A Computational Characterization of Collective Chaos

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Abstract

We suggest that the interaction of a Globally Coupled Map (GCM) with an individual element inside the system is, from a computational point of view, indistinguishable of a (μ,ϵ)-dependent noise in the turbulent region of the phase space. Therefore, we can use the framework of Computational Mechanics to give a measure that clearly separates the ordered from the turbulent phases. Furthermore, our procedure is able to detect a small ordered domain inside the turbulent phase. These results reinforce the view of GCMs as properly defined mean field models of complex nonlinear networks.

Submitted to Physical Review E

There are a plethora of systems in nature that compute, at last in a naïve sense of computation. Ant colonies and brains are some unquestionable examples: they store, transmit and manipulate information. However, research on these systems has usually emphasized pattern formation and dynamical behaviour, leaving aside their computational properties and how dynamics and computation are mutually involved. Of course, this bias in interests is a consequence of a lack of an adequate general theory of computation in dynamical systems, though some recent proposals, such as that of *Computational Mechanics* [1], aim to fill that gap. The purpose of the new field of computational mechanics is to uncover the *implicit manipulation of information embedded in natural systems* (also called *intrinsic computation*), connecting pattern discovering and pattern formation with the computational capabilities of the system [2].

The study of computation in physical/biological systems cannot be accomplished without a thorough understanding of the interplay between dynamics and computation in a more formal, and therefore more manageable, setting. Up to now, computational mechanics has been applied to formal dynamical systems (but see [3]), leading to a determination of computational features of cellular automata [4], transitions at the period-doubling route to chaos [5] and one-dimensional spin systems [2], revealing some new properties of these systems not accounted for by classical measures such as entropies and algorithmic definitions of complexity [6]. Computational mechanics has also been applied to globally coupled maps, which were taken as models of collectives of dynamical complex agents in order to speculate about the possible trade-off between collective complexity and individual complexity observed in social insects [7]. The work we introduce here aims to go further into the study of the above mentioned trade-off, through a quantitative measure of the interaction of the collective with an individual of the system. This measure will allow us to characterize properly the turbulent phase of GCMs, that is, the same region of the phase space that was found turbulent using either Kaneko's cluster distribution function $Q(k)$ [8], which is defined as the basin volume ratio for a k-cluster attractor (see below), or the mutual information between two randomly chosen elements of the system [7].

Let us recall that globally coupled maps are systems of N coupled maps. In this paper we will work with the logistic map

$$x_{n+1} = f_\mu(x_n) = 1 - \mu x_n^2$$

interacting with a sort of mean state of the system (mean field)

$$h_n = \frac{1}{N}\sum_{i=1}^{N} f_\mu(x_n^i) \tag{1}$$

by means of the "interaction parameter" ϵ, that is

$$x_{n+1}^i = (1-\epsilon) f_\mu(x_n^i) + \epsilon h_n \tag{2}$$

This apparently simple system has indeed a very complicated dynamics with a phase space (where $0 \le \epsilon \le 0.4$ and $1.4 \le \mu \le 2$, though for $\epsilon > 0.4$ interesting phenomena can be observed [9]) displaying turbulent, ordered and even glassy behaviour. These diverse dynamics are due to the interplay between μ and ϵ, that is, to the interaction of the tendency of the system to disorder because of individual chaotic behaviour with the tendency to synchronize due to global averaging. After the GCM falls on an attractor the N elements of the system split into k clusters. This allows one to define a cluster distribution function $Q(k)$ and to characterize the different phases (see [8] for details).

It is also known that h_n behaves, below some (large) N_c and in the turbulent region, as a noise with a distribution $P(h)$ close to a Gaussian form. This is not surprising, since h_n is a sum of values that fluctuate randomly and (almost) independently. However, above

N_c one can observe a deviation from the law of large numbers, which is due probably to a hidden coherence [10]. Some analysis based on the Perron-Frobenius operator have questioned this discrepance [11] by using the tent map instead of the logistic one (which is non-mixing).

