Approximating Layout Problems on Random Sparse Graphs*

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Abstract

We show that, with high probability, several layout problems are approximable within a constant for random graphs drawn from the standard $\mathcal{G}_{n,p}$ model with $p = c/n$ for some constant c. Our results establish that, in fact, any algorithm that returns a feasible solution will produce such an approximation for graphs with good expansion properties.

1 Introduction

Linear arrangement problems play an important role in Computer Science [27, 1, 8]. A *linear layout* (or *linear arrangement* or *vertex ordering*) of a graph G with n nodes is a one-to-one mapping of the vertices of G to the set $\{1, \ldots, n\}$. A layout π on $G = (V, E)$ determines in a unique way a nested sequence of vertex subsets containing those vertices placed up to the i-th position. The layout also induces an assignment of lengths to every edge in the graph: the length induced by a layout π for an edge $e = uv \in E$ is $\lambda(\pi, e) = |\pi(u) - \pi(v)|$. The complexity of a graph in terms of a linear layout is usually obtained by measuring length, crossing edges or neighbors placement.

The *bandwidth* problem asks for a layout minimizing the maximum edge length. The problem is **NP**-complete [28], even for trees with maximum degree 3 [14] or caterpillars with hair length 3 [25]. It can be approximated within a constant for some restricted classes of trees [18], but has no polynomial time approximation scheme for trees [4]. It has a constant randomized approximation algorithm for dense instances [21], and no polynomial time approximation algorithm for general graphs [20].

The *minimum cut arrangement* asks for a layout minimizing the maximum cut along the nested sequence of vertex sets. The problem is **NP**-complete [17], even for planar graphs with maximum degree 3 [26]. For trees the problem is in **P** [33] and even in **NC** [9]. It can be approximated within any constant for dense graphs [3]. A variation

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of the problem in which the cut excludes edges touching the last vertex is known as the *minimum modified cut arrangement* and is also **NP**-complete for planar graphs with maximum degree 3 [26].

The *minimum linear arrangement* problem (also known as the *minimum edge sum* [19] or the *optimal linear ordering* [1]) seeks a layout that minimizes the total edge length. This problem is also **NP**-complete [16]. For trees the problem is in **P** [31] and in **NC** [9]. It can be approximated within a $\mathcal{O}(\log^2 n)$ factor using the approximate max flow-min cut theorem [23]. A better approximation factor $\mathcal{O}(\log n \log\log n)$ can be achieved using spreading metrics [13]. This result has been improved recently to a $\mathcal{O}(\log n)$ approximation for general graphs and to a $\mathcal{O}(\log\log n)$ factor for planar graphs [30]. On the other hand, the problem can be approximated within a $1+\epsilon$ factor in time $n^{\mathcal{O}(1/\epsilon)}$ when restricted to *dense* graphs using linear programming and random rounding [3]. Nothing is known about the hardness of approximating the minimum linear arrangement problem, which is not even known to be MAX-**SNP**-hard. In [29] some heuristics algorithms to approximate this problem are empirically studied. The *maximum linear arrangement* problem that asks for a layout maximizing the total edge length is not of practical interest, but it is worth noting that its approximability properties are entirely understood: A greedy algorithm can be used to obtain an approximation within a factor of 2 [11].

The *vertex separation* problem has the same formulation as the *minimum cut arrangement* problem, but using as measure the number of vertices in the first partition connected to the second one. This measure was first introduced in [7] as the δ-operator. The problem is **NP**-complete [24], but in **P** for trees [12]. The global version in which one looks for a layout minimizing the sum of all the separations is known as the *minimal sum cut* problem [10] or the *minimal profile* problem [22]. The problem is equivalent to the *interval graph completion* problem that is also **NP**-complete [15]. For trees the problem is in **P** [22] and in **NC** [10]. An approximation factor $\mathcal{O}(\log n \log\log n)$ can be obtained using spreading metrics [13].

The above results establish the difficulty in dealing with sparse graphs. In general, considering only dense instances makes a problem easier because such graphs inherit most of the good properties of dense random graphs. In this paper we try to analyze the difficulty of approximating some of the above problems for random sparse graphs (drawn from the standard $\mathcal{G}_{n,p}$ model with $p = c/n$ [5, 2]) and expanders.

A natural question is to ask whether there is any relation between the approximability of the maximization version of the problems, and whether we can infer some consequence for the minimization version from our understanding of these maximization versions. It thus makes sense to introduce the gap between the maximum an the minimum values, the ratio between the maximum and the minimum values, and to estimate this gap value for interesting classes of graphs. Note that whenever we can bound the gap for a certain constant $r > 1$, it follows that *any* arrangement of G is r-approximate for both the minimization and maximization problems.

