{n+1}_+.$$

Fix $\gamma \in (0,1)$. Suppose that (X, g^+) is an asymptotically hyperbolic manifold with boundary M satisfying, in addition, $\mathrm{Tr}_{\hat{h}}\, h^{(1)} = 0$ when $\gamma \in (1/2, 1)$. Let ρ be the special defining function given in Proposition 1.1 and set $g = \rho^2 g^+$, $\hat{h} = g|_M$. We also define

$$\mathfrak{B}_r^+(z_0) = \{z \in X : d_g(z, z_0) < r, z_0 \in \overline{X}\},$$
$$\mathfrak{D}_r(x_0) = \{x \in M : d_{\hat{h}}(x, x_0) < r, x_0 \in M\},$$

Now, modulo the definitions of the weighted Sobolev space $W^{1,2}(X, \rho^{1-2\gamma})$ and of a Palais-Smale sequence (see section 2), the main result of this paper is the following fractional type blow up analysis theorem:

Theorem 1.3. *Let $\{u_\alpha \geq 0\}_{\alpha \in \mathbb{N}} \subset W^{1,2}(X, \rho^{1-2\gamma})$ be a Palais-Smale sequence for $\{I_g^{\gamma,\alpha}\}_{\alpha \in \mathbb{N}}$. Then there exist an integer $m \geq 1$, sequences $\{\mu_\alpha^j > 0\}_{\alpha \in \mathbb{N}}$ and $\{x_\alpha^j\}_{\alpha \in \mathbb{N}} \subset M$ for $j = 1, \cdots, m$, also a nonnegative solution $u^0 \in W^{1,2}(X, \rho^{1-2\gamma})$ to equation* (2.4) *and nontrivial nonnegative functions $U_{a_j}^{\lambda_j} \in W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$ for some $\lambda_j > 0$ and $a_j \in \mathbb{R}^n$ as given in* (1.13)*, satisfying, up to a subsequence,*

(1) *$\mu_\alpha^j \to 0$ as $\alpha \to +\infty$, for $j = 1, \cdots, m$;*
(2) *$\{x_\alpha^j\}_{\alpha \in \mathbb{N}}$ converges on M as $\alpha \to +\infty$, for $j = 1, \cdots, m$;*
(3) *As $\alpha \to +\infty$,*

$$\|u_\alpha - u^0 - \sum_{j=1}^m \eta_\alpha^j u_\alpha^j\|_{W^{1,2}(X, \rho^{1-2\gamma})} \to 0,$$

where

$$u_\alpha^j(z) = (\mu_\alpha^j)^{-\frac{n-2\gamma}{2}} U_{a_j}^{\lambda_j}((\mu_\alpha^j)^{-1} \varphi_{x_\alpha^j}^{-1}(z))$$

for $z \in \varphi_{x_\alpha^j}(B_{r_0}^+(0))$, and $\varphi_{x_\alpha^j}$ are Fermi coordinates centered at $x_\alpha^j \in M$ with $r_0 > 0$ small, and η_α^j are cutoff functions such that

$$\eta_\alpha^j \equiv 1 \quad \text{in } \varphi_{x_\alpha^j}(B_{r_0}^+(0)) \quad \text{and} \quad \eta_\alpha^j \equiv 0 \quad \text{in } M \setminus \varphi_{x_\alpha^j}(B_{2r_0}^+(0));$$

(4) *The energies*

$$I_g^{\gamma,\alpha}(u_\alpha) - I_g^\infty(u^0) - m\tilde{E}(U_{a_j}^{\lambda_j}) \to 0$$

as $\alpha \to +\infty$;
(5) *For any $1 \leq i, j \leq m$, $i \neq j$,*

$$\frac{\mu_\alpha^i}{\mu_\alpha^j} + \frac{\mu_\alpha^j}{\mu_\alpha^i} + \frac{d_{\hat{h}}(x_\alpha^i, x_\alpha^j)^2}{\mu_\alpha^i \mu_\alpha^j} \to +\infty, \quad \text{as } \alpha \to +\infty.$$

Remark 1.4. (i) *We call $\eta_\alpha^j u_\alpha^j$ a bubble for $j = 1, \cdots, m$.*
(ii) *If $u_\alpha \to u^0$ strongly in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$, then we must have $m = 0$ here.*

Although the local case $\gamma = 1$ is well known ([5, 16]), the most interesting point in the fractional case is the fact that one still has an energy decomposition into bubbles, and that these bubbles are non-interfering, which is surprising since our operator is non-local.

This paper is organized as follows: In section 2, we will first recall the definition of weighted Sobolev spaces and Palais-Smale sequences. Then we shall derive a criterion for the strong convergence of a given Palais-Smale sequence. At last, ε-regularity estimates will be established. In section 3, we shall extract the first bubble from the Palais-Smale sequence which is not strongly convergent. In section 4, we will give the proof of Theorem 1.3. Finally, some regularity estimates of the degenerate elliptic PDE are given as Appendix in Section 5.

2. Preliminary Results

Most of the arguments in this section are analogous to the results in [5] (Chapter 3). For the convenience of reader, we also prove these lemmas with the necessary modifications.

From now on we use $2^* = 2n/(n-2\gamma), \gamma \in (0,1)$ for simplicity and always assume that Palais-Smale sequences are all nonnegative. Moreover, the notation $o(1)$ will be taken with respect to to the limit $\alpha \to +\infty$.

Definition 2.1. *The weighted Sobolev space $W^{1,2}(X, \rho^{1-2\gamma})$ is defined as the closure of $\mathcal{C}^\infty(\overline{X})$ with norm*

$$\|u\|_{W^{1,2}(X,\rho^{1-2\gamma})} = \left(\int_X \rho^{1-2\gamma} |\nabla u|_g^2 \, dv_g + \int_M u^2 d\sigma_{\hat{h}} \right)^{\frac{1}{2}} \tag{2.1}$$

where dv_g is the volume form of the asymptotically hyperbolic Riemannian manifold (X, g) and $d\sigma_{\hat{h}}$ is the volume form of the conformal infinity $(M, [\hat{h}])$.

Proposition 2.2. *The norm defined above is equivalent to the following traditional norm*

$$\|u\|^*_{W^{1,2}(X,\rho^{1-2\gamma})} = \left(\int_X \rho^{1-2\gamma} (|\nabla u|_g^2 + u^2) \, dv_g \right)^{\frac{1}{2}}. \tag{2.2}$$

On one hand, $\|\cdot\|$ can be controlled by $\|\cdot\|^*$. This is a easy consequence of the following two propositions. The first one is a trace Sobolev embedding on Euclidean space.

Proposition 2.3. [12] *For any $u \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1}_+)$ we have*

$$\left(\int_{\mathbb{R}^n} |u(x,0)|^{2^*} dx \right)^{\frac{2}{2^*}} \leq S(n,\gamma) \int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma} |\nabla u(x,y)|^2 dxdy$$

where

$$S(n,\gamma) = \frac{1}{2\pi^\gamma} \frac{\Gamma(\gamma)}{\Gamma(1-\gamma)} \frac{\Gamma(\frac{n-2\gamma}{2})}{\Gamma(\frac{n+2\gamma}{2})} \left(\frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{2\gamma}{n}}.$$

Using a standard partition of unity argument one obtains a weighted trace Sobolev inequality on an asymptotically hyperbolic manifold:

Proposition 2.4. [12] *For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that*

$$\left(\int_M |u|^{2^*} d\sigma_{\hat{h}} \right)^{\frac{2}{2^*}} \leq (S(n,\gamma) + \varepsilon) \int_X \rho^{1-2\gamma} |\nabla u|_g^2 \, dv_g + C_\varepsilon \int_X \rho^{1-2\gamma} u^2 \, dv_g.$$

On the other hand, $\|\cdot\|^*$ can be controlled by $\|\cdot\|$, which is implied by the following proposition.

Proposition 2.5. *For any $u \in W^{1,2}(X, \rho^{1-2\gamma})$, there exists a constant $C > 0$ such that*

$$\int_X \rho^{1-2\gamma} u^2 \, dv_g \leq C \left(\int_X \rho^{1-2\gamma} |\nabla u|_g^2 \, dv_g + \int_M u^2 \, d\sigma_{\hat{h}} \right).$$

Proof. We use a contradiction argument. Thus, assume that for any $\alpha \geq 1$ there exists u_α satisfying
$$\int_X \rho^{1-2\gamma} u_\alpha^2\, dv_g \geq \alpha \left(\int_X \rho^{1-2\gamma} |\nabla u_\alpha|_g^2\, dv_g + \int_M u_\alpha^2\, d\sigma_{\hat{h}} \right).$$
Without loss of generality, we can assume that $\int_X \rho^{1-2\gamma} u_\alpha^2\, dv_g = 1$. Then we have
$$\int_X \rho^{1-2\gamma} (|\nabla u_\alpha|_g^2 + u_\alpha^2)\, dv_g \leq 1 + \frac{1}{\alpha}.$$
Then there exists a weakly convergent subsequence, also denoted by $\{u_\alpha\}$, such that $u_\alpha \rightharpoonup u_0$ in $W^{1,2}(X, \rho^{1-2\gamma}, \|\cdot\|^*)$.

Since
$$\lim_{\alpha\to\infty} \int_X \rho^{1-2\gamma} |\nabla u_\alpha|_g^2 dv_g = 0 \quad \text{and} \quad \lim_{\alpha\to\infty} \int_M u_\alpha^2 d\sigma_{\hat{h}} = 0,$$
then we get that $u_0 \equiv 0$. On the other hand, via the following Proposition 2.6, the embeddig $W^{1,2}(X, \rho^{1-2\gamma}, \|\cdot\|^*) \hookrightarrow L^2(X, \rho^{1-2\gamma})$ is compact. So we have
$$\int_X \rho^{1-2\gamma} u_0^2\, dv_g = 1,$$
which contradicts the fact that $u_0 \equiv 0$. Then the proof is completed. $\square$

Proposition 2.6. [12, 13, 4] *Let $1 \leq p \leq q < \infty$ with $\frac{1}{n+1} > \frac{1}{p} - \frac{1}{q}$.*

(i) *Suppose $2-2\gamma \leq p$. Then $W^{1,p}(X, \rho^{1-2\gamma}, \|\cdot\|^*)$ is compactly embedded in $L^q(X, \rho^{1-2\gamma})$ if*
$$\frac{2-2\gamma}{p(n+2-2\gamma)} > \frac{1}{p} - \frac{1}{q};$$
(ii) *Suppose $2-2\gamma > p$. Then $W^{1,p}(X, \rho^{1-2\gamma}, \|\cdot\|^*)$ is compactly embedded in $L^q(X, \rho^{1-2\gamma})$ if and only if*
$$\frac{1}{(n+2-2\gamma)} > \frac{1}{p} - \frac{1}{q}.$$

We will always use the norm in $W^{1,2}(X, \rho^{1-2\gamma})$ in the following unless otherwise stated.

Definition 2.7. $\overline{W}^{1,2}(X, \rho^{1-2\gamma})$ *is the closure of $\mathcal{C}_0^\infty(X)$ in $W^{1,2}(X, \rho^{1-2\gamma})$ with the norm*
$$\|u\|_{\overline{W}^{1,2}(X,\rho^{1-2\gamma})} = \left(\int_X \rho^{1-2\gamma} |\nabla u|_g^2\, dv_g \right)^{\frac{1}{2}}.$$

Now we define Palais-Smale sequences for the functional (1.11) precisely.

Definition 2.8. $\{u_\alpha\}_{\alpha\in\mathbb{N}} \subset W^{1,2}(X, \rho^{1-2\gamma})$ *is called a Palais-Smale sequence for $\{I_g^{\gamma,\alpha}\}_{\alpha\in\mathbb{N}}$ if:*

(i) $\{I_g^{\gamma,\alpha}(u_\alpha)\}_{\alpha\in\mathbb{N}}$ *is uniformly bounded; and*

(ii) *as $\alpha \to +\infty$,*
$$DI_g^{\gamma,\alpha}(u_\alpha) \to 0 \text{ strongly in } W^{1,2}(X, \rho^{1-2\gamma})',$$
where we have defined $W^{1,2}(X, \rho^{1-2\gamma})'$ as the dual space of $W^{1,2}(X, \rho^{2\gamma-1})$, i.e. for any $\phi \in W^{1,2}(X, \rho^{1-2\gamma})$, then
$$\begin{aligned} DI_g^{\gamma,\alpha}(u_\alpha)\cdot\phi &= \int_X \rho^{1-2\gamma} \langle \nabla u_\alpha, \nabla \phi \rangle_g\, dv_g + \int_M Q_\alpha^\gamma u_\alpha \phi\, d\sigma_{\hat{h}} \\ &\quad - \int_M u_\alpha^{2^*-1} \phi\, d\sigma_{\hat{h}} \\ &= o(\|\phi\|_{W^{1,2}(X,\rho^{1-2\gamma})}) \quad \text{as } \alpha \to +\infty. \end{aligned} \tag{2.3}$$

The main properties of Palais-Smale sequences are contained in the next several lemmas:

Lemma 2.9. *Let $\{u_\alpha\}_{\alpha\in\mathbb{N}} \subset W^{1,2}(X,\rho^{1-2\gamma})$ be a Palais-Smale sequence for the functionals $\{I_g^{\gamma,\alpha}\}_{\alpha\in\mathbb{N}}$, then $\{u_\alpha\}_{\alpha\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(X,\rho^{1-2\gamma})$.*

Proof. We can take $\phi = u_\alpha \in W^{1,2}(X,\rho^{1-2\gamma})$ as a test function in (ii) of Definition 2.8, then we get

$$\int_X \rho^{1-2\gamma}|\nabla u_\alpha|_g^2\, dv_g + \int_M Q_\alpha^\gamma u_\alpha^2\, d\sigma_{\hat h} = \int_M u_\alpha^{2^*}\, d\sigma_{\hat h} + o(\|u_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})}),$$

which yields that

$$\begin{aligned} I_g^{\gamma,\alpha}(u_\alpha) &= \frac{1}{2}\int_X \rho^{1-2\gamma}|\nabla u_\alpha|_g^2\, dv_g + \frac{1}{2}\int_M Q_\alpha^\gamma u_\alpha^2\, d\sigma_{\hat h} - \frac{1}{2^*}\int_M u_\alpha^{2^*}\, d\sigma_{\hat h} \\ &= \frac{\gamma}{n}\int_M u_\alpha^{2^*}\, d\sigma_{\hat h} + o(\|u_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})}). \end{aligned}$$

