A method to calculate generalized mixed modified semivalues: application to the Catalan Parliament (legislature 2012–2016) *

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September 4, 2014

Abstract

We focus on generalized mixed modified semivalues, a family of mixed coalitional values. They apply to games with a coalition structure by combining a (induced) semivalue in the quotient game, but share within each union the payoff so obtained by applying different (induced) semivalues to a game that concerns only the players of that union. A computation procedure in terms of the multilinear extension of the original game is also provided and an application to the Catalan Parliament (legislature 2012–2016) is shown.

Keywords: cooperative game, semivalues, coalition structure, multilinear extension.
Math. Subj. Class. (2000): 91A12.

1 Introduction

Introduced in 1974 by Aumann and Drèze [7], the notion of *game with a coalition structure* gave a new impulsion to the development of value theory. These authors extended the Shapley value to this new framework in such a manner that the game really splits into subgames played by the unions isolatedly from each other, and every player receives the payoff allocated to him by the Shapley value in the subgame he is playing within his union.

In 1977, a second approach was used by Owen [21], when introducing the first *coalitional value*, called now the *Owen value*. The Owen value is the result of a *two–step procedure*: first, the unions play a *quotient game* among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value.

In 1982, Owen [23] applied the same procedure to the Banzhaf value and obtained the *modified Banzhaf value* or *Owen–Banzhaf value*. In this case the payoffs at both levels (unions in the quotient game and players within each union) are given by the Banzhaf value.

*Research supported by Grant SGR 2009–01029 of the Catalonia Government (*Generalitat de Catalunya*) and Grant MTM 2012-34426 of the Economy and Competitiveness Spanish Ministry.

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In 2002, Alonso and Fiestras [3] suggested to modify the two–step allocation scheme and use the Banzhaf value for sharing in the quotient game and the Shapley value within unions. This gave rise to the *symmetric coalitional Banzhaf value* or *Alonso–Fiestras value*. That same year, Amer, Carreras and Giménez [6] considered a sort of "counterpart" of the Alonso–Fiestras value where the Shapley value is used in the quotient game and the Banzhaf value within unions.

Thus, the possibilities to define a coalitional value by combining the Shapley and Banzhaf values were complete at that moment. In 2011, Carreras and Puente [12] extended the binomial semivalues to games with a coalitional structure: they used these values in the quotient game and the Shapley value within unions and obtained the *symmetric coalitional binomial semivalues*, a family depending on one parameter $q \in [0,1]$ (the Alonso–Fiestras value arises for $q = 1/2$). In 2003, Amer and Giménez [5] considered coalitional values defined by using a given semivalue in both steps of the procedure. In 2004 Albizuri and Zarzuelo [1] studied coalitional values defined by *any* pair of semivalues (*mixed modified semivalues*), one in the quotient game and the other within unions. *Generalized mixed modified semivalues* were introduced by Albizuri [2] with the name of 'generalized coalitional semivalues". They apply a semivalue in the quotient game that arises once the coalition structure is actually formed, but share within each union the payoff so obtained by applying different semivalues to a game that concerns only the players of that union.

In 1972, Owen [19] introduced the *multilinear extension* and applied it to the calculus of the Shapley value. The computing technique based on the multilinear extension has been applied to many values: in 1975 to the Banzhaf value by Owen [20]; in 1992 to the Owen value by Owen and Winter [24]; in 1994 to the Owen–Banzhaf value by Carreras and Magaña [9]; in 1997 to the quotient game by Carreras and Magaña [10]; in 2000 to binomial semivalues and multinomial probabilistic indices by Puente [26]; in 2003 to coalitional semivalues by Amer and Giménez [5]; in 2004 to the α–decisiveness and Banzhaf α–indices by Carreras [8]; in 2005 to the Alonso–Fiestras value by Alonso, Carreras and Fiestras [4]; in 2011 to symmetric coalitional binomial semivalues by Carreras and Puente [12]; in 2011 to semivalues by Carreras and Giménez in [11].

The present paper focus on giving a computational procedure for generalized mixed modified semivalues by means of the multilinear extension, generalizing the method obtained by Carreras and Giménez in [11] to compute semivalues in cooperative games.

The organization of the paper is as follows. In Section 2, a minimum of preliminaries is provided. Section 3 is devoted to define generalized mixed modified semivalues and give a procedure to compute them. Section 4 contains the application of the generalized mixed modified values to the analysis of the Catalan Parliament (legislature 2012–2016).

2 Preliminaries

2.1 Games and semivalues

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* on N is a function $v : 2^N \to \mathbb{R}$, that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$ and *simple* if, moreover, $v(S) = 0$ or 1 for every $S \subseteq N$. A player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, and *null* in v if, moreover, $v(\{i\}) = 0$. Two players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. Given a nonempty coalition $T \subseteq N$, the restriction to T of a given game v on N is the game $v_{|T}$ on T that we will call a *subgame* of v and is defined by $v_{|T}(S) = v(S)$ for all $S \subseteq T$.