For instance, in the case of the minimum linear arrangement problem, it is clear that the gap is 1 for any complete graph G. Moreover, for a graph that has only one edge, the gap is n (and this is the largest possible gap). Those extremal cases suggest that the gap for this problem is related to the connectivity property of a graph, and thus it seems unlikely that we can find a bounded degree graph with a small gap value, and one would thing that, at least, *almost all* (in the probabilistic sense) sparse graphs have a large gap value. We will show that the opposite results hold.

2 Definitions and basic results

Consider an undirected graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges. We denote by $N(u)$ the set of neighbors of a vertex u including u. A *layout* for G is any bijective function that associates to each vertex a number in the range $\{1, \dots, n\} = [n]$. Given a layout π for G, for any $i \in [n]$ consider the sets $L(i) = \{v \mid \pi(v) \leq i\}$ and $R(i) = \{v \mid \pi(v) > i\}$. For a given layout π, define

$$\begin{array}{rcll} \lambda(e,\pi) & = & |\pi(u) - \pi(v)| & e = uv \in E \\ \text{cut}(i,\pi) & = & |\{uv \in E \mid u \in L(i) \wedge v \in R(i)\}| & i \in [n] \\ \text{mod-cut}(i,\pi) & = & |\{uv \in E \mid u \in L(i) - \{i\} \wedge v \in R(i)\}| & i \in [n] \\ \delta(i,\pi) & = & |\{u \in L(i) \mid \exists w \in R(i) : (u,w) \in E\}| & i \in [n] \end{array}$$

Definition 1. The formal definitions of the problems we study are the following:

- Minimum linear arrangement (MINLA). Given a graph $G = (V, E)$, find a layout π that minimizes

$$\text{LA}(G,\pi) = \sum_{i=1}^{n-1} \text{cut}(i,\pi) = \sum_{e \in E} \lambda(e,\pi).$$

- Minimum sum modified cut (MINMLA). Given a graph $G = (V, E)$, find a layout π that maximizes

$$\text{MLA}(G,\pi) = \sum_{i=1}^{n-1} \text{mod-cut}(i,\pi).$$

- Minimum sum cut (MINSC). Given a graph $G = (V, E)$, find a layout π that minimizes

$$\text{SC}(G,\pi) = \sum_{i=1}^{n-1} \delta(i,\pi).$$

We will also consider the maximization versions of such problems, namely the maximum linear arrangement (MAXLA), the maximum sum modified cut (MAXMLA), and the maximum sum cut (MAXSC). For sake of simplicity, for a given measure F, we will use the notations

$$\text{MAX}F(G) = \max_{\pi} F(G,\pi)$$
$$\text{MIN}F(G) = \min_{\pi} F(G,\pi)$$
$$\text{AV}F(G) = \frac{\sum_{\pi} F(G,\pi)}{n!}$$

to denote its maximum, minimum and average values.

Definition 2. For a measure F, we define the *gap* between the minimum and maximum values as

$$\text{GAP}F(G) = \frac{\text{MAX}F(G)}{\text{MIN}F(G)} = 1 + \frac{\text{MAX}F(G) - \text{MIN}F(G)}{\text{MIN}F(G)}.$$

Definition 3. Given a constant r, an algorithm A is an r-approximation to a minimization (maximization) problem for a function F when it holds that for any graph G

$$\frac{A(G)}{\textsc{min}F(G)} \leq 1 + r \quad \left(\frac{\textsc{max}F(G)}{A(G)} \leq 1 + r\right).$$

Equivalently, the value $\epsilon = 1 - r$ is called the approximation ratio of the r-approximate algorithm A [15]. Observe that any bound on the second expression in our definition gives a bound on the approximation ratio of any algorithm that computes a layout for G.

Basic results. Given a graph $G = (V, E)$ with n nodes and m edges, it is well known that the average length of an edge $e = uv$ is $(n+1)/3$. Taking into account that $\textsc{la}(G, \pi)$ is the sum of all edge lengths we have $\textsc{avla}(G) \geq m(n+1)/3$. To bound the average modified cut cost we use the following straightforward relationship: $\textsc{mla}(G, \pi) + \sum_{v \in V} d(v) \geq \textsc{la}(G, \pi)$. The same relationship holds for the average value and we have that $\textsc{avmla}(G) \geq m(n-5)/3$.