Kaneko [8] reported that at the turbulent phase all attractors have many ($\simeq N$) clusters, and the following condition

$$\sum_{k>\frac{N}{2}} Q(k) = 1$$

holds. Specifically, the average cluster number $R = \sum_{k=1}^{N} kQ(k)$ gives $R \simeq N$ in the turbulent phase, so that it would be reasonable to expect that any (randomly chosen) element of a GCM in the turbulent phase would not be computationally different from a logistic map perturbed by a noise with a distribution function $P(h)$ (see below).

In order to get a computational measure of an individual belonging to a certain GCM, we must get a long enough orbit of that individual and discretize it with the generating partition [5]

$$\Pi = \{x_i^j \in [-1, 0) \Rightarrow S_i^j = 0; x_i^j \in [0, 1] \Rightarrow S_i^j = 1\}$$

where x_i^j is the state of the j-th individual at the time step i and $S_1^j S_2^j ...$ will be the discretized orbit of the j-th individual. It is possible to build a stochastic finite automaton, from this bit sequence, that will give us the computational counterpart of the original dynamics and its associated *statistical complexity* C, that is, roughly, the Shannon entropy of the stationary probability distribution of the stochastic automaton, if viewed as a Markov chain (see [5] for details). C will quantify the above mentioned *intrinsic computation*, provided that the automaton was a feasible model of the dynamical behaviour of the i-th individual [1].

Now, for a given GCM, that is, given N, ϵ and μ we can choose randomly an individual from the GCM and, after a long enough transient, get a bit sequence (using the partition Π) with which to compute its complexity C_{gcm}. Besides, we can easily obtain a histogram of h_n (given a partition of the interval $[-1, 1]$ with some resolution Δx, see fig. 1), that is, an approximation of the distribution $P(h)$, with which we build the following noisy logistic map

$$x_{n+1} = (1 - \epsilon) f_\mu(x_n) + \epsilon \xi_n \tag{3}$$

where ξ_n are independent and identically distributed random variables (with distribution $P(\xi)$). From 3 we generate another bit sequence and compute its associated complexity C_ξ.

Once we have C_{gcm} and C_ξ we can argue as follows: As we have seen above, h_n behaves as a noise term in the turbulent region, so that we should expect $C_{gcm} = C_\xi$. However, in the ordered region, the system splits in a number of synchronized clusters where each cluster displays quite ordered behaviour [8], therefore one should observe the following inequality $C_{gcm} < C_\xi$, since a noisy perturbation such as that of 3 is expected to cause certain disorder on the deterministic dynamics, though not enough to "mask" the deterministic contribution. In fact, $P(h)$ will be, in the ordered phase, a discrete distribution with $P(h^*) > 0$ for a finite set of h^* values (fig. 1, B). So

- Turbulent GCM $\Rightarrow C_{gcm} = C_\xi$
- Ordered GCM $\Rightarrow C_{gcm} < C_\xi$

We have verified numerically our hypothesis using a "grid" over the phase space $0 \leq \epsilon \leq 0.2$ and $1.4 \leq \mu \leq 2$ with $\Delta\epsilon = 0.02$ and $\Delta\mu = 0.06$. We have computed C_{gcm} and C_ξ for each (μ, ϵ) (both complexities were computed with the ϵ-machine reconstruction algorithm [5] from a bit sequence of length 10^6) and we have classified the point as "ordered" or "turbulent" according to the relation between the two complexities, as stated above (we have not found any (μ, ϵ) pair in which $C_{gcm} > C_\xi$), see figs 2 and 3. Our resulting phase space (see fig. 4) is identical to that found by Kaneko (see also [7]) using the cluster distribution function $Q(k)$, but a small domain of ordered behaviour has been found inside the turbulent phase.

