Structural controllability and observability of switched linear systems

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Abstract: In the last years, the dynamic control of switched systems have attracted considerable interest, because these systems arise in many engineering fields. Controllability and observability of such systems, which are fundamental qualitative properties, have been studied by various researchers. In this work we focus on structural controllability/observability, using geometrical techniques as miniversal deformations.

Key–Words: Switched systems, structural controllability, structural observability, miniversal deformations.

1 Introduction

Switched systems constitute a particular kind of hybrid systems which have been studied with increasing interest because of the great number of areas from which they arise (electric, mechanic, robotic and hydraulic systems or communication networks and chemical processes, for example) and their interesting properties.

Roughly speaking, a switched system is a family of continuous-time (or discrete-time) dynamical subsystems and a rule that determines the switching between them.

In the last decade, many theoretical and numerical tools have been developed to study these systems: stability, controllability/observability, ... In particular, structural controllability/observability, which was first considered by Lin ([10]), in the case of standard linear time-invariant systems, have been studied in the case of switched systems in [2], [4], [6], [7], [8], [11], [12], [13], among others. In most cases, tools from graph theory are used.

In this work we study structural controllability/observability of switched linear systems using a different approach: miniversal deformations (versal deformations with minimum number of parameters), introduced in [1] in the case of square matrices. Versal deformations provide a parametrization of the space of matrices defining the family of subsystems and are an effective tool to do perturbation analysis. V.I. Arnold in [1] gave a method to find such deformations, since he proved (and the proof can be generalized to many cases) that versality is equivalent to transversality. The key point in order to apply this is to view a natural equivalence relation in the space of matrices as the one induced by the action of a Lie group.

The structure of the paper is as follows.

In Section 2, the basic definitions of switched linear systems are provided, and we recall the concepts of controllability/observability. Section 3 is devoted to recall the basic definitions of versal and miniversal deformations and compute them explicitly in our particular set-up. Section 3 is devoted to the application of miniversal deformations to the study of structural controllability/observability, illustrating this relationship with an example.

We will restrict ourselves to the case of a continuous system. The discrete case can be handled analogously.

Throughout the paper, $\mathbb{R}, \mathbb{C}$, will denote the fields of real and complex numbers, $M_{n\times m}(\mathbb{R})$ (respectively $M_{n\times m}(\mathbb{C})$) the set of matrices having n rows and m columns and entries in $\mathbb{R}$ (respectively in $\mathbb{C}$). In the case where $n = m$, we will simply write $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$. $Gl_n(\mathbb{R})$ (respectively $Gl_n(\mathbb{C})$) will denote the group of non-singular matrices in $M_n(\mathbb{R})$ (respectively in $M_n(\mathbb{C})$). Given $A \in M_{n\times m}(\mathbb{R})$, A^t will denote the transpose of A and given $A \in M_{n\times m}(\mathbb{C})$, A^* will denote the conjugate transpose of A.

2 Switched linear systems

As said in the Introduction, a switched system is a system which consists of several subsystems and a rule that orchestrates the switching between them. More concretely, its definition is as follows in the continuous case.

Definition 1 *A switched system is a system which consists of several subsystems and a and a piecewise constant map taking values into the index set* $M = \{1, ..., m\}$ *which indexes the different subsystems and determines the changes between them.*

$$\left.\begin{array}{ll}\dot{x}(t) & = A_\sigma x(t) + B_\sigma u(t) \\ y(t) & = C_\sigma x(t)\end{array}\right\} \tag{1}$$

where $A_\sigma \in M_n(\mathbb{R})$, $B_\sigma \in M_{n\times m}(\mathbb{R})$, $C_\sigma \in M_{p\times n}(\mathbb{R})$ *and* $\dot{x} = dx/dt$.

Given an initial time t_0, a *switching path* is a function of time $\theta : [t_0, T) \longrightarrow M$, $T > t_0$, $M = \{1, \ldots, \ell\}$.

