Multi-agent linear systems with noise. Solving decoupling problem

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Abstract: Dynamical multi-agent systems are being extensively studied by researchers in the field of control theory. It is due to the multi-agents appear in different study subjects as for example in the consensus problem of communication networks, formation control of mobile robots or cooperative control of unmanned aerial vehicles. The disturbance decoupling problem for linear dynamical systems with noise was the starting point for the development of a geometric approach to systems theory. The problem consists in that the disturbance not interfere with the solution of the linear dynamical system; in other words, to find a compensator such that the closed loop transfer matrix from disturbance to output is 0. Several multiagents linear systems are affected by noises, nevertheless almost all the existing results in consensus problem, do not take into account the effects of these noises. The goal of this paper is to advance in the study of the consensus problems under noise disturbances using linear algebra techniques.

Key–Words: Multi-agent systems, consensus, controllability, observability, output-controllability, disturbance decoupling problem.

1 Introduction

It is well known the great interest created in many research communities about the study of control multi-agents system, as well as the increasing interest in distributed control and coordination of networks consisting of multiple autonomous (potentially mobile) agents. There are an amount of literature as for example [7], [15], [17], [19]. It is due to the multi-agents appear in different areas as for example in consensus problem of communication networks [16], formation control of mobile robots, or cooperative control of unmanned aerial vehicles. [2].

The disturbance decoupling problem for linear dynamical systems with noise was the starting point for the development of a geometric approach to systems theory. The problem consists in that the disturbance not interfere with the solution of the linear dynamical system. In other words, to find a compensator such that the closed loop transfer matrix from disturbance to output is 0. Several multiagents linear systems are affected by noises, nevertheless almost all the existing results in consensus problem, do not take into account the effects of these noises. The main goal of this work is to advance in the study of the consensus problems under noise disturbances. Wang Lin, Liu Zhixin and Guo Le in [11] initiate the study off disturbance for the special case of the class of multi-agent systems where the state matrices are nonnegative stochastic matrices.

Jinhuan Wang, Daizhan Cheng and Xiaoming Hu in [17] study the consensus problem in the case of multiagent systems in which all agents have an identical linear dynamics and it is an stable linear system. In [7], this result is generalized to the case where the dynamic of the agents are controllable.

Wei Ni and Daizhan Cheng in [13] analyze the case where $u_1 = 0$ this particular case has practical scenarios as the flight of groups of birds. It is obvious that in this case the mechanic of the first system is independent of the others, then consensus under a fixed topology can be easily obtained and it follows from the motion of the first equation. This consensus problem is known as leader-following consensus problem [13], [9].

In this paper multiagent systems consisting of k agents having identical linear dynamic mode, with dynamics

$$\left.\begin{aligned} \dot{x}^i &= Ax^i + Bu^i + H\delta^i \\ \dot{y}^i &= Cx^i \end{aligned}\right\} \quad i = 1, \ldots, k \qquad (1)$$

with noise and ee try to study whether it is possible to decouple the noise.

Disturbance decoupling problems have been studied for time invariant linear systems under a geometrical point of view by using the concepts of some particular invariant subspaces associated to the systems (see [6, 8, 12, 14, 20], for example).

2 Preliminaries

2.1 Topology of the system

We consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order k with the set of vertices $\mathcal{V} = \{1, \ldots, k\}$ and the set of edges $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

Given an edge (i, j) i is called the parent node and j is called the child node and j is in the neighbor of i, concretely we define the neighbor of i and we denote it by $\mathcal{N}_i$ to the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

The graph is called undirected if verifies that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph we consider a matrix $G = (g_{ij})$ called (unweighted) adjacency matrix defined as follows $g_{ii} = 0$, $g_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise.

In a more general case we can consider a weighted adjacency matrix is $G = (g_{ij})$ with $g_{ii} = 0$, $g_{ij} > 0$ if $(i, j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise).

The Laplacian matrix of the graph is

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Remark 1 *i) If the graph is undirected then the matrix $\mathcal{L}$ is symmetric, then there exist an orthogonal matrix P such that $P\mathcal{L}P^t = \mathcal{D}$.*

ii) If the graph is undirected then 0 is an eigenvalue of $\mathcal{L}$ and $(1, \ldots, 1)^t$ is the associated eigenvector.

iii) If the graph is undirected and connected the eigenvalue 0 is simple.

