

## Coarse-Grained Probabilistic Automata Mimicking Chaotic Systems

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Discretization of phase space usually nullifies chaos in dynamical systems. We show that if randomness is associated with discretization dynamical chaos may survive and be indistinguishable from that of the original chaotic system, when an entropic, coarse-grained analysis is performed. Relevance of this phenomenon to the problem of quantum chaos is discussed.

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Chaos is a common characteristic of motion in continuous spaces. It is signaled by many indicators. Among these, positivity of the Kolmogorov-Sinai (KS) entropy is perhaps the most significant, both in theory and applications [1,2]. On the other hand, trajectories in *discrete* spaces are always asymptotically periodic—hence, of null KS entropy. They may arise in the discretization of continuous systems, as in the numerical simulation of differential equations, but arguably their role is most significant in the correspondence of classical and quantum dynamics.

Consider, for instance, the paradigmatic example of the quantum Arnol'd cat. Its dynamics are algorithmically equivalent to classical motion on a regular lattice, whose spacing is inversely proportional to the Planck constant [3,4]. When the spacing diminishes, the lattice becomes denser in continuous, classical phase space. Yet, it has long been recognized that chaos *cannot* be naively revived in such a limiting procedure [4–6]. A way out of this *impasse* is obtained by randomly perturbing the dynamics [7]: Is this addition enough to bring back the full algorithmic content, that is, the distinctive signature of chaos [8]? We attempt here an answer to this question in the most general and simple terms.

Deterministic-probabilistic systems (such as those occurring in cellular automata [9]) have long been investigated, with respect to their invariant measures [10] and also to the entropic content of their motion [11]. We now extend this study to the regime where continuous, discrete, and random effects are simultaneously present and intermix in nontrivial ways.

Let us therefore consider the deterministic map

$$x_{t+1} = f(x_t), \quad (1)$$

where  $t$  is discrete time,  $x$  belongs to  $[0, 1]^D$ , and  $D$  is the

dimension of the space. We embed in  $[0, 1]^D$  a uniform rectangular lattice, of spacing  $\eta$ , and we label its states by integer vectors  $n$  in  $[1, \lfloor \frac{1}{\eta} \rfloor]^D$  ( $\lfloor \cdot \rfloor$  is the integer part) [12]. We then restrict the map  $f$  to act on this lattice, and we add randomness:

$$n_{t+1} = \left\lfloor \frac{1}{\eta} f(\eta n_t) \right\rfloor + \sigma_t. \quad (2)$$

We stipulate that the uncorrelated, random “jumps”  $\sigma_t$  extend to lattice neighbors with total probability  $p$ , so that  $\sigma_t = 0$  with probability  $1 - p$ . Note that when  $p = 0$  the system is purely deterministic. For definiteness, we study the generalized tent map in one dimension:

$$x_{t+1} = \begin{cases} ax_t & \text{for } 0 \leq x \leq \frac{1}{a}, \\ a(1 - x_t)/(a - 1) & \text{for } \frac{1}{a} \leq x \leq 1; \end{cases} \quad (3)$$

with  $a = 3$ , the two-dimensional Arnol'd cat map [1]:

$$x_{t+1} = (x_t + y_t) \bmod 1, \quad y_{t+1} = (x_t + 2y_t) \bmod 1, \quad (4)$$

and their probabilistic lattice automata, Eq. (2).

We are now ready to briefly introduce our analytical tools. Let  $\{E_1, E_2, \dots\}$  be a finite partition of phase space consisting of identical hypercubic cells of side  $\epsilon$ . Let  $w_\epsilon = w_\epsilon^1, w_\epsilon^2, \dots, w_\epsilon^n$  be a finite symbolic trajectory, of length  $n = |w_\epsilon|$ :  $w_\epsilon^i = j$  if and only if  $x_i \in E_j$ . Let also  $p(w_\epsilon)$  be the frequency of  $w_\epsilon$ , defined by the physical ergodic measure. The  $n$ -block entropies  $H_n(\epsilon)$ ,  $n = 1, 2, \dots$  are defined by the sums

$$H_n(\epsilon) = - \sum_{w_\epsilon : |w_\epsilon|=n} p(w_\epsilon) \log p(w_\epsilon). \quad (5)$$