Since $\{I_g^{\gamma,\alpha}(u_\alpha)\}_{\alpha\in\mathbb{N}}$ is uniformly bounded by (i) of Definition 2.8, there exists a constant $C>0$ such that

$$\int_M u_\alpha^{2^*} d\sigma_{\hat h} \le C + o(\|u_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})}),$$

which by Hölder's inequality yields

$$\int_M u_\alpha^2\, d\sigma_{\hat h} \le C\left(\int_M u_\alpha^{2^*}\, d\sigma_{\hat h}\right)^{2/2^*} \le C + o(\|u_\alpha\|^{2/2^*}_{W^{1,2}(X,\rho^{1-2\gamma})}).$$

Note that since $|Q_\alpha^\gamma| \le C$ for some constant $C>0$, we can choose sufficiently large $C_1>0$ such that $C_1 + Q_\alpha^\gamma \ge 1$ on M. It follows

$$\begin{aligned} \|u_\alpha\|^2_{W^{1,2}(X,\rho^{1-2\gamma})} &= \int_X \rho^{1-2\gamma}|\nabla u_\alpha|_g^2\, dv_g + \int_M u_\alpha^2\, d\sigma_{\hat h} \\ &\le \int_X \rho^{1-2\gamma}|\nabla u_\alpha|_g^2\, dv_g + \int_M Q_\alpha^\gamma u_\alpha^2\, d\sigma_{\hat h} + C_1\int_M u_\alpha^2\, d\sigma_{\hat h} \\ &\le \int_M u_\alpha^{2^*}\, d\sigma_{\hat h} + o(\|u_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})}) + C + o(\|u_\alpha\|^{2/2^*}_{W^{1,2}(X,\rho^{1-2\gamma})}) \\ &\le C + o(\|u_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})}) + o(\|u_\alpha\|^{2/2^*}_{W^{1,2}(X,\rho^{1-2\gamma})}). \end{aligned}$$

which concludes that $\{u_\alpha\}_{\alpha\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(X,\rho^{1-2\gamma})$ since $2/2^* < 1$. The proof is finished. □

Remark 2.10. *From Lemma 2.9, it is easy to see that there exists a function u^0 in $W^{1,2}(X,\rho^{1-2\gamma})$ such that $u_\alpha \rightharpoonup u^0$ weakly in $W^{1,2}(X,\rho^{1-2\gamma})$ as $\alpha\to+\infty$.*

Proposition 2.11. *$u^0 \ge 0$ in $\overline{X}$.*

Proof. Using Proposition 2.4, we can easily get that $u_\alpha \to u^0$ in $L^2(M,\hat h)$ as $\alpha\to+\infty$, so furthermore we have $u_\alpha \to u^0$ almost everywhere on M. Noting that $u_\alpha \ge 0$ on M, then we obtain that $u^0 \ge 0$ on M. On the other hand, by Proposition 2.6, and the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|^*$, we have $u_\alpha \to u^0$ in $L^2(X,\rho^{1-2\gamma})$ as $\alpha\to+\infty$. For any $z\in X$, take $d_z < \mathrm{dist}(z,M)$, then we also have $u_\alpha \to u^0$ in $L^2(\mathfrak{B}^+_{d_z}(z),\rho^{1-2\gamma})$. Since $\rho^{1-2\gamma}$ is bounded below by a positive constant in $\mathfrak{B}^+_{d_z}(z)$, we get $u_\alpha \to u^0$ almost everywhere in $\mathfrak{B}^+_{d_z}(z)$ up to passing to a subsequence. Noting that $u_\alpha \ge 0$ in X, we obtain $u^0 \ge 0$ in $\mathfrak{B}^+_{d_z}(z)$. Since z is arbitrary in X, then $u^0 \ge 0$ in X. Combining the above arguments, we conclude that $u \ge 0$ in $\overline{X}$. □

Next we define the two limit functionals

$$I_g^\gamma(u)=\frac{1}{2}\int_X \rho^{1-2\gamma}|\nabla u|_g^2\,dv_g-\frac{1}{2^*}\int_M |u|^{2^*}d\sigma_{\hat h}$$

and

$$I_g^{\gamma,\infty}(u)=\frac{1}{2}\int_X \rho^{1-2\gamma}|\nabla u|_g^2\,dv_g+\frac{1}{2}\int_M Q_\infty^\gamma u^2\,d\sigma_{\hat h}-\frac{1}{2^*}\int_M |u|^{2^*}d\sigma_{\hat h}.$$

We have the following lemma:

Lemma 2.12. *Let $\{u_\alpha\}_{\alpha\in\mathbb{N}}\subset W^{1,2}(X,\rho^{1-2\gamma})$ be a Palais-Smale sequence for $\{I_g^{\gamma,\alpha}\}_{\alpha\in\mathbb{N}}$, and $u_\alpha\rightharpoonup u^0$ weakly in $W^{1,2}(X,\rho^{1-2\gamma})$ as $\alpha\to+\infty$. We also denote $\hat u_\alpha=u_\alpha-u^0\in W^{1,2}(X,\rho^{1-2\gamma})$. Then*

(i) *u^0 is a nonnegative weak solution to the limit equation*

$$\begin{cases} -\operatorname{div}(\rho^{1-2\gamma}\nabla u)=0 & \text{in } X,\\ -\lim\limits_{\rho\to 0}\rho^{1-2\gamma}\partial_\rho u+Q_\infty^\gamma u=u^{2^*-1} & \text{on } M; \end{cases}\tag{2.4}$$

(ii) *$I_g^{\gamma,\alpha}(u_\alpha)=I_g^\gamma(\hat u_\alpha)+I_g^{\gamma,\infty}(u^0)+o(1)$ as $\alpha\to+\infty$;*

(iii) *$\{\hat u_\alpha\}_{\alpha\in\mathbb{N}}$ is a Palais-Smale sequence for I_g^γ.*

Proof. (i) As $\mathcal{C}^\infty(\overline{X})$ is dense in $W^{1,2}(X,\rho^{1-2\gamma})$, we only consider the proof in $\mathcal{C}^\infty(\overline{X})$. Let $\phi\in\mathcal{C}^\infty(\overline{X})$. Since $Q_\alpha^\gamma\to Q_\infty^\gamma$ in $L^2(M,\hat h)$ as $\alpha\to+\infty$ and $u_\alpha\rightharpoonup u^0$ weakly in $W^{1,2}(X,\rho^{1-2\gamma})$ as $\alpha\to+\infty$, then

$$\int_M Q_\alpha^\gamma u_\alpha\phi\,d\sigma_{\hat h}=\int_M Q_\infty^\gamma u^0\phi\,d\sigma_{\hat h}+o(1).$$

Passing to the limit in (2.3), we get easily that

$$\int_X \rho^{1-2\gamma}\langle\nabla u^0,\nabla\phi\rangle_g\,dv_g+\int_M Q_\infty^\gamma u^0\phi\,d\sigma_{\hat h}=\int_M (u^0)^{2^*-1}\phi\,d\sigma_{\hat h},$$

i.e. u^0 is a weak solution to the limit equation (2.4).

For the proof of (ii), recall that

$$\int_M Q_\alpha^\gamma u_\alpha^2\,d\sigma_{\hat h}=\int_M Q_\infty^\gamma (u^0)^2\,d\sigma_{\hat h}+o(1),$$

and

$$\begin{aligned} I_g^{\gamma,\alpha}(u_\alpha)&=\frac{1}{2}\int_X \rho^{1-2\gamma}|\nabla u_\alpha|_g^2\,dv_g+\frac{1}{2}\int_M Q_\alpha^\gamma u_\alpha^2\,d\sigma_{\hat h}-\frac{1}{2^*}\int_M u_\alpha^{2^*}\,d\sigma_{\hat h},\\ I_g^{\gamma,\infty}(u^0)&=\frac{1}{2}\int_X \rho^{1-2\gamma}|\nabla u^0|_g^2\,dv_g+\frac{1}{2}\int_M Q_\infty^\gamma (u^0)^2\,d\sigma_{\hat h}-\frac{1}{2^*}\int_M (u^0)^{2^*}\,d\sigma_{\hat h},\\ I_g^\gamma(\hat u_\alpha)&=\frac{1}{2}\int_X \rho^{1-2\gamma}|\nabla \hat u_\alpha|_g^2\,dv_g-\frac{1}{2^*}\int_M |\hat u_\alpha|^{2^*}\,d\sigma_{\hat h}, \end{aligned}$$

where $\hat u_\alpha=u_\alpha-u^0$. Then

$$\begin{aligned} &I_g^{\gamma,\alpha}(u_\alpha)-I_g^{\gamma,\infty}(u^0)-I_g^\gamma(\hat u_\alpha)\\ &\qquad=\int_X \rho^{1-2\gamma}\langle\nabla u^0,\nabla\hat u_\alpha\rangle_g\,dv_g-\frac{1}{2^*}\int_M \Phi_\alpha\,d\sigma_{\hat h}+o(1), \end{aligned}$$

where $\Phi_\alpha = |\hat{u}_\alpha + u^0|^{2^*} - |\hat{u}_\alpha|^{2^*} - |u^0|^{2^*}$. Note that $\hat{u}_\alpha \rightharpoonup 0$ weakly in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$, thus

$$\int_X \rho^{1-2\gamma} \langle \nabla u^0, \nabla \hat{u}_\alpha \rangle_g \, dv_g \to 0, \quad \text{as } \alpha \to \infty.$$

On the other hand, it is easy to check that there exists a constant $C > 0$, independent of α, such that

$$\left| |\hat{u}_\alpha + u^0|^{2^*} - |\hat{u}_\alpha|^{2^*} - |u^0|^{2^*} \right| \leq C \left(|\hat{u}_\alpha|^{2^*-1} |u^0| + |u^0|^{2^*-1} |\hat{u}_\alpha| \right).$$

As a consequence, since $\hat{u}_\alpha \rightharpoonup 0$ weakly in $L^{2^*}(M, \hat{h})$ by Proposition 2.4, we have

$$\int_M |\Phi_\alpha| \, d\sigma_{\hat{h}} \to 0, \quad \text{as } \alpha \to +\infty.$$

The proof of (ii) is completed.

(iii) For any $\phi \in \mathcal{C}^\infty(\overline{X})$, by (i) we have

$$DI_g^{\gamma,\infty}(u^0) \cdot \phi = 0.$$

Since, in addition,

$$\int_M Q_\alpha^\gamma u_\alpha \phi \, d\sigma_{\hat{h}} = \int_M Q_\infty^\gamma u^0 \phi \, d\sigma_{\hat{h}} + o(\|\phi\|_{W^{1,2}(X, \rho^{1-2\gamma})}),$$

then

$$DI_g^{\gamma,\alpha}(u_\alpha) \cdot \phi = DI_g^\gamma(\hat{u}_\alpha) \cdot \phi - \int_M \Psi_\alpha \phi \, d\sigma_{\hat{h}} + o(\|\phi\|_{W^{1,2}(X, \rho^{1-2\gamma})}), \tag{2.5}$$

where $\Psi_\alpha = |\hat{u}_\alpha + u^0|^{2^*-2}(\hat{u}_\alpha + u^0) - |\hat{u}_\alpha|^{2^*-2}\hat{u}_\alpha - |u^0|^{2^*-2}u^0$, and it is easy to check that there exits a constant $C > 0$ independent of α such that

$$|\Psi_\alpha| \leq C \left(|\hat{u}_\alpha|^{2^*-2} |u^0| + |\hat{u}_\alpha| |u^0|^{2^*-2} \right).$$

By Hölder's inequality and the fact $\hat{u}_\alpha \rightharpoonup 0$ weakly in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$, we have

$$\begin{aligned} &\int_M \Psi_\alpha \phi \, d\sigma_{\hat{h}} \\ &\quad \leq \left(\left\| |\hat{u}_\alpha|^{2^*-2} |u^0| \right\|_{L^{2^*/(2^*-1)}(M)} + \left\| |\hat{u}_\alpha| |u^0|^{2^*-2} \right\|_{L^{2^*/(2^*-1)}(M)} \right) \|\phi\|_{L^{2^*}(M)} \\ &\quad = o(1) \|\phi\|_{L^{2^*}(M)}. \end{aligned}$$

Thus from (2.5),

$$DI_g^{\gamma,\alpha}(u_\alpha) \cdot \phi = DI_g^\gamma(\hat{u}_\alpha) \cdot \phi + o(1) \|\phi\|_{L^{2^*}(M)},$$

which implies that $DI_g^\gamma(\hat{u}_\alpha) \to 0$ in $W^{1,2}(X, \rho^{1-2\gamma})'$ as $\alpha \to +\infty$, since $\{u_\alpha\}_{\alpha \in \mathbb{N}}$ is a Palais-Smale sequence for $\{I_g^{\gamma,\alpha}\}_{\alpha \in \mathbb{N}}$.

Finally, from (ii), we know that $\{\hat{u}_\alpha\}_{\alpha \in \mathbb{N}}$ is a Palais-Smale sequence for I_g^γ. This completes the proof of the lemma. □

Now we give a criterion for strong convergence of Palais-Smale sequences. First,

Lemma 2.13. *Let $\{\hat{u}_\alpha\}_{\alpha \in \mathbb{N}}$ be a Palais-Smale sequence for I_g^γ and such that $\hat{u}_\alpha \rightharpoonup 0$ weakly in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$. If $I_g^\gamma(\hat{u}_\alpha) \to \beta$ and*

$$\beta < \beta_0 = \frac{\gamma}{n} (d_\gamma^*)^{-\frac{n}{2\gamma}} \Lambda_\gamma(M, [\hat{h}])^{\frac{n}{2\gamma}}, \tag{2.6}$$

then $\hat{u}_\alpha \to 0$ *in* $W^{1,2}(X,\rho^{1-2\gamma})$ *as* $\alpha \to +\infty$.