For more details about graph theory see [18].

2.2 Kronecker product

Given a couple o matrices $A = (a_{ij}) \in M_{n\times m}(\mathbb{C})$ and $B = (b_{ij}) \in M_{p\times q}(\mathbb{C})$, remember that the Kronecker product is defined as follows.

Definition 2 *Let $A = (a^i_j) \in M_{n\times m}(\mathbb{C})$ and $B \in M_{p\times q}(\mathbb{C})$ be two matrices, the Kronecker product of A and B, write $A \otimes B$, is the matrix*

$$A \otimes B = (a^i_j B) \in M_{np\times mq}(\mathbb{C})$$

Kronecker product verifies the following properties

1) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$

2) $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$

3) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

4) $(A \otimes B)^t = A^t \otimes B^t$

5) If $A \in Gl(n;\mathbb{C})$ and $B \in Gl(p;\mathbb{C})$), then $A \otimes B \in Gl(np;\mathbb{C})$) and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

6) If the products AC and BD are possible, then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Corollary 3 *The vector $\mathbf{1}_k \otimes v$ is an eigenvector corresponding to the zero eignevalue of $\mathcal{L} \otimes I_n$.*

Proof:

$$(\mathcal{L} \otimes I_n)(\mathbf{1}_k \otimes v) = \mathcal{L}\mathbf{1}_k \otimes v = 0 \otimes v = 0$$

□

Consequently, if $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{C}^n$, then $\mathbf{1}_k \otimes e_i$ is a basis for the nullspace of $\mathcal{L} \otimes I_n$.

Associated to the Kronecker product, can be defined the vectorizing operator that transforms any matrix A into a column vector, by placing the columns in the matrix one after another,

Definition 4 *Let $X = (x^i_j) \in M_{n\times m}(\mathbb{C})$ be a matrix, and we denote $x_i = (x^1_i, \ldots, x^n_i)^t$ for $1 \leq i \leq m$ the i-th column of the matrix X. We define the vectorizing operator vec, as*

$$\begin{aligned} vec : M_{n\times m}(\mathbb{C}) &\longrightarrow M_{nm\times 1}(\mathbb{C}) \\ X &\longrightarrow \begin{pmatrix} x_1 & x_2 & \ldots & x_m \end{pmatrix}^t \end{aligned}$$

Obviously, vec is an isomorphism.

See [10] for more information and properties.

2.3 Control Properties

Definition 5 *The dynamical system $\dot{x} = Ax + Bu$ is said to be controllable if for every initial condition $x(0)$ and every vector $x_1 \in \mathbb{R}^n$, there exist a finite time t_1 and control $u(t) \in \mathbb{R}^m$, $t \in [0, t_1]$, such that $x(t_1) = x_1$.*

This definition requires only that any initial state $x(0)$ can be steered to any final state x_1 at time t_1. However, the trajectory of the dynamical system between

0 and t_1 is not specified. Furthermore, there is no constraints posed on the control vector $u(t)$ and the state vector $x(t)$.

It is easier to compute the controllability using the following matrix

$$\mathcal{C} = \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}. \quad (3)$$

called controllability matrix, thanks to the following well-known result.

Theorem 6 *The dynamical system $\dot{x} = Ax + Bu$ is controllable if and only if* $\operatorname{rank} \mathcal{C} = n$.

As we says, controllability of the dynamical system $\dot{x} = Ax + Bu$ implies that each initial state can be steered to 0 on a finite time-interval. If only is required that this to happen asymptotically for $t \to \infty$, we have the following concept.

Definition 7 *The system $\dot{x} = Ax + Bu$ is called stabilizable if for each initial state $x(0) \in \mathrm{IR}^n$ there exists a (piece-wise continuous) control input $u : [0, \infty) \longrightarrow \mathrm{IR}^m$ such that the state-response with $x(0)$ verifies*

$$\lim_{t\to\infty} x(t) = 0.$$

Remark 8 *i) All controllable systems are stabilizable but the converse is false.*

ii) If the matrix A in the system $\dot{x} = Ax + Bu$ is Hurwitz then, the system is stabilizable.