Endowed with the natural operations for real–valued functions, *i.e.* $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set of all cooperative games on N is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$, so that $\dim(\mathcal{G}_N) = 2^n - 1$ if $n = |N|$. Each game $v \in \mathcal{G}_N$ can then be uniquely written as a linear combination of unanimity games, and its components are the *Harsanyi dividends* (Harsanyi [18]):

$$v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T, \quad \text{where} \quad \alpha_T = \alpha_T(v) = \sum_{S \subseteq T} (-1)^{t-s} v(S) \tag{1}$$

and, as usual, $t = |T|$ and $s = |S|$.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Following Weber's [28] axiomatic description, $\psi : \mathcal{G}_N \to \mathbb{R}^N$ is a *semivalue* iff it satisfies the following properties:

(i) *linearity*: $\psi[v + v'] = \psi[v] + \psi[v']$ (*additivity*) and $\psi[\lambda v] = \lambda \psi[v]$ for all $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;

(ii) *anonymity*: $\psi_{\theta i}[\theta v] = \psi_i[v]$ for all permutation θ on N, $i \in N$, and $v \in \mathcal{G}_N$;

(iii) *positivity*: if v is monotonic, then $\psi[v] \geq 0$;

(iv) *dummy player property*: if $i \in N$ is a dummy in game v, then $\psi_i[v] = v(\{i\})$.

There is an interesting characterization of semivalues, by means of *weighting coefficients*, due to Dubey, Neyman and Weber [15]. Set $n = |N|$. Then: (a) for every *weighting vector* $\{p_k\}_{k=0}^{n-1}$ such that $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$ and $p_k \geq 0$ for all k, the expression

$$\psi_i[v] = \sum_{S \subseteq N \backslash \{i\}} p_s [v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and all } v \in \mathcal{G}_N,$$

where $s = |S|$, defines a semivalue ψ; (b) conversely, every semivalue can be obtained in this way; (c) the correspondence given by $\{p_k\}_{k=0}^{n-1} \mapsto \psi$ is bijective.

Thus, the payoff that a semivalue allocates to every player in any game is a weighted sum of his marginal contributions in the game. If p_k is interpreted as the probability that a given player i joins a coalition of size k, provided that all the coalitions of a common size have the same probability of being joined, then $\psi_i[v]$ is the expected marginal contribution of that player to a random coalition he joins.

Well known examples of semivalues are the *Shapley value* φ (Shapley [27]), for which $p_k = 1/n\binom{n-1}{k}$, and the *Banzhaf value* β (Owen [20]), for which $p_k = 2^{1-n}$. The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$.

Notice that these values are defined for each N. The same happens with the *binomial semivalues*, introduced by Puente [26] (see also Giménez [17] or Amer and Giménez [5]) as follows. Let $p \in [0,1]$ and $p_k = p^k(1-p)^{n-k-1}$ for $k = 0,1,\ldots,n-1$. Then $\{p_k\}_{k=0}^{n-1}$ is a weighting vector and defines a semivalue that will be denoted as ψ^p and called the *p–binomial semivalue*. Using the convention that $0^0 = 1$, the definition makes sense also for $p = 0$ and $p = 1$, where we respectively get the *dictatorial index* ψ^0 and the *marginal index* ψ^1, introduced by Owen [22] and such that $\psi_i^0[v] = v(\{i\})$ and $\psi_i^1[v] = v(N) - v(N\backslash\{i\})$ for all $i \in N$ and all $v \in \mathcal{G}_N$. Of course, $p = 1/2$ gives $\psi^{1/2} = \beta$ —the Banzhaf value.

In fact, semivalues are defined on cardinalities rather than on specific player sets: this means that a weighting vector $\{p_k\}_{k=0}^{n-1}$ defines a semivalue ψ on all N such that $n = |N|$. When necessary, we shall write $\psi^{(n)}$ for a semivalue on cardinality n and p_k^n for its weighting coefficients. A semivalue $\psi^{(n)}$ induces semivalues $\psi^{(t)}$ for all cardinalities $t < n$, recurrently defined by the Pascal triangle (inverse) formula given by Dragan [14]:

$$p_k^t = p_k^{t+1} + p_{k+1}^{t+1} \quad \text{for} \quad 0 \le k < t, \tag{2}$$

A series $\psi = \{\psi^{(n)}\}_{n=1}^{\infty}$ of semivalues, one for each cardinality, is a *multisemivalue* if it satisfies Dragan's recurrence formula. Thus, the Shapley and Banzhaf values and all binomial semivalues are multisemivalues. By applying Eqs. (2) repeatedly, one gets the expression of the weighting coefficients of any induced semivalue in terms of the coefficients of the original semivalue, namely

$$p_s^t = \sum_{j=0}^{n-t} \binom{n-t}{j} p_{s+j}^n \quad \text{for} \quad 1 \le s < t < n. \tag{3}$$

The *multilinear extension*[1] [19] of a game $v \in \mathcal{G}_N$ is the real–valued function defined on $\mathbb{R}^N$ by

$$f_v(X_N) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N\backslash S} (1 - x_j) v(S). \tag{4}$$

where X_N denotes the set of variables x_i for $i \in N$. The following properties directly derive from the definition:

(i) If $v, w \in \mathcal{G}_N$ and $\lambda, \mu \in \mathbb{R}$, then $f_{\lambda v + \mu w} = \lambda f_v + \mu f_w$.

(ii) If $\emptyset \neq T \subseteq N$, then $f_{u_T}(X_N) = \prod_{i \in T} x_i$.

(iii) In general, if $v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T$, then $f_v(X_N) = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T \prod_{i \in T} x_i$.

(iv) If $v \in \mathcal{G}_N$ and $\emptyset \neq T \subseteq N$, then $f_{v_{|T}}(X_T) = f_v(X_T, 0_{N\backslash T})$, where $(X_T, 0_{N\backslash T})$ denotes the set X_N with $x_i = 0$ for all $i \in N\backslash T$, and $v(T) = f_v(1_T, 0_{N\backslash T})$, where 1_T means $x_i = 1$ for all $i \in T$. Moreover,

$$\frac{\partial f_{v_{|T}}}{\partial x_i}(1_T) = v(T) - v(T\backslash\{i\}).$$

In particular, $\frac{\partial f_v}{\partial x_i}(1_N) = v(N) - v(N\backslash\{i\})$ for each $i \in N$.