A switching path θ is said to be *well-defined* on $[t_0, T)$ if it is defined in $[t_0, T)$ and for all $t \in [t_0, T)$, both $\lim_{s\to t^+} \theta(s)$ and $\lim_{s\to t^-} \theta(s)$ exist and the set

$$\left\{ t \in [t_0, T) \,\middle|\, \lim_{s\to t^+} \theta(s) \neq \lim_{s\to t^-} \theta(s) \right\}$$

is finite for any finite sub-interval of $[t_0, T)$ (in the case where $t = t_0$, we will consider $\lim_{s\to t_0^-} \theta(s) = \theta(t_0)$).

A well-defined switching path is uniquely determined by a switching sequence

$$\{([t_0, t_1), \sigma(t_0^+)), \ldots, ([t_\ell, t_{\ell+1}), \sigma(t_\ell^+))\}$$

being $\sigma(t) = \sigma(t_i^+) = \lim_{s\to t_i^+} \sigma(s)$ if $t \in [t_i, t_{i+1})$ for $0 \leq i \leq \ell$, $t_{\ell+1} = T$.

We will denote by $\mathcal{M}$ the set of all triples of matrices (A_i, B_i, C_i), $i \in \{1, ..., m\}$ defining the subsystems.

The knowledge of control properties of systems are of great importance. Among them, we have the concepts of controllability/observability and structural controllability/observability, as well as stability. In short, the controllability studies the possibility of steering the state from the input and the observability studies the possibility of estimating the state from the output. The set of reachable states (uncountable union of vector subspaces) of a switched linear system can be found in [14]. Dualizing this, we can characterize observability. Both properties (controllability and observability) can be determined by the ranks of the system matrices.

A state x is said to be controllable at time t_0, if it can be transferred to the origin in a finite time starting from t_0 by appropriate choices of input u and switching path σ.

Definition 2 *The state* $x \in \mathbb{R}^n$ *is controllable at the time* t_0*, if there exist a time instant* $t_f > t_0$*, a switching path* $\sigma : [t_0, t_f] \longrightarrow M$*, and inputs* $u_k : [t_0, t_f] \longrightarrow \mathbb{R}^{r_k}$*,* $k \in M$*, such that* $x(t_f, t_0, x, u, \sigma) = 0$.

The controllable set of system 1 at t_0 is the set of states which are controllable at t_0.

Definition 3 *The switched system 1 is said to be (completely) controllable at time* t_0*, if its controllable set at* t_0 *is* $\mathbb{R}^n$.

Definition 4 *The dynamic system* $\dot{x}(t) = Ax(t), y(t) = Cxt)$ *is (completely) observable at* t_0 *if there exists a finite* $t > t_0$ *such that knowledge of the outputs* $y[t_0, t]$ *suffices to determine the value of the initial state* $x(0)$.

Remark 5 *It is obvious that in the case where one subsystem is (completely) controllable/observable, system 1 is (completely) controllable/observable.*

Let us consider the following matrices:

$\mathcal{C}_0 = (B_1 \quad \ldots \quad B_m)$,

$$\mathcal{C}_1 = (I \quad A_1 \quad \ldots \quad A_m) \begin{pmatrix} \mathcal{C}_0 & & \\ & \ddots & \\ & & \mathcal{C}_0 \end{pmatrix},$$

$$\mathcal{C}_2 = (I \quad A_1 \quad \ldots \quad A_m) \begin{pmatrix} \mathcal{C}_1 & & \\ & \ddots & \\ & & \mathcal{C}_1 \end{pmatrix},$$

$\vdots$

$$\mathcal{C}_i = (I \quad A_1 \quad \ldots \quad A_m) \begin{pmatrix} \mathcal{C}_{i-1} & & \\ & \ddots & \\ & & \mathcal{C}_{i-1} \end{pmatrix}$$

$\mathcal{O}_0 = (C_1^t \quad \ldots \quad C_m^t)$,

$$\mathcal{O}_1 = (I \quad A_1^t \quad \ldots \quad A_m^t) \begin{pmatrix} \mathcal{O}_0 & & \\ & \ddots & \\ & & \mathcal{O}_0 \end{pmatrix},$$

$$\mathcal{O}_2 = (I \quad A_1^t \quad \ldots \quad A_m^t) \begin{pmatrix} \mathcal{O}_1 & & \\ & \ddots & \\ & & \mathcal{O}_1 \end{pmatrix},$$

$\vdots$

$$\mathcal{O}_i = (I \quad A_1^t \quad \ldots \quad A_m^t) \begin{pmatrix} \mathcal{O}_{i-1} & & \\ & \ddots & \\ & & \mathcal{O}_{i-1} \end{pmatrix}$$

The following characterization of controllability/observability is well-known.

