ITERATIVE LIMIT ANALYSIS OF STRUCTURES WITHIN A SCALED BOUNDARY FINITE ELEMENT FRAMEWORK

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Summary. *This paper presents an iterative elastic analysis approach to determine the collapse load limit of structures. The proposed scheme is based on the use of a modified elastic compensation method, where the structure is modeled within a scaled boundary finite element framework. The formulation takes the general form of polygon scaled boundary finite elements, which overcomes the challenges associated with stress singularities and complex geometries. The approach provides coarse mesh accuracy and numerical stability under incompressibility conditions, and is suitable for large scale problems that often require a large number of iterations to converge to the collapse load solution. A number of successfully solved examples, one of which has been given herein, illustrate the robustness and efficiency of the proposed method to compute the collapse load of structures.*

1 INTRODUCTION

Classical limit analysis is founded on the well-known upper (kinematic) and lower (static) bound theorems to compute the collapse load of ductile structures under monotonically applied forces. Various iterative elastic analysis methods have been proposed to perform the limit analysis of structures (e.g., [1-3]). The algorithm is founded on a simple concept where an elastic analysis is iteratively performed to provide the limit load. At each iteration, the elastic stiffness properties of high loaded elements are systematically adjusted such that the stresses with high intensity are redistributed [4].

Marriott proposed an elastic modulus adjustment procedure (EMAP) that maintains a statically admissible field in a lower bound limit analysis theorem [5]. The iteration-based EMAP modifies the values of elastic modulus in the current iteration using the stress information found in the previous iteration. A so-called "gloss r-node" defines the locations or redistribution nodes of statically determinate stresses that are insensitive to the material model [6, 7]. The collapse load is calculated on the assumption that plastic collapse does not occur when the stresses at r-nodes are below yield points. Mackenzie and Boyle later further developed the method as the elastic compensation method (ECM) [4]. The ECM employs a series of standard elastic stiffness analyses with successively adjustment of the elastic modulus at each iteration. However, this conventional ECM does not guarantee convergence to the limit load of structures which contain stress singularities.

From the aforementioned comments, the present paper proposes a promising iterative limit analysis performed within a scaled boundary finite element (SBFE) framework. The approach adopts the modified ECM [8] developed recently to enhance convergence of the limit load solution. The structure is modeled using the polygon SBFE method [9], which is convenient for the discretization of structures with complex geometries. In essence, each generic polygon element is not limited by its shape, and is flexible enough to map accurately arbitrary geometries, such as curved boundaries and circular penetrations. Moreover, the polygon SBFE model advantageously provides coarse mesh accuracy, an ability to deal with discretized problems containing hanging nodes, and maintains numerical stability under incompressibility conditions.

2 FORMULATION FOR ELASTIC SBFE ANALYSIS

The SBFE method, pioneered by Song and Wolf [10-12], is a semi-analytical approach. Its applications within a polygon type construction are found in various engineering mechanics problems, such as fracture, wave propagation and unbounded media. Figure 1 shows a polygon subdomain, where a scaling center "O" is chosen such that all boundary can be visible from its center.

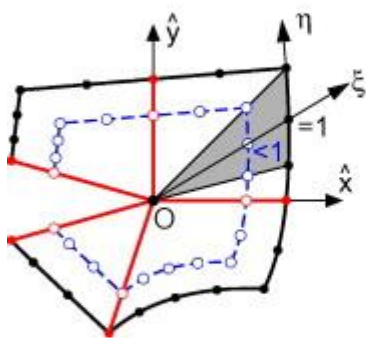

Figure 1: Generic polygon scaled boundary finite element model [13]

The boundary of a subdomain is modeled by one-dimensional line elements. The geometry of the boundary is defined using a shape function $N(\eta)$ with a local coordinate $-1 \leq \eta \leq 1$. The radial coordinate ξ pointing from the scaling center ($\xi = 0$) to the boundary ($\xi = 1$) describes the subdomain by scaling the boundary. The Cartesian coordinate of any point inside the domain is expressed as

$$x(\xi,\eta) = \xi x(\eta) = \xi[N(\eta)]\{x\} \tag{1a}$$
$$y(\xi,\eta) = \xi y(\eta) = \xi[N(\eta)]\{y\} \tag{1b}$$

where $\{x\}, \{y\}$ are the nodal coordinates.