The *partition entropy*  $h(\epsilon)$  is the limit of  $H_n(\epsilon)/n$ , or, with faster convergence, of the *information rate*  $h_n(\epsilon) = H_n(\epsilon) - H_{n-1}(\epsilon)$ , as  $n \rightarrow \infty$  [15,16]. The KS entropy  $h_K$

is the supremum of the partition entropies, with respect to all countable partitions; hence,  $h(\epsilon) \leq h_K$ , and  $h_K$  can be obtained letting  $\epsilon$  go to zero. However, the function  $h(\epsilon)$  for finite  $\epsilon$  is interesting in its own right [17]. This is because it gauges the rate of information production, for observations of finite accuracy, as a function of the resolution desired. In line with our approach in [5] we also ascribe importance to the full behavior of  $H_n(\epsilon)$  versus  $n$ , and not only to the limit  $h(\epsilon)$ .

It is well-known that the systems (3) and (4) have positive KS entropy. In contrast, the dynamics of the purely discrete systems [i.e., Eq. (2) with  $p = 0$ ] are periodic, hence of null entropy. Yet, at scales larger than the lattice spacing,  $\epsilon > \epsilon_m := \eta$ , they approximate the continuous dynamics for a finite time, roughly of the order of the logarithm of the period of the trajectory [13,14,18–20]. As a consequence, the entropies  $H_n(\epsilon)$  are also close to those of the continuous system, for  $n \leq \bar{n}$ . The upper boundary  $\bar{n}$  can be estimated requiring that at  $n = \bar{n}$  the number of different  $\epsilon$  histories of length  $n$  of the continuous system,  $\mathcal{N}_\epsilon(n) \sim \exp[h(\epsilon)n]$ , be of the same order of the number of discrete states,  $M \sim (1/\eta)^D$ . Here  $D$  is the dimension of the attractor or, in the absence of this latter, the dimension of the space. This leads to

$$\bar{n} \sim \frac{\log M}{h(\epsilon)} \sim -\frac{D \log \eta}{h(\epsilon)}. \quad (6)$$

Dependence of  $\bar{n}$  on the average period of trajectories,  $T$ , follows equally well. Since  $T \sim M^{D_2/2}$ , where  $D_2$  is the correlation dimension of the ergodic measure [20],

$$\bar{n} \sim \frac{2D \log T}{D_2 h(\epsilon)}. \quad (7)$$

In the deterministic discrete systems [Eq. (2) with  $p = 0$ ],  $\bar{n}$  may be large enough to observe the entropic growth of the continuous map and sometimes to compute the entropy  $h(\epsilon)$  of the latter to a fair accuracy. However, when  $n$  exceeds  $\bar{n}$ , the  $n$ -word entropies,  $H_n$ , quickly saturate at a constant value, of the order of  $\log T$  or  $\log(1/\eta)$  or  $\log M$ , which reveals the periodic regime of the dynamics. It is clear that while the time  $\bar{n}$  is model dependent, its logarithmic scaling with the parameters is universal: the discrete, “pseudo” chaos seems to be very short-lived [4–6]. It is at this point that the random jumps  $\sigma_i$  completely change the scenario.

First of all, the null-entropy, periodic system is turned into an aperiodic stochastic process of maximal entropy  $h_p$ . If  $\sigma_i$  extends to  $k$  neighbors with equal probability  $p/k$ , then  $h_p = -p \ln(p/k) - (1-p) \ln(1-p)$ . Since  $h_p$  totally originates from the random jumps at the lattice scale  $\epsilon_m$ , it can be fully detected only at this scale:  $h_p \approx h(\epsilon_m) \geq h(\epsilon)$ , for  $\epsilon \geq \epsilon_m$ .