It is important the following result

Theorem 9 *The system $\dot{x} = Ax + Bu$ is stabilizable if and only if there exists some feedback F such that $\dot{x} = (A - BF)x$ is asymptotically stable.*

A dual concept of controllability is the observability.

Definition 10 *The dynamical system $\dot{x} = Ax + Bu, y = Cx$ is said to be observable at t_0 if there exist a finite time $t_1 > t_0$ such that for any vector $x_0 \in \mathrm{IR}^n$, at time t_0 the knowledge of the control $u(t) \in \mathrm{IR}^m$, $t \in [t_0, t_1]$, and the output y_t over the time $[t_0, t_1]$ suffices to determine the state x_0.*

It is easier to compute the observability using the following matrix

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}. \quad (4)$$

called observability matrix, thanks to the following well-known result.

Theorem 11 *The dynamical system $\dot{x} = Ax + Bu$, $y = Cx$ is observable if and only if* $\operatorname{rank} \mathcal{O} = n$.

Another important property is the output-controllability that generally means, that the system can steer the output of dynamical system independently of its state vector.

Definition 12 *A system is functional output-controllable if and only if its output can be steered along the arbitrary given curve over any interval of time. It means that if it is given any output $y_d(t)$, $t \geq 0$, there exists t_1 and a control u_t, $t \geq 0$, such that for any $t \geq t_1$, $y(t) = y_d(t)$.*

Proposition 13 *[1, 4]*

$$\operatorname{rank} \begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix} = n + p,$$

as a polynomial matrix.

Proposition 14 *The functional output-controllability character is invariant under feedback and output injection.*

Proof.

$$\operatorname{rank} \begin{pmatrix} sI - (A + BK + WC) & B \\ C & 0 \end{pmatrix} =$$
$$\operatorname{rank} \begin{pmatrix} I_n & W \\ 0 & I_p \end{pmatrix} \begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ K & I_m \end{pmatrix} =$$
$$\operatorname{rank} \begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix}$$

□

The functional output-controllability can be computed by means of the rank of a constant matrix in the following manner

Theorem 15 ([3]) *The system (A, B, C) is functional output-controllable if and only if*

$$\operatorname{rank} oC_{\mathrm{f}}(A, B, C) =$$
$$\operatorname{rank} \begin{pmatrix} C & & & & \\ CA & CB & & & \\ CA^2 & CAB & CB & & \\ \vdots & & \ddots & & \\ CA^n & CA^{n-1}B & \dots & CAB & CB \end{pmatrix}$$
$$= (n+1)p$$

The null terms are not written in the matrix.

3 Consensus

Roughly speaking, we can define the consensus as a collection of processes such that each process starts with an initial value, where each one is supposed to output the same value and there is a validity condition that relates outputs to inputs. More concretely, the consensus problem is a canonical problem that appears in the coordination of multi-agent systems. The objective is that Given initial values (scalar or vector) of agents, establish conditions under which through local interactions and computations, agents asymptotically agree upon a common value, that is to say: to reach a consensus.

Definition 16 *Consider the system 1. We say that the consensus is achieved using local information if there is a state feedback and an output injection*

$$\begin{array}{ll} x^i & = Ax^i + WC\sum_{j\in\mathcal{N}_i}(x^i - x^j) + Bu^i \\ u^i & = K\sum_{j\in\mathcal{N}_i}(x^i - x^j) + E\delta^i,\ 1 \leq i \leq k \end{array}$$

such that

$$\lim_{t\to\infty} \|x^i - x^j\| = 0,\ 1 \leq i, j \leq k.$$

The closed-loop system obtained under this feedback is as follows

$$\left.\begin{array}{ll} \dot{\mathcal{X}} & = \mathcal{A}\mathcal{X} + (\mathcal{B}\mathcal{K} + \mathcal{W}\mathcal{C})\mathcal{Z} + \mathcal{H}\Delta, \\ \mathcal{Y} & = \mathcal{C}\mathcal{X} \end{array}\right\}$$

where

$$\mathcal{X} = \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix},\quad \dot{\mathcal{X}} = \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^k \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix},\ \mathcal{B} = \begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} C & & \\ & \ddots & \\ & & C \end{pmatrix},\ \mathcal{W} = \begin{pmatrix} W & & \\ & \ddots & \\ & & W \end{pmatrix},$$