Proof. By Lemma 2.9 (here $Q^\gamma_\alpha \equiv 0$), there exists a constant $C>0$ such that $\|\hat{u}_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})} \le C$ for all $\alpha \in \mathbb{N}$, so

$$\begin{aligned} DI^\gamma_g(\hat{u}_\alpha)\cdot \hat{u}_\alpha &= \int_X \rho^{1-2\gamma}|\nabla \hat{u}_\alpha|^2_g\, dv_g - \int_M |\hat{u}_\alpha|^{2^*}\, d\sigma_{\hat{h}} \\ &= o(\|\hat{u}_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})}) = o(1). \end{aligned}$$

Then note that $I^\gamma_g(\hat{u}_\alpha)\to\beta$ as $\alpha\to+\infty$, we have

$$\begin{aligned} \beta + o(1) &= I^\gamma_g(\hat{u}_\alpha) \\ &= \frac{1}{2}\int_X \rho^{1-2\gamma}|\nabla \hat{u}_\alpha|^2_g\, dv_g - \frac{1}{2^*}\int_M |\hat{u}_\alpha|^{2^*}\, d\sigma_{\hat{h}} \\ &= \frac{\gamma}{n}\int_X \rho^{1-2\gamma}|\nabla \hat{u}_\alpha|^2_g\, dv_g + o(1) \\ &= \frac{\gamma}{n}\int_M |\hat{u}_\alpha|^{2^*}\, d\sigma_{\hat{h}} + o(1). \end{aligned} \tag{2.7}$$

On the other hand, it was shown in [8] that in the positive curvature case, then the γ-Yamabe constant (1.9) must be positive: $\Lambda_\gamma(M,[\hat{h}])>0$. Moreover, by definition,

$$\Lambda_\gamma(M,[\hat{h}])\left(\int_M |\hat{u}_\alpha|^{2^*}\, d\sigma_{\hat{h}}\right)^{\frac{2}{2^*}} \le d^*_\gamma \int_X \rho^{1-2\gamma}|\nabla \hat{u}_\alpha|^2_g\, dv_g + \int_M Q^{\hat{h}}_\gamma \hat{u}^2_\alpha\, d\sigma_{\hat{h}}. \tag{2.8}$$

where $d^*_\gamma>0$. We also know that $|Q^{\hat{h}}_\gamma|\le C$ on M^n. Note that $\hat{u}_\alpha \rightharpoonup 0$ in $L^{2^*}(M,\hat{h})$ as $\alpha\to+\infty$ by Proposition 2.4, then $\int_M \hat{u}^2_\alpha\, d\sigma_{\hat{h}} \to 0$ as $\alpha\to+\infty$ since the embedding $L^{2^*}(M,\hat{h})\subset L^2(M,\hat{h})$ is compact. So we get from (2.7) and (2.8) that

$$\left(\frac{n}{\gamma}\beta + o(1)\right)^{\frac{2}{2^*}} \le d^*_\gamma \Lambda_\gamma(M,[\hat{h}])^{-1}\frac{n}{\gamma}\beta + o(1).$$

Taking $\alpha\to+\infty$, we must have $\beta=0$ because of our initial condition (2.6). The Lemma is proved. ∎

Note that the Palais-Smale condition (ii) is the weak form of a Dirichlet-to-Neumann problem for a degenerate elliptic PDE. In fact, as $DI^\gamma_g(\hat{u}_\alpha)\to 0$ in $W^{1,2}(X,\rho^{1-2\gamma})'$, it follows that, for any $\psi\in W^{1,2}(X,\rho^{1-2\gamma})$,

$$\int_X \rho^{1-2\gamma}\langle \nabla \hat{u}_\alpha, \nabla\psi\rangle_g\, dv_g - \int_M |\hat{u}_\alpha|^{2^*-2}\hat{u}_\alpha\psi\, d\sigma_{\hat{h}} = o(1)\|\psi\|_{W^{1,2}(X,\rho^{1-2\gamma})}. \tag{2.9}$$

In particular, for any $\bar{\psi}\in \overline{W}^{1,2}(X,\rho^{1-2\gamma})$, then

$$\int_X \rho^{1-2\gamma}\langle \nabla \hat{u}_\alpha, \nabla\bar{\psi}\rangle_g\, dv_g = o(1)\|\bar{\psi}\|_{\overline{W}^{1,2}(X,\rho^{1-2\gamma})},$$

which is is precisely the weak formulation for the asymptotic equation

$$-\operatorname{div}(\rho^{1-2\gamma}\nabla \hat{u}_\alpha) = o(1) \quad \text{in } X. \tag{2.10}$$

Multiplying both sides of (2.10) by $\psi\in W^{1,2}(X,\rho^{1-2\gamma})$ and integrating by parts, we obtain that

$$\int_M \lim_{\rho\to 0}\rho^{1-2\gamma}\partial_\rho \hat{u}_\alpha\psi\, d\sigma_{\hat{h}} + \int_X \rho^{1-2\gamma}\langle \nabla \hat{u}_\alpha, \nabla\psi\rangle_g\, dv_g = o(1)\|\psi\|_{W^{1,2}(X,\rho^{1-2\gamma})},$$

which combined with (2.9) yields that

$$\int_M \lim_{\rho\to 0}\rho^{1-2\gamma}\partial_\rho \hat{u}_\alpha \psi\, d\sigma_{\hat{h}} + \int_M |\hat{u}_\alpha|^{2^*-2}\hat{u}_\alpha \psi\, d\sigma_{\hat{h}} = o(1)\|\psi\|_{W^{1,2}(X,\rho^{1-2\gamma})},$$

and this is precisely the boundary equation in the weak sense

$$-\lim_{\rho\to 0}\rho^{1-2\gamma}\partial_\rho \hat{u}_\alpha = |\hat{u}_\alpha|^{2^*-2}\hat{u}_\alpha + o(1) \text{ on } M. \tag{2.11}$$

For the above equations (2.10) and (2.11) for $\{\hat{u}_\alpha\}_{\alpha\in\mathbb{N}}$, we have the following energy estimate, which will plays an important role in the proof of the strong convergence in the next section. We use the notation $\mathfrak{B}_r^+$ instead of $\mathfrak{B}_r^+(0)$ for convenience.

Lemma 2.14. *(ε-regularity estimates) Suppose that $\{v_\alpha\}_{\alpha\in\mathbb{N}}$ satisfies the following asymptotic boundary value problem*

$$\begin{cases} -\operatorname{div}(\rho^{1-2\gamma}\nabla v_\alpha) = o(1) & \text{in } X, \\ -\lim_{\rho\to 0}\rho^{1-2\gamma}\partial_\rho v_\alpha = |v_\alpha|^{2^*-2}v_\alpha + o(1) & \text{on } M. \end{cases} \tag{2.12}$$

If there exists small $\varepsilon > 0$ depending on n, γ such that $\int_{\partial'\mathfrak{B}_{2r}^+}|v_\alpha|^{2^*}d\sigma_{\hat{h}} \le \varepsilon$ uniformly in α for some small $r > 0$, then

$$\int_{\mathfrak{B}_r^+}\rho^{1-2\gamma}|\nabla v_\alpha|_g^2\, dv_g \le \frac{C}{r^2}\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}v_\alpha^2\, dv_g + C\int_{\partial'\mathfrak{B}_{2r}^+}v_\alpha^2\, d\sigma_{\hat{h}} + o(1)\int_{\mathfrak{B}_{2r}^+}|v_\alpha|\, dv_g,$$

where $C = C(n, \varepsilon, \gamma)$ independent of α.

Proof. Let η be a smooth cutoff function in $\overline{X}$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $\mathfrak{B}_r^+(0)$ and $\eta \equiv 0$ in $\overline{X}\setminus\mathfrak{B}_{2r}^+(0)$. Multiplying both sides of the first equation in (2.12) by $\eta^2 v_\alpha$, integrating by parts and substituting the second equation in (2.12), we get

$$\begin{aligned}
&\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}\langle\nabla v_\alpha, \nabla(\eta^2 v_\alpha)\rangle_g\, dv_g \\
&\qquad = -\int_{\partial'\mathfrak{B}_{2r}^+}\lim_{\rho\to 0}\rho^{1-2\gamma}(\partial_\rho v_\alpha)\eta^2 v_\alpha\, d\sigma_{\hat{h}} + o(1)\int_{\mathfrak{B}_{2r}^+}\eta^2 v_\alpha\, dv_g \\
&\qquad = \int_{\partial'\mathfrak{B}_{2r}^+}\eta^2|v_\alpha|^{2^*}d\sigma_{\hat{h}} + o(1)\int_{\mathfrak{B}_{2r}^+}\eta^2 v_\alpha\, dv_g,
\end{aligned}$$

so we have

$$\begin{aligned}
\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}\eta^2|\nabla v_\alpha|_g^2\, dv_g &= -\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}2\eta v_\alpha\langle\nabla v_\alpha, \nabla\eta\rangle_g\, dv_g \\
&\quad + \int_{\partial'\mathfrak{B}_{2r}^+}\eta^2|v_\alpha|^{2^*}d\sigma_{\hat{h}} + o(1)\int_{\mathfrak{B}_{2r}^+}\eta^2|v_\alpha|\, dv_g \\
&\le \frac{1}{2}\int_{\mathfrak{B}_{2r}^+}\eta^2\rho^{1-2\gamma}|\nabla v_\alpha|_g^2\, dv_g + 2\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}|\nabla\eta|_g^2\, v_\alpha^2\, dv_g \\
&\quad + \int_{\partial'\mathfrak{B}_{2r}^+}\eta^2|v_\alpha|^{2^*}\, d\sigma_{\hat{h}} + o(1)\int_{\mathfrak{B}_{2r}^+}\eta^2|v_\alpha|\, dv_g,
\end{aligned}$$

which implies that

$$\begin{aligned}\int_{\mathfrak{B}_{2r}^+} \rho^{1-2\gamma}\eta^2|\nabla v_\alpha|_g^2\,dv_g \leq& 4\int_{\mathfrak{B}_{2r}^+} \rho^{1-2\gamma}|\nabla\eta|_g^2 v_\alpha^2\,dv_g + 2\int_{\partial'\mathfrak{B}_{2r}^+} \eta^2|v_\alpha|^{2^*}\,d\sigma_{\hat{h}}\\ &+ o(1)\int_{\mathfrak{B}_{2r}^+}\eta^2|v_\alpha|\,dv_g\\ \leq& \frac{C}{r^2}\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}v_\alpha^2\,dv_g + 2\int_{\partial'\mathfrak{B}_{2r}^+}(\eta v_\alpha)^2|v_\alpha|^{2^*-2}\,d\sigma_{\hat{h}}\\ &+ o(1)\int_{\mathfrak{B}_{2r}^+}\eta^2|v_\alpha|\,dv_g.\end{aligned}$$

By Hölder's inequality and our initial hypothesis we have

$$\begin{aligned}\int_{\partial'\mathfrak{B}_{2r}^+}(\eta v_\alpha)^2|v_\alpha|^{2^*-2}\,d\sigma_{\hat{h}} &\leq \left(\int_{\partial'\mathfrak{B}_{2r}^+}|\eta v_\alpha|^{2^*}\,d\sigma_{\hat{h}}\right)^{\frac{2}{2^*}}\left(\int_{\partial'\mathfrak{B}_{2r}^+}|v_\alpha|^{2^*}\,d\sigma_{\hat{h}}\right)^{\frac{2^*-2}{2^*}}\\ &\leq \varepsilon^{\frac{2^*-2}{2^*}}\left(\int_{\partial'\mathfrak{B}_{2r}^+}|\eta v_\alpha|^{2^*}\,d\sigma_{\hat{h}}\right)^{\frac{2}{2^*}}.\end{aligned}$$

Then it follows from above that

$$\begin{aligned}\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}|\nabla(\eta v_\alpha)|_g^2\,dv_g \leq& 2\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}(|\nabla\eta|_g^2\,v_\alpha^2 + \eta^2|\nabla v_\alpha|_g^2)\,dv_g\\ \leq& \frac{C}{r^2}\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}v_\alpha^2\,dv_g + C\varepsilon^{\frac{2^*-2}{2^*}}\left(\int_{\partial'\mathfrak{B}_{2r}^+}|\eta v_\alpha|^{2^*}\,d\sigma_{\hat{h}}\right)^{\frac{2}{2^*}}\\ &+ o(1)\int_{\mathfrak{B}_{2r}^+}\eta^2 v_\alpha\,dv_g.\end{aligned}$$

The trace Sobolev inequality on our manifold setting (Proposition 2.4) gives that

$$\left(\int_{\partial'\mathfrak{B}_{2r}^+}|\eta v_\alpha|^{2^*}\,d\sigma_{\hat{h}}\right)^{\frac{2}{2^*}} \leq C\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}|\nabla(\eta v_\alpha)|_g^2\,dv_g + C\int_{\partial'\mathfrak{B}_{2r}^+}(\eta v_\alpha)^2\,d\sigma_{\hat{h}}.$$

Therefore we obtain

$$\begin{aligned}\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}|\nabla(\eta v_\alpha)|_g^2\,dv_g \leq& \frac{C}{r^2}\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}v_\alpha^2\,dv_g + C\varepsilon^{\frac{2^*-2}{2^*}}\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}|\nabla(\eta v_\alpha)|_g^2\,dv_g\\ &+ C\varepsilon^{\frac{2^*-2}{2^*}}\int_{\partial'\mathfrak{B}_{2r}^+}(\eta v_\alpha)^2\,d\sigma_{\hat{h}} + o(1)\int_{\mathfrak{B}_{2r}^+}\eta^2|v_\alpha|\,dv_g.\end{aligned}$$

Now we fix $r>0$ small such that ε small enough satisfying $C\varepsilon^{\frac{2^*-2}{2^*}} \leq 1/2$. Then we get

$$\int_{\mathfrak{B}_r^+}\rho^{1-2\gamma}|\nabla v_\alpha|_g^2\,dv_g \leq \frac{C}{r^2}\int_{\mathfrak{B}_{2r}^+}\rho^{1-2\gamma}v_\alpha^2\,dv_g + C\int_{\partial'\mathfrak{B}_{2r}^+}v_\alpha^2\,d\sigma_{\hat{h}} + o(1)\int_{\mathfrak{B}_{2r}^+}|v_\alpha|\,dv_g.$$

This completes the proof of the lemma. $\square$

3. The First Bubble Argument

In this section, we focus on the blow up analysis of a Palais-Smale sequence which is not strongly convergent. In particular, using the ε-regularity estimates (Lemma 2.14), we can figure out the first bubble. We will also show that the Palais-Smale sequence obtained by subtracting a bubble is also Palais-Smale sequence and that the energy is splitting.