It is clear that  $h_p$  has no relation with the KS entropy  $h_K$ . Then, two cases must be considered. In view of the above, when  $h_p < h_K$ , addition of randomness, and ob-

servational coarse graining, cannot achieve the full entropy content of the continuous system. This no-go rule does not apply in the opposite case. We find that when  $h_p \gg h_K$  and  $\epsilon \gg \epsilon_m$ , the partition entropies of the continuous system and of the discrete ones tend to coincide. Notice that the condition  $\epsilon \gg \epsilon_m$  requires that the number of discrete states per cell,  $M\epsilon^D$ , be large.

To prove these claims, let us first consider the tent map, Eqs. (2) and (3), with  $p = 0.05$ , and nearest-neighbor random jumps. In this case  $h_p \approx 0.233$  is much smaller than  $h_K \approx 0.636$ . Figure 1 plots  $H_n(\epsilon, M)$  versus  $n$ , for different values of  $M$ , and  $\epsilon = 1/18$ . This partition is generating, so that the partition entropy of the continuous map,  $h_{\text{cont}}(1/18)$ , is equal to  $h_K$ . We observe that for  $n < \bar{n}$ ,  $H_n(\epsilon)$  is approximately equal to  $h_K$ , the entropy of the continuous system. Later on, for  $n > \bar{n}$ , the curve  $H_n$  bends and—rather than tending to a constant, as it would if randomness were not present—redirects its growth to a different linear regime:  $H_n \approx h(\epsilon, M)(n - \bar{n}) + A \log M$ , with  $A$  a suitable constant. Numerically, we also find that  $h(\epsilon, M) \approx 0.20 \approx h_p$ , independently of  $M$ , even if  $\epsilon = 1/18$  is much larger than  $\epsilon_m$ . This is noteworthy: were the evolution driven only by the probabilistic diffusion,  $n_{t+1} = n_t + \sigma_t$ , the  $\epsilon$  entropy would have been 10 times smaller,  $h_{\text{diff}}(1/18, 252) \approx 0.02$ . The effect of randomness is strongly enhanced by the deterministic evolution [21].

By raising the value of the jump probability  $p$  [22] the entropy  $h_p$  increases, and it may exceed the KS entropy  $h_K$ , or  $h_{\text{cont}}(\epsilon)$  for a given  $\epsilon$ . Figure 2 plots  $h(\epsilon, M)$  versus  $h_p$ , for different values of  $M$ . At fixed, low  $h_p$ , the  $M$  dependence is rather mild, as was observed in Fig. 1, and  $h(\epsilon, M)$  is smaller than  $h_{\text{cont}}(\epsilon)$ . Raising  $h_p$  at fixed  $M$  to well exceed this value, and then increasing  $M$ , we obtain convergence of  $h(\epsilon, M)$  to  $h_{\text{cont}}(\epsilon)$ .

This convergence is further illustrated in Fig. 3 by fixing  $h_p = 1.5$ , by choosing  $\epsilon = 1/7$  and  $\epsilon = 1/10$ ,

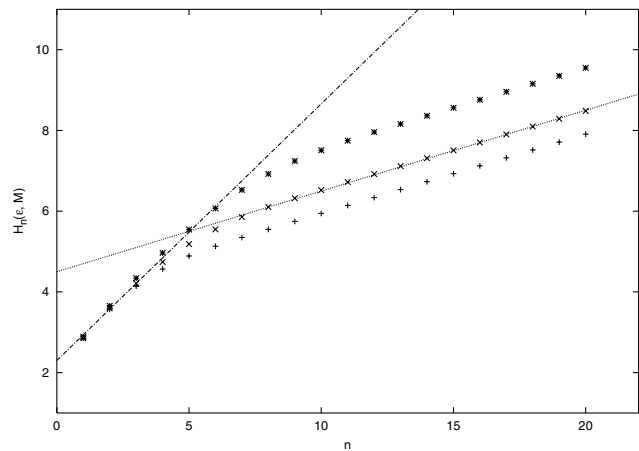


FIG. 1.  $n$ -word entropies  $H_n(\epsilon, M)$  vs  $n$  for the discretized tent map with  $\epsilon = 1/18$ , and  $M = 252$  (+), 1008 (×), and 4032 (\*).  $H_n(\epsilon, M)$  are compared to the lines with slope  $h(1/18) \approx 0.20$  and  $h_K \approx 0.636$ .

