Geometric Biplane Graphs II: Graph Augmentation *

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Abstract

We study biplane graphs drawn on a finite point set S in the plane in general position. This is the family of geometric graphs whose vertex set is S and which can be decomposed into two plane graphs. We show that every sufficiently large point set admits a 5-connected biplane graph and that there are arbitrarily large point sets that do not admit any 6-connected biplane graph. Furthermore, we show that every plane graph (other than a wheel or a fan) can be augmented into a 4-connected biplane graph. However, there are arbitrarily large plane graphs that cannot be augmented to a 5-connected biplane graph by adding pairwise noncrossing edges.

1   Introduction

In a *geometric graph* $G = (V, E)$, the vertices are distinct points in the plane in general position (that is, no three points in S are collinear) and the edges are straight line segments between pairs of vertices. A *plane graph* is a geometric graph in which no two edges cross. It is well known that every planar graph can be realized as a plane graph by Fáry's theorem [14].

We consider a generalization of plane graphs. A geometric graph $G = (V, E)$ is *k-plane* for some $k \in \mathbb{N}$ if its edge set can be partitioned into k disjoint subsets, $E = E_1 \uplus \ldots \uplus E_k$, such that $G_1 = (V, E_1), \ldots, G_k = (V, E_k)$ are all plane graphs, where $\uplus$ represents the disjoint union. For a finite point set S in the plane in general position, denote by $\mathcal{G}_k(S)$ the family of k-plane graphs with vertex set S. With this terminology, $\mathcal{G}_1(S)$ is the family of plane graphs with vertex set S, and $\mathcal{G}_2(S)$ is the family of 2-plane graphs (also known as *biplane graphs*) with vertex set S.

We contrast several combinatorial properties of plane graphs $\mathcal{G}_1(S)$ and biplane graphs $\mathcal{G}_2(S)$ for point sets S in this and a companion paper [18]. For example, it is well known that the vertex connectivity of every plane graph in $\mathcal{G}_1(S)$ is at most 5, but it is natural to expect that the larger family $\mathcal{G}_2(S)$ contains graphs of higher vertex connectivity. A graph $G = (V, E)$ in $\mathcal{G}_2(S)$ is maximal if there is no graph $G' = (V, E')$ in $\mathcal{G}_2(S)$ such that $E \subset E'$. In [18] we compare combinatorial properties of maximal biplane graphs in $\mathcal{G}_2(S)$ with triangulations in $\mathcal{G}_1(S)$, and show that there are arbitrarily large point sets S for which $\mathcal{G}_2(S)$ contains an 11-connected graph, but no biplane graph is 12-connected. In this paper, we study the maximum

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vertex connectivity of a graph in $\mathcal{G}_2(S)$ for a given point set S. We also consider closely related connectivity augmentation problems. We refer to [18] and the references therein for a broad overview of further related work.[1]

Organization. The problem of constructing a plane graph with the largest possible vertex- or edge-connectivity on a given point set has received significant attention [10, 15, 16]. The combinatorial aspect of the problem asks for characterizing the point sets that admit graphs of a certain connectivity, and the algorithmic aspect is to develop algorithms for computing highly connected graphs. These are the topics we study in Section 2, considering biplane graphs instead of plane graphs.

A closely related family of problems is the *graph augmentation*, in which one would like to add new edges, ideally as few as possible, to a given graph in such a way that some desired property is achieved. There has been extensive work on augmenting disconnected plane graphs to connected ones (see [22] for a recent survey) or achieving good connectivity properties [1, 2, 3, 12, 24, 25, 27]. For abstract graphs, this corresponds to the classical connectivity augmentation problem in combinatorial optimization and has a rich history, as well. In Section 3, we consider several problems on augmenting plane graphs to biplane graphs with higher connectivity. We conclude in Section 4 with some final remarks and open problems.

2 Drawing Biplane Graphs from Scratch

Given a set S of n points in general position, we would like to construct a graph $G \in \mathcal{G}_2(S)$ with high vertex connectivity $\kappa(G)$. We determine the maximum $\kappa(G)$ that can be attained for every (sufficiently large) point set S. We also consider the special case that the point set S is in convex position.

For comparison, we briefly review analogous results for plane graphs $\mathcal{G}_1(S)$ on a given point set S. Refer to [22] for a survey. Every set of $n \geq 3$ points in general position admits a spanning cycle (a polygonization of S), which is 2-connected. For points in convex position, every plane graph has a vertex of degree 2, so in this case $\kappa(G) = 2$ is the best possible value over all $G \in \mathcal{G}_1(S)$. Since every planar graph has a vertex of degree not greater than 5, the vertex connectivity of every graph in $\mathcal{G}_1(S)$ is at most 5. It is known that every set of $n \geq 4$ points not in convex position admits a 3-connected triangulation. Additionally, every set of $n \geq 6$ points whose convex hull is a triangle admits a 4-connected triangulation, provided that a certain condition is satisfied (see [10] for details). Characterizations for point sets in general position that admit 4-connected triangulations have only recently been proposed [11, 16, 17], and no characterization is known for 5-connectivity.

2.1 Point Sets in Convex Position

We begin with the special case of points in convex position. It turns out that all biplane graphs on a point set S in convex position are planar. Moreover, the maximum number of edges (resp., the maximum vertex connectivity) of a graph in $\mathcal{G}_2(S)$ is the same as the maximum attained in the family of planar graphs with $|S|$ vertices.

Lemma 1. *Let S be a set of n points in the plane in convex position.*

(i) *Every graph in $\mathcal{G}_2(S)$ is planar (as an abstract graph).*

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[1]**Note:** The companion paper [18] contains a larger introduction to the concept of *biplane graphs*, comparing it with other related concepts such as the *geometric thickness* and others. Ideally, both papers will appear in the same journal issue. Preliminary versions of both papers have been presented at the *Mexican Conference on Discrete Mathematics and Computational Geometry (Oaxaca, 2013)* [18, 19]. To avoid repetition, a complete introduction is presented in the companion paper, and here we include only a brief self-contained introduction.

(ii) *If $G = (V, E)$ is a Hamiltonian planar (abstract) graph with n vertices, then it has a geometric realization in $\mathcal{G}_2(S)$.*

Remark: In [6], Bernhart and Kainen show two results (Lemma 2.1 and Theorem 2.5), given in terms of book thickness, that are more general than Lemma 1. Since it is straightforward to see that Lemma 1 follows from those results, we omit its proof.

Note, however, that not every planar graph can be realized as a biplane graph on a point set in convex position. If S is in convex position, then the boundary of the convex hull $ch(S)$ forms a Hamiltonian cycle in every maximal (i.e., edge-maximal) graph in $\mathcal{G}_2(S)$. Hence every maximal graph in $\mathcal{G}_2(S)$ is planar and Hamiltonian. However, there are maximal planar graphs (triangulations) that are not Hamiltonian. (It is NP-complete to decide whether a maximal planar graphs is Hamiltonian [9, 30].) These planar graphs cannot be realized as a biplane graph on a point set in convex position.

We can now characterize the maximum vertex connectivity of a graph in $\mathcal{G}_2(S)$ when S is in convex position.

Theorem 1. *Let S be a set of n points in convex position.*

- $\mathcal{G}_2(S)$ *contains a 4-connected graph if and only if $n \geq 6$.*
- $\mathcal{G}_2(S)$ *contains a 5-connected graph if and only if $n = 12$ or $n \geq 14$.*
- $\mathcal{G}_2(S)$ *contains no 6-connected graphs for any $n \in \mathbb{N}$.*

Proof. It is well known that every 4-connected planar graph G has a Hamiltonian cycle [29]. By Lemma 1(ii), it is enough to establish the existence or nonexistence of a k-connected planar graph for a given n for $k = 4$, 5, and 6.

By Lemma 1(i), every graph in $\mathcal{G}_2(S)$ is planar. Every planar graph on $n \geq 3$ vertices has at most $3n-6$ edges, and the sum of vertex degrees is at most $6n-12$. In a k-connected graph, the degree of every vertex is at least k, and the sum of vertex degrees is at least kn. Comparing these bounds, we have $kn \leq 6n - 12$ or $12/(6 - k) \leq n$. It follows that no planar graph is 6-connected, every 5-connected planar graph has at least 12 vertices, and every 4-connected planar graph has at least 6 vertices.

It is easy to see that there is a 4-connected planar graph on n vertices for every $n \geq 6$. Specifically, the 1-skeleton of the octahedron is 4-connected with 6 vertices; and a vertex split operation can increase the number of vertices by one while maintaining 4-connectivity and planarity. In an embedding of a 4-connected planar graph on n vertices, this split operation removes an edge (u, v) and adds a new vertex connected to all the vertices of the two faces adjacent to (u, v).

Barnette [4] and Butler [8] independently designed algorithms for generating all 5-connected triangulations, using simple operations starting from the icosahedron (see also [7]). The 1-skeleton of the icosahedron is 5-connected with 12 vertices, and each operation either splits a vertex of degree 6 or higher, or simultaneously splits two adjacent vertices. Hence there is a 5-connected planar graph for $n = 12$ and for every $n \geq 14$ (but not for $n = 13$). □

Remark. We have shown that $\mathcal{G}_2(S)$ contains 4- and 5-connected graphs when $n \geq 6$ and $n \geq 14$, respectively. The existence proof in Theorem 1 can be turned into an $O(n)$-time algorithm for constructing such biplane graphs. Here we present explicit constructions for 5-connected biplane graphs for points in convex position when $n = 12$ and $n \geq 14$. The construction in Figure 1 (left) works when n is even and $n \geq 12$. It is based on two plane spanning trees, T_1 and T_2, and the edges of $ch(S)$. Each spanning tree consists of two stars with 3 leaves each, connected by a zig-zag path. Let us assume that the points are numbered clockwise. Then, the centers of the stars of T_1 (the nondashed spanning tree) are placed at opposite points, say i and $i + n/2$, the leaves of the first star are placed at points $i + 1, i + 2$

and $i + 3$ and the leaves of the second star are placed at points $i + n/2 + 1, i + n/2 + 2$ and $i + n/2 + 3$. The zig-zag path connects the centers of the stars visiting alternatively the points $\{i + 4, \ldots, i + n/2 - 1\}$ and the points $\{i - 1, i - 2, \ldots, i + n/2 + 4\}$. Tree T_2 (the dashed spanning tree) is symmetric to T_1, where point $i + n/2 + 1$ is the image of point i and $i + 1$ is the image of point $i + n/2$. Note that the only common edges to both trees are $(i, i + 1)$ and $(i + n/2, i + n/2 + 1)$. The construction in Figure 1 (right) works when n is odd and $n \geq 15$. It is analogous to the previous construction, but one of the stars in T_1 and T_2 has 4 leaves instead of 3.
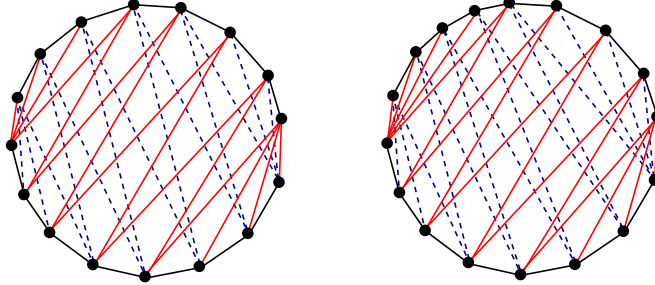

Figure 1: Two 5-connected biplane graphs for points in convex position.