Lemma 3.1. *Let $\{\hat{u}_\alpha\}_{\alpha\in\mathbb{N}}$ be a Palais-Smale sequence for I_g^γ such that $\hat{u}_\alpha \rightharpoonup 0$ weakly in $W^{1,2}(X, \rho^{1-2\gamma})$, but not strongly as $\alpha \to +\infty$. Then there exist a sequence of real numbers $\{\mu_\alpha > 0\}_{\alpha\in\mathbb{N}}$, $\mu_\alpha \to 0$ as $\alpha \to +\infty$, a converging sequence of points $\{x_\alpha\}_{\alpha\in\mathbb{N}} \subset M$ and a nontrivial solution u to the equation*

$$\begin{cases} -\operatorname{div}(y^{1-2\gamma}\nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{y\to 0} y^{1-2\gamma}\partial_y u = |u|^{2^*-2}u & \text{on } \mathbb{R}^n, \end{cases} \tag{3.1}$$

such that, up to a subsequence, if we take

$$\hat{v}_\alpha(z) = \hat{u}_\alpha(z) - \eta_\alpha(z)\mu_\alpha^{-\frac{n-2\gamma}{2}} u(\mu_\alpha^{-1}\varphi_{x_\alpha}^{-1}(z)), \quad z \in \varphi_{x_\alpha}(B^+_{2r_0}(0))$$

where r_0, $\eta_\alpha(z)$ and $\varphi_{x_\alpha}(z)$ are as same as in the Theorem 1.3, then we have the following three conclusions

(i) *$\hat{v}_\alpha \rightharpoonup 0$ weakly in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$;*
(ii) *$\{\hat{v}_\alpha\}_{\alpha\in\mathbb{N}}$ is also a Palais-Smale sequence for I_g^γ;*
(iii) *$I_g^\gamma(\hat{v}_\alpha) = I_g^\gamma(\hat{u}_\alpha) - \tilde{E}(u) + o(1)$ as $\alpha \to +\infty$.*

Proof. Without loss of generality, we assume that $\hat{u}_\alpha \in \mathcal{C}^\infty(\overline{X})$. By the proof of Lemma 2.13,

$$I_g^\gamma(\hat{u}_\alpha) = \frac{\gamma}{n}\int_X \rho^{1-2\gamma}|\nabla\hat{u}_\alpha|_g^2\, dv_g + o(1) = \frac{\gamma}{n}\int_M |\hat{u}_\alpha|^{2^*}\, d\sigma_{\hat{h}} + o(1).$$

Note that $\{\hat{u}_\alpha\}_{\alpha\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(X, \rho^{1-2\gamma})$ by Lemma 2.9, so there exist a subsequence, also denoted by $\{\hat{u}_\alpha\}_{\alpha\in\mathbb{N}}$ and a nonnegative constant β, such that

$$I_g^\gamma(\hat{u}_\alpha) = \beta + o(1), \quad \text{as } \alpha \to +\infty.$$

Since $\hat{u}_\alpha \rightharpoonup 0$ weakly in $W^{1,2}(X, \rho^{1-2\gamma})$ but not strongly as $\alpha \to +\infty$, by Lemma 2.13 again we get

$$\lim_{\alpha\to+\infty}\int_M |\hat{u}_\alpha|^{2^*}\, d\sigma_{\hat{h}} = \frac{n}{\gamma}\beta \geq \frac{n}{\gamma}\beta_0.$$

We will decompose the rest of the proof into several steps:

Step 1. Pick up the likely blow up points. First we show the following claim.

Claim 1. *For any $t_0 > 0$ small, there exist $x_0 \in M$ and $\varepsilon_0 > 0$ such that, up to a subsequence*

$$\int_{\mathfrak{D}_{t_0}(x_0)} |\hat{u}_\alpha|^{2^*}\, d\sigma_{\hat{h}} \geq \varepsilon_0.$$

Proof. If the Claim is not true, there exists $t > 0$ small, such that for any $x \in M$ it holds

$$\int_{\mathfrak{D}_t(x)} |\hat{u}_\alpha|^{2^*}\, d\sigma_{\hat{h}} \to 0, \quad \alpha \to +\infty.$$

On the other hand, since $(M,\hat{h})$ is compact and $M \subset \cup_{x\in M}\mathfrak{D}_t(x)$, there exists an integer $N(\geq 1)$ such that $M \subset \cup_{i=1}^{N}\mathfrak{D}_t(x_i)$. Thus

$$\int_M |\hat{u}_\alpha|^{2^*}\,d\sigma_{\hat{h}} \leq \sum_{i=1}^{N}\int_{\mathfrak{D}_t(x_i)} |\hat{u}_\alpha|^{2^*}\,d\sigma_{\hat{h}} \to 0, \quad \alpha \to +\infty,$$

which is a contradiction. $\square$

For $t > 0$, we set

$$\omega_\alpha(t) = \max_{x\in M}\int_{\mathfrak{D}_t(x)} |\hat{u}_\alpha|^{2^*}\,d\sigma_{\hat{h}}.$$

Then by Claim 1, there exists $x_\alpha \in M$ such that

$$\omega_\alpha(t_0) = \int_{\mathfrak{D}_{t_0}(x_\alpha)} |\hat{u}_\alpha|^{2^*} d\sigma_{\hat{h}} \geq \varepsilon_0.$$

Note that

$$\int_{\mathfrak{D}_t(x_\alpha)} |\hat{u}_\alpha|^{2^*} d\sigma_{\hat{h}} \to 0, \quad \text{as } t \to 0.$$

Hence for any $\varepsilon \in (0,\varepsilon_0)$, there exists $t_\alpha \in (0,t_0)$ such that

$$\varepsilon = \int_{\mathfrak{D}_{t_\alpha}(x_\alpha)} |\hat{u}_\alpha|^{2^*}\,d\sigma_{\hat{h}}. \tag{3.2}$$

Step 2. At each likely blow up point, we will establish weak convergence of a Palais-Smale sequence after properly rescaling.

For $r_0 > 0$ small, consider the Fermi coordinates at the likely blow up point $x_\alpha \in M$, $\varphi_{x_\alpha} : B^+_{2r_0}(0) \to X$. Here we restrict r_0 to $r_0 \leq i_g(X)/2$, where $i_g(X)$ is the injectivity radius of X. Then for any $0 < \mu_\alpha \leq 1$, we define

$$\tilde{u}_\alpha(z) = \mu_\alpha^{\frac{n-2\gamma}{2}}\,\hat{u}_\alpha(\varphi_{x_\alpha}(\mu_\alpha z)), \quad \tilde{g}_\alpha(z) = (\varphi^*_{x_\alpha} g)(\mu_\alpha z), \quad \tilde{h}_\alpha(x) = (\varphi^*_{x_\alpha}\hat{h})(\mu_\alpha x),$$

if $z \in B^+_{\mu_\alpha^{-1}r_0}(0)$ and $x \in \partial' B^+_{\mu_\alpha^{-1}r_0}(0)$.

Given $z_0 \in \mathbb{R}^{n+1}_+$ and $r > 0$ such that $|z_0| + r < \mu_\alpha^{-1} r_0$, we have

$$\int_{B^+_r(z_0)} \tilde{\rho}_\alpha^{1-2\gamma}|\nabla \tilde{u}_\alpha|^2_{\tilde{g}_\alpha}\,dv_{\tilde{g}_\alpha} = \int_{\varphi_{x_\alpha}(\mu_\alpha B^+_r(z_0))} \rho^{1-2\gamma}|\nabla \hat{u}_\alpha|^2_g\,dv_g$$

where

$$\tilde{\rho}_\alpha(z) = \mu_\alpha^{-1}\rho(\varphi_{x_\alpha}(\mu_\alpha z))$$

and $|d\tilde{\rho}_\alpha|_{\tilde{g}_\alpha} = 1$ on $\partial' B^+_r(z_0)$ since $|d\rho|_g = 1$ on M.

On the other hand, if $z_0 \in \mathbb{R}^n$, and $|z_0| + r < \mu_\alpha^{-1} r_0$, then

$$\begin{aligned}\int_{D_r(z_0)} |\tilde{u}_\alpha|^{2^*}\,d\sigma_{\tilde{h}_\alpha} &= \int_{\varphi_{x_\alpha}(\mu_\alpha D_r(z_0))} |\hat{u}_\alpha|^{2^*}\,d\sigma_{\hat{h}} \\ &\leq \int_{\mathfrak{D}_{2\mu_\alpha r}(\varphi_{x_\alpha}(\mu_\alpha z_0))} |\hat{u}_\alpha|^{2^*}\,d\sigma_{\hat{h}}.\end{aligned}$$

Here we have used that $\varphi_{x_\alpha}(\mu_\alpha D_r(z_0)) = \varphi_{x_\alpha}(D_{\mu_\alpha r}(\mu_\alpha z_0))$, and that for $|x| < r_0, |y| < r_0$, $x, y \in \mathbb{R}^n$, we have $1/2|x-y| \leq d_g(\varphi_{x_\alpha}(x), \varphi_{x_\alpha}(y)) \leq 2|x-y|$.

Next, take $r \in (0, r_0)$ and choose t_0 in Claim 1 such that $0 < t_0 \le 2r$. For any $\varepsilon \in (0, \varepsilon_0)$, ε to be determined later, and $t_\alpha \in (0, t_0)$, let $0 < \mu_\alpha = \frac{1}{2} r^{-1} t_\alpha \le \frac{1}{2} r^{-1} t_0 \le 1$, then by the definition of ε from (3.2), if $|z_0| + r < \mu_\alpha^{-1} r_0$, we have

$$\int_{\partial' B_r^+(z_0)} |\tilde{u}_\alpha|^{2^*} d\sigma_{\tilde{h}_\alpha} \le \varepsilon. \tag{3.3}$$

Note that $\varphi_{x_\alpha}(\partial' B^+_{2r\mu_\alpha}(0)) = \mathfrak{D}_{t_\alpha}(x_\alpha)$, we have

$$\begin{aligned} \varepsilon = & \int_{\mathfrak{D}_{t_\alpha}(x_\alpha)} |\hat{u}_\alpha|^{2^*} \, d\sigma_{\hat{h}} = \int_{\varphi_{x_\alpha}(\partial' B^+_{2r\mu_\alpha}(0))} |\hat{u}_\alpha|^{2^*} \, d\sigma_{\hat{h}} \\ = & \int_{\varphi_{x_\alpha}(\mu_\alpha \partial' B^+_{2r}(0))} |\hat{u}_\alpha|^{2^*} \, d\sigma_{\hat{h}} = \int_{\partial' B^+_{2r}(0)} |\tilde{u}_\alpha|^{2^*} \, d\sigma_{\tilde{h}_\alpha}. \end{aligned}$$

This $r_0 > 0$ can be chosen smaller again, such that for any $0 < \mu \le 1$ and any $x_0 \in M$, we can assume that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma} |\nabla u|^2 \, dxdy \le & \int_{\mathbb{R}^{n+1}_+} \tilde{\rho}^{1-2\gamma}_{x_0,\mu} |\nabla u|^2_{\tilde{g}_{x_0,\mu}} \, dv_{\tilde{g}_{x_0,\mu}} \\ \le & 2 \int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma} |\nabla u|^2 \, dxdy, \end{aligned} \tag{3.4}$$

where $u \in \overline{W}^{1,2}(\mathbb{R}^{n+1}_+, \tilde{\rho}^{1-2\gamma}_{x_0,\mu})$, $\mathrm{supp}(u) \subset B^+_{2\mu^{-1} r_0}(0)$, $\tilde{\rho}_{x_0,\mu}(z) = \mu^{-1} \rho(\varphi_{x_0}(\mu z))$ and $\tilde{g}_{x_0,\mu}(z) = (\varphi^*_{x_0} g)(\mu z)$. And for $u \in L^1(\mathbb{R}^n)$ such that $\mathrm{supp}(u) \subset \partial' B^+_{2\mu^{-1} r_0}(0)$, we can also assume that

$$\frac{1}{2} \int_{\mathbb{R}^n} |u| \, dx \le \int_{\mathbb{R}^n} |u| \, d\sigma_{\tilde{h}_{x_0,\mu}} \le 2 \int_{\mathbb{R}^n} |u| \, dx,$$

where $\tilde{h}_{x_0,\mu}(x) = (\varphi^*_{x_0} \hat{h})(\mu x)$.

Let $\tilde{\eta} \in \mathcal{C}^\infty_0(\mathbb{R}^{n+1}_+)$ be a cutoff function satisfying $0 \le \tilde{\eta} \le 1$, $\tilde{\eta} \equiv 1$ in $B^+_{1/4}(0)$ and $\tilde{\eta} \equiv 0$ in $\mathbb{R}^{n+1}_+ \setminus B^+_{3/4}(0)$, and we set $\tilde{\eta}_\alpha(z) = \tilde{\eta}(r_0^{-1} \mu_\alpha z)$.

Claim 2. *$\{\tilde{\eta}_\alpha \tilde{u}_\alpha\}_{\alpha \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$.*

Proof. Note that

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}_+} \tilde{\rho}_\alpha^{1-2\gamma} |\nabla(\tilde{\eta}_\alpha \tilde{u}_\alpha)|^2_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha} + \int_{\mathbb{R}^{n+1}_+} \tilde{\rho}_\alpha^{1-2\gamma} (\tilde{\eta}_\alpha \tilde{u}_\alpha)^2 \, dv_{\tilde{g}_\alpha} \\ \le & \int_{\mathbb{R}^{n+1}_+} \tilde{\rho}_\alpha^{1-2\gamma} (2|\nabla \tilde{\eta}_\alpha|^2_{\tilde{g}_\alpha} + \tilde{\eta}_\alpha^2) \tilde{u}_\alpha^2 \, dv_{\tilde{g}_\alpha} + 2 \int_{\mathbb{R}^{n+1}_+} \tilde{\rho}_\alpha^{1-2\gamma} \tilde{\eta}_\alpha^2 |\nabla \tilde{u}_\alpha|^2_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha} \\ \le & C \int_X \rho^{1-2\gamma} \hat{u}_\alpha^2 \, dv_g + C \int_X \rho^{1-2\gamma} |\nabla \hat{u}_\alpha|^2_g \, dv_g \le C, \end{aligned}$$

since $\{\hat{u}_\alpha\}_{\alpha \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(X, \rho^{1-2\gamma})$. Combining this with (3.4), we obtain that $\{\tilde{\eta}_\alpha \tilde{u}_\alpha\}_{\alpha \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$, as desired. $\square$

Due to the weak compactness of $W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$, there exists some u in $W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$ such that $\tilde{\eta}_\alpha \tilde{u}_\alpha \rightharpoonup u$ in $W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$ as $\alpha \to +\infty$.