Proposition 6 *The switched system 1 is (completely) controllable if and only if*

$$\operatorname{rank} \mathcal{C}_{n-1} = n.$$

And it is (completely) observable when

$$\operatorname{rank} \mathcal{O}_{n-1} = n.$$

Controllability and observability are generic properties. That is to say, the sets of matrices defining the subsystems of a (completely) controllable/observable switched linear system are open dense sets in the space of all matrices.

3 Miniversal deformations

When tackling the problem of how small perturbations of the system may lead to different structures a classical approach is to consider miniversal deformations, which provide all possible structures which can arise from small perturbations. Moreover, they can be applied to the study of singularities and bifurcations. Note that the number of parameters in any miniversal deformation is always equal to the codimension of orbits. We recall here the definition of deformation and their characterization through versality (see [1], [15]).

Definition 7 *A deformation $\varphi(\lambda)$ of $x_0 \in \mathcal{M}$ is a smooth mapping*

$$\varphi : \mathcal{U}_0 \longrightarrow \mathcal{M}$$

such that $\mathcal{U}_0 \subseteq \mathbb{R}^l$ is an open neighborhood of the origin and $\varphi(0) = x_0$. The vector $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{U}_0$ is called the parameter vector.

Let $\mathcal{G}$ be a Lie group acting over $\mathcal{M}$ via an action α, that is to say, for all $g \in \mathcal{G}$, $x \in \mathcal{M}$, $\alpha_x(g) = g \circ x \in \mathcal{M}$.

Definition 8 *A deformation $\varphi(\lambda)$ of x_0 is called versal if any deformation $\varphi'(\xi)$ of x_0, where $\xi = (\xi_1, \ldots, \xi_k) \in \mathcal{U}'_0 \subset \mathbb{R}^k$ is the parameter vector, can be represented in some neighborhood of the origin as*

$$\varphi'(\xi) = g(\xi) \circ \varphi(\phi(\xi)), \quad \xi \in \mathcal{U}''_0 \subset \mathcal{U}'_0, \qquad (2)$$

where $\phi : \mathcal{U}''_0 \longrightarrow \mathbb{R}^\ell$ and $g : \mathcal{U}''_0 \longrightarrow \mathcal{G}$ are differentiable mappings such that $\phi(0) = 0$ and $g(0)$ is the identity element of $\mathcal{G}$.

When a versal deformation has the minimal number of parameters, it is called *miniversal*.

Locally, in $x \in \mathcal{M}$, $\mathcal{M}$ is isomorphic to the cartesian product of $\varphi(\mathcal{U})$ and a submanifold of $\mathcal{G}$. This can be stated as follows.

Theorem 9 ([1])

1. *A deformation $\varphi(\lambda)$ of x_0 is versal if, and only if, it is transversal to the orbit $\mathcal{O}(x_0)$ at x_0.*
2. *Minimal number of parameters of a versal deformation is equal to the codimension of the orbit of x_0 in $\mathcal{M}$, $d = \operatorname{codim} \mathcal{O}(x_0)$.*

Let $\{v_1, \ldots, v_d\}$ be a basis of any arbitrary complementary subspace $(T_{x_0}\mathcal{O}(x_0))^c$ to $T_{x_0}\mathcal{O}(x_0)$ (for example, $(T_{x_0}\mathcal{O}(x_0))^\perp$).

Corollary 10 *The deformation*

$$x : \mathcal{U}_0 \subset \mathbb{R}^d \longrightarrow M, \quad x(\lambda) = x_0 + \sum_{i=1}^{d} \lambda_i v_i \quad (3)$$

is a miniversal deformation.

If v_j, $j = 1, \ldots, d$, is a basis for $(T_{x_0}\mathcal{O}(x_0))^\perp$ then the corresponding miniversal deformation is called *orthogonal*.

Taking into account the Closed Orbit Lemma (see [9]), the following statement holds.