2.2 Point Sets in General Position

In this section we find the largest $k \in \mathbb{N}$ such that every sufficiently large point set S in general position admits a k-connected biplane graph. Hutchinson et al. [23] proved that every biplane graph in $\mathcal{G}_2(S)$ has at most $6n - 18$ edges for $n \geq 8$. In particular, this implies that every biplane graph contains a vertex of degree 11 or less, hence $k \leq 11$. Theorem 1 directly improves the bound to $k \leq 5$. We now show that this bound is tight, that is, every sufficiently large point set S admits a 5-connected biplane graph.

Theorem 2. *Let S be a set of n points in the plane in general position. If S contains at least 14 points in convex position, then there is a 5-connected graph in $\mathcal{G}_2(S)$.*

Erdős and Szekeres proved that for every $k \in \mathbb{N}$ there is an integer $f(k)$ such that every set of at least $f(k)$ points in the plane in general position contains a subset of k points in convex position. They conjectured $f(k) = 2^{k-2} + 1$, and showed $f(k) \geq 2^{k-2} + 1$. The currently best upper bound [28] for $k \geq 7$ is $f(k) \leq \binom{2k-5}{k-2} + 1$. When $k = 14$, this result implies that every set S of n points in general position has a subset of at least 14 points in convex position for $n \geq \binom{2 \cdot 14 - 5}{14 - 2} + 1 = 1352079$. Therefore, every sufficiently large point set S in general position satisfies the condition in Theorem 2.

Corollary 1. *If S is a set of $n \geq 1352079$ points in the plane in general position, then there is a 5-connected biplane graph in $\mathcal{G}_2(S)$.*

Outline. The remainder of Section 2.2 is devoted to the proof of Theorem 2. Our approach is as follows: given a point set S, let $S_0 \subset S$ be a largest subset of points in convex position (in case of ties, choose a set S_0 whose convex hull has the largest area). By assumption, we have $|S_0| \geq 14$. By Theorem 1, S_0 admits a 5-connected biplane graph G_0. We increment G_0 with new vertices from $S \setminus S_0$, maintaining a 5-connected biplane graph, in 3 phases: we first insert the points lying in the interior of $\text{ch}(S_0)$, then the vertices of $\text{ch}(S)$, and finally all remaining points (which lie in the exterior of $\text{ch}(S_0)$). We continue with the details.

4

Preliminaries. To ensure that we maintain 5-connectivity, we use the following well-known properties of graphs.

Property 1. *Let $G = (V, E)$ be a k-connected (abstract) graph. Augment G with a new vertex x joined to k vertices of G. Then the new graph on vertex set $V \cup \{x\}$ is also k-connected.*

Property 2. *Let $G = (V, E)$ be a k-connected (abstract) graph in which vw is an edge. Remove edge vw from G, and augment it with a new vertex x joined to both v, w and to $k-2$ additional vertices. Then the new graph with vertex set $V \cup \{x\}$ is also k-connected.*

Since adding edges to a graph can only increase the vertex connectivity, we may assume in each phase of our algorithm that we have started with a 5-connected maximal biplane graph. Thus, we can rely on the following two structural results for maximal biplane graphs from the companion paper [18].

Lemma 2. *[18] Let $G = (S, E)$ be a maximal biplane graph in $\mathcal{G}_2(S)$. Then there are two triangulations $T_1 = (S, E_1)$ and $T_2 = (S, E_2)$ such that $E = E_1 \cup E_2$.*

Given an edge $e \in E$ in a triangulation $T = (S, E)$, we denote by $Q(e)$ the quadrilateral formed by the two triangles adjacent to e. Note that $Q(e)$ is not defined if e is an edge of ch(S). An edge e is *flippable* if and only if $Q(e)$ is a convex quadrilateral.

Lemma 3. *[18] Let $G = (S, E)$ be a maximal biplane graph in $\mathcal{G}_2(S)$ such that $E = E_1 \cup E_2$, where $T_1 = (S, E_1)$ and $T_2 = (S, E_2)$ are two triangulations. Every edge of $E_1 \cap E_2$ is flippable in neither T_1 nor T_2. Furthermore, every maximal biplane graph with $n \geq 4$ vertices is 3-connected.*

The following tool (Lemma 4) is crucial for increasing the vertex degree of a vertex in a triangulation. This tool is applicable to all triangulations other than the wheel. (A *wheel* is a triangulation on n points such that $n - 1$ points are in convex position and one point lies in the interior of ch(S), the points on the convex hull induce a cycle on the boundary of ch(S) and the interior point is joined to all other $n - 1$ points.)

Lemma 4. *Let $T = (S, E)$ be a triangulation other than the wheel. Let $s \in S$ be a point in the interior of ch(S) such that it is adjacent to a vertex on the boundary of ch(S), and the graph induced by its neighbors in T is a cycle. Then T contains a triangle incident to s in which the edge opposite to s is flippable.*

Proof. Denote the neighbors of s by $v_1, v_2, \ldots, v_k \in S$, for some $k \geq 3$, in counterclockwise order. Since T is not a wheel, some edges of the cycle $(v_1, \ldots, v_k)$ are not on the boundary of ch(S). Without loss of generality, assume that v_1 is a vertex of ch(S) but $v_1 v_2$ is not on the boundary of ch(S). Note that $Q(v_1 v_2)$ is defined, and it has a convex vertex at v_1.

Starting with $i = 1$ we use the following iterative argument: we know that $Q(v_i v_{i+1})$ is defined and has a convex vertex at v_i. If $v_i v_{i+1}$ is flippable, we are done. Otherwise, $Q(v_i v_{i+1})$ is a nonconvex quadrilateral and has a reflex vertex at v_{i+1}. It follows that v_{i+1} is in the interior of ch(S), and thus $Q(v_{i+1} v_{i+2})$ is defined. Since the neighbors of s induce a cycle, we have $v_i v_{i+2} \notin E$. Therefore $v_i v_{i+1}$ and $v_{i+1} v_{i+2}$ are not adjacent to a common triangle. Since v_{i+1} is a reflex vertex of $Q(v_i v_{i+1})$, it must be a convex vertex of $Q(v_{i+1} v_{i+2})$. Thus, we can increment the value of i and repeat the same argument. This process ends as soon as we find a flippable edge or when we conclude that none of the edges $v_i v_{i+1}$, $i = 1, \ldots, k - 1$ is flippable. However, in the latter case, the above argument implies that v_1 is in the interior of ch(S), contradicting our initial assumption. $\qquad\square$

Inserting interior vertices. The following lemma allows augmenting a 5-connected biplane graph with an *interior* point.
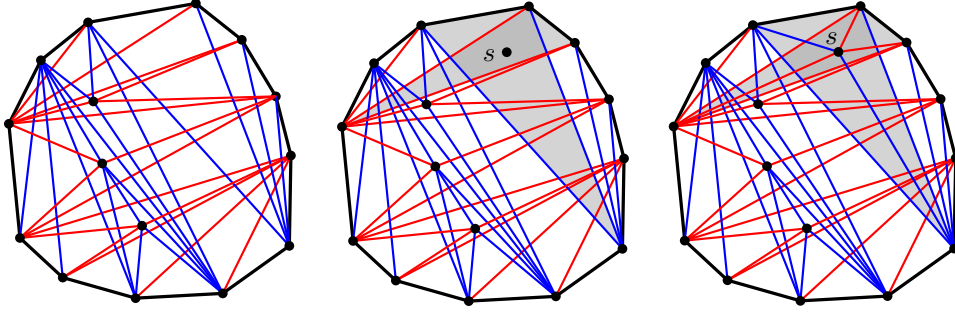
Figure 2: Left: 5-connected biplane graph on 15 points. Middle: point s lies in the interior of two gray triangles, which jointly have 5 distinct vertices. Right: point s is now part of the 5-connected biplane graph.

Lemma 5. *Let $G = (S, E)$ be a 5-connected biplane graph. Denote by $S_{\text{int}} \subset S$ the points lying in the interior of $\text{ch}(S)$, and let $s \notin S$ be a point such that s is in the interior of $\text{ch}(S)$ but in the exterior of $\text{ch}(S_{\text{int}})$. Then a 5-connected biplane graph on $S \cup \{s\}$ can be constructed from $G = (S, E)$ by adding at least 5 new edges incident to s and deleting at most one edge of E.*

Proof. Augment $G = (S, E)$ to a 5-connected maximal biplane graph $\widehat{G} = (S, \widehat{E})$ by adding dummy edges, if necessary. By Lemma 2, $\widehat{G}$ is the union of two triangulations, T_1 and T_2. Point s lies in the interior of some triangles Δ_1 and Δ_2 in the two triangulations (Δ_1 and Δ_2 may share vertices and edges). Since s lies in the exterior of $\text{ch}(S_{\text{int}})$, at least one vertex of Δ_1 (resp., Δ_2) is on the boundary of $\text{ch}(S)$. Let us augment T_1 (resp., T_2) with vertex s and three edges joining s to the vertices of Δ_1 (resp., Δ_2) to a new triangulation T_1' (resp., T_2') in which s is adjacent to a vertex of $\text{ch}(S)$. We distinguish three cases based on the total number of distinct vertices of Δ_1 and Δ_2.

Case 1: Δ_1 and Δ_2 jointly have 5 or 6 distinct vertices. We have joined s to at least 5 distinct vertices of G (Figure 2). The union of T_1' and T_2' is biplane and 5-connected by Property 1.

Case 2: Δ_1 and Δ_2 jointly have 4 distinct vertices. In this case, Δ_1 and Δ_2 share an edge (see Figure 3), say $\Delta_1 = v_1 v_2 v_3$ and $\Delta_2 = v_1 v_2 v_4$. Since s is in the interior of both Δ_1 and Δ_2, the points v_3 and v_4 are on the same side of the line $v_1 v_2$. Since v_4 is in the exterior of Δ_1 and v_3 is in the exterior of Δ_2, the convex hull $\text{ch}(v_1, v_2, v_3, v_4)$ is a convex quadrilateral. Without loss of generality, we may assume $\text{ch}(v_1, v_2, v_3, v_4) = (v_1, v_2, v_3, v_4)$ in counterclockwise order. By Lemma 4, T_1' (resp., T_2') has a flippable edge e_1' (resp., e_2') in a triangle opposite to s. If flipping edge e_1' in T_1' or edge e_2' in T_2' increases the degree of s to 5, then perform the edge flip. By Property 2, the union of the two triangulations is a 5-connected biplane graph.

