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APPLICATION OF THE PROPER GENERALIZED DECOMPOSITION TO ELASTO-PLASTIC FINITE ELEMENT ANALYSIS

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Abstract. The aim of this paper is to develop the PGD for time dependent elasto-plastic problems with cyclic loadings. The objective is to show that the PGD can be well suited for simulating a ratchet effect by separating space and time, and using a finite element approximation for both variables. The first part of this paper is dedicated to the mechanical formulation of the problem and especially the nonlinear behavior of the material. Then, the PGD approximation and the finite element discretization are presented, as well as the computational strategy. Finally, two examples are presented to show the capability of the method.

1 Introduction

This work falls within the context of the determination of asymptotic state of mechanical structures undergoing cyclic loadings. Three different states can be observed:

- the first one is purely elastic and named elastic shakedown,
- the second one is stabilized and partially plastic but the hardening is not enough to lead to an elastic state; it is named plastic shakedown,
- the third one leads to a progressive strain which increases at each cycle and is named plastic ratcheting [1].

This last phenomenon is due to a specific elasto-plastic behavior coupled with a non proportional loading. In practice, it can generate cracks. The ratchet effect is not easily predictable and induces an incremental increase of the plastic strain at each loading cycle. After a certain number of loading cycles, an asymptotic state can be observed when the increment becomes constant. Thus, the simulation of an asymptotic state requires the computation of a lot of loading cycles (often hundred and even thousands of cycles) and the cost of this calculation is very high and sometimes unaffordable. Consequently, one important industrial goal consists in developing a efficient computational strategy able to determine asymptotic states due to plastic ratcheting.

In order to get the asymptotic states, various methods have been used in literature. Inglebert *et al.* [4] have proposed a simplified method to study directly the asymptotic state of a structure without considering the transient stage. Maitournam *et al.* [5] have proposed an approach based on a truncated Fourier serie. Cognard and Ladeveze [6] have adapted the LArge Time INcrement Method (LATIN). These methods are very similar to the harmonic balance method [7] used by structural dynamicists to find approximated solutions of periodic nonlinear problems. Unfortunately these methods do not take into account the ratcheting phenomenon which is one of the goals of this work. Moreover, the problem with these methods is that they lead to linear systems of size n times greater (where n is the number of harmonics or basic functions considered) than classical step by step methods. Harmonics are coupled and convergence is often difficult.

The use of Reduced Order Model methods [8] seems to be a promising way to deal with these problems. The Proper Generalized Decomposition (PGD) [9] is one of them and is well known for its good results in the case of linear problems. The PGD is an *a priori* method and this means it involves no computation of a reference base contrary to the Proper Orthogonal Decomposition introduced by Lumley [10]. The approximation base is enriched on the fly and the solution is then obtained as successive sums of functions products also called PGD modes. For example, in the case of a 3D space problem, the PGD allows a space decomposition and then the approximation is written as $u(x, y, z) = \sum_{i=1}^{N} X_i(x)Y_i(y)Z_i(z)$ where X, Y, Z are the functions associated to each space dimension. This leads to solve three one dimensional problems instead of a three dimension one. In the same way, PGD can also be applied to problems where additional parameters are considered as new dimensions and the solution is given by $u(x, \theta^1, \ldots, \theta^k) = \sum_{i=1}^{N} X_i(x)\Theta_i^1(\theta) \ldots \Theta_i^k(\theta^k)$ where θ^1,. . ., θ^k could be material, geometric, stochastic or other type of parameters. This allows to compute abbacus taking into account the parameter variations [11, 12]. In a lot of cases, the PGD approximation of linear problem needs the calculation of few modes and leads to efficient computations. To deal with non-linear problems, the LATIN method presented by Ladeveze [13] suggests to separate the space and the time by considering that the spatial problem is linear whereas the non-linearity due to elasto-plasticity is treated in the 1D problem in time.

The aim of this paper is to develop the PGD for time dependent elasto-plastic problems with cyclic loadings. The objective is to show that the PGD can be well suited for simulating a ratchet effect by separating space and time, and using a finite element approximation for both variables. The first part of this paper is dedicated to the mechanical formulation of the problem and especially the nonlinear behavior of the material. Then, the PGD approximation and the finite element discretization are presented, as well as the computational strategy. Finally, two examples are presented to show the capability of the method.