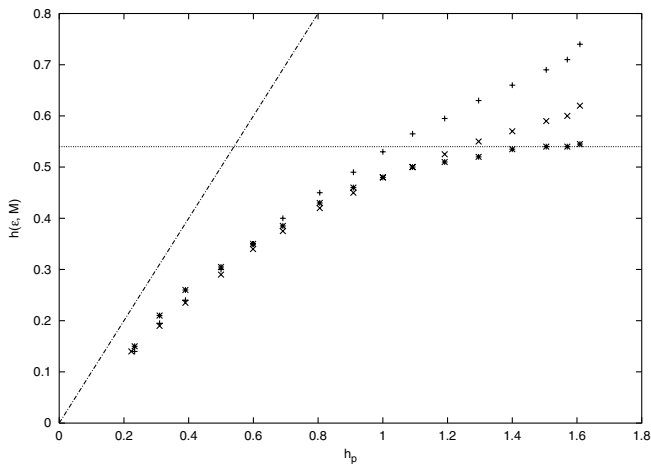


FIG. 2. Partition entropies of the discretized tent map  $h(\epsilon, M)$  vs  $h_p$ , with  $k = 4$ ,  $\epsilon = 1/7$ , and  $M = 63$  (+),  $M = 168$  (x) and  $M = 4032$  (\*). The dotted line is drawn at  $h_{\text{cont}}(1/7)$ , the partition entropy of the continuous system for  $\epsilon = 1/7$ .

and by plotting  $h(\epsilon, M)$  versus  $z = M\epsilon$ , the number of lattice points enclosed in a cell of the partition. The different curves tend to coincide for small  $z$ , where the entropies overshoot: when coarse graining is too fine (too few states in a cell) the direct action of randomness is dominating. This effect fades in the opposite direction: randomness becomes a germ that gets scale amplified by the dynamics, and  $h(\epsilon, z/\epsilon)$  tends to  $h_{\text{cont}}(\epsilon)$ , the partition entropy of the continuous system.

This behavior appears to be generic, as indicated by the results for the two-dimensional Arnol'd cat map, subject to a random perturbation with  $k = 4$  and  $h_p = 1.5$ , plotted in Fig. 4. Since  $\epsilon = 1/4, 1/16, 1/64$ , all provide generating partitions, the corresponding values of  $h_{\text{cont}}(\epsilon)$  are equal to  $h_K \approx 0.962$ . As a consequence the data fall on a single line, starting from the entropy of the random

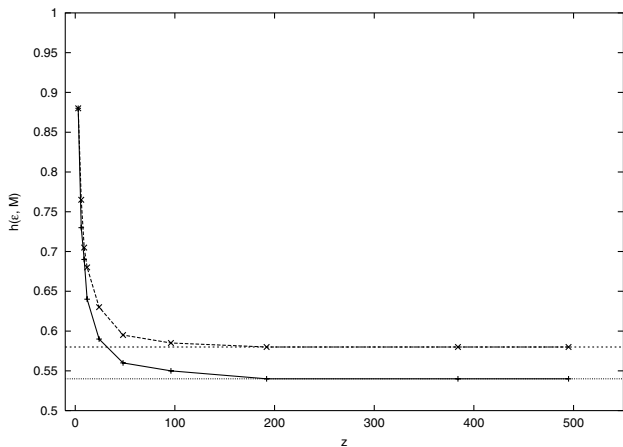


FIG. 3.  $\epsilon$  entropies  $h(\epsilon, M)$  vs  $z = M\epsilon$ , for the discretized tent map, with  $\epsilon = 1/7$  and  $\epsilon = 1/10$  (x). Horizontal lines are drawn at  $h_{\text{cont}}(1/7) \approx 0.54$ , and  $h_{\text{cont}}(1/10) \approx 0.58$ .