[1]The term "multilinear" means that, for each $i \in N$, the function is linear in x_i, that is, of the form $f_v(x_1, x_2, \ldots, x_n) = g_i(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_n) x_i + h_i(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_n)$.

As is well known, both the Shapley and Banzhaf values of any game v can be easily obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0,1]^N$ (Owen [19]), while the partial derivatives of that multilinear extension evaluated at point $(1/2, 1/2, \ldots, 1/2)$ give $\beta[v]$ (Owen [20]. This latter procedure extends well to any p–binomial semivalue (see Puente [26], Freixas and Puente [16] or Amer and Giménez [5]) by evaluating the derivatives at point $(p, p, \ldots, p)$.

2.2 Games with coalition structure

Given $N = \{1, 2, \ldots, n\}$, we will denote by $B(N)$ the set of all partitions of N. Each $B \in B(N)$ is called a *coalition structure* in N, and a *union* each member of B. The so–called *trivial coalition structures* are $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ (individual coalitions) and $B^N = \{N\}$ (grand coalition). A *cooperative game with a coalition structure* is a pair $[v; B]$, where $v \in \mathcal{G}_N$ and $B \in B(N)$ for a given N. Each partition B gives a pattern of cooperation among players. We denote by $\mathcal{G}_N^{cs} = \mathcal{G}_N \times B(N)$ the set of all cooperative games with a coalition structure and player set N.

If $[v; B] \in \mathcal{G}_N^{cs}$ and $B = \{B_1, B_2, \ldots, B_m\}$, the *quotient game* v^B is the cooperative game played by the unions or, rather, by the *quotient set* $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^B(R) = v(\bigcup_{r \in R} B_r) \quad \text{for all } R \subseteq M.$$

By a *coalitional value* on $\mathcal{G}_N^{cs}$ we will mean a map $g : \mathcal{G}_N^{cs} \to \mathbb{R}^N$, which assigns to every pair $[v; B]$ a vector $g[v; B]$ with components $g_i[v; B]$ for each $i \in N$.

If f is a value on $\mathcal{G}_N$ and g is a coalitional value on $\mathcal{G}_N^{cs}$, it is said that g is a *coalitional value of f* iff $g[v; B^n] = f[v]$ for all $v \in \mathcal{G}_N$. For instance, the Owen value is a *coalitional value of the Shapley value* φ in the sense that $\Phi[v; B^n] = \varphi[v]$ for all $v \in \mathcal{G}_N$. Besides, $\Phi[v; B^N] = \varphi[v]$.

2.3 Mixed modified semivalues

Given two semivalues ψ and ϕ defined on games with n players (with possibly $\phi = \psi$), Albizuri and Zarzuelo [1] defined the concept of *mixed modified semivalue for games with coalition structure*. They also proved that every mixed modified semivalue is a coalitional semivalue and, reciprocally, every coalitional semivalue has this form.

If $[v; B] \in \mathcal{G}_N^{cs}$ and $B = \{B_1, B_2, \ldots, B_m\}$, the *modified quotient game* $v_{B_j|K}$ is defined as follows:

$$v_{B_j|K}(L) = v(\bigcup_{l \in L} B_l \setminus K') \quad \text{for all } L \subseteq M,$$

where $K' = B_j \setminus K$. This is the game played by the partition classes with the exception of B_j, that is replaced by the subset K. Given a semivalue $\psi \in \mathcal{G}_{\mathcal{N}}$, since the game $u_{B_j|K}$ is defined on a set M with m players ($1 \leq m \leq n$), we can apply the induced semivalue $\psi^{(m)}$:

$$w_j(K) = (\psi^{(m)})_j[u_{B_j|K}] \quad \forall K \subseteq B_j. \tag{5}$$

The value $w_j(K)$ shows the strategic position of the subset $K \subseteq B_j$ if this subset directly negotiate with the other classes as players in the quotient game –according to the semivalue induced by ψ– in absence of $K' = B_j \setminus K$.

Next, since the game w_j is defined on B_j, a set with $b_j = |B_j|$ players ($1 \leq b_j \leq n$), we can apply the induced semivalue $\phi^{(b_j)}$ and we define *the mixed semivalue* ψ/ϕ *modified* by the coalition structure B as

$$\big(\psi/\phi\big)_i[v;B] = (\phi^{(b_j)})_i[w_j] \quad \forall i \in B_j.$$

The mixed semivalue ψ/ϕ yields the result of a *two–step bargaining procedure* analogous to that used in [21, 23] and also in [3, 12]. Indeed, here we first apply the semivalue induced by ψ in the quotient game to get a payoff for each union; next, we use within union B_j the semivalue induced by ϕ to share the payoff by applying it to a *reduced game* played in this union.

If ψ and ϕ are respectively defined by the weighting coefficients $(q_s^n)_{s=0}^{n-1}$ and $(p_s^n)_{s=0}^{n-1}$, according to [1], the payoff given by the mixed modified ψ/ϕ semivalue to every player i in $B_j \in B$, is

$$\big(\psi/\phi\big)_i[v;B] = \sum_{R \subseteq M \setminus \{j\}} q_r^m \sum_{S \subseteq B_j \setminus \{i\}} p_s^{b_j} [v(Q \cup S \cup \{i\}) - v(Q \cup S)] \tag{6}$$

where $Q = \bigcup_{r \in R} B_r$, $b_j = |B_j|$, and $s = |S|$.