Step 3. The weak convergence is in fact strong via ε-regularity estimates.

Claim 3. *Let $r_1 = r_0/8$, then there exists $\varepsilon_1 = \varepsilon_1(\gamma, n)$ such that for any $0 < r < r_1$, $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$, we have $\tilde{\eta}_\alpha \tilde{u}_\alpha \to u$ in $W^{1,2}(B^+_{2r}(0), y^{1-2\gamma})$ as $\alpha \to +\infty$.*

Proof. Given r sufficiently small, to be determined later, for any $z_0 \in \mathbb{R}^{n+1}_+$, let $\psi \in \mathcal{C}^\infty_0(B^+_r(z_0)) \cap W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$. Let $\hat{\psi}_\alpha(z) = \mu_\alpha^{-\frac{n-2\gamma}{2}} \psi(\mu_\alpha^{-1}\varphi^{-1}_{x_\alpha}(z))$ for $z \in \varphi_{x_\alpha}(B^+_r(z_0))$. Since $\{\hat{u}_\alpha\}$ satisfies the asymptotic equation (2.10), then we have

$$\begin{aligned} o(1)\|\psi\|_{\overline{W}^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})} &= o(1)\|\hat{\psi}_\alpha\|_{\overline{W}^{1,2}(X,\rho^{1-2\gamma})} \\ &= \int_{\varphi_{x_\alpha}(\mu_\alpha B^+_r(z_0))} \rho^{1-2\gamma}\langle \nabla \hat{u}_\alpha, \nabla\hat{\psi}_\alpha\rangle_g \, dv_g \\ &= \int_{B^+_r(z_0)} (\mu_\alpha^{-1}\rho)^{1-2\gamma}\langle \nabla(\tilde{\eta}_\alpha \tilde{u}_\alpha), \nabla\psi\rangle_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha}, \end{aligned}$$

since $\tilde{\eta}$ is supported in $B^+_{3/4}(0)$ and $\tilde{\eta} \equiv 1$ in $B^+_{1/4}(0)$. Also note that $\tilde{\eta}_\alpha(z) = \tilde{\eta}(\mu_\alpha r_0^{-1} z)$, so $\tilde{\eta}_\alpha \equiv 1$ in $B^+_{1/4\mu_\alpha^{-1}r_0}$, and thus we need $|z_0| + r < 1/4\mu_\alpha^{-1} r_0$.

It is easy to check that $\mu_\alpha^{-1}\rho \to y$ as $\alpha \to +\infty$ since $|d(\mu_\alpha^{-1}\rho)|_{\tilde{g}_\alpha} = 1$ on $\mathbb{R}^n$ and $\tilde{g}_\alpha \to (|dx|^2 + dy^2)$. Then we have the asymptotic equation

$$-\operatorname{div}(y^{1-2\gamma}\nabla(\tilde{\eta}_\alpha \tilde{u}_\alpha)) = o(1) \quad \text{in} \quad B^+_r(z_0). \tag{3.5}$$

Since $\tilde{\eta}_\alpha \tilde{u}_\alpha \rightharpoonup u$ weakly in $W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$, we simultaneously get that

$$-\operatorname{div}(y^{1-2\gamma}\nabla u) = 0 \quad \text{in} \quad B^+_r(z_0). \tag{3.6}$$

Now let $\psi \in W^{1,2}(B^+_r(z_0), y^{1-2\gamma})$. Then multiplying both sides of equation (3.5) by ψ and integrating by parts, we get

$$\begin{aligned} o(1)\|\psi\|_{W^{1,2}(B^+_r(z_0), y^{1-2\gamma})} &= \int_{\partial' B^+_r(z_0)} \lim_{y\to 0} y^{1-2\gamma}\partial_y(\tilde{\eta}_\alpha\tilde{u}_\alpha)\psi \, d\sigma_{\tilde{h}_\alpha} \\ &\quad + \int_{B^+_r(z_0)} y^{1-2\gamma}\langle \nabla(\tilde{\eta}_\alpha\tilde{u}_\alpha), \nabla\psi\rangle_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha}. \end{aligned} \tag{3.7}$$

On the other hand, using (2.10) and (2.11), and the definition of $\hat{\psi}_\alpha$, we have

$$\begin{aligned} &\int_{B^+_r(z_0)} y^{1-2\gamma}\langle \nabla(\tilde{\eta}_\alpha\tilde{u}_\alpha), \nabla\psi\rangle_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha} \\ &\quad = \int_{\varphi_{x_\alpha}(\mu_\alpha B^+_r(z_0))} \rho^{1-2\gamma}\langle \nabla\hat{u}_\alpha, \nabla\hat{\psi}_\alpha\rangle_g \, dv_g \\ &\quad = -\int_M \lim_{\rho\to 0} \rho^{1-2\gamma}(\partial_\rho \hat{u}_\alpha)\psi_\alpha \, d\sigma_{\hat{h}} + o(1)\|\hat{\psi}_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})} \\ &\quad = \int_M |\hat{u}_\alpha|^{2^*-2}\hat{u}_\alpha\hat{\psi}_\alpha \, d\sigma_{\hat{h}} + o(1)\|\hat{\psi}_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})} \\ &\quad = \int_{\partial' B^+_r(z_0)} |\tilde{\eta}_\alpha\tilde{u}_\alpha|^{2^*-2}(\tilde{\eta}_\alpha\tilde{u}_\alpha)\psi \, d\sigma_{\tilde{h}_\alpha} + o(1)\|\hat{\psi}_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})}. \end{aligned} \tag{3.8}$$

Since $\|\psi\|_{W^{1,2}(B^+_r(z_0), y^{1-2\gamma})} = \|\hat{\psi}_\alpha\|_{W^{1,2}(X,\rho^{1-2\gamma})}$, combining expressions (3.7) and (3.8) then we have

$$\begin{aligned} o(1)\|\psi\|_{W^{1,2}(B^+_r(z_0), y^{1-2\gamma})} &= \int_{\partial' B^+_r(z_0)} \lim_{y\to 0} y^{1-2\gamma}\partial_y(\tilde{\eta}_\alpha\tilde{u}_\alpha)\psi \, d\sigma_{\tilde{h}_\alpha} \\ &\quad + \int_{\partial' B^+_r(z_0)} |\tilde{\eta}_\alpha\tilde{u}_\alpha|^{2^*-2}(\tilde{\eta}_\alpha\tilde{u}_\alpha)\psi \, d\sigma_{\tilde{h}_\alpha}, \end{aligned}$$

i.e.

$$-\lim_{y\to 0} y^{1-2\gamma}\partial_y(\tilde{\eta}_\alpha\tilde{u}_\alpha) = |\tilde{\eta}_\alpha\tilde{u}_\alpha|^{2^*-2}(\tilde{\eta}_\alpha\tilde{u}_\alpha) + o(1) \quad \text{on} \quad \partial' B_r^+(z_0).$$

Meanwhile, since $\tilde{\eta}_\alpha\tilde{u}_\alpha \rightharpoonup u$ weakly in $W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$, the same argument as above gives that

$$-\lim_{y\to 0} y^{1-2\gamma}\partial_y u = |u|^{2^*-2}u \quad \text{on} \quad \partial' B_r^+(z_0).$$

If we denote by

$$\Gamma_\alpha := |\tilde{\eta}_\alpha\tilde{u}_\alpha|^{2^*-2}(\tilde{\eta}_\alpha\tilde{u}_\alpha) - |u|^{2^*-2}u - |\tilde{\eta}_\alpha\tilde{u}_\alpha - u|^{2^*-2}(\tilde{\eta}_\alpha\tilde{u}_\alpha - u),$$

then

$$\begin{cases} -\operatorname{div}(y^{1-2\gamma}\nabla(\tilde{\eta}_\alpha\tilde{u}_\alpha - u)) = o(1) & \text{in} \quad B_r^+(z_0), \\ -\lim_{y\to 0} y^{1-2\gamma}\partial_y(\tilde{\eta}_\alpha\tilde{u}_\alpha - u) = |\tilde{\eta}_\alpha\tilde{u}_\alpha - u|^{2^*-2}(\tilde{\eta}_\alpha\tilde{u}_\alpha - u) + \Gamma_\alpha + o(1) & \text{on} \quad \partial' B_r^+(z_0). \end{cases} \tag{3.9}$$

We have proved in (3.3) that for any $r > 0$ and $\varepsilon_1 > 0$, there exists a sequence $\{\mu_\alpha\}_{\alpha\in\mathbb{N}}$ such that, if $|z_0| + r \le r_0 \le \mu_\alpha^{-1} r_0$, it holds that

$$\int_{\partial' B_r^+(z_0)} |\tilde{u}_\alpha|^{2^*} dx \le \varepsilon_1.$$

Therefore we can also choose r small enough such that, if $|z_0| + 3r < r_0$,

$$\int_{\partial' B_r^+(z_0)} |\tilde{\eta}_\alpha\tilde{u}_\alpha - u|^{2^*} dx \le \varepsilon_1.$$

We claim that $\Gamma_\alpha = o(1)$ in the sense that for any $\phi \in W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})'$, we have

$$\int_{\partial' B_r^+(z_0)} |\Gamma_\alpha \phi| d\sigma_{\hat{h}} = o(1)||\phi||_{L^{2^*}(\partial' B_r^+(z_0))} \quad \text{as } \alpha \to +\infty.$$

We can use the same arguments as in the proof of Lemma 2.12 to show this claim.

Then by the ε-regularity estimates and the compact embedding of the weighted Sobolev space, we can prove that $\tilde{\eta}_\alpha\tilde{u}_\alpha \to u$ in $W^{1,2}(B_r^+(z_0), y^{1-2\gamma})$, then by the finite covering we can prove that $\tilde{\eta}_\alpha\tilde{u}_\alpha \to u$ in $W^{1,2}(B_{2r}^+(0), y^{1-2\gamma})$.

□

Applying Claim 3, noting that $\tilde{\eta}_\alpha\tilde{u}_\alpha \to u$ in $W^{1,2}(B_{2r}^+(0), y^{1-2\gamma})$, and that $\tilde{\eta}_\alpha \equiv 1$ in $\partial' B^+_{1/4\mu_\alpha^{-1} r_0}$, since $0 < \mu_\alpha \le 1$ and $2r < r_0/4$, we have

$$\begin{aligned} \varepsilon = \int_{\partial' B_{2r}^+(0)} |\tilde{u}_\alpha|^{2^*} \, d\sigma_{\tilde{h}_\alpha} &= \int_{\partial' B_{2r}^+(0)} |\tilde{\eta}_\alpha\tilde{u}_\alpha|^{2^*} \, d\sigma_{\tilde{h}_\alpha} \\ &\le 2\int_{\partial' B_{2r}^+(0)} |u|^{2^*} \, dx + o(1), \end{aligned}$$

where we used $\tilde{\eta}_\alpha\tilde{u}_\alpha \to u$ in $L^{2^*}(\partial' B_{2r}^+(0), |dx|^2)$ as $\alpha \to +\infty$ by Proposition 2.4. So $u \ne 0$.

Claim 4. $\lim_{\alpha\to+\infty} \mu_\alpha = 0$.

In fact, if $\mu_\alpha \to \mu_0 > 0$, then $\tilde{\eta}_\alpha\tilde{u}_\alpha \rightharpoonup 0$ in $W^{1,2}(B_{2r}^+(0), y^{1-2\gamma})$ since $\hat{u}_\alpha \rightharpoonup 0$ in $W^{1,2}(X, \rho^{1-2\gamma})$. But $u \ne 0$, which is a contradiction.

Claim 5. *For any $0 < \mu_0 \le 1$, $\tilde{u}_\alpha \to u$ strongly in $W^{1,2}(B^+_{\mu_0^{-1}}(0), y^{1-2\gamma})$ as $\alpha \to +\infty$, and u is a weak solution of equation (3.1).*

Proof. Let $0 < \mu_0 \le 1$, by Claim 4, we know $0 < \mu_\alpha \le \mu_0$ for α large. Then (3.3) holds for $|z_0| + r < \mu_0^{-1} r_0$. By the same arguments, it is easy to check that

$$\tilde{\eta}_\alpha \tilde{u}_\alpha \to u \ \text{ in } \ W^{1,2}(B^+_{2r\mu_0^{-1}}(0), y^{1-2\gamma}).$$

For α large, we have $\tilde{\eta}_\alpha \equiv 1$ in $B^+_{2r\mu_0^{-1}}(0)$, so we have

$$\tilde{u}_\alpha \to u \ \text{ in } \ W^{1,2}(B^+_{2r\mu_0^{-1}}(0), y^{1-2\gamma})$$

strongly as $\alpha \to +\infty$.

We finally claim that u solves the following boundary problem.

$$\begin{cases} -\operatorname{div}(y^{1-2\gamma}\nabla u) = 0 \ \text{ in } \ \mathbb{R}^{n+1}_+, \\ -\lim\limits_{y\to 0} y^{1-2\gamma}\partial_y u = |u|^{2^*-2}u \ \text{ on } \ \mathbb{R}^n. \end{cases} \tag{3.10}$$

Since $0 < \mu_0 \le 1$ is arbitrary, we have $\tilde{u}_\alpha \to u$ strongly in $W^{1,2}(B^+_R(0), y^{1-2\gamma})$ for any large $R > 0$. Without loss of generality, let $\psi \in \mathcal{C}^\infty_0(\mathbb{R}^{n+1}_+)$ and $\operatorname{supp}\psi \subset B^+_0(R_0)$ for some $R_0 > 0$. Set

$$\psi_\alpha(z) = \mu_\alpha^{-\frac{n-2\gamma}{2}} \psi(\mu_\alpha^{-1}\varphi_{x_\alpha}^{-1}(z)).$$

For α large enough, we have

$$\int_X \rho^{1-2\gamma}\langle \nabla \hat{u}_\alpha, \nabla\psi_\alpha\rangle_g dv_g = \int_{\mathbb{R}^{n+1}_+} \tilde{\rho}_\alpha^{1-2\gamma}\langle \nabla(\tilde{\eta}_\alpha\tilde{u}_\alpha), \nabla\psi\rangle_{\tilde{g}_\alpha} dv_{\tilde{g}_\alpha},$$

and

$$\int_M |\hat{u}_\alpha|^{2^*-2}\hat{u}_\alpha\psi_\alpha dv_g = \int_{\mathbb{R}^n} |\tilde{\eta}_\alpha\tilde{u}_\alpha|^{2^*-2}(\tilde{\eta}_\alpha\tilde{u}_\alpha)\psi\, dv_{\tilde{g}_\alpha}.$$

Note that $\tilde{g}_\alpha \to |dx|^2 + dy^2$ in $\mathcal{C}^1(B^+_R(0))$ as $\alpha \to +\infty$, $\{\hat{u}_\alpha\}$ is a Palais-Smale sequence for I_g^γ and $\tilde{\eta}_\alpha\tilde{u}_\alpha \to u$ in $W^{1,2}(B^+_R(0))$ for any $R > 0$. Then we have

$$\int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}\langle \nabla u, \nabla\psi\rangle\, dxdy - \int_{\mathbb{R}^n} |u|^{2^*-2}u\psi\, dxdy = 0,$$

which yields our desired result.