Assume now that neither flipping e_1' in T_1' nor e_2' in T_2' increases the degree of s to 5. This implies that the third vertex of the triangles adjacent to e_1' and e_2', respectively, are v_4 and v_3. That is, the 4-cycle (v_1, v_2, v_3, v_4) is part of both triangulations T_1' and T_2'. After flipping e_1' in T_1', the cycle (v_1, v_2, v_3, v_4) has a flippable edge $\hat{e}$ by Lemma 4. We can flip $\hat{e}$ in T_1' to increase the degree of s to 5, while retaining edge $\hat{e}$ in the other triangulation T_2'. By Property 2, the union of these two triangulations is a 5-connected biplane graph.

Case 3: Δ_1 and Δ_2 jointly have 3 distinct vertices. In this case, $\Delta_1 = \Delta_2$, say $\Delta_1 = \Delta_2 = v_1 v_2 v_3$. By Lemma 4, T_1' (resp., T_2') has a flippable edge e_1' (resp., e_2') in a triangle opposite to s. If $e_1' \neq e_2'$, then we can flip each edge in its corresponding triangulation, while
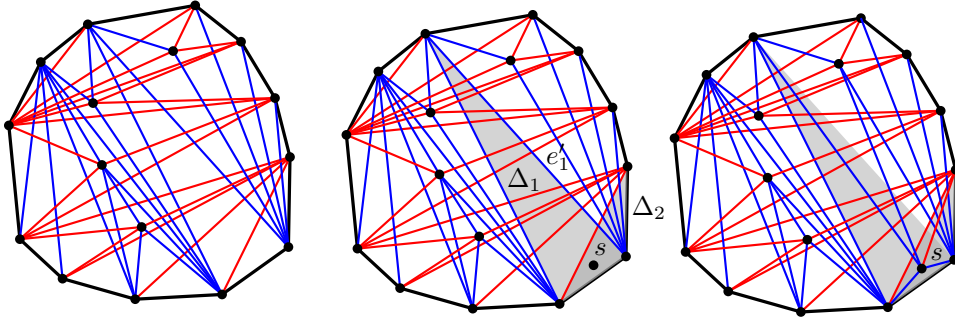
6

Figure 3: Left: 5-connected biplane graph on 16 points. Middle: point s lies in the interior of triangles Δ_1 and Δ_2, which share an edge, and e_1' is a flippable edge adjacent to Δ_1. Right: point s is now part of the 5-connected biplane graph.

keeping it in the other triangulation. Thus the degree of s increased to 5, and the union of the two triangulations forms a 5-connected biplane graph by Property 1. If $e_1' = e_2'$ but the two flips together increase the degree of s to 5, then the union of these two triangulations is a 5-connected biplane graph by Property 2 (since the two flips together remove at most one edge $e_1' = e_2'$ from G).

It remains to consider the case that $e_1' = e_2'$, say $e_1' = e_2' = v_2 v_3$, and the two flips together would only increase the degree of s to 4. That is, $\Delta_1 = \Delta_2 = v_1 v_2 v_3$ is adjacent to the same triangle, say $v_2 v_3 v_4$, in both T_1 and T_2. In particular, the 4-cycle (v_1, v_2, v_4, v_3) is part of both triangulations. We claim that the 4-cycle (v_1, v_2, v_4, v_3) has no external chords in at least one of T_1 and T_2. Clearly, the claim is true when (v_1, v_2, v_4, v_3) is a convex quadrilateral. Let us assume to the contrary that $\text{ch}(v_1, v_2, v_3, v_4)$ is a triangle, say $\Delta = v_1 v_2 v_4$, and suppose that the external chord $v_1 v_4$ belongs to both T_1 and T_2, so $\Delta = v_1 v_2 v_4$ is part of both triangulations. If $\Delta = \text{ch}(S)$, then the path $v_4 v_3 v_1$ separates v_2 from the rest of the vertices in G, and if $\Delta \neq \text{ch}(S)$, then v_3 lies in the interior of Δ and some point of S lies in its exterior. It follows that either the path $v_4 v_3 v_1$ or Δ is a 3-vertex cut in G, contradicting our initial assumption that G is 5-connected, and proving the claim. Without loss of generality, we may now assume that the 4-cycle (v_1, v_2, v_4, v_3) has no external chord in T_1, and hence in T_1' either.

After flipping $e_1' = v_2 v_3$ in T_1', the 4-cycle (v_1, v_2, v_3, v_4) is an induced subgraph in the resulting triangulation T_1'', and this cycle has a flippable edge $\hat{e}$ by Lemma 4. Flipping $\hat{e}$ in T_1'' increases the degree of s to 5, while the edge $\hat{e}$ remains part of the triangulation T_2'. By Property 1, the union of these two triangulations is a 5-connected biplane graph. This completes the proof in case 3.

In all three cases, we have augmented $\widehat{G} = (S, \widehat{E})$ with a new vertex s by adding at least 5 new edges incident to s and deleting at most one edge of $\widehat{E}$. Finally, delete all remaining dummy edges (that have not been flipped in the above procedure). Since the original graph G was 5-connected without the dummy edges, Properties 1 and 2 imply that the resulting biplane graph on $S \cup \{s\}$ is also 5-connected. $\qquad \square$

Inserting vertices of the convex hull. We now introduce a method to augment a 5-connected biplane graph $G = (S_a, E)$ with a set S_b of points in the exterior of $\text{ch}(S_a)$. We would like the new edges to be disjoint from the interior of $\text{ch}(S_a)$ (although some edge flips will be necessary). For this purpose, we introduce the concept of *visibility*. We say that a point s in the exterior of $\text{ch}(S_a)$ *sees* an edge uv of $\text{ch}(S_a)$ if the triangle suv is also in the exterior of $\text{ch}(S_a)$. A line segment st in the exterior of $\text{ch}(S_a)$ *sees* uv if both s and t see uv. Note that every exterior point s must see a subset of consecutive edges of $\text{ch}(S_a)$, but cannot see all edges of $\text{ch}(S_a)$.

We show below (Lemma 7) that a 5-connected biplane graph $G = (S_a, E)$ can be augmented with a set S_b of exterior points if S_a and S_b satisfy the following property (see Fig. 4).

Property 3. *Let S_a and S_b be disjoint point sets such that*

- $|\mathrm{ch}(S_a)| \geq 4$;
- *every point $s \in S_b$ is a vertex of $\mathrm{ch}(S_a \cup S_b)$;*
- *if k points in S_b are consecutive vertices of $\mathrm{ch}(S_a \cup S_b)$, then they jointly see at least $k+2$ consecutive edges of $\mathrm{ch}(S_a)$, for all positive integers $k < |\mathrm{ch}(S_a)|$.*

Property 3 implies a similar property for edges (rather than vertices) under some additional conditions. This will allow the application of Hall's theorem to match the edges of $\mathrm{ch}(S_b)$ to some edges of $\mathrm{ch}(S_a)$.

Lemma 6. *Assume that S_a and S_b satisfy Property 3, S_a lies in the interior of $\mathrm{ch}(S_b)$, and every two consecutive vertices of $\mathrm{ch}(S_b)$ see some common edge of $\mathrm{ch}(S_a)$. Then every k edges of $\mathrm{ch}(S_b)$ jointly see at least k edges of $\mathrm{ch}(S_a)$.*

Proof. When $k = 1$, the claim holds by our assumption that every two consecutive vertices of $\mathrm{ch}(S_b)$ see some common edge of $\mathrm{ch}(S_a)$. Suppose, to the contrary, that there is a counterexample for some $k > 1$. That is, there is a set H_b of $k \geq 2$ edges of $\mathrm{ch}(S_b)$ that jointly only see a set H_a of edges of $\mathrm{ch}(S_a)$ with $|H_a| < k$. Consider a counterexample where $|H_a|$ is minimal. We may assume that H_b is the maximal set of edges of $\mathrm{ch}(S_b)$ that jointly see exactly the edges in H_a. If two edges $h_1, h_2 \in H_b$ see the same $h \in H_a$, then every edge along $\mathrm{ch}(S_b)$ between h_1 and h_2 (say, in counterclockwise order) can see only edges of $\mathrm{ch}(S_a)$ that are already visible to h_1 or h_2. Thus every edge in H_a is visible from a sequence of *consecutive* edges in H_b. It is clear that every edge $h \in H_b$ sees a set of consecutive edges of $\mathrm{ch}(S_a)$. Consequently, we may assume that both H_b and H_a consist of consecutive edges (along $\mathrm{ch}(S_b)$ and $\mathrm{ch}(S_a)$, respectively). The k consecutive edges in H_b form a path P. By Property 3, the $k - 1$ interior vertices of this path jointly see at least $k + 1$ consecutive edges of $\mathrm{ch}(S_a)$. These edges of $\mathrm{ch}(S_a)$ are each visible by at least two (consecutive) vertices of the path P: either by two interior vertices or by one interior vertex and an endpoint of P. Hence $|H_a| \geq k + 1$, contradicting our initial assumption $|H_a| < k$. $\qquad\square$

Using the preceding observations, we present our main tool for augmenting a 5-connected biplane graph with exterior points.

Lemma 7. *Let S_a and S_b be two point sets satisfying Property 3, and let $G = (S_a, E)$ be a 5-connected biplane graph in $\mathcal{G}_2(S_a)$. Then there exists a 5-connected biplane graph $G' = (S_a \cup S_b, E')$ such that $E \subset E'$.*

Proof. We may assume, by adding dummy edges if necessary, that G is a 5-connected maximal biplane graph. We consider two cases depending on whether the conditions of Lemma 6 are satisfied or not.

Case 1: S_a lies in the interior of $\mathrm{ch}(S_b)$ and every two consecutive vertices of $\mathrm{ch}(S_b)$ see some common edge of $\mathrm{ch}(S_a)$. Denote the vertices of $\mathrm{ch}(S_a)$ by $a_1, \ldots, a_p$ in counterclockwise order, and let $b_1, \ldots, b_q$ denote the vertices of $\mathrm{ch}(S_b)$ in counterclockwise order. By Lemma 6, every set of k edges of $\mathrm{ch}(S_b)$ jointly see at least k edges of $\mathrm{ch}(S_a)$. Thus, using Hall's theorem, we can assign every edge of $\mathrm{ch}(S_b)$ to a unique visible edge of $\mathrm{ch}(S_a)$. We have $q \geq 3$, and Property 3 yields $p \geq 5$. Given an index $i \in \{1, \ldots, q\}$, let j be the index such that the edge $b_i b_{i+1}$ is assigned to $a_j a_{j+1}$. By hypothesis, the quadrilateral $b_i b_{i+1} a_{j+1} a_j$ must be convex. We look for an assignment in which these quadrilaterals have pairwise disjoint interiors. If two such quadrilaterals, say $b_i b_{i+1} a_{j+1} a_j$ and $b_{i'} b_{i'+1} a_{j'+1} a_{j'}$,
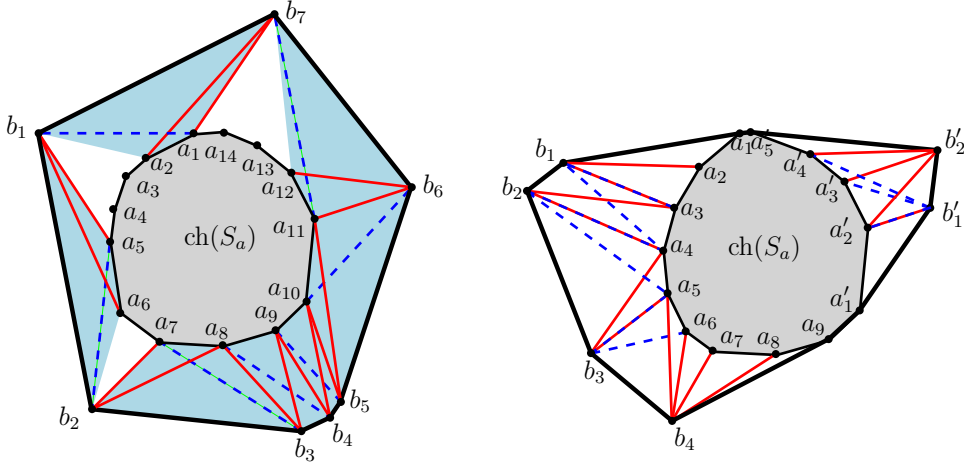
Figure 4: Left: the boundaries of $\mathrm{ch}(S_a)$ and $\mathrm{ch}(S_b)$ are disjoint. Right: the points in S_b are partitioned into two treatable chains.