$$\mathcal{K} = \begin{pmatrix} K & & \\ & \ddots & \\ & & K \end{pmatrix},\ \mathcal{Z} = \begin{pmatrix} \sum_{j\in\mathcal{N}_1} x^1 - x^j \\ \vdots \\ \sum_{j\in\mathcal{N}_k} x^k - x^j \end{pmatrix}.$$

$$\mathcal{H} = \begin{pmatrix} H & & \\ & \ddots & \\ & & H \end{pmatrix},\ \Delta = \begin{pmatrix} \delta^1 & & \\ & \ddots & \\ & & \delta^k \end{pmatrix},$$

Following this notation we can conclude the following.

Proposition 17 ([17]) *The closed-loop system can be described as*

$$\begin{array}{ll} \dot{\mathcal{X}} & = ((I_k \otimes A) + (I_k \otimes (BK + WC))(\mathcal{L} \otimes I_n))\mathcal{X} \\ & +(I_k \otimes H)\Delta \\ \mathcal{Y} & = \mathcal{C}\mathcal{X} \end{array} \tag{5}$$

Assuming $\mathcal{X}(0) = 0$, the equation 5 can be solved as

$$\begin{array}{l} \mathcal{Y}(t) = \\ \int_0^t \mathcal{C}e^{((I_k\otimes A)+(I_k\otimes(BK+WC))(\mathcal{L}\otimes I_n))(t-s)}\mathcal{X}(s)ds \\ +\int_0^t \mathcal{C}e^{((I_k\otimes A)+(I_k\otimes(BK+WC))(\mathcal{L}\otimes I_n))(t-s)}\mathcal{H}\Delta(s)ds. \end{array} \tag{6}$$

3.1 Disturbance decoupling problem

The requirement for the noise to be decoupled is that the last term in 6 be zero for any Δ. Or equivalently, in any derivative $\mathcal{Y}^{(i)}$, $i = 1, 2, \ldots$, Δ should not appear.

$$\begin{array}{l} \mathcal{Y}^{(1)} = \mathcal{C}\dot{\mathcal{X}} = \\ \mathcal{C}((I_k \otimes A) + (I_k \otimes (BK + WC))(\mathcal{L} \otimes I_n))\mathcal{X} \\ +\mathcal{C}(I_k \otimes H)\Delta \end{array}$$

we must have $\mathcal{C}(I_k \otimes H) = 0$, equivalently $CH = 0$, that is to say $\operatorname{Im} H \subset \operatorname{Ker} C$.

Inductively, and calling

$$M = (I_k \otimes A) + (I_k \otimes (BK + WC))(\mathcal{L} \otimes I_n)$$

$$\mathcal{Y}^{(i)} = \mathcal{C}M^i\mathcal{X} + \mathcal{C}M^{i-1}(I_k \otimes H)\Delta$$

we must have

$$\mathcal{C}M^{i-1}(I_k \otimes H) = 0. \tag{7}$$

which is equivalent to

$$\begin{array}{l} S = \langle M \mid \operatorname{Im}\mathcal{H}\rangle = \\ \operatorname{Im}\mathcal{H} + M\operatorname{Im}\mathcal{H} + \ldots + M^{nk-1}\operatorname{Im}\mathcal{H} \subseteq \operatorname{Ker}\mathcal{C} \end{array}$$

Cayley-Hamilton theorem ensures that the space S is M-invariant and permit us to conclude

Theorem 18 *The system 5 is disturbance decoupled if and only if*

$$Im\,\mathcal{H} \subseteq S_{max}(Ker\,\mathcal{C}),$$

where $S_{max}(Ker\,\mathcal{C})$ is the maximal M-invariant subspace contained in $Ker\,\mathcal{C}$.

One way to get the subspace $S_{\max}(\operatorname{Ker}\mathcal{C})$ is as follows.

$$\begin{aligned} W_0 &= \operatorname{Ker}\mathcal{C} \\ W_{\ell+1} &= \operatorname{Ker}\mathcal{C} \cap \{x \in \mathbb{C}^{nk} \mid Mx \in W_\ell\} \end{aligned}$$

Limit of recursion exists and we will denote by $S_{\max}(\operatorname{Ker}\mathcal{C})$. This subspace is the supremal M-invariant subspace contained in $\operatorname{Ker}\mathcal{C}$.