For a sector covered by a scaling boundary element, the displacement field $\{u(\xi,\eta)\}$ is expressed by

$$\{u(\xi,\eta)\} = [N_u(\eta)]\{u(\xi)\} = [N_1(\eta)[I], N_2(\eta)[I], \ldots]\{u(\xi)\} \tag{2}$$

where $[I]$ is a 2 × 2 identity matrix and $\{u(\xi)\}$ is the displacement along a radial line between a scaling center and the boundary. For each element, the displacement $\{u(\xi)\}$ can be calculated from:

$$[E^0]\xi^2\{u(\xi)\}_{,\xi\xi} + ([E^0] - [E^1] + [E^1]^T)\xi\{u(\xi)\}_{,\xi} - [E^2]\{u(\xi)\} = 0 \tag{3}$$

where $[E^i]$ for i = 0, 1 and 2 are the coefficient matrices based on its geometry and material property. The assembly of these matrices is similar to the implementation of a standard finite element method.

The internal nodal forces are described as

$$\{q(\xi)\} = [E^0]\xi\{u(\xi)\}_{,\xi} + [E^1]^T\{u(\xi)\} \tag{4}$$

Rewriting Eqs. (3) and (4) as the first-order ordinary differential equations gives:

$$\xi\begin{Bmatrix}\{u(\xi)\}\\ \{q(\xi)\}\end{Bmatrix}_{,\xi} = -[Z]\begin{Bmatrix}\{u(\xi)\}\\ \{q(\xi)\}\end{Bmatrix} \tag{5}$$

where [Z] is a Hamiltonian matrix equal to

$$[Z] = \begin{bmatrix}[E^0]^{-1}[E^1]^T & -[E^0]^{-1}\\ -[E^2] + [E^1][E^0]^{-1}[E^1]^T & -[E^1][E^0]^{-1}\end{bmatrix} \tag{6}$$

A block-diagonal Schur decomposition of matrix Z [14] is adopted. The Schur block and transformation matrices for a polygon are:

$$[Z][\Psi] = [\Psi][S] \tag{7}$$

where [S] is a block-diagonal matrix (viz. its diagonal entries are the real parts of the eigenvalues that are sorted in ascending order), and [Ψ] is a transformation matrix. Partitioning the two matrices [S] and [Ψ] into 2*N*-1 and 2*N* block matrices gives:

$$[S] = \text{diag}([S_1] \ldots [S_{N-1}]\begin{bmatrix}0 & [I]\\ 0 & 0\end{bmatrix}[S_{N+2}] \ldots [S_{2N}]) \tag{8}$$

$$[\Psi] = \begin{bmatrix}[\Psi_1^{(u)}] & \cdots & [\Psi_{N-1}^{(u)}] & [\Psi_N^{(u)}] & [\Psi_{N+1}^{(u)}] & [\Psi_{N+2}^{(u)}] & \cdots & [\Psi_{2N}^{(u)}]\\ [\Psi_1^{(q)}] & \cdots & [\Psi_{N-1}^{(q)}] & [\Psi_N^{(q)}] & [\Psi_{N+1}^{(q)}] & [\Psi_{N+2}^{(q)}] & \cdots & [\Psi_{2N}^{(q)}]\end{bmatrix} \tag{9}$$

where $[\Psi_i^{(u)}]$ and $[\Psi_i^{(q)}]$ are transformation matrices for nodal displacements and forces, respectively.

The stiffness matrix for a polygon is computed by:

$$[K] = \left[\left[\Psi_1^{(q)}\right]\ldots\left[\Psi_{N-1}^{(q)}\right]\left[\Psi_N^{(q)}\right]\right]\left[\left[\Psi_1^{(u)}\right]\ldots\left[\Psi_{N-1}^{(u)}\right]\left[\Psi_N^{(u)}\right]\right]^{-1} \tag{10}$$

The stiffness matrix for a polygon is assembled with the stiffness matrices of the other subdomains to construct the total stiffness matrix for the domain, which is then solved for the nodal displacements of the system. The solution which satisfies the displacements finiteness at the scaling center is expressed as

$$\{u(\xi)\} = \left[\left[\Psi_1^{(u)}\right]\ldots\left[\Psi_{N-1}^{(u)}\right]\left[\Psi_N^{(u)}\right]\right]\{C\} = \sum_{i=1}^{N-1}\left[\Psi_i^{(u)}\right]\xi^{-[S_i]}\{C_i\} + \left[\Psi_N^{(u)}\right]\{C_N\} \tag{11}$$

where the integration constants $\{C\}$ are determined from displacements on the boundary derived from the nodal displacements of the global system

$$\{C\} = \left[\left[\Psi_1^{(u)}\right]\ldots\left[\Psi_{N-1}^{(u)}\right]\left[\Psi_N^{(u)}\right]\right]\{u(\xi = 1)\} \tag{12}$$

Similar to FEM, the stresses are calculated element by element. For a given polygon, the stresses at the local coordinate η is expressed as

$$\{\sigma(\xi,\eta)\} = [D]([B^1(\eta)]\{u(\xi)\}_{,\xi} + [B^2(\eta)]\{u(\xi)\}/\xi) \tag{13}$$

in which $[B^1(\eta)]$ and $[B^2(\eta)]$ represent the relationship between strains and displacements. D is the elasticity matrix. Substituting Eqs. (11) and (13) gives the stresses for each subdomain as

$$\{\sigma(\xi,\eta)\} = \sum_{i=1}^{N-1}[\Psi_{\sigma i}(\eta)]\xi^{-[S_i]-[I]}\{C_i\} \tag{14}$$

where

$$[\Psi_{\sigma i}(\eta)] = [D](-[B^1(\eta)][\Psi_i^{(u)}][S_i] + [B^2(\eta)][\Psi_i^{(u)}] \tag{15}$$

exhibits the stress modes which correspond to the displacement modes, $[\Psi_i^{(u)}]$.

3 ITERATIVE SBFE APPROACH FOR LIMIT ANALYSIS

This section presents an iterative elastic analysis procedure that captures the maximum load capacity of ductile structures at plastic collapse. The main advantage of the proposed scheme is its numerical efficiency; for each iteration, the computational effort is as small as that required to perform a standard elastic analysis. Current computing technologies enable a large number of elastic analysis computations to be completed efficiently, and thus make it eminently suitable to handle large scale structures.

The proposed scheme is a (lower bound) modified ECM [8] of structures that are discretized by polygon SBFEs [9-12]. We start with a structural system that has been discretized into *n* (polygon) scaled boundary subdomains. The general idea underpinning the modified ECM [8] is simple. For the applied forces **f**, a series of elastic SBFE analyses are successively perfomed with the elastic stiffness properties (namely Young's modulus E_j^i for all j = 1 to n) of the structure being modified in the current iteration i based on the stress results computed in the previous analysis step i–1. The elastic modulus of a subdomain j, which has in step i–1 the equivalent (e.g. von Mises in plane strain $\sigma_j^{i-1} = \sqrt{3/4\{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2\}}$ and in plane stress $\sigma_j^{i-1} = \sqrt{\{(\sigma_x - \sigma_y)^2 + \sigma_x\sigma_y + 3\tau_{xy}^2\}}$) stress σ_j^{i-1} greater than the nominal value $\bar{\sigma}^{i-1}$, is adjusted to allow inelastic stress redistribution in the current analysis step i by:

$$E_j^i = \begin{cases} E_j^{i-1}\dfrac{\bar{\sigma}^{i-1}}{\sigma_j^{i-1}}, & \text{for} \quad \sigma_j^{i-1} > \bar{\sigma}^{i-1} \\ E_j^{i-1}, & \text{for} \quad \sigma_j^{i-1} \leq \bar{\sigma}^{i-1} \end{cases} \tag{16}$$

where

$$\bar{\sigma}^{i-1} = \sigma_{max}^{i-1} - \lambda(\sigma_{max}^{i-1} - \sigma_{min}^{i-1}), \tag{17}$$

λ is a modification factor; and $\sigma_{max}^{i-1} = \max(\sigma_1^{i-1}, \dots, \sigma_n^{i-1})$ and $\sigma_{min}^{i-1} = \min(\sigma_1^{i-1}, \dots, \sigma_n^{i-1})$ are the maximum and minimum values of stresses σ_j^{i-1} for all j, respectively. The factor λ plays an important role. A small value of λ provides good convergence to the collapse load solution, but requires more iterations. On the other hand, a large value may not lead to convergence. From our numerical experience, the value of $\lambda = 0.05$ leads to an efficient and robust iterative modified ECM algorithm.