To analyze the average sum cut cost we consider a simplified measure. Given a graph $G = (V, E)$ each vertex u selects a neighbor $s(u) \neq u$ if any, for an isolated vertex set $s(u) = u$. We will use an arbitrary (but fixed) selection s. Given a layout π define $D(G, \pi) = \sum_{v \in V} \max(0, \pi(v) - \pi(s(v)))$. Notice that for all π, $\textsc{sc}(G, \pi) \geq D(G, \pi)$. Furthermore the expected contribution of the edge $(v, s(v))$ is 0 with probability 1/2 and $(n+1)/3$ with probability 1/2, that is $(n+1)/6$. Adding up for all vertices we get $\textsc{avsc}(G) \geq n(n+1)/6$.

3 The graphs

We introduce now two graph classes that capture the properties needed to bound the gap.

Definition 4 (Mixing graphs). Let $0 < \gamma, \epsilon < 1$ and $c > 0$. A graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ is said to be (ϵ, γ, c)-*mixing* if for any two disjoint sets $A, B \subseteq V$ such that $|A| \geq \epsilon n$, $|B| \geq \epsilon n$, it is the case that

$$\left|\theta(A, B) - \frac{c}{n} \cdot |A||B|\right| \leq \gamma \frac{c}{n} \cdot |A||B|,$$

where $\theta(A, B)$ is the number of edges of G having one endpoint in A and another in B.

Definition 5 (Disperser graphs). Let $0 < \epsilon < 1$. A graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ is said to be an ϵ-*disperser* if for any two disjoint sets $A, B \subseteq V$ such that $|A| \geq \epsilon n$, and $|B| \geq \epsilon n$ there is at least an edge having an endpoint in A and an endpoint in B.

It is known that explicit constructions of expander graphs imply efficient construction of mixing graphs. In particular, the following result holds.

Theorem 1 (See e.g. [6]). *A constant α exists such that for any $\epsilon, \gamma > 0$, for any n and any $d \geq \alpha/(\epsilon^2\gamma^2)$, an $(\epsilon, \gamma, m/n)$-mixing graph with maximum degree at most d can be constructed in poly(n) time.*

Remark that from the definition of mixing and disperser graphs, it follows that, any (ϵ, γ, c)-mixing graph is also an ϵ-disperser, and so Theorem 1 also gives an explicit construction of disperser graphs.

Theorem 2. *A constant β exists such that for any $\epsilon > 0$, for any n and any $d \geq \beta/\epsilon^2$, an ϵ-disperser graph with n vertices and maximum degree at most d can be constructed in poly(n) time.*

Definition 6 (Random sparse graphs [5, 2]). We consider the standard class of random graphs $\mathcal{G}_{n,p}$ which have n nodes and each potential edge exists with probability p.

Although the random sparse graphs considered in this paper are expected to be non connected, with high probability they have good mixing properties.

Lemma 1 (Chernoff bounds). *Let $X_1, \dots, X_n$ be independent random variables whose range is $\{0,1\}$. Let $\mu = \mathbf{E}\left[\sum_{i=1}^n X_i\right]$. Then for any $0 < \gamma < 1$ it is the case that*

$$\Pr\left[(1-\gamma)\mu \leq \sum_{i=1}^n X_i \leq (1+\gamma)\mu\right] \geq 1 - 2\exp(-\gamma^2\mu/3).$$

Theorem 3 (Random graphs are mixing). *For any $\epsilon, \gamma > 0$, for any $c \geq \dfrac{3.296}{\epsilon^2\gamma^2}$, random graphs drawn from $\mathcal{G}_{n,p}$ with $p = c/n$ are (ϵ, γ, c)-mixing with probability at least $1 - 2^{-\Omega(n)}$.*

Proof. Consider any two sets $A, B \subseteq V$ such that $|A|, |B| \geq \epsilon n$. There are $k = |A||B|$ possible edges having an endpoint in A and an endpoint in B. Let us call $Y_1, \dots, Y_k$ the random variables such that $Y_i = 1$ if the i-th (in lexicographical order) of such edges is in the graph, and $Y_i = 0$ otherwise. The average of $\sum_{i=1}^k Y_i$ is clearly $\mu = c|A||B|/n$. Then we have that

$$\Pr\left[(1-\gamma)\mu \leq \sum_{i=1}^k Y_i \leq (1+\gamma)\mu\right] \geq 1 - \exp(\frac{\gamma^2\mu}{3} + 1).$$

Since there are at most 3^n choices for the sets A and B, it follows that the probability that the graph is not mixing is at most

$$\exp\left((\ln 3)n - \gamma^2 c \frac{|A|\,|B|}{n}\frac{1}{3} - 1\right).$$

Note that the term in the exponent is

$$n\left(\ln 3 - c\gamma^2\epsilon^2\frac{1}{3} - 1/n\right) \leq -\Omega(n).$$

□

As mixing graphs are dispersers, we also have that random graphs are disperser graphs with high probability.