In order to obtain conditions for 7 directly from the linear mode of the agent and taking into account that the graph is undirected, following remark 1, we have that there exists an orthogonal matrix $P \in Gl(k;\mathbb{R})$ such that $P\mathcal{L}P^t = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$, $(\lambda_1 \geq \ldots \geq \lambda_k)$.

Corollary 19 *The closed-loop system can be described in terms of the matrices A, B, the feedback K and the eigenvalues of $\mathcal{L}$ in the following manner*

$$\begin{aligned} \dot{\widehat{\mathcal{X}}} &= \begin{pmatrix} A+\lambda_1(BK+WC) & & \\ & \ddots & \\ & & A+\lambda_k(BK+WC) \end{pmatrix} \widehat{X} \\ &+ \begin{pmatrix} H & & \\ & \ddots & \\ & & H \end{pmatrix} \widehat{\Delta} \\ \mathcal{Y} &= \mathcal{C}(P^t \otimes I_n)\widehat{X}. \end{aligned} \tag{8}$$

Proof:

$$\begin{aligned} &(I_k \otimes BK)(\mathcal{L} \otimes I_n) = \\ &(I_k \otimes (BK+WC))(P^tDP \otimes I_n) = \\ &(I_k \otimes (BK+WC))(P^t \otimes I_n)(D \otimes I_n)(P \otimes I_n) = \\ &\quad (P^t \otimes (BK+WC))(D \otimes I_n)(P \otimes I_n) = \\ &(P^t \otimes I_n)(I_k \otimes (BK+WC))(D \otimes I_n)(P \otimes I_n) = \\ &\quad (P^t \otimes I_n)(D \otimes (BK+WC))(P \otimes I_n) \end{aligned}$$

$$\begin{aligned} (I_k \otimes A) &= (P^t \otimes I_n)(I_k \otimes A)(P \otimes I_n) \\ (I_k \otimes H) &= (P^t \otimes I_n)(I_k \otimes H)(P \otimes I_n) \end{aligned}$$

Then,

$$\begin{aligned} &(P^t \otimes I_n)(P \otimes I_n)\dot{\mathcal{X}} = \\ &(P^t \otimes I_n)(I_k \otimes A)(P \otimes I_n)\mathcal{X} + \\ &\quad (P^t \otimes I_n)(D \otimes (BK+WC))(P \otimes I_n)\mathcal{X} \end{aligned}$$

so,

$$\begin{aligned} &(P \otimes I_n)\dot{\mathcal{X}} = (I_k \otimes A)(P \otimes I_n)\mathcal{X} + \\ &(D \otimes (BK+WC))(P \otimes I_n)\mathcal{X} + \\ &(I_k \otimes H)(P \otimes I_n)\Delta \end{aligned}$$

and calling $(P \otimes I_n)\mathcal{X} = \widehat{\mathcal{X}}$ and $(P \otimes I_n)\Delta = \widehat{\Delta}$ we have the result. □

Now, calling

$$\widehat{M} = \begin{pmatrix} A+\lambda_1(BK+WC) & & \\ & \ddots & \\ & & A+\lambda_k(BK+WC) \end{pmatrix},$$

for the system to be disturbance decoupled, we must have

$$\mathcal{C}(P^t \otimes I_n)\widehat{M}^{i-1}\mathcal{H} = 0$$

Proposition 20 *A necessary condition to ensure that the system 6 is disturbance decoupled is*

$$C(A + \lambda_j(BK+WC))^i H = 0.$$

Proof:

First of all we observe that $\mathcal{C}(P^t \otimes I_n) = (P^t \otimes C)$.