Proposition 11 *Any equivalence class is a locally closed differentiable submanifold of $\mathcal{M}$ and its boundary is a union of equivalence classes or orbits of strictly lower dimension. In particular, equivalence classes or orbits of minimal. dimension are closed.*

4 Application to the study of structural controllability/observability

We consider a natural equivalence relations in the space $\mathcal{M}$ of matrices defining switched linear systems, generalizing those defined in [1], [15] (see [5]).

From now on, for simplicity's sake, we will consider the case of two subsystems ($M = \{1, 2\}$). The case where the system consists of more than two subsystems can be handled in an analogous way.

Then a non-singular switched linear system consists of two subsystems defined by a 6−tuple of matrices $(A_1, A_2, B_1, B_2, C_1, C_2) \in \mathcal{M} = M_n(\mathbb{R}) \times M_n(\mathbb{R}) \times M_{n\times m}(\mathbb{R}) \times M_{n\times m}(\mathbb{R}) \times M_{p\times n}(\mathbb{R}) \times M_{p\times n}(\mathbb{R})$.

We will consider the following natural equivalence relation in $\mathcal{M}$.

Definition 12 *Two systems*

$$(A_1, A_2, B_1, B_2, C_1, C_2) \sim (A'_1, A'_2, B'_1, B'_2, C'_1, C'_2) \qquad (4)$$

are equivalent if and only if there exist $T \in Gl_n(\mathbb{R})$, $V \in Gl_m(\mathbb{R})$, $W \in Gl_p(\mathbb{R})$, such that

$(A_1', A_2', B_1', B_2', C_1', C_2') =$
$(TA_1T^{-1}, TA_2T^{-1}, TB_1V, TB_2V,$
$WC_1T^{-1}, WC_2T^{-1})$

In order to make use of geometrical properties and, in particular, miniversal deformations, we need to identify the equivalence classes as orbits under a suitable Lie group action.

Proposition 13 *This equivalence relation above is the one induced by the action of the Lie group:*

$$\mathcal{G} = \{(T, V, W) \in Gl_n(\mathrm{I\!R}) \times Gl_m(\mathrm{I\!R}) \times Gl_p(\mathrm{I\!R})$$

over $\mathcal{M}$
being the action:

$$\alpha : \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}$$

where if $G = (T, V, W) \in \mathcal{G}$ *and* $(A_\sigma, B_\sigma, C_\sigma) \in \mathcal{M}$*, then:*

$$\alpha(G, X) = (TA_1T^{-1} \; TA_2T^{-1}, TB_1V, TB_2V, WC_1T^{-1}, WC_2T^{-1})$$

and $\alpha_X : \mathcal{G} \longrightarrow \mathcal{M}$*,* $\alpha_X(G) = \alpha(G, X)$*.*

The Proposition below provides a way to obtain a parametric description of the tangent space in a point of a orbit and a implicit description of the normal space by using any Euclidean scalar product in $\mathcal{M}$. In particular, when we consider the following standard scalar products:

$$\langle (A_\sigma, B_\sigma, C_\sigma), (Y_\sigma, Z_\sigma, T_\sigma) \rangle_1 = \mathrm{tr}(A_1Y_1^t) + \mathrm{tr}(A_2Y_2^t) + \mathrm{tr}(B_1Z_1^t) + \mathrm{tr}(B_2Z_2^t) + \mathrm{tr}(C_1T_1^t) + \mathrm{tr}(C_2Z_2^t)$$

and

$$\langle (T, V, W), (T', V', W') \rangle_2 = \mathrm{tr}(TT'^t) + \mathrm{tr}(VV'^t) + \mathrm{tr}(WW'^t)$$

where $\langle , \rangle$ is defined on $\mathcal{M}$ and $\langle , \rangle_2$ is defined on $M_n(\mathrm{I\!R}) \times M_m(\mathrm{I\!R}) \times M_p(\mathrm{I\!R})$, then the following characterization is obtained.