4 Bounding the Gap

Now we bound the gap between the maximum and minimum costs for mixing graphs.

Lemma 2. *Let $G=(V,E)$ be an (ϵ,γ,c)-mixing graph (with $0<\epsilon,\gamma<1$), then*

$$\textsc{gapla}(G) \le 1 + \frac{1}{1-\gamma}\left(\frac{6\epsilon}{1-6\epsilon}(1+\gamma)+2\gamma\right) = 1+\mathcal{O}(\epsilon+\gamma).$$

Proof. Let π be any layout of G. We will bound its cost from above and from below:

$$\textsc{la}(G,\pi) = \sum_{i=1}^{n-1} \mathrm{cut}(i,\pi) \ge \sum_{i=\epsilon n}^{n-\epsilon n} \mathrm{cut}(i,\pi) \ge (1-\gamma)\frac{c}{n}\sum_{i=\epsilon n}^{n-\epsilon n} i(n-i).$$

For the lower bound,

$$\textsc{la}(G,\pi) = \sum_{i=1}^{n-1} \mathrm{cut}(i,\pi) \le 2\epsilon mn + \sum_{i=\epsilon n}^{n-\epsilon n} \mathrm{cut}(i,\pi) \le 2\epsilon mn + (1+\gamma)\frac{c}{n}\sum_{i=\epsilon n}^{n-\epsilon n} i(n-i).$$

Therefore, letting $S=\frac{c}{n}\sum_{i=\epsilon n}^{n-\epsilon n} i(n-i)$, we have

$$\begin{aligned}\textsc{maxla}(G) &\le 2\epsilon mn + (1+\gamma)S,\\ \textsc{minla}(G) &\ge (1-\gamma)S,\\ \textsc{maxla}(G)-\textsc{minla}(G) &\le 2\epsilon mn + 2\gamma S.\end{aligned}$$

As $\textsc{avla} \ge m(n+1)/3$ there is a layout that gives at least this value so,

$$2\epsilon mn + (1+\gamma)S \ge \frac{m(n+1)}{3} > \frac{mn}{3}$$

therefore $2\epsilon mn \le \frac{6\epsilon}{1-6\epsilon}(1+\gamma)S$ and we get

$$\textsc{gapla}(G) \le 1 + \frac{\left(\frac{6\epsilon}{1-6\epsilon}(1+\gamma)S + 2\gamma S\right)}{(1-\gamma)S}.$$

□

A similar result holds for the minimum sum modified cut problem.

Lemma 3. *Let $G=(V,E)$ be an (ϵ,γ,c)-mixing graph (with $0<\epsilon,\gamma<1$) with $|V|>9$. Then*

$$\textsc{gapmla}(G) \le 1 + \frac{1}{1-\gamma}\left(\frac{12\epsilon}{1-12\epsilon}(1+\gamma)+2\gamma\right) = 1+\mathcal{O}(\epsilon+\gamma).$$

Proof. Let π be any arrangement. We will bound its cost from above and from below:

$$\textsc{mla}(G,\pi) = \sum_{i=1}^{n-1} \text{mod-cut}(i,\pi) \ge \sum_{i=\epsilon n+1}^{n-\epsilon n-1} \text{mod-cut}(i,\pi) \ge (1-\gamma)\frac{c}{m}\sum_{i=\epsilon n+1}^{n-\epsilon n-1} (i-1)(n-i).$$

For the lower bound,

$$\begin{aligned}\textsc{mla}(G,\pi) = \sum_{i=1}^{n-1} \mathrm{cut}(i,\pi) &\le 2(\epsilon n+1)m + \sum_{i=\epsilon n+1}^{n-\epsilon n-1} \text{mod-cut}(i,\pi)\\ &\le 2\epsilon nm + (1+\gamma)\frac{c}{n}\sum_{i=\epsilon n+1}^{n-\epsilon n-1}(i-1)(n-i)\end{aligned}$$

where the last inequality holds because mod-cut$(1,\pi)$ = mod-cut$(n,\pi) = 0$ for any layout π. Therefore, letting $T = \frac{c}{n}\sum_{i=\epsilon n+1}^{n-\epsilon n-1}(i-1)(n-i)$, we have

$$\begin{aligned} \text{MAXMLA}(G) &\leq 2\epsilon nm + (1+\gamma)T, \\ \text{MINMLA}(G) &\geq (1-\gamma)T, \\ \text{MAXMLA}(G) - \text{MINMLA}(G,\pi) &\leq 2\epsilon nm + 2\gamma T. \end{aligned}$$