Calling $A_i = A + \lambda_i(BK+WC)$ and denoting $P^t = (p_{ji})$ we have that

$$\begin{aligned} &\mathcal{C}(P^t \otimes I_n)\widehat{M}^{i-1}\mathcal{H} = \\ &\begin{pmatrix} p_{11}C & \ldots & p_{k1}C \\ \vdots & & \vdots \\ p_{1k} & & p_{kk} \end{pmatrix} \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix} \begin{pmatrix} H & & \\ & \ddots & \\ & & H \end{pmatrix} \\ &= \begin{pmatrix} p_{11}C(A_1)^iH & \ldots & p_{k1}C(A_k)^iH \\ \vdots & & \vdots \\ p_{1k}C(A_1)^iH & \ldots & p_{kk}C(A_k)^iH \end{pmatrix} \\ &= 0. \end{aligned}$$

Finally, it is sufficient to observe that in the same column of the matrix

$$\begin{pmatrix} p_{11}C(A_1)^iH & \ldots & p_{k1}C(A_k)^iH \\ \vdots & & \vdots \\ p_{1k}C(A_1)^iH & \ldots & p_{kk}C(A_k)^iH \end{pmatrix}$$

not all p_{ij} may be zero because the matrix P is invertible. □

Taking into account that $\lambda_1 = 0$ we have that

Corollary 21 *A necessary condition to ensure that the system 6 is disturbance decoupled is*

$$CA^iH = 0.$$

Equivalently

Proposition 22 *A necessary condition to ensure that the system 6 is disturbance decoupled is*

$$\mathit{Im}\, H \subset \mathit{Ker}\, \mathcal{O}(A, C).$$

Corollary 23 *A necessary condition to ensure that the system 6 is disturbance decoupled is*

$$\langle A \mid H\rangle \subset C.$$

Proposition 24 *A necessary condition to ensure that the system 6 is disturbance decoupled is*

$$\begin{pmatrix} C & & & & \\ CA & CB & & & \\ CA^2 & CAB & CB & & \\ \vdots & & \ddots & & \\ CA^n & CA^{n-1}B & \dots & CAB & CB \end{pmatrix}\begin{pmatrix} H \\ \lambda_i KH \\ \vdots \\ \lambda_i KA^nH \end{pmatrix} = 0$$

That is to say

$$\mathrm{Im}\begin{pmatrix} H \\ \lambda_i KH \\ \vdots \\ \lambda_i KA^nH \end{pmatrix} \subseteq \mathrm{Ker}\, oC_f(A,B,C).$$

Proof:

From $C(A+\lambda_j(BK+WC))H = 0$ $CAH + C\lambda_j BKH + C\lambda_j CWCH = 0$ and taking into account 21 we have

$$\begin{pmatrix} C & \\ CA & CB \end{pmatrix}\begin{pmatrix} H \\ \lambda_j KH \end{pmatrix} = 0$$

Following inductively we get the result. □

3.1.1 Disturbance decoupling under feedback

The problem now is to find a state feedbacks K, $\overline{K}$ and an output injection

$$\begin{aligned} x^i &= Ax^i + WC\textstyle\sum_{j\in\mathcal{N}_i}(x^i - x^j) + Bu^i \\ u^i &= K\textstyle\sum_{j\in\mathcal{N}_i}(x^i - x^j) + \overline{K}x^i + E\delta^i,\ 1\leq i\leq k \end{aligned}$$

such that

$$\lim_{t\to\infty}\|x^i - x^j\| = 0,\ 1\leq i,j\leq k.$$

and

$$\begin{pmatrix} C & & & & \\ C\overline{A} & CB & & & \\ C\overline{A}^2 & C\overline{A}B & CB & & \\ \vdots & & \ddots & & \\ C\overline{A}^n & C\overline{A}^{n-1}B & \dots & C\overline{A}B & CB \end{pmatrix}\begin{pmatrix} H \\ \lambda_i KH \\ \vdots \\ \lambda_i KA^nH \end{pmatrix} = 0$$

where $\overline{A} = A + B\overline{K}$

Remark 25 *rank* $oC_f(A,B,C)$ *is invariant under feedback and output injection.*

4 Solving consensus problem

Let us consider a group of k identical agents. The dynamic of each agent is given by the following linear dynamical systems

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ &\vdots \\ \dot{x}^k &= Ax^k + Bu^k \end{aligned} \tag{9}$$

with the topology defined in 2.1. (In this case $y = x$ and no noise is considered.

It would seem that if the graph is connected the consensus problem would be solvable of there is a K such that the system 8 is stabilized. But taking into account that $\lambda_1 = 0$ this system is only stabilized if $\dot{x}^1 = Ax^1$ is stable.