3 Generalized mixed modified semivalues

Generalized mixed modified semivalues, from now on GMMS, were introduced by Albizuri [2] with the name of 'generalized coalitional semivalues". They apply a (induced) semivalue in the quotient game that arises once the coalition structure is actually formed, but share within each union the payoff so obtained by applying different (induced) semivalues to a game that concerns only the players of that union.

Definition 3.1 Let $[v;B] \in \mathcal{G}_N^{cs}$ and $B = \{B_1, B_2, \ldots, B_m\}$ a coalition structure in N. If ψ, φ_1, $\ldots$,φ_m are $m+1$ semivalues in $\mathcal{G}_N$, we define the GMMS $\psi/\varphi_1 \cdots \varphi_m$ by

$$\big(\psi/\varphi_1 \cdots \varphi_m\big)_i[v;B] = \big(\psi/\varphi_j\big)_i[v;B] \quad \forall i \in B_j \ (j = 1, \ldots, m). \tag{7}$$

From now on we will denote $\big(\psi/\varphi_1 \cdots \varphi_m\big)_i[v;B]$ by $\Phi_i[v;B]$.

Lemma 3.2 *Let $[v;B] \in \mathcal{G}_N^{cs}$, $B = \{B_1, B_2, \ldots, B_m\}$ a coalition structure in N and Φ a GMMS defined by $m+1$ semivalues in $\mathcal{G}_N$. The allocations given by Φ to players belonging to a union B_j can be obtained as a linear combination of the allocations to unanimity games u_T, where $T = V \cup W$, $V \subseteq B_j$ and $W \in 2^{B \setminus B_j}$.*

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Proof Each game $v \in \mathcal{G}_N$ can be uniquely written as linear combination of unanimity games

$$v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T, \quad \text{where} \quad \alpha_T = \alpha_T(v) = \sum_{S \subseteq T} (-1)^{t-s} v(S).$$

For all $i \in B_j$, by linearity, $\Phi_i[v;B] = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T \Phi_i[u_T]$ and it suffices consider unanimity games u_T with

$$T = V \cup A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_p},\ V \subseteq B_j,\ \{i_1, i_2, ..., i_p\} \subseteq M \setminus \{j\},\ \emptyset \neq A_{i_q} \subseteq B_{i_q},\ q = 1, ..., p.$$

According to Definition 3.1 and by using expression (6), it is easy to check that the allocations to players in B_j only depend on the allocations in the unanimity games defined on inside coalitions in B_j and entire unions outside B_j. That is,

$$\Phi_i[u_T;B] = \Phi_i[u_{V \cup A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_p}};B] = \Phi_i[u_{V \cup B_{i_1} \cup B_{i_2} \cup \ldots \cup B_{i_p}};B]. \qquad \square$$

Notice that the number of unanimity games of this form is $(2^{b_j} - 1)2^m$ with $b_j = |B_j|$ and $m = |M|$.

Proposition 3.3 *Let $B = \{B_1, B_2, \ldots, B_m\}$ be a coalition structure in N and Φ a GMMS defined by $m+1$ semivalues in $\mathcal{G}_N$, ψ, φ_1, $\ldots, \varphi_m$. Fixed a union B_j, the allocation to a player i belonging to B_j in a unanimity game u_T, $T = V \cup B_{i_1} \cup \cdots \cup B_{i_h}$, $V \subseteq B_j$ and $\{i_1, ..., i_h\} \subseteq M \setminus \{j\}$ is given by*

$$\Phi_i[u_T;B] = \left(\psi/\varphi_j\right)_i [u_T;B] = \begin{cases} q_h^{h+1} p_{v-1}^{v} & i \in T \\ 0 & i \notin T \end{cases}$$

where the weighting coefficients of the induced semivalues $(q_s^{h+1})_{s=0}^{h}$ and $(p_s^v)_{s=0}^{v-1}$ are obtained following (2) from the weighting coefficients $(q_s^n)_{s=0}^{n-1}$ and $(p_s^n)_{s=0}^{n-1}$ of the semivalues ψ and φ_j respectively.

Proof For $i \in T$, according to (6) we have

$$\Phi_i[u_T;B] = \sum_{R \subseteq M \setminus \{j\}} q_r^m \sum_{S \subseteq B_j \setminus \{i\}} p_s^{b_j} [u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)]$$

where $Q = \bigcup_{r \in R} B_r$, $b_j = |B_j|$, and $s = |S|$.

Only $u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)$ does not vanish for coalitions R such that $\{i_1, ..., i_h\} \subseteq R \subseteq M \setminus \{j\}$ and for coalitions S such that $V \setminus \{i\} \subseteq S \subseteq B_j \setminus \{i\}$. Then, according to Eq.(3) for induced weights, we have

$$\Phi_i[u_T;B] = \sum_{r=h}^{m-1} \binom{m-1-h}{r-h} q_r^m \sum_{s=v-1}^{b_j-1} \binom{b_j - v}{s-v+1} p_s^{b_j} = q_h^{h+1} p_{v-1}^{v}$$

In case of $i \notin T$, all marginal contributions $u_T(Q \cup S \cup \{i\}) - u_T(Q \cup S)$ vanish. $\square$

Example 3.4 On the players set $N = \{1,2,3,4,5,6\}$, let $B = \{\{1,2,3\},\{4,5\},\{6\}\}$ be a coalition structure on N and Φ a GMMS defined by the semivalues in $\mathcal{G}_N$, ψ, φ_1, φ_2, and φ_3, where $\psi = \varphi$ is the Shapley value and φ_1 is defined by the weighting coefficients $(p_s^6)_{s=0}^6$=(1/24, 3/80, 1/30, 7/240, 1/40, 1/48).