Proposition 14 *Let us denote by* $T\mathcal{O}(A_\sigma, B_\sigma, C_\sigma)$ *the tangent space and by* $N\mathcal{O}(A_\sigma, B_\sigma, C_\sigma)$ *the normal space to the orbit of* $(A_\sigma, B_\sigma, C_\sigma)$ *at the point* $(A_\sigma, B_\sigma, C_\sigma)$*. Then:*

(a) $T\mathcal{O}(A_\sigma, B_\sigma, C_\sigma)$ *is the set* $\{([T, A_1], [T, A_2], TB_1 + B_1V, TB_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T) \mid (T, V, W) \in T_I\mathcal{G}\}$ *where* $T_I\mathcal{G}$ *is* $\{(T, V, W) \in M_n(\mathrm{I\!R}) \times M_m(\mathrm{I\!R}) \times M_p(\mathrm{I\!R})\}$.

(b) $N\mathcal{O}(A_\sigma, B_\sigma, C_\sigma)$ *is the vector subspace consisting of* $(Y_\sigma, Z_\sigma, T_\sigma) \in \mathcal{M}$ *such that*

$$\begin{aligned} [A_1, Y_1^t] + B_1Z_1^t - T_1^tC_1 + [A_2, Y_2^t] \;+ & \\ B_2Z_2^t - T_2^tC_2 &= 0 \\ Z_1^tB_1 + Z_2^tB_2 &= 0 \\ C_1T_1^t + C_2T_2^t &= 0 \end{aligned} \quad (5)$$

Proof:

(a) It suffices to compute the differential at the identity of the action map: $d\alpha_X(I + \varepsilon G) = = (A_\sigma, B_\sigma, C_\sigma) + \varepsilon(TA_1 - A_1T, TA_2 - A_2T, TB_1 + B_1V, TB_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T) + \varepsilon^2(\ldots)$ and the statement follows.

(b) For any $(Y_\sigma, Z_\sigma, T_\sigma) \in \mathcal{M}$, this switched system is in $N\mathcal{O}(A_1, A_2, B_1, B_2, C_1, C_2)$ if and only if, $\langle (TA_1 - A_1T, TA_2 - A_2T, TB_1 + B_1V, TB_2 + B_2V, WC_1 - C_1T, WC_2 - C_2T), (Y_1, Y_2, Z_1, Z_2, T_1, T_2) \rangle = = \mathrm{tr}((TA_1 - A_1T)Y_1^t) + \mathrm{tr}((TA_2 - A_2T)Y_2^t) + \mathrm{tr}((TB_1 + B_1V)Z_1^t) + \mathrm{tr}((TB_2 + B_2V)Z_2^t) + \mathrm{tr}((WC_1 - C_1T)T_1^t) + \mathrm{tr}((WC_2 - C_2T)T_2^t) = 0$
Computing this product we obtain
$= \langle (A_1Y_1^t - Y_1^tA_1 + A_2Y_2^t - Y_2^tA_2 + B_1Z_1^t + B_2Z_2^t - T_1^tC_1 - T_2^tC_2, Z_1^tB_1 + Z_2^tB_2, C_1T_1^t + C_2T_2^t), (T^t, V^t, W^t) \rangle_2 = 0, \quad \forall (T^t, V^t, W^t)$
Then, we obtain the equations:

$$\begin{aligned} [A_1, Y_1^t] + B_1Z_1^t - T_1^tC_1 + [A_2, Y_2^t] \;+ & \\ B_2Z_2^t - T_2^tC_2 &= 0 \\ Z_1^tB_1 + Z_2^tB_2 &= 0 \\ C_1T_1^t + C_2T_2^t &= 0 \end{aligned}$$

□

Miniversal deformations can then be obtained.

Definition 15 *A deformation of* $(A_\sigma, B_\sigma, C_\sigma) \in \mathcal{M}$ *is a differentiable map* $\varphi : U \longrightarrow \mathcal{M}$*, with* U *an open neighborhood of the origin in* $\mathrm{I\!R}^d$*, such that* $\varphi(0) = (A_\sigma, B_\sigma, C_\sigma)$.

A deformation $\varphi : U \longrightarrow \mathcal{M}$ of $(A_\sigma, B_\sigma, C_\sigma)$ is called *versal* at 0 if for any other deformation of $(A_\sigma, B_\sigma, C_\sigma)$, $\psi : V \longrightarrow \mathcal{M}$, there exists a neighborhood $V' \subseteq V$ with $0 \in V'$, a differentiable map $\gamma : V' \longrightarrow U$ with $\gamma(0) = 0$ and a deformation of the identity $I \in \mathcal{G}$, $\theta : V' \longrightarrow \mathcal{G}$, such that $\psi(\mu) = \alpha(\theta(\mu), \varphi(\gamma(\mu)))$ for all $\mu \in V'$.