cross (have intersecting interiors), then $b_i b_{i+1}$ also sees $a_{j'+1} a_{j'}$ and $b_{i'} b_{i'+1}$ also sees $a_{j+1} a_j$, thus we can exchange the edges assigned to $b_i b_{i+1}$ and $b_{i'} b_{i'+1}$, reducing the total number of crossing quadrilaterals by at least one. We can now assume that the edges of $\mathrm{ch}(S_b)$ and the assigned edges of $\mathrm{ch}(S_a)$ form interior-disjoint convex quadrilaterals.

We describe how to augment G with the vertices $b_1, \ldots, b_q$. In one layer, add all edges of the cycle $(b_1, \ldots, b_q)$. If edge $b_i b_{i+1}$ is assigned to $a_j a_{j+1}$, then join b_i to a_j and a_{j+1} in one layer, and b_{i+1} to a_j in the other layer (where $a_{p+1} = a_1$ and $b_{q+1} = b_1$). Denote the resulting graph by G' (Figure 4, left). All new edges are disjoint from the interior of $\mathrm{ch}(S_a)$, and the edges in each layer are noncrossing, thus G' is biplane. Each b_i is joined to at least three vertices of the cycle $(a_1, \ldots, a_p)$, which is part of the 5-connected graph G, and to its two neighbors in the cycle $(b_1, \ldots, b_q)$. In particular, each b_i has vertex-independent paths to five distinct vertices of the 5-connected graph G. It follows that G' is 5-connected.

Case 2: S_a does not lie in the interior of $\mathrm{ch}(S_b)$ or two consecutive vertices of $\mathrm{ch}(S_b)$ see disjoint sets of edges of $\mathrm{ch}(S_a)$. In this case, we partition the vertices S_b into maximal chains of consecutive vertices along $\mathrm{ch}(S_a \cup S_b)$ such that every two consecutive vertices of a chain see a common edge of $\mathrm{ch}(S_a)$; and then successively augment G with the vertices of the chains. We say that a counterclockwise chain $c = (b_1, \ldots, b_q)$ along the boundary of $\mathrm{ch}(S_a \cup S_b)$ is *treatable* if $b_i \in S_b$ for $i = 1, \ldots, q$ and every edge of c sees some edge of $\mathrm{ch}(S_a)$. A treatable chain c is *maximal* if it is not contained in a longer treatable chain.

Let $c = (b_1, \ldots, b_q)$ be a maximal treatable path, and let $(a_1, \ldots, a_p)$ be the counterclockwise sequence of vertices of $\mathrm{ch}(S_a)$ jointly visible from c. Property 3 implies $p \geq q + 3$ (in particular, $p \geq 4$, since b_1 alone sees at least 3 edges, hence at least 4 vertices of $\mathrm{ch}(S_a)$). We distinguish two subcases depending on the length of c.

Case 2(a): $q \geq 2$. To each b_i $(i = 1, \ldots, q)$ we assign a sequence of visible edges of $(a_1, \ldots, a_p)$ such that the sequences are disjoint and cover all edges of $(a_1, \ldots, a_p)$. Assign to b_1 the edges $a_1 a_2$, $a_2 a_3$, and any subsequent edge of $(a_1, \ldots, a_p)$ that is *not* visible to b_2. For $i = 2, \ldots, q$, assign to vertex b_i the counterclockwise first edge of $\mathrm{ch}(S_a)$ that is visible to b_i and has not been assigned to any previous vertex b_j, $i < j$; furthermore, assign to b_i any subsequent edge of $(a_1, \ldots, a_p)$ that is not visible to any subsequent vertex b_j, $j > i$. (See Figure 4, right.) We have assigned at least 2 edges to b_1 by construction, and at least one edge to all other vertices b_i $(i = 2, \ldots, q)$ by Property 3. By the maximality of the sequence $b_1, \ldots, b_q$, every two consecutive vertices see at least one edge of $\mathrm{ch}(S_a)$. Using this fact,

9

observe that b_i sees the first edge assigned to b_{i+1}, for $i = 1, \ldots, q-1$. Note that at least two edges are assigned to b_q. Indeed, if vertices $b_2, \ldots, b_{q-1}$ are each assigned exactly one edge, then b_q is assigned at least 2 edges since $p \geq q + 3$. Otherwise consider the last vertex b_x, $1 < x < q$, assigned to two or more edges. Then by construction, all edges visible by $b_{x+1}, \ldots, b_q$ are assigned to these vertices. By Property 3, the vertices $b_{x+1}, \ldots, b_q$ jointly see at least $(q - x) + 2$ edges of $\mathrm{ch}(S_a)$. Since each of $b_{x+1}, \ldots, b_{q-1}$ is assigned to exactly one edge, b_q is assigned to at least 3 edges.

We can now augment G with the vertices $b_1, \ldots, b_q$. We distinguish two cases based on the number of edges assigned to b_q. Assume first that 3 or more edges are assigned to b_q (refer to Figure 4, right). Then, in one layer, add all edges of the path $(b_1, \ldots, b_q)$ and join each b_i to the endpoints of all edges assigned to b_i. In the second layer, join every vertex b_i $(i = 1, \ldots, q-1)$ to both endpoints of the first edge assigned to b_{i+1}. Denote the resulting graph by G'. All new edges are disjoint from the interior of $\mathrm{ch}(S_a)$, and the edges in each layer are noncrossing, and so G' is biplane. Vertices b_1 and b_q are each joined to at least 4 consecutive vertices of $(a_1, \ldots, a_p)$, which is part of the 5-connected graph G, and b_1 and b_q are also joined by the path $(b_1, \ldots, b_q)$. Vertices b_i $(i = 2, \ldots, q-1)$ are joined to at least 3 vertices of the arc $(a_1, \ldots, a_p)$ (two adjacent vertices in the first layer, and one additional vertex in the second layer) and they each have two neighbors in the path $(b_1, \ldots, b_q)$. It follows that G' is 5-connected.

Assume now that exactly two edges are assigned to b_q, namely $a_{p-2}a_{p-1}$ and $a_{p-1}a_p$ (refer to Figure 4, right). Thus, the edge $a_{p-3}a_{p-2}$ is assigned to b_{q-1}. Now, in one layer, add all edges of the path $(b_1, \ldots, b_q)$, join each b_i $(i = 1, \ldots, q-2)$ to the endpoints of all edges assigned to b_i, join b_{q-1} to a_{p-3} and join b_q to the endpoints of edges $a_{p-3}a_{p-2}$, $a_{p-2}a_{p-1}$ and $a_{p-1}a_p$. In the second layer, join every vertex b_i $(i = 1, \ldots, q-1)$ to both endpoints of the first edge assigned to b_{i+1}. Similarly to the previous case, the resulting graph is biplane and 5-connected.

Case 2(b): $q = 1$. If $p \geq 5$, then we augment G with vertex b_1 and join it to all visible vertices of $\mathrm{ch}(S_a)$. The resulting graph is biplane, and 5-connected by Property 1.

Otherwise, $p = 4$. We augment G with vertex b_1 and join it to a_1, a_2, a_3, a_4 and a fifth vertex of G as follows. Since G is 5-connected, a_2 is incident to at least 5 edges in G, at least 3 of which are interior edges of $\mathrm{ch}(S_a)$. By Lemma 2, G is the union of two triangulations, T' and T''. We may assume without loss of generality that a_2 is incident to at least 2 interior edges in T'. Denote the triangles of T' adjacent to a_1a_2, a_2a_3 and a_3a_4, by Δ_1, Δ_2 and Δ_3, respectively (some of these triangles may be the same). Clearly, $\Delta_1 \neq \Delta_3$, because the two triangles involve at least four different vertices, and $\Delta_1 \neq \Delta_2$, because a_2 is incident to some interior edges. We consider two cases depending on whether Δ_2 and Δ_3 are equal or not. If Δ_1, Δ_2, and Δ_3 are pairwise distinct, then one of them forms a convex quadrilateral with b_1, since a_1b_1 and a_4b_1 are tangent to $\mathrm{ch}(S_a)$. If, say, $\Delta_i = a_i a_{i+1} v_i$, forms a convex quadrilateral with b_1, we can join b_1 to v_i and the resulting graph is biplane and 5-connected by Property 1.

Suppose that $\Delta_2 = \Delta_3$, that is, $a_2 a_3 a_4 = \Delta_2 = \Delta_3$ is a triangle in T'. Denote the other triangle adjacent to $a_2 a_4$ by $\Delta' = a_2 a_4 v'$. Since a_2 is incident to at least 2 interior edges, we have $\Delta_1 \neq \Delta'$. Again, either Δ_1 or Δ' forms a convex quadrilateral with b_1. If $\Delta_1 = a_1 a_2 v_1$ forms a convex quadrilateral with b_1, then we join b_1 to v_1 and if Δ' forms a convex quadrilateral with b_1, then we delete edge $a_2 a_4$ and we add edge $b_1 v'$. The resulting graph is biplane and 5-connected by Properties 1 or 2. $\hspace{1em}\square$

Proof of Theorem 2.

Proof. Let S be a set of n points in general position, with at least 14 points in convex position. Consider the subsets of S that are in convex position and have maximum cardinality. Among

all these sets, let S_0 be one whose convex hull has the largest area. Clearly, we have $|S_0| \geq 14$ and there is a 5-connected graph $G_0 \in \mathcal{G}_2(S_0)$ by Theorem 1.

Partition the points of $S \setminus S_0$ into three sets S_{int}, S_{bou}, and S_{ext} containing the points in the interior of $\text{ch}(S_0)$, those on the boundary of $\text{ch}(S)$, and the remaining points, respectively. The choice of S_0 ensures that $S_a = S_0 \cup S_{\text{int}}$ and $S_b = S_{\text{bou}}$ satisfy Property 3.