$\square$

Step 4. The Palais-Smale sequence subtracted by a bubble is still a Palais-Smale sequence. Define

$$\begin{cases} \hat{w}_\alpha(z) = \hat{\eta}_\alpha(z)\mu_\alpha^{-(n-2\gamma)/2}u(\mu_\alpha^{-1}\varphi_{x_\alpha}^{-1}(z)), & z \in \varphi_{x_\alpha}(B^+_{2r_0}(0)), \\ \hat{w}_\alpha(z) = 0, & \text{otherwise}, \end{cases} \tag{3.11}$$

where $\hat{\eta}_\alpha$ is a cut-off function satisfying $\hat{\eta}_\alpha = 1$ in $\varphi_{x_\alpha}(B^+_{r_0}(0))$ and $\hat{\eta}_\alpha = 0$ in $M \setminus \varphi_{x_\alpha}(B^+_{2r_0}(0))$. Here we have $\mathfrak{B}^+_{2r_0}(x_\alpha) = \varphi_{x_\alpha}(B^+_{2r_0}(0))$. Let $\hat{v}_\alpha = \hat{u}_\alpha - \hat{w}_\alpha$. We claim:

(i) $\hat{v}_\alpha \rightharpoonup 0$ in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$;
(ii) $DI_g^\gamma(\hat{v}_\alpha) \to 0$ in $W^{1,2}(X, \rho^{1-2\gamma})'$ as $\alpha \to +\infty$;
(iii) $I_g^\gamma(\hat{v}_\alpha) = I_g^\gamma(\hat{u}_\alpha) - \tilde{E}(u) + o(1)$ as $\alpha \to +\infty$;
(iv) $\{\hat{v}_\alpha\}_{\alpha\in\mathbb{N}}$ is also a Palais-Smale sequence for I_g^γ.

The proof of these claims follows from: (i) Since $\hat{u}_\alpha \rightharpoonup 0$ in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$, it suffices to prove $\hat{w}_\alpha \rightharpoonup 0$ in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$. First, we prove that $\int_M \hat{w}_\alpha \psi d\sigma_{\hat{h}} = o(1)$ as $\alpha \to +\infty$ for any $\psi \in \mathcal{C}^\infty(\overline{X})$. Given $R > 0$, then

$$\int_M \hat{w}_\alpha \psi \, d\sigma_{\hat{h}} = \int_{\mathfrak{D}_{\mu_\alpha R}(x_\alpha)} \hat{w}_\alpha \psi \, d\sigma_{\hat{h}} + \int_{M \setminus \mathfrak{D}_{\mu_\alpha R}(x_\alpha)} \hat{w}_\alpha \psi \, d\sigma_{\hat{h}}. \tag{3.12}$$

Note that $\tilde{h}_\alpha(x) = (\varphi_{x_\alpha}^* \hat{h})(\mu_\alpha x)$. Using (3.11) we have

$$\begin{aligned}
\int_{\mathfrak{D}_{\mu_\alpha R}(x_\alpha)} \hat{w}_\alpha \psi \, d\sigma_{\hat{h}} &= \int_{\mathfrak{D}_{\mu_\alpha R}(x_\alpha)} \hat{\eta}_\alpha(x) \mu_\alpha^{-\frac{n-2\gamma}{2}} u(\mu_\alpha^{-1} \varphi_{x_\alpha}^{-1}(x)) \psi(x) \, d\sigma_{\hat{h}} \\
&= \mu_\alpha^{\frac{n+2\gamma}{2}} \int_{D_R(0)} \hat{\eta}_\alpha(\varphi_{x_\alpha}(\mu_\alpha x)) u(x) \psi(\varphi_{x_\alpha}(\mu_\alpha x)) \, d\sigma_{\tilde{h}_\alpha} \\
&\leq C \|\psi\|_{L^\infty(M)} \mu_\alpha^{\frac{n+2\gamma}{2}} \int_{D_R(0)} |u(x)| \, dx.
\end{aligned}$$

Similarly, we can deal with the second term in the right hand side of (3.12):

$$\begin{aligned}
&\int_{M \setminus \mathfrak{D}_{\mu_\alpha R}(x_\alpha)} \hat{w}_\alpha \psi \, d\sigma_{\hat{h}} = \int_{\mathfrak{D}_{2r_0}(x_\alpha) \setminus \mathfrak{D}_{\mu_\alpha R}(x_\alpha)} \hat{w}_\alpha \psi \, d\sigma_{\hat{h}} \\
&\quad \leq C \|\psi\|_{L^\infty(M)} \mu_\alpha^{\frac{n+2\gamma}{2}} \int_{D_{2r_0 \mu_\alpha^{-1}}(0) \setminus D_R(0)} |u(x)| \, dx \\
&\quad \leq C \|\psi\|_{L^\infty(M)} \mu_\alpha^{\frac{n+2\gamma}{2}} \left(\int_{D_{2r_0 \mu_\alpha^{-1}}(0) \setminus D_R(0)} |u(x)|^{2^*} \, dx \right)^{\frac{1}{2^*}} \left(\int_{D_{2r_0 \mu_\alpha^{-1}}(0) \setminus D_R(0)} dx \right)^{\frac{n+2\gamma}{2n}} \\
&\quad \leq C \|\psi\|_{L^\infty(M)} \left(\int_{D_{2r_0 \mu_\alpha^{-1}}(0) \setminus D_R(0)} |u(x)|^{2^*} \, dx \right)^{\frac{1}{2^*}}.
\end{aligned}$$

Since $u \in L^{2^*}(\mathbb{R}^n, |dx|^2)$ and $\mu_\alpha \to 0$ as $\alpha \to +\infty$, taking R large enough we get $\int_M \hat{w}_\alpha \psi d\sigma_{\hat{h}} = o(1)$ as $\alpha \to +\infty$.

Next, we will show that $\int_X \rho^{1-2\gamma} \langle \nabla \hat{w}_\alpha, \nabla \psi \rangle_g dv_g = o(1)$ as $\alpha \to +\infty$ for any $\psi \in \mathcal{C}^\infty(\overline{X})$. Let $\tilde{\eta}_\alpha(z) = \hat{\eta}_\alpha(\varphi_{x_\alpha}(\mu_\alpha z))$, $\tilde{\rho}_\alpha(z) = \mu_\alpha^{-1} \rho(\varphi_{x_\alpha}(\mu_\alpha z))$. Noting that $\hat{w}_\alpha \equiv 0$ in $X \setminus \mathfrak{B}_{2r_0}^+(x_\alpha)$, then for any $R > 0$ and α large, we have

$$\begin{aligned}
\int_X \rho^{1-2\gamma} \langle \nabla \hat{w}_\alpha, \nabla \psi \rangle_g \, dv_g &= \int_{\mathfrak{B}_{2r_0}^+(x_\alpha)} \rho^{1-2\gamma} \langle \nabla \hat{w}_\alpha, \nabla \psi \rangle_g \, dv_g \\
&= \int_{\mathfrak{B}_{2r_0}^+(x_\alpha) \setminus \mathfrak{B}_{R\mu_\alpha}^+(x_\alpha)} \rho^{1-2\gamma} \langle \nabla \hat{w}_\alpha, \nabla \psi \rangle_g \, dv_g + \int_{\mathfrak{B}_{R\mu_\alpha}^+(x_\alpha)} \rho^{1-2\gamma} \langle \nabla \hat{w}_\alpha, \nabla \psi \rangle_g \, dv_g \\
&=: I_1 + I_2.
\end{aligned} \tag{3.13}$$

By Hölder's inequality and that $u \in W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$, we have

$$\begin{aligned} I_1 &\le \left(\int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus \mathfrak{B}^+_{R\mu_\alpha}(x_\alpha)} \rho^{1-2\gamma} |\nabla \hat{w}_\alpha|_g^2 \, dv_g \right)^{\frac{1}{2}} \left(\int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus \mathfrak{B}^+_{R\mu_\alpha}(x_\alpha)} \rho^{1-2\gamma} |\nabla \psi|_g^2 \, dv_g \right)^{\frac{1}{2}} \\ &= \left(\int_{B^+_{2r_0\mu_\alpha^{-1}}(0)\setminus B^+_R(0)} \tilde{\rho}_\alpha^{1-2\gamma} |\nabla (\tilde{\eta}_\alpha u)|^2_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha} \right)^{\frac{1}{2}} \left(\int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus \mathfrak{B}^+_{R\mu_\alpha}(x_\alpha)} \rho^{1-2\gamma} |\nabla \psi|_g^2 \, dv_g \right)^{\frac{1}{2}} \\ &=: \beta(R), \end{aligned}$$

where

$$\lim_{R\to+\infty} \lim_{\alpha\to+\infty} \sup \beta(R) = 0. \tag{3.14}$$

The previous limit is estimated because $u \in W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$, so we have for any α, R

$$\left(\int_{B^+_{2r_0\mu_\alpha^{-1}}(0)\setminus B^+_R(0)} \tilde{\rho}_\alpha^{1-2\gamma} |\nabla (\tilde{\eta}_\alpha u)|^2_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha} \right)^{\frac{1}{2}} \le C ||u||_{W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})},$$

and for any $\varepsilon > 0$ and any α large, there exists $R_0 > 0$ such that for $R > R_0$, we have

$$\left(\int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus \mathfrak{B}^+_{R\mu_\alpha}(x_\alpha)} \rho^{1-2\gamma} |\nabla \psi|_g^2 \, dv_g \right)^{\frac{1}{2}} \le \varepsilon.$$

Meanwhile we have

$$\begin{aligned} I_2 &\le \left(\int_{\mathfrak{B}^+_{R\mu_\alpha}(x_\alpha)} \rho^{1-2\gamma} |\nabla \hat{w}_\alpha|_g^2 \, dv_g \right)^{\frac{1}{2}} \left(\int_{\mathfrak{B}^+_{R\mu_\alpha}(x_\alpha)} \rho^{1-2\gamma} |\nabla \psi|_g^2 \, dv_g \right)^{\frac{1}{2}} \\ &= \left(\int_{B^+_R(0)} \tilde{\rho}_\alpha^{1-2\gamma} |\nabla (\tilde{\eta}_\alpha u)|^2_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha} \right)^{\frac{1}{2}} \left(\int_{\mathfrak{B}^+_{R\mu_\alpha}(x_\alpha)} \rho^{1-2\gamma} |\nabla \psi|_g^2 \, dv_g \right)^{\frac{1}{2}} \\ &= o(1), \end{aligned}$$

uniformly in R as $\alpha \to +\infty$. To see this, for any $R > 0$,

$$\left(\int_{B^+_R(0)} \tilde{\rho}_\alpha^{1-2\gamma} |\nabla (\tilde{\eta}_\alpha u)|^2_{\tilde{g}_\alpha} \, dv_{\tilde{g}_\alpha} \right)^{\frac{1}{2}} \le C ||u||_{W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})},$$

also in Claim 4 we have proved that

$$\lim_{\alpha\to+\infty} \mu_\alpha = 0$$

and note that $\psi \in W^{1,2}(X, \rho^{1-2\gamma})$. Since $R > 0$ is arbitrary, (3.13) implies that

$$\int_X \rho^{1-2\gamma} \langle \nabla \hat{w}_\alpha, \nabla \psi \rangle_g \, dv_g = o(1)$$

as $\alpha \to +\infty$.

(ii) For any $\psi \in W^{1,2}(X, \rho^{1-2\gamma})$, the proof of (i), and Propositions 2.4 and 2.6 imply that

$$DI_g^\gamma(\hat{w}_\alpha) \cdot \psi = \int_X \rho^{1-2\gamma} \langle \nabla \hat{w}_\alpha, \nabla \psi \rangle_g \, dv_g - \int_M |\hat{w}_\alpha|^{2^*-2} \hat{w}_\alpha \psi \, d\sigma_{\hat{h}} \to 0, \quad \text{as} \quad \alpha \to +\infty.$$

On the other hand, we have

$$
\begin{aligned}
DI_g^\gamma(\hat v_\alpha)\cdot\psi &= \int_X \rho^{1-2\gamma}\langle\nabla\hat v_\alpha,\nabla\psi\rangle_g\,dv_g - \int_M |\hat v_\alpha|^{2^*-2}\hat v_\alpha\psi\,d\sigma_{\hat h}\\
&= DI_g^\gamma(\hat u_\alpha)\cdot\psi - DI_g^\gamma(\hat w_\alpha)\cdot\psi - \int_M \Phi_\alpha\psi\,d\sigma_{\hat h},
\end{aligned}
$$

where

$$
\Phi_\alpha = |\hat u_\alpha-\hat w_\alpha|^{2^*-2}(\hat u_\alpha-\hat w_\alpha) + |\hat w_\alpha|^{2^*-2}\hat w_\alpha - |\hat u_\alpha|^{2^*-2}\hat u_\alpha.
$$

Following the same argument of [5] (pp. 39-40), we can prove that

$$
\int_M \Phi_\alpha\psi\,d\sigma_{\hat h}\to 0 \quad \text{as } \alpha\to+\infty.
$$

Then we get that $DI_g^\gamma(\hat v_\alpha)\to 0$ in $W^{1,2}(X,\rho^{1-2\gamma})'$ as $\alpha\to+\infty$, since $\{\hat u_\alpha\}_{\alpha\in\mathbb{N}}$ is a Palais-Smale sequence for I_g^γ.