Theorem 16 *The mapping*

$$\begin{aligned} \mathbb{R}^d &\longrightarrow \mathcal{M} \\ (\eta_1, \dots, \eta_d) &\mapsto (A_\sigma, B_\sigma, C_\sigma) + \sum_{i=1}^{i=d} \eta_i V_i \end{aligned}$$

where $\{V_1, \dots, V_d\}$ *is any basis of the vectorial subspace* $N\mathcal{O}(A_\sigma, B_\sigma, C_\sigma)$ *is a miniversal deformation of* $(A_\sigma, B_\sigma, C_\sigma)$*, with regard to any equivalence relation defined on* $\mathcal{M}$*.*

We can also derive the dimension of orbits or equivalence classes.

Proposition 17 *The dimension of orbits* $\mathcal{O}(A_\sigma, B_\sigma, C_\sigma)$ *is given by* rank($M(A_\sigma, B_\sigma, C_\sigma)$) where $M(A_\sigma, B_\sigma, C_\sigma)$ is the matrix associated to the linear system yielding the normal space to the orbit of the given switched system.

We recall now the concepts of structural controllability/observability.

Structural controllability is a generalization of the controllability concept. It is of great interest because many times we know the entries of the matrices only approximately. Roughly speaking, a switched linear system is said to be structurally controllable if one can find a set of values for the parameters in the matrices such that the corresponding switched system is controllable. More concretely, the definition is as follows.

Definition 18 *The switched system 1 is structurally controllable if and only if* $\forall \varepsilon > 0$*, there exists a completely controllable switched system* $\dot{x}(t) = \overline{A}_\sigma x(t) + \overline{B}_\sigma u(t)$*, of the same structure as* $\dot{x}(t) = Ax(t) + Bu(t)$ *such that* $\|\overline{A}_i - A\| < \varepsilon$ *and* $\|\overline{B}_i - B\| < \varepsilon$*,* $\forall i \in M$*.*

Recall that, a switched dynamic system $\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t)$ has the same structure as another switched system $\dot{x}(t) = \overline{A}_\sigma x(t) + \overline{B}_\sigma u(t)$, of the same dimensions, if for every fixed zero entry of the triple of matrices (A_i, B_i, C_i) for all $i \in M$, the corresponding entry of the triple of matrices $(\overline{A}_i, \overline{B}_i, \overline{C}_i)$ is fixed zero and vice versa.

Dualizing, we have the definition of structural observability.

Definition 19 *The switched system 1 is structurally observable if and only if* $\forall \varepsilon > 0$*, there exists a completely observable system* $\dot{x}(t) = \overline{A}_\sigma x(t), y(t) = \overline{C}_\sigma x(t)$ *of the same structure as* $\dot{x}(t) = A_\sigma x(t), y(t) = C_\sigma x(t)$ *such that* $\|\overline{A}_i - A_i\| < \varepsilon$ *and* $\|\overline{C}_i - C\| < \varepsilon$*, for all* $i \in M$*.*

Remark 20 *It is immediate that any controllable system is structurally controllable, but the converse is not true. Analogously, an observable system is structurally observable, but the converse is not true.*

Then, to study the structural controllability/observability of the switched system it suffices to analyze if a system in the family consisting of those systems in the miniversal deformation intersected with the set of systems having the same structure than the giving switched system is (completely) controllable/observable.

Example 1 Let us consider a switched linear system consisting of two subsystems defined by matrices $(A_1, A_2, B_1, B_2, C_1, C_2)$ with $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 2 & 2 \end{pmatrix}$ is not completely controlable and it is not completely observable. But the system is structutaly controllable and structuraly observable, it suffices to take: $(\overline{A}_1, \overline{A}_2, \overline{B}_1, \overline{B}_2, \overline{C}_1, \overline{C}_2)$ with $\overline{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{pmatrix}$, $\overline{A}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2+\varepsilon \end{pmatrix}$, $\overline{B}_1 = B_1$ and $\overline{C}_1 = C_1$

A miniversal deformation of this system is the following one.