2 Mechanical formulation

The mechanical analysis is based on the momentum equation where inertial effects are neglected.

$$\begin{cases} \boldsymbol{div}(\boldsymbol{\sigma}) = \mathbf{0} & \text{in } \Omega \quad (1a) \\ \boldsymbol{\sigma}\,.\mathbf{n} = \mathbf{T}^p & \text{on } \Gamma_T \quad (1b) \\ \mathbf{u} = \mathbf{u}^p & \text{on } \Gamma_u \quad (1c) \end{cases}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor; $\mathbf{n}$ is the unit outward normal vector to the boundary $\partial\Omega$ of the study domain Ω; $\mathbf{T}^p$ is a prescribed "stress vector"; $\mathbf{u}^p$ is an imposed value of the displacement $\mathbf{u}$. Furthermore, $\Gamma_T \cup \Gamma_u = \partial\Omega$ and $\Gamma_T \cap \Gamma_u = \emptyset$.

It is performed with the infinitesimal strain theory. Therefore, $\boldsymbol{\sigma}$ is assumed to depend linearly on the elastic strain tensor $\boldsymbol{\varepsilon}^e$ by means of the fourth rank elastic tensor $\mathbf{C}$ as follows

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}^e \qquad (2)$$

Just like the majority of classical elasto-plastic models for solid metals, the proposed approach is based on the assumption of additive decomposition of the strain $\boldsymbol{\varepsilon}$ into an elastic part $\boldsymbol{\varepsilon}^e$ and a plastic part $\boldsymbol{\varepsilon}^p$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \qquad (3)$$

In this work, the material is assumed to follow a von Mises plastic behavior described by a non-linear kinematic strain-hardening model. The main objective is to simulate a ratchet effect due to a cyclic loading applied during a time interval T. Then the kinematic stain-hardening variable $\boldsymbol{\chi}$ is governed by the Armstrong-Frederick law:

$$\dot{\boldsymbol{\chi}} = \frac{2}{3}\, c\, \dot{\boldsymbol{\varepsilon}}^p - \gamma\, \boldsymbol{\chi}\, \dot{\varepsilon}^p_{eq} \qquad (4)$$

where $\dot{\varepsilon}^p_{eq} = \sqrt{\frac{3}{2}\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p}$ denotes the equivalent plastic strain rate; c and γ are two material constants.

To apply the finite element procedure in space and time, the weak formulation for the mechanical problem is obtained by multiplying equation (1a) by weighting function $\delta\boldsymbol{u}$. Integrating by parts over the domain Ω, accounting for the boundary condition (1b) and integrating over the time domain T, one thus obtains the following weak formulation of the problem:

$$\int_T \int_\Omega \boldsymbol{\varepsilon}\left(\delta\boldsymbol{u}\right) : \boldsymbol{\sigma}\,\mathrm{d}v\,\mathrm{d}t = \int_T \int_{\Gamma_T} \delta\boldsymbol{u}\,\boldsymbol{T^p}\,ds\,\mathrm{d}t \tag{5}$$

with

$$\boldsymbol{\varepsilon}\left(\delta\boldsymbol{u}\right) = \frac{1}{2}\left(\mathbf{grad}\,\delta\boldsymbol{u} + \mathbf{grad}^t\,\delta\boldsymbol{u}\right) \tag{6}$$

3 Computational scheme

The main objective of this work is to solve the previous non-linear mechanical problem with a cyclic loading for simulating a ratchet effect with the PGD. Then we assume that the approximate solution $\boldsymbol{u}\left(\boldsymbol{x},t\right)$ can be written as follows:

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{i=1}^{\infty} \boldsymbol{d_i}(\boldsymbol{x}) f_i(t) \tag{7}$$

where each PGD mode is build as a product of two functions $\boldsymbol{d_i}(\boldsymbol{x})$ and $f_i(t)$ by separating the space and the time.