We will obtain the allocations to players $i \in B_1$ according to Φ for the unanimity games $u_{\{1,2,4,6\}}$ and $u_{\{1,2,4,5,6\}}$. They are

$$\Phi_i[u_{\{1,2,4,6\}};B] = q_2^3\, p_1^2 = \frac{1}{3}\frac{7}{15} = \frac{7}{45}, \text{ for } i = 1,2 \text{ and}$$
$$\Phi_3[u_{\{1,2,4,6\}};B] = 0,$$

where $p_1^2 = \sum_{k=0}^{4} \binom{4}{k} p_{1+k}^6 = 7/15$ is the corresponding weighting coefficient of the induced semivalue.

In a similar way and according to Lemma 3.2, for $u_{\{1,2,4,5,6\}}$ we obtain

$$\Phi_i[u_{\{1,2,4,5,6\}};B] = q_2^3\, p_1^2 = \frac{1}{3}\frac{7}{15} = \frac{7}{45}, \text{ for } i = 1,2 \text{ and}$$
$$\Phi_3[u_{\{1,2,4,5,6\}};B] = 0,$$

Notice that the allocations in both games are the same because coalitions $\{1,2,4,6\}$ and $\{1,2,4,5,6\}$ intersect the same unions B_2 and B_3.

For players in B_1, their allocations does not depend on the remaining semivalues φ_2 and φ_3.

The computing technique based on the multilinear extension has been applied to many coalitional values: in 1992 to the Owen value by Owen and Winter [24]; in 1994 to the Owen–Banzhaf value by Carreras and Magaña [9]; in 1997 to the quotient game by Carreras and Magaña [10]; in 2003 to coalitional semivalues by Amer and Giménez [5]; in 2005 to the Alonso–Fiestras value by Alonso, Carreras and Fiestras [4]; in 2011 to symmetric coalitional binomial semivalues by Carreras and Puente [12]; and to coalitional multinomial probabilistic values iby Carreras and Puente [13]. In next theorem we present a method to compute any GMMS by means of the multilinear extension of the game.

Theorem 3.5 *Let $[v;B] \in \mathcal{G}_N^{cs}$, $B = \{B_1,B_2,\ldots,B_m\}$ a coalition structure in N and Φ a GMMS defined by $m+1$ semivalues in $\mathcal{G}_N$, ψ, $\varphi_1,\ldots,\varphi_m$ with weighting coefficients $(q_s^n)_{s=0}^{n-1}$ and $(p_s^{r,n})_{s=0}^{n-1}$, $r = 1,\ldots,m$, respectively.*

Then the following steps lead to the GMMS value of any player $i \in B_j$ in $[v;B]$.

1. *Obtain the multilinear extension $f(x_1,x_2,\ldots,x_n)$ of game v.*
2. *For every $r \neq j$ and all $h \in B_r$, replace the variable x_h with y_r. This yields a new function of x_k for $k \in B_j$ and y_r for $r \in M\backslash\{j\}$.*
3. *In this new function, reduce to 1 all higher exponents, i.e. replace with y_r each y_r^q such that $q > 1$. This gives a new multilinear function denoted as $g_j((x_k)_{k\in B_j},(y_r)_{r\in M\backslash\{j\}})$ (The modified multilinear extension of union B_j).*

4. *After some calculus, the obtained modified multilinear extension reduces to*

$$g_j((x_k)_{k\in B_j},(y_r)_{r\in M\setminus\{j\}}) = \sum_{V\subseteq B_j}\sum_{W\subseteq M\setminus\{j\}} \lambda_{V\cup W}\prod_{k\in V}x_k\prod_{r\in W}y_r$$

5. *Multiply each product $\prod_{k\in V}x_k$ by $p^{j,v}_{v-1}$ and each product $\prod_{r\in W}y_r$ by q^{w+1}_w obtaining a new multilinear function called $\overline{g}_j$.*

6. *Obtain the partial derivative of $\overline{g}_j$ with respect to x_i evaluated at point $(1,\ldots,1)$ and*

$$\Phi_i[v;B] = \frac{\partial \overline{g}_j}{\partial x_i}(1_{B_j},1_{M\setminus\{j\}}).$$

Proof Steps 1–3 have been already used in [24, 9, 26, 16, 4, 12, 13] to obtain the modified multilinear extension of union B_j. Step 4 shows the modified multilinear extension as a linear combination of multilinear extensions of unanimity games. Step 5 weights each unanimity game according to Proposition 3.3 so that step 6 gives as usual the marginal contribution of player i and his allocation $\Phi_i[v;B]$ is obtained. □

The following corollary shows as this procedure can be applied to compute semivalues allocations from the MLE of a cooperative game, in a similar way given by Carreras and Giménez [11].

Corollary 3.6 *Let $v\in\mathcal{G}_N$ and Ψ a semivalue defined in $\mathcal{G}_N$ by the weighting coefficient p^n_s, $s=0,...,n-1$. Then the following steps lead to the allocation of any player $i\in N$ in game v.*

1. *Obtain the multilinear extension $f(x_1,x_2,\ldots,x_n)$ of game v as a sum of products*

$$f(x_1,x_2,\ldots,x_n) = \sum_{S\subseteq N}\alpha_S\prod_{k\in S}x_k$$

2. *Multiply each product $\prod_{k\in V}x_k$ by p^v_{v-1} obtaining a new multilinear function called $\overline{f}$.*

3. *Obtain the partial derivative of $\overline{f}$ with respect to x_i evaluated at point $(1,\ldots,1)$ and*

$$\Psi_i[v] = \frac{\partial \overline{f}}{\partial x_i}(1_N).$$

Proof Considerer the coalition structure given by the grand coalition $B=B^N=\{N\}$. In this case, semivalue $\Psi[v]$ coincides with the GMMS $\Phi[v;B^N]$. Then, applying Theorem 3.5 and taking into account that coefficient $q^1_0=1$, we easily obtain $\Psi_i[v]=\frac{\partial\overline{f}}{\partial x_i}(1_N)$. □

Example 3.7 Let $v\equiv[68;50,21,20,19,13,9,3]$ be the 7–person weighted majority game and the coalition structure $B=\{\{1\},\{2,3,5\},\{4\},\{6\},\{7\}\}$. We will compute $\Phi[v;B]$, where the GMMS Φ is defined by 6 semivalues in $\mathcal{G}_N$, ψ, $\varphi_1,\ldots,\varphi_5$ with weighting coefficients $(q^7_s)^6_{s=0}$ and $(p^{r,7}_s)^6_{s=0}$, $r=1,...,5$, respectively.

The set of minimal winning coalitions of the game is

$$W^m(v) = \{\{1,2\},\{1,3\},\{1,4\},\{1,5,6\}\},$$

so that players 2, 3 and 4 on one hand, and 5 and 6 on the other, are symmetric in v. Moreover, player 7 is null and the multilinear extension of v is

$$\begin{aligned} f(X_N) =& x_1x_2 + x_1x_3 + x_1x_4 - x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4 + x_1x_5x_6 + x_1x_2x_3x_4 \\ & - x_1x_2x_5x_6 - x_1x_3x_5x_6 - x_1x_4x_5x_6 + x_2x_3x_4x_5 + x_2x_3x_4x_6 - x_1x_2x_3x_4x_5 \\ & - x_1x_2x_3x_4x_6 + x_1x_2x_3x_5x_6 + x_1x_2x_4x_5x_6 + x_1x_3x_4x_5x_6 - x_2x_3x_4x_5x_6. \end{aligned}$$

The coalition structure is $B = \{\{1\},\{2,3,5\},\{4\},\{6\},\{7\}\}$ and steps 1–4 in Theorem 3.5 give the modified multilinear extension of each union B_j, for $j = 1,2,3,4$ (notice that player 7 is null in v and it is not necessary to compute g_5).

$$g_1(x_1,y_2,y_3,y_4,y_5) = x_1y_2 + x_1y_3 - 2x_1y_2y_3 + y_2y_3,$$

$$\begin{aligned} g_2(x_2,x_3,x_5,y_1,y_3,y_4,y_5) =& x_2y_1 + x_3y_1 + y_1y_3 - x_2x_3y_1 - x_2y_1y_3 - x_3y_1y_3 + x_5y_1y_4 + x_2x_3y_1y_3 \\ & - x_2x_5y_1y_4 - x_3x_5y_1y_4 - x_5y_1y_3y_4 + x_2x_3x_5y_3 + x_2x_3y_3y_4 - x_2x_3x_5y_1y_3 \\ & - x_2x_3y_1y_3y_4 + x_2x_3x_5y_1y_4 + x_2x_5y_1y_3y_4 + x_3x_5y_1y_3y_4 - x_2x_3x_5y_3y_4, \end{aligned}$$

$$\begin{aligned} g_3(x_4,y_1,y_2,y_4,y_5) &= y_1y_2 + x_4y_1 + x_4y_2 - 2x_4y_1y_2, \\ g_4(x_6,y_1,y_2,y_3,y_5) &= y_1y_2 + y_1y_3 + y_2y_3 - 2y_1y_2y_3. \end{aligned}$$

Step 5 leads to $\overline{g}_j$ for each $j = 1,2,3,4$.

$$\overline{g}_1(x_1,y_2,y_3,y_4,y_5) = p_0^{1,1}q_1^2x_1y_2 + p_0^{1,1}q_1^2x_1y_3 - 2p_0^{1,1}q_2^3x_1y_2y_3 + q_2^3y_2y_3,$$

$$\begin{aligned} \overline{g}_2(x_2,x_3,x_5,y_1,y_3,y_4,y_5) =& p_0^{2,1}q_1^2x_2y_1 + p_0^{2,1}q_1^2x_3y_1 + q_2^3y_1y_3 - p_1^{2,2}q_1^2x_2x_3y_1 - p_0^{2,1}q_2^3x_2y_1y_3 \\ & - p_0^{2,1}q_2^3x_3y_1y_3 + p_0^{2,1}q_2^3x_5y_1y_4 + p_1^{2,2}q_2^3x_2x_3y_1y_3 - p_1^{2,2}q_2^3x_2x_5y_1y_4 - p_1^{2,2}q_2^3x_3x_5y_1y_4 \\ & - p_0^{2,1}q_3^4x_5y_1y_3y_4 + p_2^{2,3}q_1^2x_2x_3x_5y_3 + p_1^{2,2}q_2^3x_2x_3y_3y_4 - p_2^{2,3}q_2^3x_2x_3x_5y_1y_3 - p_1^{2,2}q_3^4x_2x_3y_1y_3y_4 \\ & + p_2^{2,3}q_2^3x_2x_3x_5y_1y_4 + p_1^{2,2}q_3^4x_2x_5y_1y_3y_4 + p_1^{2,2}q_3^4x_3x_5y_1y_3y_4 - p_2^{2,3}q_2^3x_2x_3x_5y_3y_4, \end{aligned}$$

$$\begin{aligned} \overline{g}_3(x_4,y_1,y_2,y_4,y_5) &= q_2^3y_1y_2 + p_0^{3,1}q_1^2x_4y_1 + p_0^{3,1}q_1^2x_4y_2 - 2p_0^{3,1}q_2^3x_4y_1y_2, \\ \overline{g}_4(x_6,y_1,y_2,y_3,y_5) &= q_2^3y_1y_2 + q_2^3y_1y_3 + q_2^3y_2y_3 - 2q_3^4y_1y_2y_3. \end{aligned}$$

Step 6 yields

$$\begin{aligned} &\Phi_1[v;B] = 2p_0^{1,1}q_1^2 - 2p_0^{1,1}q_2^3, \\ &\Phi_i[v;B] = p_0^{2,1}q_1^2 - p_1^{2,2}q_1^2 - p_0^{2,1}q_2^3 + p_1^{2,2}q_2^3 + p_2^{2,3}q_1^2 - p_2^{2,3}q_2^3, \quad \text{for } i = 2,3, \\ &\Phi_4[v;B] = 2p_0^{3,1}q_1^2 - 2p_0^{3,1}q_2^3, \\ &\Phi_5[v;B] = p_0^{2,1}q_2^3 - 2p_1^{2,2}q_2^3 - p_0^{2,1}q_3^4 + p_2^{2,3}q_1^2 - p_2^{2,3}q_2^3 + 2p_1^{2,2}q_3^4, \\ &\Phi_6[v;B] = 0, \\ &\Phi_7[v;B] = 0. \end{aligned}$$

4 The Catalan Parliament. Legislature 2012–2016

In this section we will revisit Example 3.7 and will complete, by applying different GMMS's, the study initiated there to an interesting political structure: the Catalan Parliament (Legislature 2012–2016). All allocations we will need have been computed using Theorem 3.5, most of them in Example 3.7.

Seven parties elected members to the Catalonia Parliament (135 seats) in the elections held on 25 November 2012. The seat distribution of the parties are as follows.

1: CiU (Convergència i Unió), Catalan nationalist middle–of–the–road coalition of two federated parties: 50 seats.

2: ERC (Esquerra Republicana de Catalunya), Catalan nationalist left–wing party: 21 seats.

3: PSC (Partit dels Socialistes de Catalunya), moderate left–wing socialist party, federated to the Partido Socialista Obrero Español: 20 seats.

4: PPC (Partit Popular de Catalunya), conservative party, Catalan delegation of the Partido Popular: 19 seats.

5: ICV (Iniciativa per Catalunya–Verds), coalition of Catalan eurocommunist parties, federated to Izquierda Unida, and ecologist groups ("Verds"): 13 seats.

6: C's (Ciutadans), Spanish nationalist liberal party: 9 seats.

7: CUP (Candidatura d'unitat popular), radical left–wing catalanist party organized by assemblies: 3 seats.

Under the standard absolute majority rule, and assuming voting discipline within parties, the structure of this parliamentary body can be represented by the weighted majority game

$$v \equiv [68; 50, 21, 20, 19, 13, 9, 3].$$

Therefore, the strategic situation is given by

$$W^m(v) = \{\{1,2\},\{1,3\},\{1,4\},\{1,5,6\}\},$$

so that players 2, 3 and 4 on one hand, and 5 and 6 on the other, are symmetric in v. Moreover, player 7 is null.

The calculation of $\Psi[v]$ for an arbitrary semivalue with weighting coefficients $(p_s^7)_{s=0}^6$ derives from Corollary 3.6 considering the coalition structure given by the grand coalition $B^N = \{\{1,2,3,4,5,6,7\}\}$. We get:

$$\begin{aligned}
\Psi_1[v] &= 3p_1^2 - 2p_2^3 - 2p_3^4 + p_4^5,\\
\Psi_i[v] &= p_1^2 - 2p_2^3 + 2p_3^4 - p_4^5, \quad \text{for } i = 2,3,4,\\
\Psi_i[v] &= p_2^3 - 2p_3^4 + p_4^5, \quad \text{for } i = 5,6,\\
\Psi_7[v] &= 0.
\end{aligned}$$

We show in Table 1 the allocations for every player in game v according to the Shapley (φ) and Banzhaf (β) values and the p–binomial value with $p = 1/3$ ($\psi^{1/3}$). Notice that the weighting coefficients p^s_{s-1} for these values are respectively $p^s_{s-1} = \frac{1}{s}$, $p^s_{s-1} = \frac{1}{2^{s-1}}$ and $p^s_{s-1} = p^{s-1}$.

Table 1. Initial power distribution in the Catalonia Parliament 2012–2016

Value	1. CiU	2. ERC	3. PSC	4. PPC	5. ICV	6. C's	7. CUP
$\varphi_i[v]$	0.5333	0.1333	0.1333	0.1333	0.0333	0.0333	0.0000
$\beta_i[v]$	0.8125	0.1875	0.1875	0.1875	0.0625	0.0625	0.0000
$\psi^{1/3}$	0.7160	0.1728	0.1728	0.1728	0.0494	0.0494	0.0000

We will not attempt to give here a full description of the complexity of the Catalan politics, a task more suitable for a political science article. We wish only to state that, in view of all these components, the politically most likely coalitions to form, and the corresponding coalition structures, are the following:

- ERC + PSC + ICV, the left alliance: $B_L = \{\{1\},\{2,3,5\},\{4\},\{6\},\{7\}\}$.
- CiU + ERC, the Catalanist majority alliance: $B_C = \{\{1,2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$.