(iii) Note that $\hat v_\alpha=\hat u_\alpha-\hat w_\alpha$ and $\hat w_\alpha\equiv 0$ in $X\setminus\mathfrak{B}^+_{2r_0}(x_\alpha)$. Given $R>0$, for α large, we have

$$
\begin{aligned}
&\int_X \rho^{1-2\gamma}|\nabla\hat v_\alpha|_g^2\,dv_g\\
&\quad= \int_{\mathfrak{B}^+_{2r_0}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat v_\alpha|_g^2\,dv_g + \int_{X\setminus\mathfrak{B}^+_{2r_0}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat u_\alpha|_g^2\,dv_g\\
&\quad= \int_{\mathfrak{B}^+_{\mu_\alpha R}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat v_\alpha|_g^2\,dv_g + \int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus\mathfrak{B}^+_{\mu_\alpha R}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat v_\alpha|_g^2\,dv_g\\
&\qquad+ \int_{X\setminus\mathfrak{B}^+_{2r_0}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat u_\alpha|_g^2\,dv_g\\
&\quad=: I_1+I_2+\int_{X\setminus\mathfrak{B}^+_{2r_0}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat u_\alpha|_g^2\,dv_g.
\end{aligned}
\tag{3.15}
$$

Since $\tilde\eta_\alpha\tilde u_\alpha\to u$ in $W^{1,2}(\mathbb{R}^{n+1}_+,y^{1-2\gamma})$ as $\alpha\to+\infty$ because of Claim 5, then

$$
\begin{aligned}
I_1 &= \int_{\mathfrak{B}^+_{\mu_\alpha R}(x_\alpha)} \rho^{1-2\gamma}|\nabla(\hat u_\alpha-\hat w_\alpha)|_g^2\,dv_g = \int_{B_R^+(0)} \tilde\rho_\alpha^{1-2\gamma}|\nabla(\tilde u_\alpha-u)|^2_{\tilde g_\alpha}\,dv_{\tilde g_\alpha}\\
&\le 2\int_{B_R^+(0)} y^{1-2\gamma}|\nabla(\tilde u_\alpha-u)|^2\,dxdy = o(1), \quad \text{as } \alpha\to+\infty,
\end{aligned}
$$

where we have used that $\tilde\eta_\alpha\equiv 1$ in $B_R^+(0)$ for α large.

On the other hand, direct computations give that

$$
\begin{aligned}
\int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus\mathfrak{B}^+_{\mu_\alpha R}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat w_\alpha|_g^2\,dv_g &= \int_{B^+_{2r_0\mu_\alpha^{-1}}(0)\setminus B_R^+(0)} \tilde\rho_\alpha^{1-2\gamma}|\nabla u|^2_{\tilde g_\alpha}\,dv_{\tilde g_\alpha}\\
&\le 2\int_{B^+_{2r_0\mu_\alpha^{-1}}(0)\setminus B_R^+(0)} y^{1-2\gamma}|\nabla u|^2\,dxdy = \beta(R),
\end{aligned}
$$

since $u \in W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$ and $\mu_\alpha \to 0$ as $\alpha \to +\infty$, where $\beta(R)$ is defined as in (3.14). Hence we get that

$$\begin{aligned} I_2 &= \int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus\mathfrak{B}^+_{\mu_\alpha R}(x_\alpha)} \rho^{1-2\gamma}(|\nabla\hat{u}_\alpha|^2_g + |\nabla\hat{w}_\alpha|^2_g - 2\langle\nabla\hat{u}_\alpha, \nabla\hat{w}_\alpha\rangle_g)\, dv_g \\ &= \int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus\mathfrak{B}^+_{\mu_\alpha R}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat{u}_\alpha|^2_g\, dv_g + \beta(R). \end{aligned}$$

Here we have used Hölder's inequality and the fact that $\{\hat{u}_\alpha\}$ is uniformly in $W^{1,2}(X, \rho^{1-2\gamma})$ to get

$$\int_{\mathfrak{B}^+_{2r_0}(x_\alpha)\setminus\mathfrak{B}^+_{\mu_\alpha R}(x_\alpha)} \rho^{1-2\gamma}\langle\nabla\hat{u}_\alpha, \nabla\hat{w}_\alpha\rangle_g\, dv_g = \beta(R).$$

Therefore, noting that $\tilde{u}_\alpha \to u$ in $W^{1,2}(\mathbb{R}^{n+1}_+, y^{1-2\gamma})$ as $\alpha \to +\infty$, we have from (3.15) that

$$\begin{aligned} &\int_X \rho^{1-2\gamma}|\nabla\hat{v}_\alpha|^2_g\, dv_g \\ &\quad= \int_X \rho^{1-2\gamma}|\nabla\hat{u}_\alpha|^2_g\, dv_g - \int_{\mathfrak{B}^+_{\mu_\alpha R}(x_\alpha)} \rho^{1-2\gamma}|\nabla\hat{u}_\alpha|^2_g\, dv_g + \beta(R) + o(1) \\ &\quad= \int_X \rho^{1-2\gamma}|\nabla\hat{u}_\alpha|^2_g\, dv_g - \int_{B^+_R(0)} \tilde{\rho}_\alpha^{1-2\gamma}|\nabla\tilde{u}_\alpha|^2_{\tilde{g}_\alpha}\, dv_{\tilde{g}_\alpha} + \beta(R) + o(1) \\ &\quad= \int_X \rho^{1-2\gamma}|\nabla\hat{u}_\alpha|^2_g\, dv_g - \int_{B^+_R(0)} y^{1-2\gamma}|\nabla u|^2\, dxdy + \beta(R) + o(1) \\ &\quad= \int_X \rho^{1-2\gamma}|\nabla\hat{u}_\alpha|^2_g\, dv_g - \int_{\mathbb{R}^{n+1}_+} y^{1-2\gamma}|\nabla u|^2\, dxdy + \beta(R) + o(1). \end{aligned}$$

In a similar way, we can get that

$$\int_M |\hat{v}_\alpha|^{2^*}\, d\sigma_{\hat{h}} = \int_M |\hat{u}_\alpha|^{2^*} d\sigma_{\hat{h}} - \int_{\mathbb{R}^n} |u|^{2^*}\, dx + \beta(R) + o(1).$$

These imply that

$$I^\gamma_g(\hat{v}_\alpha) = I^\gamma_g(\hat{u}_\alpha) - \tilde{E}(u) + \beta(R) + o(1).$$

Since $R > 0$ is arbitrary, we get conclusion (iii).

(iv) It is a direct consequence of (ii) and (iii).

□

4. Proof of the main Results

Proof of Theorem 1.3. From Remark 2.10, we have $u_\alpha \rightharpoonup u^0$ in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$. And $u_\alpha \to u^0$ a.e. on M as $\alpha \to +\infty$. Then $u^0 \geq 0$ on M since $u_\alpha \geq 0$. Also $\hat{u}_\alpha = u_\alpha - u^0$ satisfies the Palais-Smale condition and

$$I^\gamma_g(\hat{u}_\alpha) = I^{\gamma,\alpha}_g(u_\alpha) - I^{\gamma,\infty}_g(u^0) + o(1).$$

If $\hat{u}_\alpha \to 0$ in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$, then the theorem is proved. If $\hat{u}_\alpha \rightharpoonup 0$ but not strongly in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$, using Lemma 3.1, we can obtain a new Palais-Smale sequence $\{\hat{u}^1_\alpha\}_{\alpha\in\mathbb{N}}$ satisfying

$$I^\gamma_g(\hat{u}^1_\alpha) = I^\gamma_g(\hat{u}_\alpha) - \tilde{E}(u) + o(1).$$

Now again, either $\hat{u}^1_\alpha \to 0$ in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$, in which case the theorem holds, or $\hat{u}^1_\alpha \rightharpoonup 0$ but not strongly in $W^{1,2}(X, \rho^{1-2\gamma})$ as $\alpha \to +\infty$, in which case we again use Lemma 3.1. Since $\{I^{\gamma,\alpha}_g(u_\alpha)\}_{\alpha\in\mathbb{N}}$ is uniformly bounded, after a finite number of induction steps, we get the last Palais-Smale sequence $\{\hat{u}^m_\alpha\}_{\alpha\in\mathbb{N}}$ $(m > 1)$ with $I^\gamma_g(\hat{u}^m_\alpha) \to \beta < \beta_0$. Then by Lemma 2.13, we can get that $\hat{u}^m_\alpha \to 0$ in $W^{1,2}(X, \rho^{2\gamma-1})$ as $\alpha \to +\infty$. Applying Lemma 3.1 in the process, we can get $\{u^j\}^m_{j=1}$ are solutions to (3.1). We will prove the positivity of u^j, $j = 1, \cdots, m$, in Lemma 4.2, and the relation (5) of Theorem 1.3 in Lemma 4.1.

For the regularity of u^j we can use Lemma 5.2 in the Appendix. Then the proof of the theorem is finished.

Lemma 4.1. *For any integer k in $[1, m]$, and any integer l in $[0, k-1]$, there exist an integer s and sequences $\{y^j_\alpha\}_{\alpha\in\mathbb{N}} \subset M$ and $\{\lambda^j_\alpha > 0\}_{\alpha\in\mathbb{N}}$, $j = 1, \cdots, s$, such that $d_{\hat{h}}(x^k_\alpha, y^j_\alpha)/\mu^k_\alpha$ is bounded and $\lambda^j_\alpha/\mu^k_\alpha \to 0$ as $\alpha \to +\infty$, and for any $R, R' > 0$,*

$$\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)\setminus\cup^s_{j=1}\mathfrak{D}_{R'\lambda^j_\alpha}(y^j_\alpha)} |\hat{u}_\alpha - \sum_{i=1}^{l} u^i_\alpha - u^k_\alpha|^{2^*} d\sigma_{\hat{h}} = o(1) + \epsilon(R'), \tag{4.1}$$

where

$$\lim_{R'\to+\infty} \lim_{\alpha\to+\infty} \sup \epsilon(R') = 0,$$

and $\{u^i_\alpha\}$ is derived from the rescaling of u^i we obtained in the above proof of Theorem 1.3, and $\{x^i_\alpha\}$ is the i-th likely blow up points sequence.

Proof. We prove this lemma by iteration on l. For any integer k $(1 \le k \le m)$, if $l = k-1$, combining the above proof of Theorem 1.3 with Lemma 3.1 and Proposition 2.4, we have

$$\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)} |\hat{u}_\alpha - \sum_{i=1}^{k-1} u^i_\alpha - u^k_\alpha|^{2^*} d\sigma_{\hat{h}} = o(1),$$

so (4.1) holds for $s = 0$.

Suppose that (4.1) holds for some l, $1 \le l \le k-1$, we need to show that (4.1) holds for $l-1$.

<u>Case 1</u> $d_{\hat{h}}(x^l_\alpha, x^k_\alpha) \nrightarrow 0$ as $\alpha \to +\infty$. Then for any $\bar{R} > 0$, up to a subsequence, $\mathfrak{D}_{\bar{R}\mu^l_\alpha}(x^l_\alpha) \cap \mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha) = \emptyset$, so we have

$$\begin{aligned}\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)\setminus\cup^s_{j=1}\mathfrak{D}_{R'\lambda^j_\alpha}(y^j_\alpha)} |u^l_\alpha|^{2^*} d\sigma_{\hat{h}} &\le \int_{M\setminus\mathfrak{D}_{\bar{R}\mu^l_\alpha}(x^l_\alpha)} |u^l_\alpha|^{2^*} d\sigma_{\hat{h}} \\ &\le C\int_{\mathbb{R}^n\setminus D_{\bar{R}}(0)} |u^l|^{2^*} d\sigma_{\tilde{h}_\alpha} \le C\int_{\mathbb{R}^n\setminus D_{\bar{R}}(0)} |u^l|^{2^*} dx.\end{aligned}$$

Since $\bar{R} > 0$ is arbitrary and $u^l \in L^{2^*}(\mathbb{R}^n)$, we get

$$\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)\setminus\cup^s_{j=1}\mathfrak{D}_{R'\lambda^j_\alpha}(y^j_\alpha)} |u^l_\alpha|^{2^*} d\sigma_{\hat{h}} = o(1), \quad \text{as } \alpha \to +\infty. \tag{4.2}$$

So by the induction hypothesis for l and (4.2) we obtain

$$
\begin{aligned}
&\int_{\mathfrak{D}_{R\mu_\alpha^k}(x_\alpha^k)\setminus\cup_{j=1}^s\mathfrak{D}_{R'\lambda_\alpha^j}(y_\alpha^j)} |\hat{u}_\alpha - \sum_{i=1}^{l-1} u_\alpha^i - u_\alpha^k|^{2^*}\, d\sigma_{\hat{h}} \\
&\le 2^{2^*-1}\int_{\mathfrak{D}_{R\mu_\alpha^k}(x_\alpha^k)\setminus\cup_{j=1}^s\mathfrak{D}_{R'\lambda_\alpha^j}(y_\alpha^j)} |\hat{u}_\alpha - \sum_{i=1}^{l} u_\alpha^i - u_\alpha^k|^{2^*}\, d\sigma_{\hat{h}} \\
&\quad + 2^{2^*-1}\int_{\mathfrak{D}_{R\mu_\alpha^k}(x_\alpha^k)\setminus\cup_{j=1}^s\mathfrak{D}_{R'\lambda_\alpha^j}(y_\alpha^j)} |u_\alpha^l|^{2^*}\, d\sigma_{\hat{h}} \\
&= o(1) + \epsilon(R').
\end{aligned}
$$

Thus we have proven that (4.1) holds for $l-1$.