3.1 Progressive construction of the PGD approximation

As PGD approximation is a sum of modes, the approximation of the displacement is enriched at each resolution step by adding a new mode which is computed by considering the n previous known:

$$\boldsymbol{u}_{n+1}(\boldsymbol{x},t) = \sum_{i=1}^{n} \boldsymbol{d_i}(\boldsymbol{x}) f_i(t) + \boldsymbol{D}\left(\boldsymbol{x}\right)F\left(t\right) = \boldsymbol{u}_n(\boldsymbol{x},t) + \boldsymbol{D}\left(\boldsymbol{x}\right)F\left(t\right) \tag{8}$$

where $\boldsymbol{D}(\boldsymbol{x}) = \boldsymbol{d}_{n+1}(\boldsymbol{x})$ and $F(t) = f_{n+1}(t)$ are the parts of the new mode to compute.

For calculating the new mode $\boldsymbol{D}(\boldsymbol{x})\,F(t)$, the weighting function is written from equation (8) as:

$$\delta\boldsymbol{u} = \delta\boldsymbol{D}\,F + \boldsymbol{D}\,\delta F \tag{9}$$

Thus, the weak form (5) becomes:

$$\forall\delta\boldsymbol{D}, \int_{\partial\Omega} \delta\boldsymbol{D} \int_T F\,\mathbf{T}^p dt\,ds - \int_\Omega \boldsymbol{\varepsilon}(\delta\boldsymbol{D}) \int_T F\boldsymbol{\sigma}(\boldsymbol{u}_n + \boldsymbol{D}F)dt\,dx = \mathbf{0} \tag{10}$$

$$\forall\delta F, \int_T \delta F \int_{\partial\Omega} \boldsymbol{D}\,\mathbf{T}^p ds\,dt - \int_T \delta F \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{D})\boldsymbol{\sigma}(\boldsymbol{u}_n + \boldsymbol{D}F)dx\,dt = \mathbf{0} \tag{11}$$

3.2 Space and time finite element discretization

Following the usual finite element procedure, the weak formulations (10) and (11) are applied to functions $\boldsymbol{D}$ and F of the form

$$\boldsymbol{D}(\boldsymbol{x}) = \sum_{p=1}^{n_D} \boldsymbol{D}_p \, N_p(\boldsymbol{x}) \ ; \ F(t) = \sum_{q=1}^{n_F} F_q \, N_q(t). \tag{12}$$

In these expressions n_D and n_F denote respectively the number of nodes of the 3D mesh of the structure and the 1D mesh used for the time integration. $\boldsymbol{D}_p$ and F_p are the values of the functions $\boldsymbol{D}$ and F at nodes p in space and q in time. $N_p(\boldsymbol{x})$ and $N_q(t)$ the shape functions associated to these nodes. Following Galerkin's standard approach, the weighting functions $\delta\boldsymbol{D}$ and δF are taken of the same form.

3.3 Computational procedure

To compute $D(x)$ and $F(t)$ we use a two steps algorithm. The first one is based on the LATIN concept [6]: $\boldsymbol{D}(\boldsymbol{x})$ is obtained from an elastic calculation considering $f_n(t)$ as a satisfactory approximation of $F(t)$ in expression (8). Thus, the first finite element step consists in solving the following linear problem to build $\boldsymbol{D}$:

$$\begin{aligned} \forall \delta\boldsymbol{D}, \int_\Omega \boldsymbol{\varepsilon}(\delta\boldsymbol{D}) : \left(\int_T f_n \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{D}) \, dt \right) dx \\ = \int_{\partial\Omega} \delta\boldsymbol{D} \int_T f_n \, \mathbf{T}^p \, dt \, ds - \int_\Omega \boldsymbol{\varepsilon}(\delta\boldsymbol{D}) : \left(\int_T f_n \boldsymbol{\sigma}(\boldsymbol{u}_n) \, dt \right) dx \end{aligned} \tag{13}$$

One can note that this first calculation is solved on the whole mesh of the structure. It leads to an efficient computation time since it is linear. The second step deals with the non-linear aspect of the problem which is due to elasto-plasticity. It consists in computing F by solving the non-linear equation (11) written as follows:

$$\forall \delta F, \int_T \delta F \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{D}) \boldsymbol{\sigma}(\boldsymbol{u}_n + \boldsymbol{D}F) \, dx \, dt = \int_T \delta F \int_{\partial\Omega} \boldsymbol{D} \, \mathbf{T}^p ds \, dt \tag{14}$$

Equation (14) is solved by means of a Newton-Raphson algorithm on the 1D mesh used for the finite element approximation of F in time. One can note that this approach leads to solve linear systems involving band matrices since F is one dimensional. We thus obtain efficient computing time.