Suppose now, that the system (A,B) is controllable, so there exist $\overline{K}$ such that the close loop system $\dot{x} = (A + B\overline{K})x = \overline{A}x$ is asymptotically stable.

$$\begin{aligned} \dot{x}^1 &= \overline{A}x^1 + Bu^1 \\ &\vdots \\ \dot{x}^k &= \overline{A}x^k + Bu^k, \end{aligned} \tag{10}$$

$x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$, $1\leq i\leq k$.

Lemma 26 ([17]) *Let (A,B) be a controllable pair of matrices and we consider the set of k-linear systems*

$$\dot{x}^i = Ax^i + \lambda_i Bu^i,\ 1\leq i\leq k$$

with $\lambda_i > 0$. Then, there exist a feedback K which simultaneously assigns the eigenvalues of the systems as negative as possible.

More concretely, for any $M > 0$, there exist $u^i = Kx^i$ for $1\leq i\leq k$ such that

$$Re\,\sigma(A + \lambda_i BK) < -M,\ 1\leq i\leq k.$$

($\sigma(A+\lambda_i BK)$ denotes de spectrum of $A+\lambda_i BK$ for each $1\leq i\leq k$).

As a corollary, we can consider the consensus problem.

Corollary 27 ([7]) *If (A,B) is a controllable pair then, the consensus is achieved by means the feedback of lemma 26 and a feedback $\overline{K}$ stabilizing (A,B).*

Proof:

Taking into account that the adjacent topology is connected we have that $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k$ and $(1,\ldots,1)^t = \mathbf{1}_k$ is the eigenvector corresponding to the simple eigenvalue $\lambda_1 = 0$.

On the other hand we can find $\overline{K}$ stabilizing (A, B) and then we can find K stabilizing the associate system 8, then we find $\mathcal{Z}$ such that $\lim_{t\to\infty} \mathcal{Z} = 0$.

Using $\mathcal{Z} = (\mathcal{L} \otimes I_n)\mathcal{X}$ we have that $\lim_{t\to\infty} \mathcal{X} = \mathbf{1}_k \otimes v$ for some vector $v \in \mathbb{R}^n$ and the consensus is obtained.

Example 1.

We consider 3 identical agents with the following dynamics of each agent

$$\begin{aligned} \dot{x}^1 &= Ax^1 + Bu^1 \\ \dot{x}^2 &= Ax^2 + Bu^2 \\ \dot{x}^3 &= Ax^3 + Bu^3 \end{aligned} \tag{11}$$

with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Where the communication topology is defined by the graph $(\mathcal{V}, \mathcal{E})$:

$\mathcal{V} = \{1, 2, 3\}$

$\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}\} = \{(1, 2), (1, 3)\} \subset \mathcal{V}\times\mathcal{V}$

Taking $\overline{K} = (-0.1\ -0.5)$, and $K = -(0.5\ -0.2)$ the eigenvalues are

$\lambda_1 = -0.2500 + 0.1936i$,
$\lambda_2 = -0.2500 - 0.1936i$,
$\lambda_3 = -0.3500 + 0.6910i$,
$\lambda_4 = -0.3500 - 0.6910i$,
$\lambda_5 = -0.5500 + 1.1391i$,
$\lambda_6 = -0.5500 - 1.1391i$,

so, the system has been stabilized.

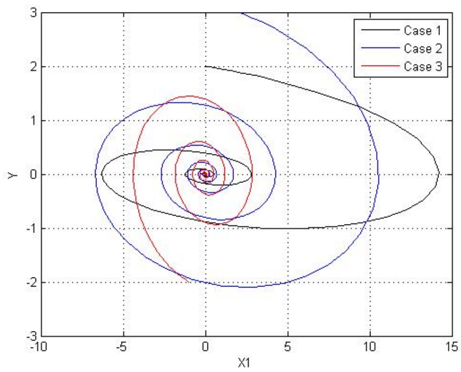

Figure 1. Trajectories

The graphic shows that the three trajectories arrive at a common point.

5 Conclusions

In this paper the disturbance decoupling problem for multi-agent systems having identical dynamical mode has been considered. Necessary conditions ensuring that the system achieving consensus is disturbance decoupled.

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