We increment G_0 with the vertices in S_{int}, S_{bou}, and S_{ext} in this order. First, consider the interior points in S_{int}. Order the points in S_{int} in lexicographic order (i.e., sort by x-coordinate, and break ties by y-coordinates). We use Lemma 5 for successively inserting the points in S_{int} into G_0, maintaining a 5-connected biplane graph. Denote by G_0' the resulting 5-connected biplane graph in $\mathcal{G}_2(S_0 \cup S_{\text{int}})$.

Since $S_0 \cup S_{\text{int}}$ and S_{bou} satisfy Property 3, we can use Lemma 7 to augment G_0' into a 5-connected biplane graph G_0'' on $S_0 \cup S_{\text{int}} \cup S_{\text{bou}}$. Finally, we use Lemma 5 again for successively inserting the points in S_{ext} into G_0'' in a suitable order. Since S_{ext} lies in the exterior of $\text{ch}(S_0)$, there exists at least one point in S_{ext} that lies on the boundary of $\text{ch}(S_0 \cup S_{\text{ext}})$. Choose an arbitrary such point, and remove it from S_{ext}. Repeat this process until we have removed all points of S_{ext}. Let $p_1, \ldots, p_k$ be the order in which points of S_{ext} are removed. By construction, point p_i belongs to the convex hull of $S_0 \cup \{p_i, \ldots, p_k\}$, for all $i = 1, \ldots, k$. Thus, if we successively insert them into G_0'' in reverse order they each satisfy the conditions of Lemma 5. When all three phases of the algorithm are complete, we obtain a 5-connected biplane graph in $\mathcal{G}_2(S)$, as claimed. $\qquad\square$

3 Graph Augmentation

In this section we study how the addition of a second layer can improve the vertex connectivity of a plane graph. Suppose we are given a plane graph $G = (S, E)$ in $\mathcal{G}_1(S)$, and wish to augment it with a set E' of new edges to obtain a biplane graph $G = (S, E \cup E') \in \mathcal{G}_2(S)$ such that the vertex connectivity $\kappa(G)$ is maximal. The more edges G has, the more constrained the problem becomes. In the worst case, we may assume that $G \in \mathcal{G}_1(S)$ is a triangulation, and it is augmented with another triangulation.

In Section 3.1 we show how to augment a triangulation with a plane graph in a second layer in order to make it 4-connected, whenever that is possible. In Section 3.2 we consider the *minimal augmentation* problem, that is, finding a minimum set of new edges that augments a given plane graph into a k-connected biplane graph.

3.1 Increasing the connectivity of triangulations

Our first observation is that there are arbitrarily large point sets S and triangulations $T \in \mathcal{G}_1(S)$ such that T cannot be augmented into a 4-connected biplane graph on S by adding a plane graph. These special triangulations are the wheels and fans. Recall that a *wheel* is a triangulation on n points such that $n - 1$ points are in convex position and one point (the *center* of the wheel) lies in the interior of $\text{ch}(S)$, the points on the convex hull induce a cycle on the boundary of $\text{ch}(S)$ and the interior point is joined to all other $n - 1$ points. A *fan* is a triangulation on n points in convex position, where one point (called *center*) is joined to all other $n - 1$ points. Every wheel is 3-connected and every fan is 2-connected. We show below that their connectivity cannot be increased to 4 by augmenting them with a set of pairwise noncrossing edges. Moreover, a fan cannot be augmented into a 4-connected biplane graph, even in the case that the added graph is not plane.

Lemma 8. *Let $T = (S, E)$ be a fan. Then the vertex-connectivity of every biplane graph $G = (S, E \cup E')$ is at most 3.*

Proof. Let $T = (S, E)$ be a fan centered at v_0, with $\text{ch}(S) = (v_0, \ldots, v_n)$. Consider a biplane graph $G = (S, E \cup E')$ and suppose it is 4-connected. We may assume that G is maximal

biplane, and by Lemma 2, it is the union of two triangulations $T_1 = (S, E_1)$ and $T_2 = (S, E_2)$. Note that the two triangulations share all convex hull edges.

First suppose that all edges of the fan are in one of the two triangulations, say $T_1 = T$. Then every vertex v_i, except for the center of the fan, has degree at most 3 in T_1, because the only adjacent edges to v_i are the convex hull edges (v_i, v_{i-1}) and (v_i, v_{i+1}), and the chord (v_i, v_0) (if it exists). On the other hand, any triangulation on a convex point set has at least two vertices of degree 2, so the only edges adjacent to these vertices of degree 2 are convex hull edges. Hence, the triangulation T_2 has at least two vertices of degree 2. At most one of these vertices may be v_0, but the other one must be a vertex of degree at most 3 in T. Since the convex hull edges are common to both triangulations, we conclude that at least one vertex in S has degree at most 3 in G.

Now assume that not all edges of the fan are in T_1. Then the fan has consecutive edges that belong to different triangulations, say $v_0 v_i \in E_1$ and $v_0 v_{i+1} \in E_2$ for $1 < i < n-1$. Then $\{v_0, v_i\}$ is a 2-vertex cut in T_1, and $\{v_0, v_{i+1}\}$ is a 2-vertex cut in T_2. No edge of G can cross both $v_0 v_i \in E_1$ and $v_0 v_{i+1} \in E_2$. Therefore $\{v_0, v_i, v_{i+1}\}$ is a 3-vertex cut of G. $\qquad \square$

Lemma 9. *Let $T = (S, E)$ be a wheel. For every triangulation $T' = (S, E')$, the vertex-connectivity of the biplane graph $G = (S, E \cup E')$ is at most 3.*

Proof. Let pq be an edge in $E' \setminus E$. Necessarily, p and q are nonconsecutive vertices on the boundary of $\text{ch}(S)$ because T is a wheel. Since T' is a plane graph, no edge in E' crosses pq, and so $\{p, q\}$ is a 2-vertex cut in T'. Every path in G between two vertices on opposite sides of the line pq must pass through p, q, or the center of the wheel. Hence G has a 3-vertex cut. $\qquad \square$

Remark. Note that in some cases, a wheel $T = (S, E)$ can be augmented into a 4- or even 5-connected biplane graph $G = (S, E \cup E')$, but then some of the new edges in E' must cross each other (that is, $T' = (S, E')$ is not a plane graph). For example, consider a wheel $T = (S, E)$ on $2n + 1$ points where $\text{ch}(S)$ is a regular $2n$-gon and the center of the wheel is located at the centroid of the polygon. Then T can be augmented into a 5-connected biplane graph $G = (S, E \cup E')$, where E' consists of two cycles, each of n vertices, formed by the odd and even points of $2n$-gon, respectively.

Recall that every maximal biplane graph on $n \geq 3$ points is 3-connected (Lemma 3), hence Lemmas 8 and 9 are tight. We next show that wheels and fans are the only exceptions, that is, any other triangulation on $n \geq 4$ vertices can be augmented to a 4-connected biplane graph by adding a plane graph. To that end, we use the following auxiliary lemma.

Lemma 10. *Let S be a set of $n \geq 5$ points in general position such that u, v, and w, are consecutive vertices of $\text{ch}(S)$. Let T be a triangulation on S containing the empty triangles $\Delta = (u, v, w)$ and $\Delta_1 = (u, v', w)$. Then, edge uw is flippable in T and, after flipping uw, one of the edges uv' or $v'w$ is also flippable in the new triangulation.*

Proof. Since u, v and w are consecutive vertices of $\text{ch}(S)$, all points in S lie in the closed angular domain $\angle(u, v, w)$. Hence, the points u, v, w and v' are in convex position, and edge uw is flippable. Replace edge uw with the edge vv' to obtain a new triangulation T'.

In T', the triangles (u, v, v') and (v, w, v') are adjacent to edges uv' and $v'w$, respectively. Let $\Delta_2 = (u, x, v')$ and $\Delta_3 = (w, y, v')$ be (if they exist) the other two triangles of T' adjacent to edges uv' and $v'w$, respectively (possibly $x = y$). As T' is a triangulation and $n \geq 5$, at least one of Δ_2 and Δ_3 must exist. Assume first that only one of Δ_2 and Δ_3, say Δ_2, exists. Then v' is a vertex of the convex hull, and so u, v, v' and x are in convex position. Consequently, edge uv' is flippable. Next, assume that both Δ_2 and Δ_3 exist. Notice that the edges $v'v$, $v'u$, $v'x$, $v'y$, and $v'w$ appear in this circular order around v' (possibly $v'x = v'y$). Let ℓ be the

line spanned by v and v'. If x and u are in the same halfplane defined by ℓ, then (v, v', x, u) is a convex quadrilateral and, consequently, edge uv' is flippable. Otherwise, x, y and w must be in the same halfplane defined by ℓ. In this case, (v, v', w, y) is a convex quadrilateral and edge wv' is then flippable.

$\square$

We also recall the characterization of 2- and 3-vertex cuts in a triangulation $T = (S, E)$ on $n \geq 5$ vertices. A set $\{u, v\} \subset S$ is a 2-vertex cut in T if and only if uv is a chord of $\mathrm{ch}(S)$. A *separating triangle* of T is a triangle containing a vertex in its interior and another vertex in its exterior. We define a *bichord* of T as a path of length two between two vertices of $\mathrm{ch}(S)$ such that no edge of the path lies on the boundary of $\mathrm{ch}(S)$ and there is a vertex on each side of the path. If the middle vertex of the bichord is an interior vertex, then the bichord splits S into two nonempty sets. If the middle vertex is a vertex on $\mathrm{ch}(S)$, then the bichord is the union of two chords and splits S into three nonempty sets. A set $\{u, v, w\} \subset S$ is a 3-vertex cut in T if and only if they form a separating triangle, a bichord or some of them form a chord.

In all the cases (2- and 3-vertex cuts), S is split into several nonempty subsets by a chord, a separating triangle or a bichord, such that no edge of T connects vertices in distinct subsets. When we augment a triangulation T into a 4-connected biplane graph G, we must add edges connecting vertices belonging to these subsets. For example, a separating triangle consisting of three chords splits S into four nonempty subsets. In this case, if we remove the triangle, then at least three new edges crossing the chords must be added to reconnect the graph. Observe that the edges of this triangle also form three bichords.

We say that an edge *properly* crosses a bichord or a separating triangle when the edge intersects it in exactly one internal point (a chord is always crossed properly). We wish to augment a triangulation $T = (S, E)$ on $n \geq 5$ vertices into a 4-connected biplane graph $G = (S, E \cup E')$ by adding a plane graph $T' = (S, E')$ (not necessarily a triangulation). Since T is 2-connected, any 2- or 3-vertex cut of an augmentation $G = (S, E \cup E')$ is a 2- or 3-vertex cut of T. Therefore, it is not difficult to check that G is 4-connected if and only if (i) every separating triangle and every bichord of T are properly crossed by at least one new edge, (ii) every chord of T is properly crossed by at least two new edges, and (iii) when S contains at least two points on each side of a chord of T, then the chord is crossed by at least two nonadjacent new edges (otherwise the chord and a common endpoint of crossing edges would be a 3-vertex cut).

We are now ready to augment any triangulation, other than the wheel or the fan, by a plane graph to a 4-connected biplane graph.