perturbation,  $h_p = 1.5$ , and converging to the KS entropy  $h_K \approx 0.962$ . The convergence of  $h_n(\epsilon, M)$  to the asymptotic values is shown in the inset of Fig. 4, together with the partition entropies obtained for the purely discrete system ( $p = 0$ ), and for the purely stochastic motion. This comparison proves that  $h(\epsilon, z/\epsilon)$  are truly asymptotic values and provides further evidence in support of our explanation of the phenomenon, with which we now conclude.

Imposing a finite lattice to the otherwise continuum set of states of a dynamical system, inevitably bounds the algorithmic complexity of its trajectories, and the value of its partition entropies, to the logarithm of the number of states. If the lattice is perceived with some fuzziness—or if random errors are allowed, following the approach of this Letter—one expects that on large scales continuum properties and chaos might reemerge and be indistinguishable from that of the original system [23]. We have determined the conditions for this to happen. First, observational coarse graining must be invoked. Second, the action of the external randomness must be confined to the “microscopic,” unresolved scales. The instability of deterministic dynamics amplifies these microscopic, random errors and carries them over to the large, observation scales. Finally, by a sort of conservation law, the *flow of information* supplied by

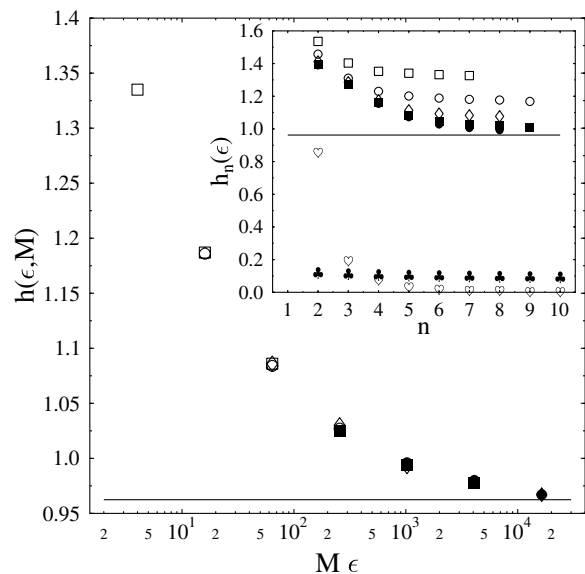


FIG. 4. Entropies  $h(\epsilon, M)$  vs  $M\epsilon$  in the perturbed Arnol'd cat map. Here,  $M = 16^2$  (squares),  $32^2$  (circles),  $64^2$  (diamonds),  $128^2$  (filled squares),  $256^2$  (filled circles), and  $512^2$  (filled diamonds);  $\epsilon = 1/4, 1/16, 1/64$ . The horizontal line is drawn at  $h_K \approx 0.962$ . In the inset,  $h_n(\epsilon, M)$  vs  $n$  for  $\epsilon = 1/64$  and  $M = 16^2, \dots, 256^2$  (coded as before). At the largest value of  $M$ ,  $M = 256^2$ , the purely discrete system (hearts) shows rapid convergence to zero, in accordance with Eqs. (6) and (7). The purely diffusive system (clubs) on its part is consistently close to  $h_{\text{diff}}(1/64, 256^2) \approx 0.085$ .

the microscopic *zitterbewegung* must not be less than  $h_K$ , the maximum entropy production rate of the continuous system.

For a long time, research in quantum chaos has looked for quantum characteristics related to classical chaotic motion. The fact that none of these could be properly called chaos led to the concept of *pseudochaos* and cast doubts on the very existence of chaos in nature. The results presented in this Letter suggest that one might try to reverse this approach and consider classical dynamics as an effective theory that, via truly chaotic deterministic dynamical systems, models a randomly perturbed quantum motion under observational coarse graining.

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