To sum up, in this paper we have explored the logistic GCM from a computational mechanics point of view, by means of a generating partition Π and the corresponding binary time series. We have particularly analyzed the turbulent phase, where nonstatistical behavior due to statistical dependence of different lattice sites has been reported. It is not difficult to show that the fluctuations of the field h are given by [12]

$$\begin{aligned} \left\langle (h - \langle h \rangle)^2 \right\rangle &= \left\langle \left\{ \frac{1}{N} \sum_{i=1}^{N} \left[f_\mu(x^i) - \left\langle f_\mu(x^i) \right\rangle \right] \right\}^2 \right\rangle \\ &= \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N^2} \sum_{i \neq j} \alpha_{ij} \end{aligned} \quad (4)$$

where σ_i^2 is the variance for the single map $f_\mu(x^i)$ and α_{ij} is the covariance between $f_\mu(x^i)$ and $f_\mu(x^j)$. The non-statistical behaviour comes from the last term in the right hand side, which is zero only in some cases (such as the tent map [11,12]). Clearly there is a relevant difference between both systems: for the single noisy map the random field ξ obtained by randomly sampling $P(h)$ is uncorrelated in time and this is not the case in the GCM. So we might suspect that some relevant structure should be present at the turbulent phase. However, in terms of computational mechanics, no real difference arises between the GCM at the turbulent phase and a single map with a noise term with the same statistical structure. In spite of the presence of a hidden coherence in the GCM, this underlying structure does not contribute to the statistical complexity (approximated with the reconstructed ϵ-machines), implying that the system lacks any information processing capability [2] beyond the "trivial" one associated to the noisy map.

In this context, it has been suggested that GCMs are a formal model of neural-network-like structures [8]. Real neural ensembles do show complex dynamics and spatiotemporal patterns [13] and it has been conjectured that chaos would play a key role in brain dynamics. If the GCM analogy is appropriate the present study suggests that neural networks should avoid highly-dimensional chaotic phases in order to retain their basic information-processing features, which would not be supported by the hidden coherence.

Acknowledgments

This work was started during a research visit at the Santa Fe Institute (July 1997). The authors thank Jim Crutchfield and Brian Goodwin for several useful discussions. Besides, Jim Crutchfield had the original idea of comparing the individual inside the collective and the noisy individual, as we have done in this paper. We also thank our friends at the CSRG: Susanna C. Manrubia, Bartolo Luque and Jordi Bascompte. This work has been supported by a grant DGYCIT PB94-1195 and the Santa Fe Institute.

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Figure Captions

[**1**] Probability distributions $P(h)$ for a GCM in the turbulent phase (A, $\mu = 1.76$, $\epsilon = 0.04$) and in the ordered phase (B, $\mu = 1.58$, $\epsilon = 0.08$). Insets: in each case we show the dynamics of a single element in the GCM (above) and the corresponding single noisy logistic map (below, see text).

[**2**] C_{gcm} and C_ξ computed for two (μ, ϵ) pairs belonging to the turbulent phase, according to [8]. As we can see, $C_{gcm} \simeq C_\xi$.

[**3**] C_{gcm} and C_ξ computed for two (μ, ϵ) pairs belonging to the ordered phase [8]. In these cases $C_{gcm} < C_\xi$.

[**4**]. Phase space of GCM ($N = 500$). C_{gcm} and C_ξ were computed for 90 (μ, ϵ)-pairs (44 belonging to Kaneko's turbulent phase, 46 to the ordered phase) and classified according to either $C_{gcm} = C_\xi$ ("turbulent") or $C_{gcm} < C_\xi$ ("ordered"). The resulting phase space is almost identical to that computed by Kaneko [8] (see text). C_{gcm} was computed choosing at random one element of the GCM (after a transient of 10^4 steps).