Theorem 3. *Let S be a set of $n \geq 6$ points in convex position or a set of $n \geq 5$ points not in convex position. For every triangulation $T = (S, E)$ other than the wheel and the fan, there is a plane graph $T' = (S, E')$ such that $G = (S, E \cup E')$ is 4-connected.*

Proof. We consider two cases depending on the vertex connectivity of T.

Case 1: T is 3-connected. In this case, since there are no chords, it is enough augment T such that every separating triangle and every bichord of T is properly crossed by at least one new edge. If there is a vertex v that does not belong to any 3-vertex cut, then augment T with a star $T' = (S, E')$ centered at v. Then in $(S, E \cup E')$, every separating triangle and bichord is properly crossed by an edge incident to v.

Assume now that every vertex in S is part of a 3-vertex cut of T. In this case, we start by showing that T has no separating triangle. Suppose, to the contrary, that there is a separating triangle in T. Let Δ be a separating triangle that contains the minimum number of vertices in its interior, and let v be a vertex in the interior of Δ. Then v cannot be part of any separating triangle by the minimality of Δ. If v is part of a bichord, then the two endpoints of the bichord

must be vertices of Δ, and so an edge of Δ is a chord of $\mathrm{ch}(S)$, contradicting our assumption that T is 3-connected. We conclude that T has no separating triangle.
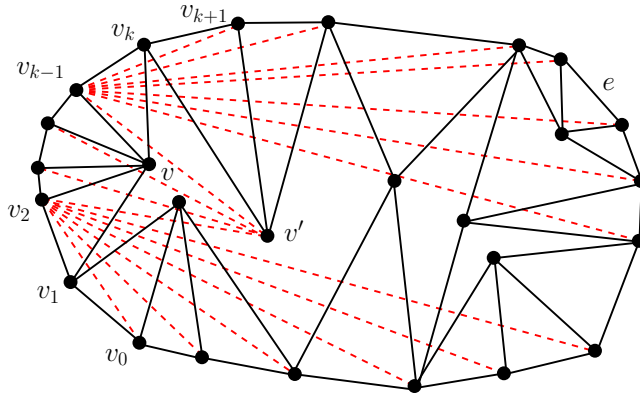

Figure 5: A spanning subgraph H formed by the union of bichords. All new edges (dashed edges) are incident to v_2, v_{k-1} or v'.

We are left with the case in which every 3-vertex cut is a bichord and every point belongs to some bichord. Denote by H the subgraph of T formed by the union of all bichords. The graph H is the union of stars centered at interior points (each star is the union of all bichords with a common middle vertex). Refer to Figure 5. Each star divides $\mathrm{ch}(S)$ into several *sectors*. There are at least two interior points, since T is not a wheel. Let e be an arbitrary edge of $\mathrm{ch}(S)$, and let v be an interior vertex such that the sector of v containing e has maximal area. The maximality implies that no other sector of v contains any bichord. Denote by $v_1, \ldots, v_k$ the neighbors of v in H in clockwise order such that all other interior vertices of H lie in the sector bounded by vv_1 and vv_k. Note that $k \geq 4$, since if $k \in \{2, 3\}$, then v_2 cannot belong to a bichord, contradicting the assumption that every vertex of G is part of some bichord.

We can now define a plane graph T' as follows. Connect $v_2, \ldots, v_{k-1}$ to an arbitrary interior vertex v', $v' \neq v$. Partition the sector bounded by $v'v_2$ and $v'v_{k-1}$ that contains v_k into two convex parts along the angle bisector of $\angle(v_2, v', v_{k-1})$. Connect v_2 and v_{k-1}, respectively, to all vertices of $\mathrm{ch}(S)$ that lie in the same convex part of the sector (see Figure 5). If the angle bisector hits a vertex w of $\mathrm{ch}(S)$ then w can be connected to either v_2 or v_{k-1}. The graph T' contains an edge properly crossing every bichord. Therefore $G = (S, E \cup E')$ is 4-connected, as required.
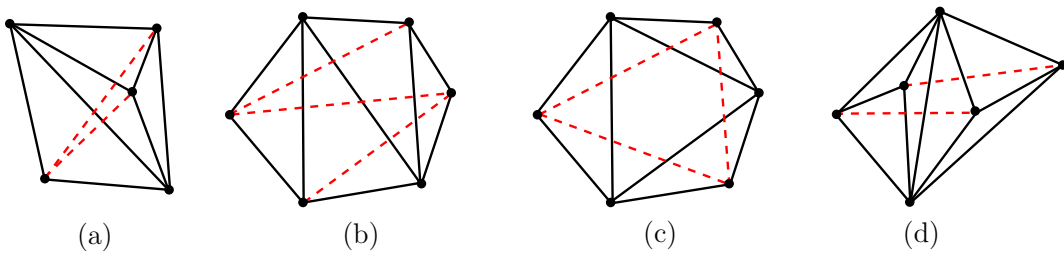

Figure 6: (a–c) Base cases when T in not 3-connected. Dashed edges indicate edges in E'. (d) A triangulation of 6 points with two interior points on opposite sides of a chord.

Case 2: T is not 3-connected. Since every triangulation on $n \geq 3$ vertices is 2-connected, then T must have a 2-vertex cut. Each 2-vertex cut $\{u, v\}$ corresponds to a chord $uv \in E$ of $\mathrm{ch}(S)$. The chords of $\mathrm{ch}(S)$ decompose $\mathrm{ch}(S)$ into convex regions, called *cells*. We define a dual graph H on the cells as follows: the nodes of H correspond to the cells, and two nodes are joined by an edge in H if and only if the corresponding cells share a chord. Clearly, H

14

is a tree. The cells corresponding to the leaves of H are called *leaf cells*. For a leaf ℓ of H, denote by $S_\ell \subset S$ the set of vertices that lie in the interior or on the boundary of the leaf cell ℓ (including the endpoints of the chord defining the leaf). The *size* of a leaf cell ℓ is the cardinality of S_ℓ. Note that the subgraph T_ℓ of T induced by the vertices in S_ℓ for a leaf cell ℓ, is either a triangle (if $|S_\ell| = 3$) or a 3-connected triangulation (if $|S_\ell| \geq 4$). We proceed by induction on n, the total number of vertices.

Base cases. Suppose $n = 5$ and the points are not in convex position. Since T has a chord, $\mathrm{ch}(S)$ has 4 vertices, and there is one interior point (Figure 6 (a)). Since $\mathrm{ch}(S)$ is a quadrilateral, there is exactly one chord, which determines two leaf cells. The vertices in the two cells induce K_3 and K_4, respectively, with 1 and 2 vertices disjoint from the chord. Let E' contain the two edges between the vertices on opposite sides of the chord. Then $G = (S, E \cup E')$ is isomorphic to K_5, which is 4-connected.

Suppose $n = 6$ and the points are in convex position. Every triangulation T, other than a fan, is composed of three chords of $\mathrm{ch}(S)$ forming a triangle or a path. In both cases, there is a set E' of 3 noncrossing edges such that $(S, E \cup E')$ is 4-connected, as indicated in Figures 6(b) and 6(c).

Induction step. To prove the induction step we will distinguish two cases, based on whether a leaf cell of size 3 exists or not.

Case 2.1: There is a leaf cell L of size 3. First, assume $n \geq 6$ ($n \geq 7$ for points in convex position) and suppose that a leaf cell of size 3 exists, i.e. there is a triangle $\Delta = (u, v, w)$ in T, with uw being a chord and v being the only vertex (on the boundary of $\mathrm{ch}(S)$) to the left of $\overrightarrow{uw}$. By removing v from T, we obtain a triangulation T_1 on the $n - 1$ remaining points. Three subcases arise depending on whether T_1 is a wheel, a fan or neither.

If T_1 is a wheel, then let $T' = (S, E')$ be a star centered at v. The biplane graph obtained $(S, E \cup E')$ is 4-connected because all bichords of T (which are the bichords of T_1) are properly crossed by T', and the only chord uw is crossed by at least two edges of T'.

If T_1 is a fan, then, instead of removing v, we can remove a different vertex of degree 2 from T, obtaining a new 2-connected triangulation other than a fan (otherwise T would be a fan).
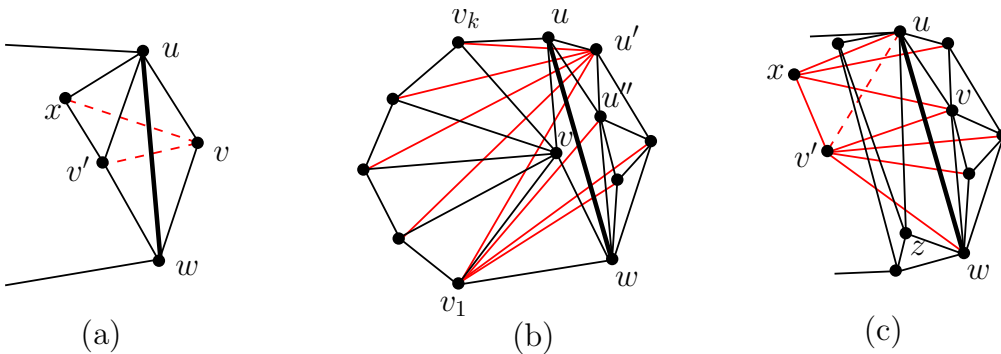

Figure 7: Augmenting 2-connected triangulations to 4-connected biplane graphs by crossing chords. Chord uw is shown as a bold segment.

Otherwise, T_1 is a 3-connected triangulation other than a wheel or a 2-connected triangulation other than a fan. In this case, by induction, there is a plane graph T_2 on the $n - 1$ remaining points such that $T_1 \cup T_2$ is biplane and 4-connected. We may assume that T_2 is a triangulation.

15

Now, we construct a triangulation from T_2 by adding vertex v and some incident edges, and possibly deleting some of the edges of T_2, as explained next. Refer to Figure 7(a). Let $\Delta_1 = (u, w, v')$ be the triangle of T_2 adjacent to uw and let $\Delta_2 = (u, x, v')$ and $\Delta_3 = (w, y, v')$ be (if they exist) the two triangles of T_2 adjacent to triangle Δ_1. Note that at least one of them must exist. First, we add Δ to T_2, obtaining a triangulation T_2' on the n points. By Lemma 10, we can flip edge uw in T_2' and, after flipping it, we can flip one of the edges uv' and $v'w$. Assume without loss of generality that we flip uv'. Then, from T_2', we remove edges uw and uv' and we add edges vv' and vx, obtaining a new triangulation T' on the n points. By Property 2, $T \cup T'$ is 4-connected. Indeed, we have obtained $T \cup T'$ from the 4-connected graph $T_1 \cup T_2$ by removing at most one edge, uv' (since edge uw is always in both T_1 and T_2) and adding a new vertex v joined to u, v' and two additional vertices, x and w. This completes the proof of Case 2.1.

Case 2.2: There is no leaf cell of size 3. Our proof in this case is similar to Case 2.1, but the constructions are now a bit more complicated. Since E contains at least one chord of $\mathrm{ch}(S)$, the convex hull has at least 4 vertices and there are at least two leaf cells. Since no leaf cell is a triangle, there are at least two points in the interior of $\mathrm{ch}(S)$.