4 Applications

4.1 Traction on a holed plate

In this first application, a squared plate of 4mm length and 0.1mm thickness with a hole of diameter 1mm in its center is subjected to a monotonous traction loading as shown in fig. 1. The loading is applied from 0 to 58 MPa in 6s and the computation time step is taken equal to 0.5s. For symmetry reasons, only a quarter of the plate has been modeled in 3D. The mesh is plotted in figure 1.

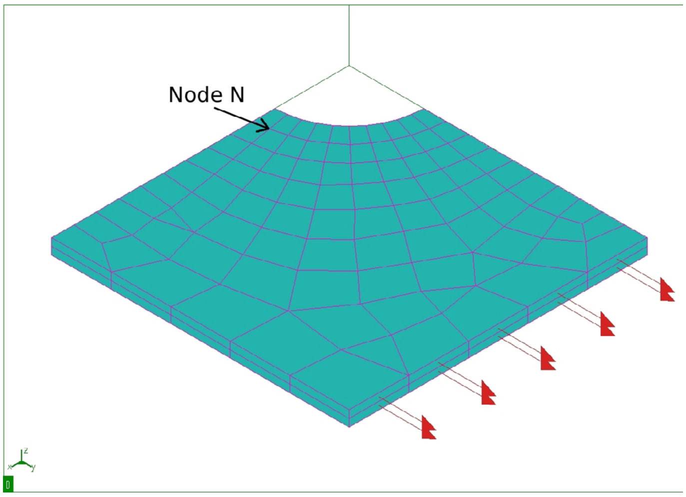

Figure 1: Mesh and loading

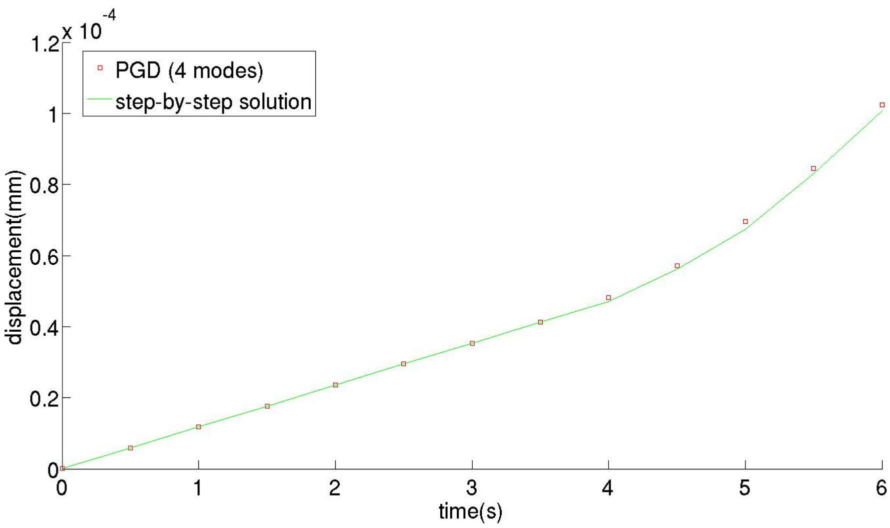

Figure 2: Computed displacement at point N at time t=6s

The plastic behavior is described by an isotropic strain-hardening. The Young's modulus E, the Poisson's coefficient ν, the yield stress and $\frac{\partial \sigma}{\partial \epsilon^p}$ are respectively taken equal to 210 000 MPa, 0.3, 100 MPa and 5000 MPa. The results given by the the PGD approach are compared to the ones obtained with a standard step-by-step finite element approach. All the computations have been performed using the computer code Systus®[14]. Figure 2 shows the displacement using 4 PGD modes. The comparison with the reference solution shows a good agreement.

4.2 Cyclic loading of an elasto-plastic cube

A second test is performed to simulate the cyclic loading of a cubic metal piece of 2mm length side. The loading cycle is composed of a non-proportionnal stress including a traction step followed by a compression step. The time step is taken equals to 0.1s which corresponds to the length of the 1D finite elements used for the approximation of the function F. Fig. 3 shows the evolution of the prescribed stress.