As AVMLA $= m(n-5)/3$ there is a layout that gives at least this value so,

$$2\epsilon nm + (1+\gamma)T \geq \frac{m(n-5)}{3}$$

and as $n > 9$ it holds that $n-5 \geq n/2$. Therefore $2\epsilon nm + (1+\gamma)T \geq \frac{mn}{6}$ and we get $2\epsilon nm \leq \frac{12\epsilon}{1-12\epsilon}(1+\gamma)T$. □

A similar results holds for the minimum sum cut problem.

Lemma 4. *Let $G = (V,E)$ be an ϵ-disperser graph (with $\epsilon < 1$), then*

$$\text{GAPSC}(G) \leq \frac{1}{1-4\epsilon}.$$

Proof. We find lower and upper bounds for the value SC(G). We first notice that in an ϵ-disperser graph it is the case that $\delta(\pi,i) \geq i - \epsilon n$ for every $\epsilon n < i < n - \epsilon n$. This is because there cannot be ϵn vertices on the left of i and ϵn vertices on the right of i without any connection.

$$\begin{aligned} \text{SC}(G,\pi) = \sum_{i=1}^{n-1}\delta(\pi,i) \geq \sum_{i=\epsilon n}^{n-\epsilon n}\delta(\pi,i) \geq \sum_{i=1}^{n-\epsilon n}(i-\epsilon n) \\ > (n-\epsilon n)^2/2 - \epsilon n(n-\epsilon n) > n^2/2 - 2\epsilon n^2. \end{aligned}$$

For the upper bound, we get

$$\text{SC}(G,\pi) = \sum_{i=1}^{n-1}\delta(\pi,i) \leq \sum_{i=1}^{n-1} i = (n-1)n/2 \leq n^2/2.$$

Thus, we have MINSC$(G) \geq n^2/2 - 2\epsilon n^2$ and MAXSC$(G) \leq n^2/2$ and thus GAPSC $\leq \frac{1}{1-4\epsilon}$. □

Consequently, we have established the following theorem:

Theorem 4. *The problems* MINLA *and* MINMLA *can be approximated within a constant on mixing graphs. The* MINSC *problem can be approximated within a constant on disperser graphs. Furthermore, there exists a constant c such that for any $\alpha > 0$, for any n,*

$$\Pr[\text{GAP}F(G) > 1+\alpha] \leq 2^{-n}$$

where G is a random graph from the $\mathcal{G}_{n,p}$ model with $p = \frac{c}{\alpha^4 n}$, and F is any of the three measures LA, MLA, SC.

5 Conclusions

Similar results can be achieved for the *local* problems such as bandwidth, mincut layout and vertex separation. For the bandwidth problem fixing any layout and taking the sets formed by the first ϵn vertices and the last ϵn vertices, in an (ϵ, γ, c)-mixing graph we have at least one edge connecting both partitions, and therefore a lower bound for the layout bandwidth of $(1 - 2\epsilon)n$. In the case that $\epsilon < 1/3$ we have a 3 approximation. A similar result for the bandwidth minimization is given in [32].

In an (ϵ, γ, c)-mixing graph we have at least $(1 - \gamma)cn/4$ edges in the central cut. This is a lower bound for the problem MINCUT. The bound also applies to the BISECTION problem, because the central cut splits the graph into two equal sized sets. It also applies to the MAX CUT problem. Therefore we can approximate those problems within a constant, for such graphs.

In an (ϵ)-disperser mixing graph we have at least $(1/2 - \epsilon)n$ nodes in the central cut. So, for n large enough we get a constant approximation for the vertex separation problem.

It is worth to remark that the obtained results give the approximation regardless the connectivity of the graph. For this class of graphs, to get a constant approximation, it is not necessary to finish off a connected component before starting a new one.

A standard way of evaluating the *real* efficiency (from a practical point of view) of an algorithm is to evaluate its performance on random instances. Our results show that *any* algorithm computing a layout, no matter how bad (or good), will perform very well on random sparse graphs, pointing out that such evaluations may be unworthy for some problems.

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