Assume first that $n = 6$. In this case, there are exactly four hull vertices, hence there is a unique chord, and exactly one interior vertex on each side of the chord (Figure 6(d)). Consider the four vertices disjoint from the chord, on each side: they admit two disjoint edges between vertices on opposite sides of the chord. These edges augment T into a 4-connected biplane graph, as required.

Assume now $n \geq 7$. Let ℓ be a leaf cell of minimal size (i.e., with minimum number of vertices). This leaf cell is defined by a chord uw, and its size $|S_\ell|$ is at least 4. If we remove the points in S_ℓ except for u and w, then all vertices of the second smallest leaf cell survive, and so we are left with at least $|S_\ell| \geq 4$ vertices. We obtain a triangulation T_1 on the $n - |S_\ell| + 2$ remaining points. Note that, if $|S_\ell| = 4$, then $n - |S_\ell| + 2 \geq 5$ because $n \geq 7$, and if $|S_\ell| \geq 5$, then $n - |S_\ell| + 2 \geq |S_\ell| \geq 5$. Therefore, if T_1 is neither a wheel nor a fan, we can apply induction to the triangulation T_1.

First, observe that T_1 cannot be a fan, otherwise it would have two leaves of size 3, and one of them would be a leaf cell of size 3 in T, contradicting the assumption that there is no such leaf cell. Assume that T_1 is a wheel. Refer to Figure 7(b). Let v denote the center of the wheel T_1, and let the neighbors of v be denoted $w, v_1, \ldots, v_k, u$ in clockwise order. Note that $k \geq 2$ because $n - |S_\ell| + 2 \geq 5$ and so the degree of v is at least 4. Let h be the line through v_1 and u, and let $u' \neq w$ be the first point in S_ℓ hit when we rotate h clockwise about v_1. Now, let T' contain edges connecting u' to $v_1, \ldots, v_k$; and v_1 to all the vertices in $S_\ell \setminus \{u, w\}$. Since T_ℓ (the graph induced by S_ℓ in T) is 3-connected, then by connecting each vertex in S_ℓ (except for u and w) to a vertex outside of S_ℓ, each separating triangle and bichord of T_ℓ is properly crossed. In addition, every bichord incident to v is properly crossed by an edge of type $u'v_j$. Finally, the only chord of T, the chord uw, is properly crossed by two new edges, for example, edges v_1u'' and v_ku' (where u'' is an arbitrary vertex of ℓ other than u, w and u', which must exist because $|S_\ell| \geq 4$). Therefore, $(S, E \cup E')$ is 4-connected.

We can now assume that T_1 is a 3-connected triangulation other than a wheel or a 2-connected triangulation other than a fan. By induction, there is a plane graph T_2 on the $n - |S_\ell| + 2$ remaining points such that $T_1 \cup T_2$ is a 4-connected biplane graph. We may assume that T_2 is a triangulation. We modify T_2 to construct a new plane graph T' on all n points as follows (see Figure 7(c)). Similarly to Case 2.1, let $\Delta_1 = (u, w, v')$ be the triangle of T_2 adjacent to the chord uw and let $\Delta_2 = (u, x, v')$ and $\Delta_3 = (w, y, v')$ be (if they exist) the two triangles of T_2 adjacent to triangle Δ_1. Note that at least one of them must exist. Let $\Delta = (u, w, v)$ be the triangle of T_ℓ adjacent to edge uw. Vertex v must be an interior vertex because T_ℓ is 3-connected. By adding Δ to T_2, we obtain a new triangulation T_2' and,

again, by Lemma 10, we can flip edge uw in T_2'. We can then flip one of the edges uv' and $v'w$. Assume without loss of generality that we flip uv'. We construct a plane graph T' from T_2 on the n points by removing edges uw and uv', adding edges xv and $v'v$, and connecting all remaining points in S_ℓ to one of x and v', depending on which side of the angle bisector of $\angle(x, v, v')$ they lie on. We have added an edge of type $v'v''$ or xv'' adjacent to each vertex v'' in S_ℓ, where $v'' \notin \{u, w\}$.

We claim that $T \cup T'$ is 4-connected. We need to show that every chord, every bichord and every separating triangle of T is properly crossed. By induction, $T_1 \cup T_2$ is 4-connected. In the case that edge uv' is removed from the 4-connected graph $T_1 \cup T_2$, the path (u, v, v') establishes a new connection from u to v'. Note that, since triangle Δ_1 is empty, any chord properly crossed by edge uv' is also properly crossed by edge $v'v$. Consequently, every separating triangle, every bichord and every chord of T_1, is properly crossed at least once or twice, as required. However, it is possible that a bichord of T consists of two edges of T_1 but it is not a bichord in T_1. One possibility for such a bichord is (u, z, w), where $\Delta_0 = (u, w, z)$ is the triangle of T_1 adjacent to the chord uw and z is an interior point (possibly, $z = v'$). See Figure 7(c). This bichord (if it exists) is properly crossed by the edge xv. The other possibility is that edge uw belongs to a bichord of T consisting of two chords, say wu and uw'. In this case, since there are at least two points on each side of uw' (there are no leaf cells of size 3 on one of the sides, and w and z are on the other side), by induction, there are at least two disjoint edges crossing uw', one of them properly crossing the bichord. Moreover, every separating triangle and every bichord in T_ℓ is properly crossed by an edge of type $v'v''$ or xv'', where v'' is in S_ℓ. Finally, we show that the chord uw is properly crossed by two disjoint edges. For an arbitrary vertex v'' in $S_\ell \setminus \{u, v, w\}$, if v'' is connected to x, then both xv'' and $v'v$ cross uw, and if v'' is connected to v', then both xv and $v'v''$ cross uw. $\qquad \square$

We have seen (Lemmas 8 and 9) that a fan and a wheel, respectively, are 2- and 3-connected triangulations that cannot be augmented to a 4-connected biplane graph by adding a second triangulation. We now show that there are 4-connected triangulations that cannot be augmented to 5-connected biplane graphs by adding a second triangulation.

Theorem 4. *There exist arbitrarily large point sets S and 4-connected triangulations $T = (S, E)$ such that for every triangulation $T' = (S, E')$, the biplane graph $(S, E \cup E')$ is not 5-connected.*

Proof. Our construction is shown in Figure 8, where the initial triangulation T appears in Figure 8(a) drawn with black edges. The point set has two main clusters. The top cluster consists of $4k - 3$ points in convex position, with $2k$ points in a lower chain $x_1, \ldots, x_{2k}$, and $2k - 3$ additional in an upper chain $x_1, y_1, \ldots, y_{2k-3}, x_{2k}$. The parameter $k \in \mathbb{N}$ can be made arbitrarily large. The lower cluster consists of 7 points as shown in Figure 8(a). The bottom cluster is sufficiently far below the top cluster so that any new edge between a point y_i and a point in the bottom cluster crosses the edge $x_{i+1}x_{i+2}$. Intuitively, the bottom cluster is a "big dot" far below the top cluster. It is not difficult to verify that T is 4-connected (i.e., it has no chords, bichords, or separating triangles).

Our argument crucially depends on four 4-vertex cuts in T, each consists of the vertices of a path with 3 edges. We call these the four *critical trichords* of T. Two such critical trichords are highlighted in Figure 8(b): one from u to v, and one from s to x_1. The other two are symmetric (one also goes from u to v, and the other from s to x_{2k}).

Suppose, to the contrary, that there is a triangulation $T' = (S, E')$ such that $G = (S, E \cup E')$ is 5-connected. Then the vertices on opposite sides of each critical trichord must be connected by at least one edge in E'. If an edge $e \in E'$ crosses the trichord from s to x_1, then e must be incident to u, and it must cross either edge a or edge b. If e crosses b, it is easy to verify that, no matter what the other endpoint of e is (there are only three possibilities), it is impossible to add pairwise noncrossing edges that cross both trichords between u and v, and the trichord
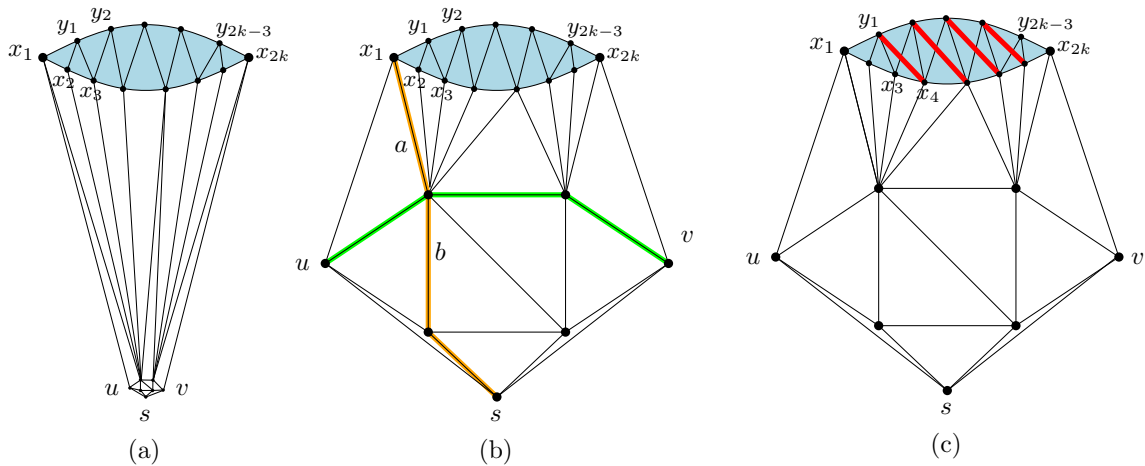
Figure 8: (a) Overview of the construction. The top convex cluster is sufficiently far from the bottom part. (b) Schematic view of the construction, highlighting two of the four critical trichords of T. (c) To increase the degree of the vertices in the convex cluster, more edges are needed. However, vertex y_{2k-3} cannot connect to any other vertex.

from s to x_{2k}, but do not cross e. We conclude that e must cross a and hence is incident to some vertex in the top cluster.

If G is 5-connected, every vertex must have degree at least 5 in G, and so every vertex in $\{x_1, y_1, y_2, \ldots, y_{2k-3}, x_{2k}\}$ must be incident to at least one new edge in $E' \setminus E$. Consider an edge $e_1 \in E' \setminus E$ incident to y_1. If e_1 connects y_1 to any vertex in $\{y_3, \ldots, y_{2k-3}\}$, then the vertices on or above e_1 form a convex polygon within in the top cluster whose triangulation in T' would necessarily contain two ears; therefore at least one vertex y_i would not be incident to any new edge. If e_1 connects y_1 to any vertex in the bottom cluster, then e_1 crosses edge $x_2 x_3$. Since e and e_1 do not cross, we have either $e = ux_2$ or $e = e_1 = uy_1$: In any case, x_1 cannot be incident to any new edge in $E' \setminus E$. Therefore, e_1 must connect y_1 to some vertex in the chain $(x_4, x_5, \ldots, x_{2k})$.