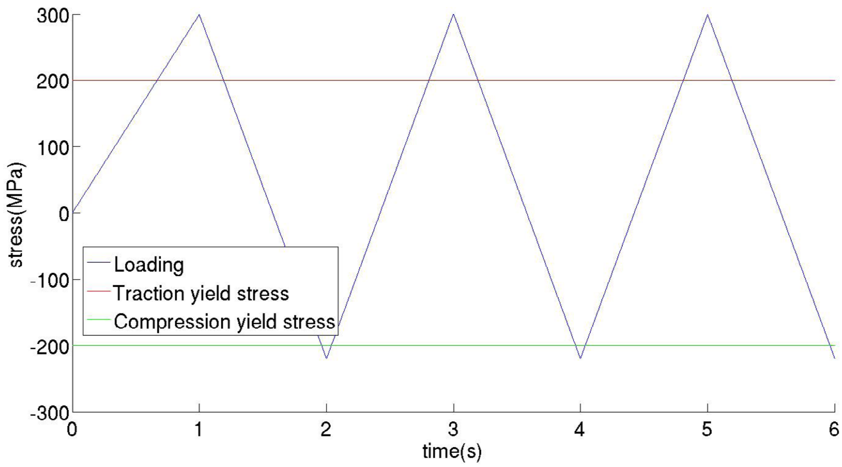

Figure 3: Uniaxial cyclic loading

The mesh is composed of one finite element of type $Q1$ with selective integration. The plastic behavior is described by a nonlinear strain-hardening Armstrong-Frederick's law which is able to model a ratchet phenomenon. The Young's modulus E, the Poisson's coefficient ν, the yield stress, c and γ are respectively taken equal to 210 000 MPa, 0.3, 200 MPa, 25 000 and 125. The results given by the the PGD approach are compared to the ones obtained with a standard step-by-step finite element computation. To accelerate the modal convergence for PGD, the whole first mode composed of $\boldsymbol{d}_1$ and f_1 has been computed in a purely elastic way. Fig. 4 shows the longitudinal displacement using 2 PGD modes. The results are in good agreement. One can note the increase of the displacement which induces an incremental increase of the plastic strain at each cycle. This corresponds to plastic ratcheting.

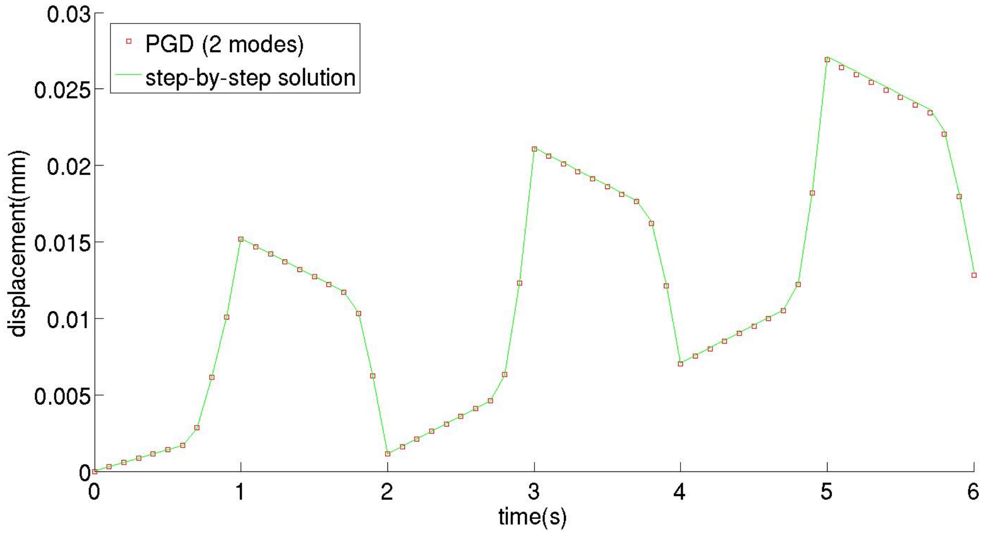

Figure 4: Computed elongation of the elasto-plastic cube

5 Conclusion

In this work, the PGD has been applied to simulate elasto-plastic problems by separating space and time as suggested by the LATIN approach. The finite element procedure has been used for both separated variables. The examples evidence the capability of the PGD approach to deal with non-linear material behaviors, and especially the plastic ratcheting.

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