In order to evaluate the strategic possibilities of each party we apply $\Phi[v;B]$ in cases $B = B_L$ (see Example 3.7) and $B = B_C$ according to the Owen value, the Owen–Banhaf value, the Alonso and Fiestras value, the symmetric coalitional binomial value Ω^p with $p = 1/3$ and the mixed semivalue μ defined by the coefficients

$$(q^7_s)^6_{s=0} = \left(\frac{6-s}{192}\right)^6_{s=0} \text{ and } (p^7_s)^6_{s=0} = \left(\frac{2^{6-s}}{3^6}\right)^6_{s=0}.$$

Notice that the first coefficients refer to an arithmetic semivalue with increase $-1/192$ and the second ones to the binomial semivalue with $p = 1/3$. Now the last weighting coefficients for the induced arithmetic semivalue are:

$$q^6_5 = 1/192, \quad q^5_4 = 1/48, \quad q^4_3 = 1/16, \quad q^3_2 = 1/6, \quad q^2_1 = 5/12 \quad \text{and} \quad q^1_0 = 1.$$

In this case player 7 (CUP) is null in the quotient game and then $\Phi_7[v;B] = 0$. The results obtained in case of the left alliance for the remaining players are given in Table 2.

Table 2. Different GMMS in the Catalonia Parliament 2012–2016 (the left–wing alliance)

Value	1. CiU	2. ERC	3. PSC	4. PPC	5. ICV	6. C's
Owen	0.3333	0.1389	0.1389	0.3333	0.0556	0.0000
Owen–Banzhaf	0.5000	0.1875	0.1875	0.5000	0.0625	0.0000
Alonso–Fiestras	0.5000	0.2083	0.2083	0.5000	0.0833	0.0000
Sym. coalitional ($p = 1/3$)	0.4444	0.1852	0.1852	0.4444	0.0741	0.0000
Mixed semivalue μ	0.5000	0.1944	0.1944	0.5000	0.0625	0.0000

In case $B = B_C$ players 3, 4, 5, 6 and 7 became null in the quotient game and then $\Phi_i[v;B] = 0$ for all of them. We only need to calculate the modified multilinear extension g_1.

$$\begin{aligned} g_1(x_1,x_2,y_2,\dots,y_6) = & x_1x_2 + x_1y_2 + x_1y_3 - x_1x_2y_2 - x_1x_2y_3 - x_1y_2y_3 + x_1y_4y_5 + x_1x_2y_2y_3 \\ & - x_1x_2y_4y_5 - x_1y_2y_4y_5 - x_1y_3y_4y_5 + x_2y_2y_3y_4 + x_2y_2y_3y_5 - x_1x_2y_2y_3y_4 \\ & - x_1x_2y_2y_3y_5 + x_1x_2y_2y_4y_5 + x_1x_2y_3y_4y_5 + x_1y_2y_3y_4y_5 - x_2y_2y_3y_4y_5. \end{aligned}$$

Following the same procedure as the previous alliance, the results obtained for the Catalanist majority alliance for players 1 and 2 are given in Table 3.

Table 3. Different GMMS in the Catalonia Parliament 2012–2016 (the Catalanist majority alliance)

Value	1. CiU	2. ERC
Owen	0.7000	0.3000
Owen–Banzhaf	0.8125	0.1875
Alonso–Fiestras	0.8125	0.1875
Symmetric coalitional binomial ($p = 1/3$)	0.7716	0.2284
Mixed semivalue μ	0.7847	0.1597

In order to form a government coalition in the two last Legislatures (2003–2007 prematurely finished in 2006 and 2006–2010) the role of ERC was crucial. Thus, in both cases ERC was faced to the dilemma of choosing among either a Catalanist majority coalition with CiU or a left–wing majority coalition with PSC and ICV, which was finally formed in 2003 and was repeated in 2006.

In the present Legislature, studied here, the politic scenario and parties' strategic possibilities have changed. Among the different possibilities, the two alliances above considered have been the most commented in the Parliament and in the media. According to the left–wing alliance and comparing the results obtained in Table 1 and Table 2, ERC does not get a significant benefit regarding PSC. When considering the Catalanist majority alliance, the outside parties are reduced to a null position and the power of ERC increases regarding the initial power in v (see Table 1 and Table 3). However, ERC's current decision has not been to form a government coalition with CiU.

5 Concluding remarks

The present work is focussed on the calculus of generalized modified mixed semivalues. More precisely, the computation of allocations to the players can be obtained from the multilinear extension by using a common procedure for all GMMS. As is well known, both the Shapley and Banzhaf values of any game v can be easily obtained from its multilinear extension. This latter procedure extends well to any p–binomial semivalue (see Puente [26], Freixas and Puente [16] or Amer and Giménez [5]).

In the context of games with a coalition structure, the multilinear extension technique has been also applied to computing the Owen value (Owen and Winter [24]), as well as the

Owen–Banzhaf value (Carreras and Magaña [9]), the symmetric coalitional Banzhaf value (Alonso, Carreras and Fiestras [4]), the symmetric coalitional binomial semivalue (Carreras and Puente [12]) and the multinomial coaltional probabilistic values (Carreras and Puente [13]). In all these cases, (including the GMMS) the first three steps of the procedure are the same.

Instead, the consideration of the modified MLE g_j for the union B_j obtained from the initial one has changed the procedure: first, we weight the terms of g_j multiplying each product $\prod_{k\in V} x_k$ by p^v_{v-1} and each product $\prod_{r\in W} y_r$ by q^{w+1}_w obtaining a new multilinear function called $\overline{g}_j$. Second, we obtain players' marginal contributions by partial differentiation of $\overline{g}_j$. This new procedure has an advantage with respect to the traditional method: the allocations given by any coalitional semivalue are available since the weighting coefficients p^{k-1}_k and q^{k+1}_k can be always obtained.

Finally, the procedure presented here is a generalization of the method given by Carreras and Giménez [11] to compute semivalues allocations from the MLE of a cooperative game.

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