The yield conditions (e.g. for von Mises $\sigma_j^i \leq \sigma_{y,j}$ for all j) are confined solely at a number of predefined locations (i.e. Gauss's points $\xi = 1$ and $\eta = 0$), where $\sigma_{y,j}$ denotes the yield stress and σ_j^i the maximum stress developed within a generic SBFE subdomain j. For each analysis iteration i, the algorithm ensures yield conformity by adjusting the applied forces α^i**f** with a (positive) load multiplier α^i:

$$\alpha^i = min\left(\frac{\sigma_{y,1}}{\sigma_1^i}, \dots, \frac{\sigma_{y,n}}{\sigma_n^i}\right). \tag{18}$$

The solutions contain admissible stresses σ_j^i for all j and load multiplier α^i.

The SBFE analysis is performed for i = 1 to a iterations, and selects the solution set associated with the maximum value of load multipliers α^i for all a, namely the plastic collapse load factor α_{col}:

$$\alpha_{col} = max(\alpha^1, \dots, \alpha^a). \tag{19}$$

The pseudocode summarizing the iterative SBFE procedure to obtain the collapse load α_{col} is as follows:

Step 0: Initialization

- Set maximum number of iterations (a).
- Set initial Young's modulus (i.e. $E_j^0 = E_j$ for j = 1 to n), where E_j are original elastic material properties.
- Generate (polygon) SBFE discretization model.

Step 1: Iterative SBFE analyses

- For i = 1 to a

  Solve the elastic SBFE problem for stresses using equations (11), (14) and (15).

  Calculate load multiplier α^i from Eq. (18).

  Update Young's modulus E_j^i for all n subdomains using Eqs. (16) and (17) with $\lambda = 0.05$.

  End

Step 2: Termination

- Calculate the plastic collapse load α_{col} using Eq. (19).
- Collect the corresponding stresses to predict a feasible collapse mechanism.

4 ILLUSTRATIVE EXAMPLE

A number of benchmarks and numerical examples have been successfully processed using the proposed iterative SBFE analysis approach to capture the collapse load of structures. One of these is provided herein to illustrate application of the scheme. It concerns the classical plane strain Prandtl's punch problem shown in Figure 2a [15]. The presence of strong discontinuities at the two edges of the applied loads leads to high computational costs [15], and can pose numerical challenges to standard finite element methods.

In Figure 2, the uniform (dimensionless) force of 2α was applied and lumped at an appropriate number of nodes. The material (dimensionless) properties used throughout were a Young's modulus of $E = 10^4$ and Poison's ratio of 0.4999 (viz. giving close to an incompressibility condition in an elastic analysis). Two perfectly plastic material, namely Tresca and von Mises, cases were considered, where the (dimensionless) yield stress of $\sigma_y = 2$ was adopted throughout. The specific yield functions are
for plane strain von Mises:

$$\sigma_j^i = \sqrt{3/4\{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2\}} \le \sigma_{y,j} \text{ for all } j = 1 \text{ to } n; \tag{20}$$

for plane strain Tresca:

$$\sigma_j^i = \sqrt{\{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2\}} \le \sigma_{y,j} \text{ for all } j = 1 \text{ to } n. \tag{21}$$

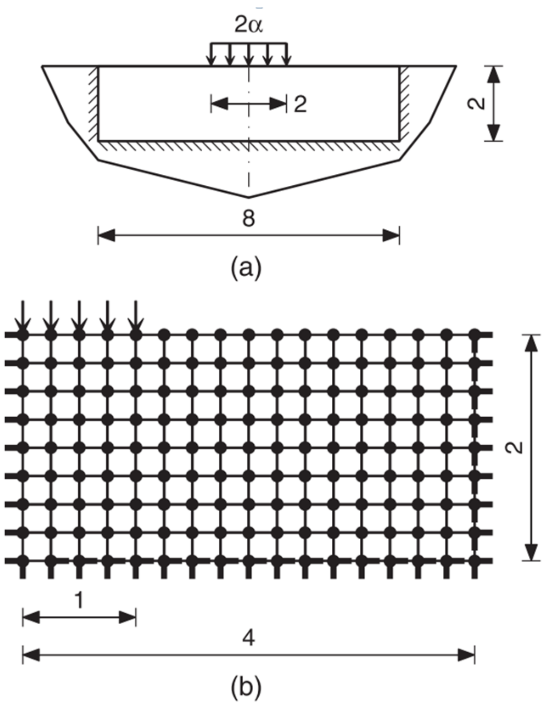

Figure 2: Prandtl's punch (a) geometry and loading, (b) discrete model; thick solid lines denote restrained directions

In view of symmetry in geometry and loading, only half of the punch structure was modeled using the SBFE discretization. We adopted a simple four-node bilinear SBFE for each subdomain. The discrete structural model consists of 128 subdomains and 512 Gauss's points.