$(A_1+Y_1, A_2+Y_2, B_1+Z_1, B_2+Z_2, C_1+T_1, C_2+T_2)$ with $A_1 + Y_1 = \begin{pmatrix} 1+y_1 & y_2 \\ y_3 & 1+y_4 \end{pmatrix}$, $A_2 + Y_2 = \begin{pmatrix} 2+y_5 & y_6 \\ y_7 & 2+y_8 \end{pmatrix}$, $B_1 + Z_1 = \begin{pmatrix} 1+z_1 \\ 1+z_2 \end{pmatrix}$, $B_2 + Z_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $C_1 + T_1 = \begin{pmatrix} 1+t_1 & 1+t_2 \end{pmatrix}$ and $C_2 + T_2 = \begin{pmatrix} 2 & 2 \end{pmatrix}$.

Intersecting the miniversal family with the variety of fixed zeros of the switched system we have

$(\overline{A}_1, \overline{A}_2, \overline{B}_1, \overline{B}_2, \overline{C}_1, \overline{C}_2)$ with $\overline{A}_1 = \begin{pmatrix} 1+y_1 & 0 \\ 0 & 1+y_4 \end{pmatrix}$, $\overline{A}_2 = \begin{pmatrix} 2+y_5 & 0 \\ 0 & 2+y_8 \end{pmatrix}$, $\overline{B}_1 = \begin{pmatrix} 1+z_1 \\ 1+z_2 \end{pmatrix}$, $\overline{B}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\overline{C}_1 = \begin{pmatrix} 1+t_1 & 1+t_2 \end{pmatrix}$ and $\overline{C}_2 = \begin{pmatrix} 2 & 2 \end{pmatrix}$.

Straightforward calculations show that

$$\mathcal{C}_1 = (\mathcal{C}_0 \mid A_1\mathcal{C}_0 \mid A_2\mathcal{C}_0)$$

being

$$\mathcal{C}_0 = \begin{pmatrix} 1+z_1 & 2 \\ 1+z_2 & 2 \end{pmatrix}$$

$$A_1\mathcal{C}_0 = \begin{pmatrix} (1+y_1)(1+z_1) & 2+2y_1 \\ (1+y_4)(1+z_2) & 2+2y_2 \end{pmatrix}$$

$$A_2\mathcal{C}_0 = \begin{pmatrix} (2+y_5)(1+z_1) & 4+2y_5 \\ (2+y_8)(1+z_2) & 4+2y_8 \end{pmatrix}$$

In any of the cases $z_1 \neq z_2, y_1 \neq y_4, y_5 \neq y_8$, for example, the corresponding systems is (completely) controllable. Therefore the system in the statement is structurally controllable.

Notice that using miniversal deformations we reduce the number of parameters to consider in the study of structural controllability and structural observability.

Example 2 Let us consider a switched linear system consisting of two subsystems defined by matrices $(A_1, A_2, B_1, B_2, C_1, C_2)$ with $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $C_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$ is not completely controlable and it is not completely observable. In this case the system is not structuraly controllable and not structuraly observable.

A miniversal deformation of this system is the following one.

$(A_1+Y_1, A_2+Y_2, B_1+Z_1, B_2+Z_2, C_1+T_1, C_2+T_2)$ with $A_1 + Y_1 = \begin{pmatrix} 1+y_1 & y_2 \\ y_3 & 1+y_4 \end{pmatrix}$, $A_2 + Y_2 = \begin{pmatrix} 1+y_5 & y_6 \\ y_7 & 1+y_8 \end{pmatrix}$, $B_1 + Z_1 = \begin{pmatrix} z_1 \\ 1+z_2 \end{pmatrix}$, $B_2 + Z_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $C_1 + T_1 = \begin{pmatrix} t_1 & 1 \end{pmatrix}$ and $C_2 + T_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

Intersecting the miniversal family with the variety of fixed zeros of the switched system we have