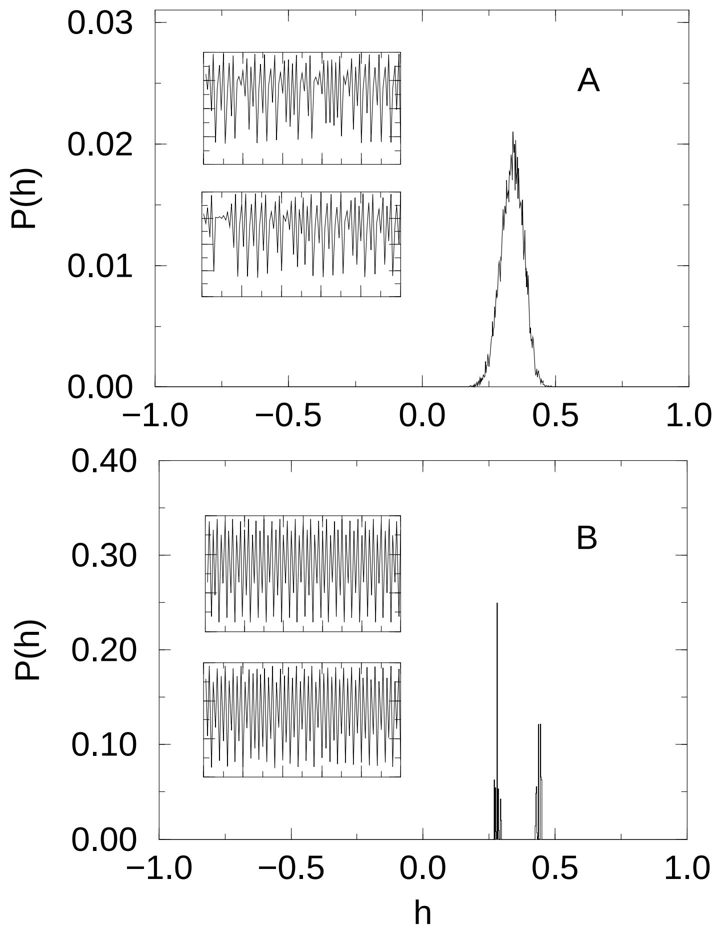
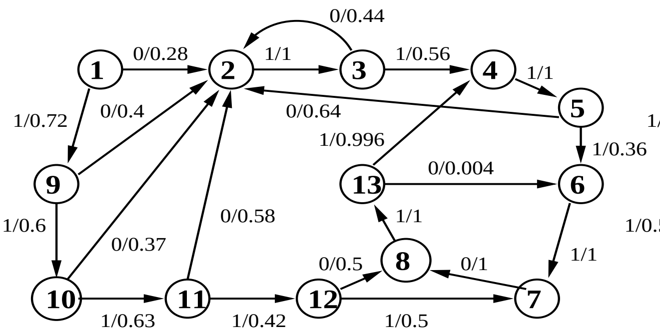
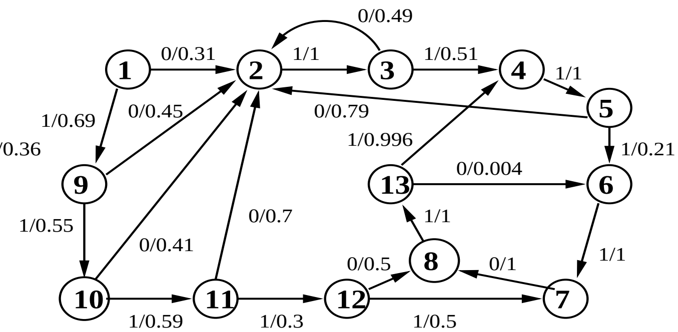


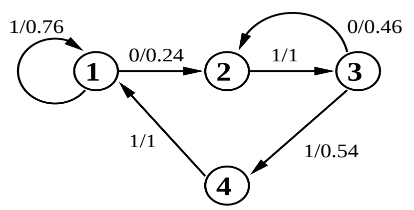
Figure 1:



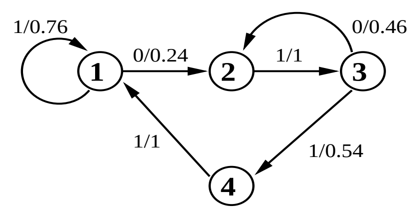
Globally Coupled Map
$N = 500$
C = 2.6 $\mu = 1.5436$
$\varepsilon = 0.025$



Logistic map with noise
$N = 500$
C = 2.61 $\mu = 1.5436$
$\varepsilon = 0.025$



Globally Coupled Map
$N = 500$
C = 1.82 $\mu = 1.85$
$\varepsilon = 0.1$



Logistic map with noise
$N = 500$
C = 1.82 $\mu = 1.85$
$\varepsilon = 0.1$

Figure 2:

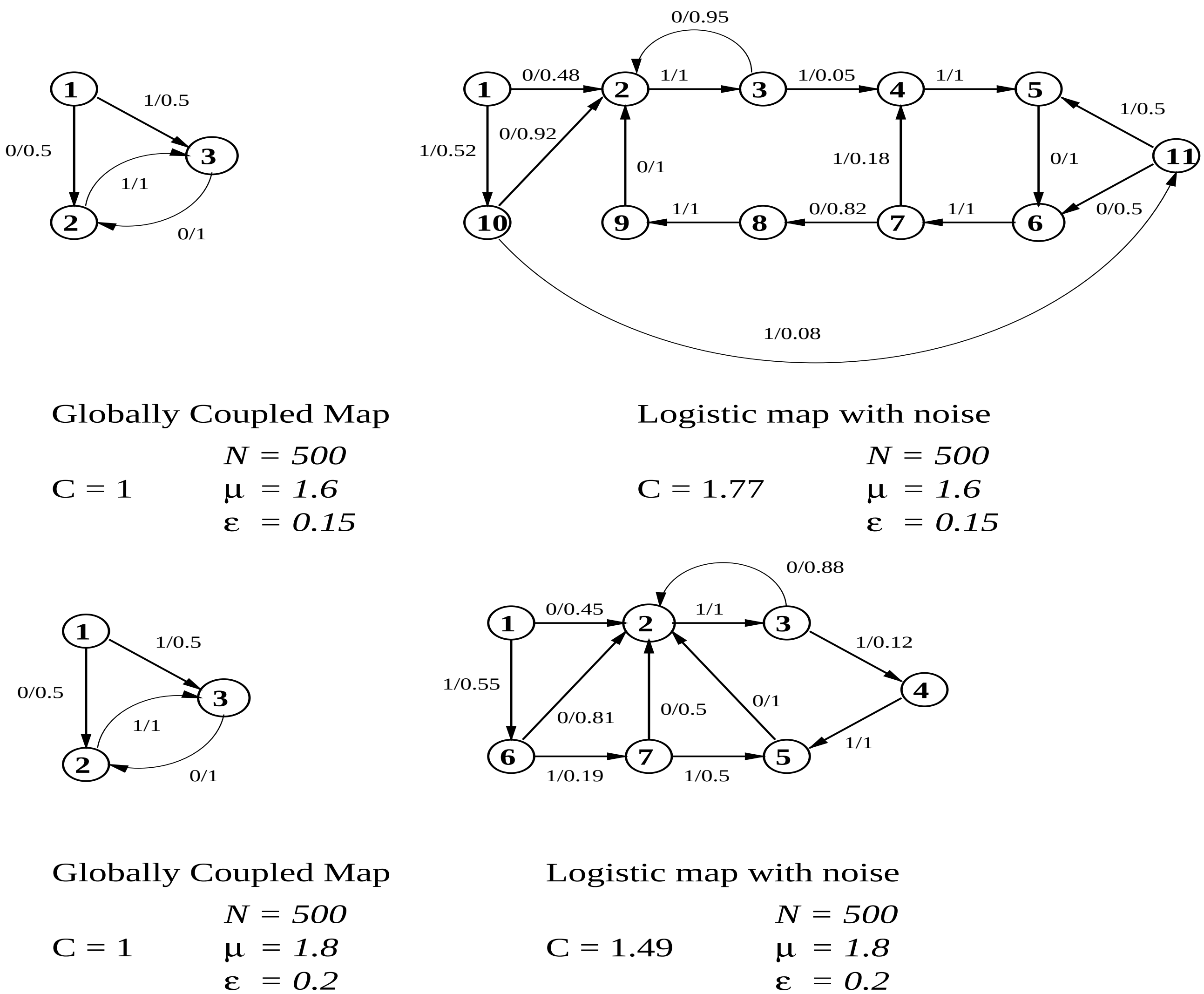

1/0.5
0/0.5
1/1
0/1
0/0.95
0/0.48
1/1
1/0.05
1/1
1/0.5
1/0.52
0/0.92
0/1
1/0.18
0/1
1/1
0/0.82
1/1
0/0.5
1/0.08
Globally Coupled Map
N = 500
C = 1
μ = 1.6
ε = 0.15
Logistic map with noise
N = 500
C = 1.77
μ = 1.6
ε = 0.15
1/0.5
0/0.5
1/1
0/1
0/0.88
0/0.45
1/1
1/0.12
1/0.55
0/0.81
0/0.5
0/1
1/1
1/0.19
1/0.5
Globally Coupled Map
N = 500
C = 1
μ = 1.8
ε = 0.2
Logistic map with noise
N = 500
C = 1.49
μ = 1.8
ε = 0.2