Iterative SBFE analyses, namely 300 iterations, were successfully performed by adopting the proposed procedure. The values of load multiplier α^i obtained for all i = 1 to 300 iterations are depicted in Figure 3 for both Tresca and von Mises materials. The modification factor of $\lambda = 0.05$ provided numerical stability; α^i values increased and converged to the maximum load factor. The collapse load solutions reported were: $\alpha_{col} = 5.249$ (i.e. some 2% higher than the reference value of 5.142, directly plotted in Figure 3 [16]) for Tresca, and $\alpha_{col} = 6.06$ for von Mises.

It is worthwhile mentioning that the collapse load result does not provide a strict lower bound solution. The adopted polygon SBFE discretization, as with displacement finite elements, does not ensure equilibrium pointwise. The yield conditions are, however, enforced everywhere in the domain.

We also performed limit analyses of this plane strain Tresca structure for five discrete structural models (namely a = 8 for 2×4; a = 32 for 4×8; a = 512 for 16×32; a = 2048 for 32×64; and a = 8192 for 64×128). The collapse load results computed for all analysis cases are summarized in Table 1, where the percentage differences (Diff %) as compared to the reference value are reported.

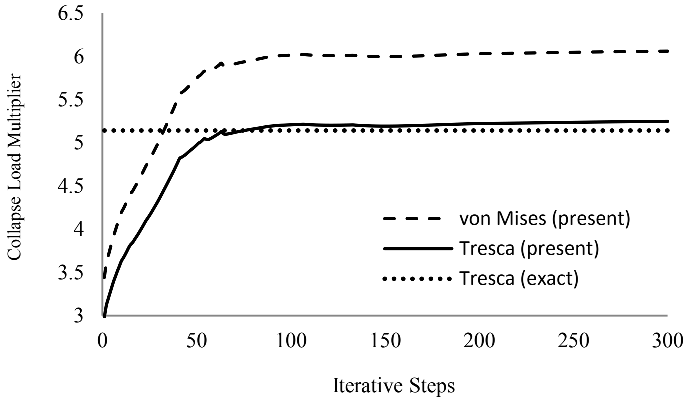

Figure 3: Iterative procedures of the modified ECM using SBFEM

Table 1: Collapse load solutions for different SBFE discretization

a	Discretization	α_{col}	Diff %
8	2×4	5.981	16.32
32	4×8	5.404	5.09
128	8×16	5.249	2.08
512	16×32	5.177	0.67
2048	32×64	5.122	–0.38
8192	64×128	5.107	–0.67

The value of collapse load α_{col} was 5.981 when the SBFE discretization was small (namely 8), and converged to a lower bound result of 5.107 at the higher (namely 8192) discretization. The diagram in Figure 4 illustrates this trend.

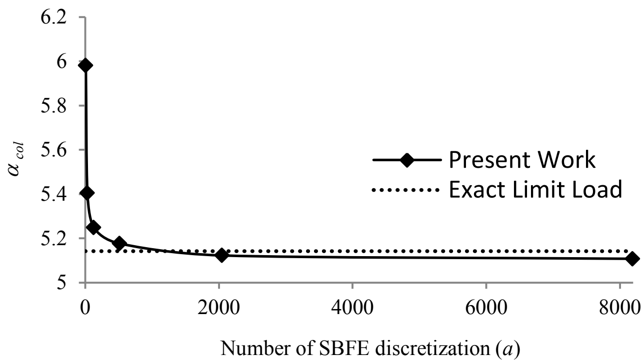

Figure 4: Variation of the collapse load limit with different SBFE discretization

5 CONCLUDING REMARKS

An iterative SBFE analysis approach has been presented to perform the limit analysis of ductile structures. The approach adopts a modified ECM for a structural system that is modeled using polygon SBFEs. The use of the polygon SBFE method offers various computational advantages. The main ones are: its ability to overcome the numerical challenges related to stress singularities, coarse mesh accuracy, and a locking free scheme under incompressibility conditions. The analysis at each iteration only involves the formulation and solution of a standard elastic SBFE problem, and hence brings with it all the inherent and additional advantages of an elastic SBFE scheme, including efficient adaptivity.

A number of numerical examples, one of which has been given, illustrate the efficiency and robustness of the proposed iterative SBFE approach to compute the collapse load limit of structures.

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