$(\overline{A}_1, \overline{A}_2, \overline{B}_1, \overline{B}_2, \overline{C}_1, \overline{C}_2)$ with $\overline{A}_1 = \begin{pmatrix} 1+y_1 & 0 \\ 0 & 1+y_4 \end{pmatrix}$, $\overline{A}_2 = \begin{pmatrix} 1+y_5 & 0 \\ 0 & 1+y_8 \end{pmatrix}$, $\overline{B}_1 = \begin{pmatrix} 0 \\ 1+z_2 \end{pmatrix}$, $\overline{B}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\overline{C}_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $\overline{C}_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

Straightforward calculations show that

$$\mathcal{C}_1 = (\mathcal{C}_0 \mid A_1\mathcal{C}_0 \mid A_2\mathcal{C}_0)$$

being

$$\mathcal{C}_0 = \begin{pmatrix} 0 & 0 \\ 1+z_2 & 2 \end{pmatrix}$$

$$A_1\mathcal{C}_0 = \begin{pmatrix} 0 & 2+2y_1 \\ (1+y_4)(1+z_2) & 2+2y_2 \end{pmatrix}$$

$$A_2\mathcal{C}_0 = \begin{pmatrix} 0 & 0 \\ (1+y_8)(1+z_2) & 2+2y_8 \end{pmatrix}$$

There is no value of y_1, y_4, y_5, y_8 z_2 for which the switched system is (completely) controllable and/or observable, then the corresponding systems is not structurally controllable and not structurally observable.

5 Conclusions

In this paper, structural controllability and observability properties of switched linear systems have been studied. The techniques used for this study are geometrical techniques as miniversal deformations.

References:

[1] V.I. Arnold, *O*n matrices depending on parameters. UspekhiMat. Nauk.26, 1971.

[2] W. Borutzky, *B*ond Graph Model-based Fault Diagnosis of Hybrid Systems. Springer, Switzerland, 2014.

[3] C.T. Chen, *Introduction to Linear System Theor*. Holt, Rinehart and Winston Inc., New York, 1970.

[4] J. Clotet, M.I. García-Planas, M. D. Magret, *Controllability of second-order switched linear systems* Conference Proceedings of 5th International Conference on Physics and Control (PhysCon 2011). IPACS Electronic library, 2011.

[5] J. Clotet, H. Diez-Macho, M.I. Garca-Planas, M.D. Magret, M.E. Montoro, *Singular switched linear systems: some geometric aspects* , International Journal of Modern Physics B **26**, no. 25, pp. 1-10, 2012.

[6] M.I. García-Planas, *Sensivity and stability of singular systems under proportional and derivative feedback*, Wseas Transactions on Mathematics, **8**, (11), pp. 635-644, 2009.

[7] M.I. García-Planas, *Analysis of the disturbance decoupling problem for singular switched linear systems via feedback and output injection*. Recent Advances in Circuits, Systems, Signal Processing and Communications, pp. 41-45. Wseas Press, 2014.

[8] H. Hihi, *Structural Observability of Controlled Switching Linear Systems*. International Journal of Control Science and Engineering **2** (5), pp. 127-135, 2012.

[9] J. E. Humphreys, *Linear Algebraic Groups*, "Graduate Texts in Mathematics2 21, Springer-Verlag, Berlin, (1981).

[10] C.T. Lin, *Structural Controllability*. IEEE Trans. Automatic Control. AC-19, pp. 201–208, 1974.

[11] X. Liu, H. Lin, B.M. Chen, *Graphic interpretations of structural controllability for switched linear systems*, 11th International Conference on Control Automation Robotics and Vision (ICARCV), pp. 549 - 554, 2010.

[12] M. Perera, S. Anushka, B. Lie, C.F. Pfeiffer, *Structural Observability Analysis of Large Scale Systems Using Modelica and Python*, Modeling, Identification and Control, **36**, pp. 53-65, 2015.

[13] C. Sueur, G. Dauphin-Tanguy, *Structural controllability/observability of linear systems represented by bond graphs*. Journal of the Franklin Institute **326**, Issue 6, pp. 869-883, 1989.

[14] Z. Sun, S.S. Ge, *S*witched Linear Systems. Springer, London, 2005.

[15] A. Tannebaum, *I*nvariance and System Theory: Algebraic and Geometric Aspects, Lecture Notes in Mathematics 845, Springer-Verlag, Berlin, 1981.