Assume that e_1 connects y_1 to $x_{j(1)}$ for some $j(1) \geq 4$. Using a similar reasoning, y_2 should connect to $x_{j(2)}$ for $j(2) \geq 5$, and in general, y_i should connect to $x_{j(i)}$ for some $j(i) \geq i + 3$, see Figure 8(c). Since the top chain has three fewer vertices than the bottom chain, then y_{2k-3} cannot be connected to any other vertex by a new edge, and its vertex degree is 4 in G, contradicting our assumption that G is 5-connected. □

3.2 Minimal Augmentation

Given a plane graph $G = (S, E)$, we wish to augment G with a minimal set of new edges E' such that we obtain a k-connected biplane graph $G = (S, E \cup E')$ for some target value k. In this section we present an efficient solution (Lemma 12) when $k = 3$ and G is a triangulation. We start with a helpful lemma about augmenting a plane tree to 2-edge-connectivity.

Lemma 11. *Given a plane tree $H = (S, E)$ with n vertices and m leaves, let $L \subset S$ be the set of m leaves of H. In $O(n + m \log m)$ time, one can find a set E' of $\lceil m/2 \rceil$ pairwise noncrossing edges among the leaves of H such that $(S, E \cup E')$ is a 2-edge-connected biplane graph. Moreover, if the m leaves are in convex position, then E' can be found in $O(n)$ time, provided that the clockwise ordering of the leaves along their convex hull is given.*

It is well known (see for example [13]) that an abstract tree with m leaves can be augmented to a 2-edge-connected graph by adding $\lceil m/2 \rceil$ new edges among its leaves. However, establishing the noncrossing condition requires a proof.

Proof. Choose a root $r \in S$ arbitrarily, and let T be the rooted tree obtained from H. We denote by $a \prec b$ if a is a descendent of b in the rooted tree T. For two leaves $u, v \in L$, denote by $\operatorname{lca}(u, v)$ their *lowest common ancestor* in T. For a rooted tree with $O(n)$ vertices, there is a data structure that can report $\operatorname{lca}(u, v)$ for any query vertex pair $\{u, v\}$ in $O(1)$ time after $O(n)$ time preprocessing [5, 20, 26].

We construct the set of new edges recursively. Initialize E' to be the empty set. If $|L| \in \{2, 3\}$, then add an arbitrary spanning tree of L with $\lceil |L|/2 \rceil$ edges to E'. While $|L| > 3$, repeat the following loop: Let v be an arbitrary vertex of $\operatorname{ch}(L)$ other than the root, and let $u \in L$ and $w \in L$ be the two vertices of $\operatorname{ch}(L)$ adjacent to v. If $\operatorname{lca}(u, v) \succeq \operatorname{lca}(v, w)$, then put edge uv into E', and remove both u and v from L. Otherwise, when $\operatorname{lca}(u, v) \prec \operatorname{lca}(v, w)$, put edge vw into E', and remove both v and w from L.

In each loop, the algorithm selects an edge from the boundary of the convex hull of L, which cannot cross any edge selected in a later loop. It follows that E' consists of pairwise noncrossing edges. The algorithm adds one edge for each pair of vertices until $|L|$ drops below 4, and then it uses $\lceil |L|/2 \rceil$ edges. So we have $|E'| = \lceil m/2 \rceil$.

It remains to show that $(S, E \cup E')$ is 2-edge-connected. Let ab be an edge of T with $a \prec b$, and let L_{ab} denote the set of leaves that are the descendants of b. Consider the loop of the algorithm in which the last leaf of L_{ab} is removed from L. The algorithm connects a leaf in $L_{a,b}$ to another leaf, which cannot be in $L_{a,b}$ (recall that the algorithm compares two alternatives). Thus in this loop, the algorithm adds an edge that induces a cycle containing ab. Therefore, every edge ab is contained in a cycle in $(S, E \cup E')$, thus it is 2-edge-connected. The convex hull $\operatorname{ch}(L)$ can be maintained by the semi-dynamic data structure by Hershberger and Suri [21] in $O(m \log m)$ time, hence the total running time is $O(n + m \log m)$.

Finally, observe that, when the m leaves are in convex position, $\operatorname{ch}(L)$ can be updated in constant time after removing two consecutive vertices of $\operatorname{ch}(L)$, without using the semi-dynamic data structure by Hershberger and Suri. Therefore, in this case, the set E' of pairwise noncrossing edges can be found in $O(n)$ time, after computing $\operatorname{ch}(L)$ in $O(m \log m)$ preprocessing time. $\square$

Lemma 12. *Given a triangulation $G = (S, E)$, with $n \geq 3$ vertices, one can find a minimal set of edges E' such that $G' = (S, E \cup E')$ is a 3-connected biplane graph in $O(n)$ time, after computing $\operatorname{ch}(S)$ in $O(n \log n)$ preprocessing time.*

Proof. If the given triangulation G is not 3-connected, then it must contain 2-vertex cuts. Recall that a set $\{u, v\} \subset S$ is a 2-vertex cut if and only if $uv \in E$ is a chord of $\operatorname{ch}(S)$. The biplane graph $G' = (S, E \cup E')$ will be 3-connected if each chord of $\operatorname{ch}(S)$ in E is crossed by at least one edge in E'.

Similar to the proof of Theorem 3, we construct a dual graph H. The chords of $\operatorname{ch}(S)$ in E decompose $\operatorname{ch}(S)$ into convex cells. The nodes of H correspond to the cells, and two nodes are joined by an edge in H if and only if the corresponding cells share a chord. Clearly, H is a tree (see the thick edges in Figure 9), and can be easily constructed in $O(n)$ time from G. The leaves of H correspond to leaf cells. Each leaf ℓ of H is associated with the set of vertices $R_\ell \subset S$ that lie in the interior or on the boundary of the cell, excluding the endpoints of the chord on the boundary of the cell. Note that distinct leaves of H are associated with disjoint vertex sets (i.e., $R_\ell \cap R_{\ell'} = \emptyset$ for $\ell \neq \ell'$). Consider the chords that lie on the boundaries of the leaf cells. These chords are in convex position, thus any new edge can cross at most two of them. It follows that we need to add at least $\lceil m/2 \rceil$ new edges, where m denotes the number of leaves of H.

We now show that $\lceil m/2 \rceil$ new edges suffice, and can be computed in linear time. For each leaf ℓ of H, pick a point $v_\ell \in R_\ell$ on the boundary of $\operatorname{ch}(S)$. We refer to this point as the *representative* of ℓ. Embed H in the plane such that every leaf ℓ is embedded at point v_ℓ, and every nonleaf node is embedded at an arbitrary point in the interior of its cell. Clearly, this embedding can be constructed in $O(n)$ time, after computing $\operatorname{ch}(S)$.
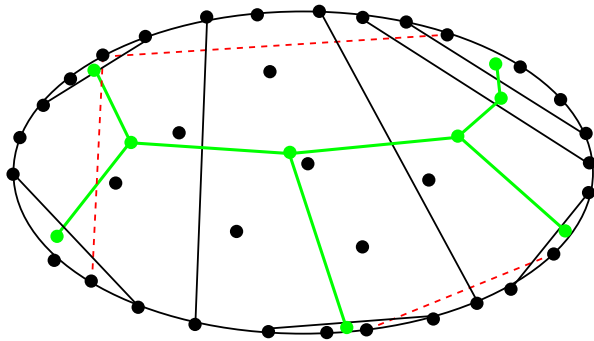
Figure 9: A point set, the associated graph H (thick edges), and the additional edges to obtain 3-connectivity (dashed edges).

By Lemma 11, the embedding of H can be augmented to a 2-edge-connected graph H' by a set E' of $\lceil m/2 \rceil$ noncrossing edges (dashed edges in Figure 9) between the leaves (that are representatives from S) in $O(n)$ time, because all the representatives are in convex position. We claim that $(S, E \cup E')$ is 3-connected. Every 2-vertex cut of G is a chord c of $\mathrm{ch}(S)$ that corresponds to an edge e_c of the embedding of H. When the embedding of H is augmented to H', e_c becomes part of at least one cycle in H', and each of these cycles contains exactly one edge from E'. Observe that, if C is a cycle in H' that contains both e_c and a new edge $e \in E'$, then all edges of C (except for e) correspond to chords, which are crossed by edge e. In particular, e crosses chord c. Since H' is 2-edge-connected, every edge of the embedding of H belongs to a cycle in H', so every chord of $\mathrm{ch}(S)$ is crossed by some edge of E'. $\qquad\square$

In the above argument, we have added one edge for every two leaf cells of H. Associate to each leaf ℓ of H all vertices in R_ℓ and the two endpoints of the chord. Every vertex of $\mathrm{ch}(S)$ is the endpoint of at most two chords that bound leaf cells. Therefore, one can assign to each leaf cell at least two vertices (a vertex of R_ℓ and half of each endpoint of the chord on the boundary of the leaf). It follows that there are at most $\lfloor n/2 \rfloor$ leaf cells. This bound is tight, since the chords of the leaf cells may form a cycle of $n/2$ edges. We obtain an upper bound for the total number of edges added in Lemma 12.

Corollary 2. *Every triangulation $G = (S, E)$ on $n \geq 3$ points can be augmented to a 3-connected biplane graph by adding at most $\lceil \lfloor n/2 \rfloor /2 \rceil = \lfloor \frac{n+2}{4} \rfloor$ new edges.*

4 Conclusions

We have presented several results on the maximum vertex-connectivity attained by biplane graphs on a point set S, either by constructing a graph from scratch (starting from the empty graph) or by combining a given plane graph with a new plane graph. Our proofs are constructive and lead to polynomial-time algorithms for constructing the new edges. Moreover, when the starting graph is a triangulation, we have also presented an efficient algorithm for finding the *minimum* number of edges needed to augment the triangulation to a 3-connected biplane graph.

Our Theorem 2 shows that $\mathcal{G}_2(S)$ contains a 5-connected graph if the point set S contains 14 points in convex position, which is guaranteed only for very large sets (roughly $1.3 \cdot 10^6$ points or more). Recent research indicates that the threshold can be reduced to 137 and perhaps to 27 using a so-called *U-condition* [16] or combining 4-connected plane graphs [17] with stars when no 14 points are in convex position. The improved bound heavily relies on our Theorem 2, and will be the subject of a future paper.

In Theorem 3, we have shown that every triangulation (other than the wheel and the fan) can be augmented to a 4-connected biplane graph by adding a second plane graph on the

same point set; but a second layer is not always sufficient to augment a plane graph to a 5-connected biplane graph. In general, we do not know whether every sufficiently large plane triangulation, other than the fan, can be augmented to a 5-connected biplane graph when the new edges are not required to form a plane graph.

Several computational problems related to our results remain open. Is there a polynomial-time algorithm that, given a point set S, finds a 5-connected biplane graph with the *minimum* number of edges or reports that none exists? Is there a polynomial-time algorithm for finding the *minimum* number of edges to augment a given 3-connected plane graph with $n \geq 6$ vertices into a 4-connected biplane graph?

Acknowledgements

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