Case 2 $d_{\hat{h}}(x_\alpha^l, x_\alpha^k) \to 0$ as $\alpha \to +\infty$. Let r_0 be sufficiently small such that for any $P \in M$, $x, y \in \mathbb{R}^n$ and $|x|, |y| \le r_0$,

$$1/2|x-y| \le d_{\hat{h}}(\varphi_P(x), \varphi_P(y)) \le 2|x-y|.$$

Let $\tilde{x}_\alpha^l = (\mu_\alpha^k)^{-1}\varphi_{x_\alpha^k}^{-1}(x_\alpha^l)$, $\tilde{y}_\alpha^j = (\mu_\alpha^k)^{-1}\varphi_{x_\alpha^k}^{-1}(y_\alpha^j)$, then

$$
\begin{aligned}
&D_{\frac{R}{2}\frac{\mu_\alpha^l}{\mu_\alpha^k}}\left(\tilde{x}_\alpha^l\right) \subset (\mu_\alpha^k)^{-1}\varphi_{x_\alpha^k}^{-1}(\mathfrak{D}_{R\mu_\alpha^l}(x_\alpha^l)) \subset D_{2R\frac{\mu_\alpha^l}{\mu_\alpha^k}}\left(\tilde{x}_\alpha^l\right), \\
&D_{\frac{R}{2}\frac{\lambda_\alpha^j}{\mu_\alpha^k}}\left(\tilde{y}_\alpha^j\right) \subset (\mu_\alpha^k)^{-1}\varphi_{x_\alpha^k}^{-1}(\mathfrak{D}_{R\lambda_\alpha^j}(y_\alpha^j)) \subset D_{2R\frac{\lambda_\alpha^j}{\mu_\alpha^k}}\left(\tilde{y}_\alpha^j\right).
\end{aligned}
\tag{4.3}
$$

Given $\tilde{R} > 0$, from Lemma 3.1, Proposition 2.4 and proof of Theorem 1.3 we have

$$\int_{\mathfrak{D}_{\tilde{R}\mu_\alpha^l}(x_\alpha^l)} |\hat{u}_\alpha - \sum_{i=1}^{l} u_\alpha^i|^{2^*}\, d\sigma_{\hat{h}} = o(1). \tag{4.4}$$

By the assumption for $1 \le l \le k-1$, i.e.

$$\int_{\mathfrak{D}_{R\mu_\alpha^k}(x_\alpha^k)\setminus\cup_{j=1}^s\mathfrak{D}_{R'\lambda_\alpha^j}(y_\alpha^j)} |\hat{u}_\alpha - \sum_{i=1}^{l} u_\alpha^i - u_\alpha^k|^{2^*}\, d\sigma_{\hat{h}} = o(1) + \epsilon(R'),$$

combined with (4.4) then we get that

$$\int_{[\mathfrak{D}_{R\mu_\alpha^k}(x_\alpha^k)\setminus\cup_{j=1}^s\mathfrak{D}_{R'\lambda_\alpha^j}(y_\alpha^j)]\cap\mathfrak{D}_{\tilde{R}\mu_\alpha^l}(x_\alpha^l)} |u_\alpha^k|^{2^*}\, d\sigma_{\hat{h}} = o(1) + \epsilon(R'),$$

so using (4.3) we arrive at

$$\int_{[D_R(0)\setminus\cup_{j=1}^s D_{2R'\lambda_\alpha^j/\mu_\alpha^k}(\tilde{y}_\alpha^j)]\cap D_{1/2\tilde{R}\mu_\alpha^l/\mu_\alpha^k}(\tilde{x}_\alpha^l)} |u^k|^{2^*}\, d\sigma_{\tilde{h}_\alpha} = o(1) + \epsilon(R'). \tag{4.5}$$

Next, we consider two scenarios: first, assume $d_{\hat{h}}(x_\alpha^l, x_\alpha^k)/\mu_\alpha^k \to +\infty$ as $\alpha \to +\infty$. We claim that $d_{\hat{h}}(x_\alpha^l, x_\alpha^k)/\mu_\alpha^l \to +\infty$ as $\alpha \to +\infty$. If not, then (4.5) with $\tilde{R}$ large enough yields that $\mu_\alpha^l/\mu_\alpha^k \to 0$ as $\alpha \to +\infty$. Moreover,

$$\frac{d_{\hat{h}}(x_\alpha^l, x_\alpha^k)}{\mu_\alpha^l} = \frac{d_{\hat{h}}(x_\alpha^l, x_\alpha^k)}{\mu_\alpha^k}\frac{\mu_\alpha^k}{\mu_\alpha^l},$$

so we can choose $\tilde{R} > 0$ such that $\mathfrak{D}_{\tilde{R}\mu_\alpha^k}(x_\alpha^k) \cap \mathfrak{D}_{\tilde{R}\mu_\alpha^l}(x_\alpha^l) = \emptyset$, which reduces to the previous case 1 and, as a consequence, (4.1) holds for $l-1$.

Second, if $d_{\hat{h}}(x^l_\alpha, x^k_\alpha)/\mu^k_\alpha \nrightarrow +\infty$ as $\alpha \to +\infty$, then up to a subsequence, $d_{\hat{h}}(x^l_\alpha, x^k_\alpha)/\mu^k_\alpha$ converges. Then (4.5) implies that $\mu^l_\alpha/\mu^k_\alpha \to +\infty$. Set $y^{s+1}_\alpha = x^l_\alpha$ and $\lambda^{s+1}_\alpha = \mu^l_\alpha$, then

$$\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)\setminus\cup_{j=1}^{s+1}\mathfrak{D}_{R'\lambda^j_\alpha}(y^j_\alpha)} |\hat{u}_\alpha - \sum_{i=1}^{l} u^i_\alpha - u^k_\alpha|^{2^*}\, d\sigma_{\hat{h}} = o(1) + \epsilon(R')$$

and

$$\begin{aligned}\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)\setminus\cup_{j=1}^{s+1}\mathfrak{D}_{R'\lambda^j_\alpha}(y^j_\alpha)} |u^l_\alpha|^{2^*}\, d\sigma_{\hat{h}} &\le \int_{M\setminus\mathfrak{D}_{R'\mu^l_\alpha}(x^l_\alpha)} |u^l_\alpha|^{2^*}\, d\sigma_{\hat{h}} \\ &\le C\int_{\mathbb{R}^n\setminus D_{R'}(0)} |u^l|^{2^*}\, dx \le \epsilon(R'),\end{aligned}$$

which yield that

$$\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)\setminus\cup_{j=1}^{s+1}\mathfrak{D}_{R'\lambda^j_\alpha}(y^j_\alpha)} |\hat{u}_\alpha - \sum_{i=1}^{l-1} u^i_\alpha - u^k_\alpha|^{2^*}\, d\sigma_{\hat{h}} = o(1) + \epsilon(R').$$

In particular, 4.1 holds for $l-1$, as desired. The iteration process is thus completed.

Moreover, we have also shown that for any $i \neq j$

$$\frac{\mu^i_\alpha}{\mu^j_\alpha} + \frac{\mu^j_\alpha}{\mu^i_\alpha} + \frac{d_{\hat{h}}(x^i_\alpha, x^j_\alpha)^2}{\mu^i_\alpha\mu^j_\alpha} \to +\infty$$

as $\alpha \to +\infty$ (c.f. [1],[5],[16]). Note that this convergence contains two kinds of bubbles: one case is that when $\mu^i_\alpha = O(\mu^j_\alpha)$ when $\alpha \to +\infty$, then the two blow up points are far away from each other. The other case is that $\mu^i_\alpha = o(\mu^j_\alpha)$ or $\mu^j_\alpha = o(\mu^i_\alpha)$ when $\alpha \to +\infty$, then the distance of the two blow up point cannot be determined. Also we get that $\lambda^j_\alpha/\mu^k_\alpha \to 0$ as $\alpha \to +\infty$. $\square$

Lemma 4.2. *The u^i $(i = 0, 1, \cdots, m)$ we get in the Theorem 1.3 are all nonnegative.*

Proof. First of all, note that $u^0 \ge 0$ in $\overline{X}$ by Proposition 2.11. So we just need to prove the positivity of u^i for $i \ge 1$. For any $k \in [1, m]$, taking $l = 0$ in Lemma 4.1, we have

$$\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)\setminus\cup_{j=1}^{s}\mathfrak{D}_{R'\lambda^j_\alpha}(y^j_\alpha)} |\hat{u}_\alpha - U^k_\alpha|^{2^*}\, d\sigma_{\hat{h}} = o(1) + \epsilon(R') \tag{4.6}$$

where

$$U^k_\alpha(x) = (\mu^k_\alpha)^{-\frac{n-2\gamma}{2}} u^k((\mu^k_\alpha)^{-1}\varphi^{-1}_{x^k_\alpha}(x)), \quad \text{for} \quad x \in \mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)$$

is called a bubble. Since $u_\alpha = \hat{u}_\alpha + u^0$, then for $x \in D_{r_0/\mu^k_\alpha}(0) \subset \mathbb{R}^n$, where the r_0 is the same as the one mentioned in Theorem1.3, we have

$$u^k_\alpha(x) = \tilde{u}^k_\alpha(x) + \tilde{u}^{0,k}_\alpha(x),$$

where

$$\begin{aligned}u^k_\alpha(x) &= (\mu^k_\alpha)^{\frac{n-2\gamma}{2}} u_\alpha(\varphi_{x^k_\alpha}(\mu^k_\alpha x)),\\ \tilde{u}^k_\alpha(x) &= (\mu^k_\alpha)^{\frac{n-2\gamma}{2}} \hat{u}_\alpha(\varphi_{x^k_\alpha}(\mu^k_\alpha x)),\\ \tilde{u}^{0,k}_\alpha(x) &= (\mu^k_\alpha)^{\frac{n-2\gamma}{2}} u^0(\varphi_{x^k_\alpha}(\mu^k_\alpha x)).\end{aligned}$$

Then (4.6) implies that

$$\int_{D_R(0)\setminus\cup_{j=1}^{s}D_{2R'\lambda^j_\alpha/\mu^k_\alpha}(\tilde{y}^j_\alpha)} |\tilde{u}^k_\alpha - u^k|^{2^*}\, dx = o(1) + \epsilon(R'), \tag{4.7}$$

where $\tilde{y}^j_\alpha = (\mu^k_\alpha)^{-1}\varphi^{-1}_{x^k_\alpha}(y^j_\alpha)$. Noting that $\{d_{\hat{h}}(x^k_\alpha, y^j_\alpha)/\mu^k_\alpha\}_{\alpha\in\mathbb{N}}$ is uniformly bounded by Lemma 4.1, therefore $\{\tilde{y}^j_\alpha\}_{\alpha\in\mathbb{N}}$ is bounded and there exists a subsequence, also denoted by $\{\tilde{y}^j_\alpha\}$, such that $\tilde{y}^j_\alpha \to \tilde{y}^j$ as $\alpha \to +\infty$ for $j = 1, \dots, s$. Combining (4.7) with $\lambda^j_\alpha/\mu^k_\alpha \to 0$ as $\alpha \to +\infty$, we get

$$\tilde{u}^k_\alpha \to u^k, \quad \text{in } L^{2^*}_{loc}(D_R(0) \setminus Y)$$

as $\alpha \to +\infty$ for $Y = \{\tilde{y}^j\}^s_{j=1}$, so

$$\tilde{u}^k_\alpha \to u^k \ \text{ a.e. in } \ \mathbb{R}^n,$$

since $R > 0$ is arbitrary.

Also note that

$$\int_{\mathfrak{D}_{R\mu^k_\alpha}(x^k_\alpha)} |u^0|^{2^*}\, d\sigma_{\hat{h}} = \int_{D_R(0)} |\tilde{u}^{0,k}_\alpha|^{2^*}\, d\sigma_{\tilde{h}^k_\alpha},$$

where $\tilde{h}^k_\alpha(x) = (\varphi^*_{x^k_\alpha}\hat{h})(\mu^k_\alpha x)$. Then $\mu^k_\alpha \to 0$ as $\alpha \to +\infty$ and $u^0 \in L^{2^*}(M, \hat{h})$ yield that

$$\tilde{u}^{0,k}_\alpha \to 0, \quad \text{in } L^{2^*}(D_R(0), |dx|^2)$$

as $\alpha \to +\infty$, so

$$\tilde{u}^{0,k}_\alpha \to 0 \ \text{ a.e. in } \ \mathbb{R}^n$$

since $R > 0$ is arbitrary.

In particular, we have shown that $u^k_\alpha \to u^k$ almost everywhere on $\mathbb{R}^n$ as $\alpha \to +\infty$. Note that u_α is nonnegative by definition, so $u^k_\alpha \geq 0$ on $\mathbb{R}^n$. We conclude that $u^k \geq 0$ on $\mathbb{R}^n$. □

5. Appendix

We would prove the $\mathcal{C}^\infty$ estimates from the L^∞ estimates by Harnack inequality. The two important lemmas are given here.

Lemma 5.1. [8] *Let $R > 0$ and u be a weak solution of*

$$\begin{cases} -\operatorname{div}(y^{1-2\gamma}\nabla u) = 0 & \text{in } B^+_{2R}(0), \\ -\lim\limits_{y\to 0} y^{1-2\gamma}\partial_y u = f(x)u + g(x)|u|^{2^*-2}u & \text{on } D_{2R}(0). \end{cases} \tag{5.1}$$

Here f and g are smooth functions on $D_{2R}(0)$. Assume that $\lambda = \int_{D_{2R}(0)} |u|^{2^*} dx < \infty$. Then for any $p > 1$, there exists a constant $C_p = C(p, \lambda)$ such that

$$\sup_{B^+_R(0)} |u| + \sup_{D_R(0)} |u| \leq C_p \left\{ R^{-\frac{n+2-2\gamma}{p}} \|u\|_{L^p(B^+_{2R}(0))} + R^{-\frac{n}{p}} \|u\|_{L^p(D_{2R}(0))} \right\}.$$

Lemma 5.2. [11] *Let $a(x), b(x) \in \mathcal{C}^\alpha(D_2(0))$ for some $0 < \alpha \notin \mathbb{N}$ and $u \in W^{1,2}(\partial' B^+_2, y^{1-2\gamma})$ be a weak solution of*

$$\begin{cases} -\operatorname{div}(y^{1-2\gamma}\nabla u) = 0 & \text{in } B^+_2(0), \\ -\lim\limits_{y\to 0} y^{1-2\gamma}\partial_y u = a(x)u + b(x) & \text{on } D_2(0). \end{cases} \tag{5.2}$$

If $2\gamma + \alpha \notin \mathbb{N}$, then $u(\cdot, 0)$ is of $\mathcal{C}^{2\gamma+\alpha}(D_1(0))$, and

$$\|u(\cdot, 0)\|_{C^{2\gamma+\alpha}(D_1(0))} \leq C(\|u\|_{L^\infty(B^+_2(0))} + \|b\|_{C^\alpha(D_2(0))})$$

where $C > 0$ depends only on n, γ, α and $\|a\|_{\mathcal{C}^\alpha(D_2(0))}$.

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