Figure 3:

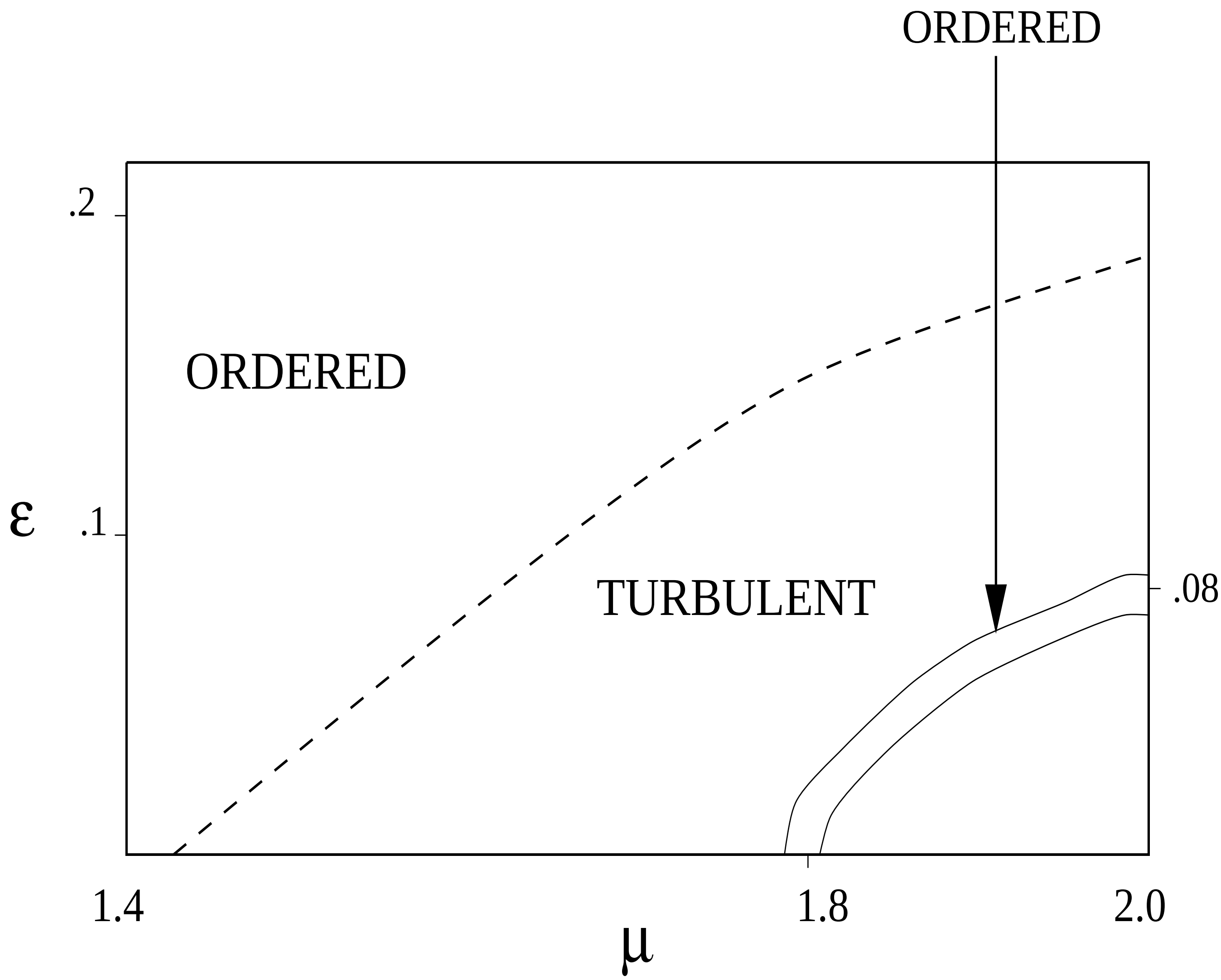


Figure 4: