XIII International Conference on Computational Plasticity. Fundamentals and Applications
COMPLAS XIII
E. Oñate, D.R.J. Owen, D. Peric and M. Chiumenti (Eds)

XFEM formulation for discontinuities in fractured rock masses

Laura Crusat, Daniel Garolera and Ignacio Carol

ETSECCPB (School of Civil Engineering)
UPC (Universitat Politècnica de Catalunya)
08034 Barcelona

Key words: XFEM, geomechanics, quasi-brittle materials, discontinuities, fracture mechanics.

Abstract.
The paper describes some aspects of the application of XFEM to represent Geomechanical discontinuities, including the choice of additional nodal variables and the appearance and remedies to the oscillations that may take place depending on the mesh layout. An example of application to recover the stresses along a discontinuity line emanating from a tunnel cross-section is presented together with the comparison to an analytical solution. The formulation is developed in terms of the "overhang" displacement variables on the other side of the discontinuity (instead of more traditional displacement jump variables), and the oscillations associated to nodes too close to the discontinuity are solved by moving those nodes onto the discontinuity (instead of moving them away as seems more common in current practice).

1 INTRODUCTION

Discontinuities play an important role in many types of geomechanical problems, especially in those involving rock masses. The Extended Finite Element Method (XFEM) is a relatively recent method used to represent discontinuities, which was introduced as an enriched finite element approach that combines the standard FEM and the Partition of Unity Method (PUM) [1]. However, most of the XFEM literature describes the application to model opening tensile cracks [2], although the approach should be in principle also capable of modelling shear-compression mechanisms as well, as it corresponds for instance to faults or fractures in rock masses. To this end, the XFEM discontinuity should incorporate general standard traction-separation/sliding constitutive laws similar to those used for instance in traditional zero-thickness interface elements in rock mechanics.

In the paper, one such formulation is described and the results are verified with examples. The fundamental assumption of the displacement field, including the jumps across the discontinuity, is normally formulated in terms of the jump variables at the nodes, which are then interpolated to the discontinuity location. In this case, instead, an alternative approach has been followed that was proposed originally by Jirásek [1], which considers as new variables the "overhang" displacements that would correspond to the nodes across the discontinuity to obtain the desired deformation with the standard continuum interpolation functions. The proposed formulation, described in more detail in [3] together with some additional theoretical considerations, seem to lead to smooth results in general, although that requires to solve a number of implementation aspects that otherwise generate numerical oscillations in the results [4].

2 XFEM FORMULATION AND IMPLEMENTATION

The most characteristic assumption of the Extended Finite Element Method (XFEM) is an interpolation of the displacement field within the elements including a first term with the classical FEM shape functions, plus a second term for the discontinuity jump, as given by expression:

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}^{CONT} + \boldsymbol{u}^{DISC} = \sum_{k=1}^{n} \boldsymbol{N'}_k(\boldsymbol{x})\boldsymbol{u}_k + \sum_{i=1}^{m} \boldsymbol{N'}_i(\boldsymbol{x})\varphi_i(\boldsymbol{x})\boldsymbol{a}_i \tag{1}$$

where $\boldsymbol{u}(\boldsymbol{x})$ is the total displacement at the point of location $\boldsymbol{x}$ within the element, $\boldsymbol{u}_k$ the regular nodal displacements of the element, $\boldsymbol{a}_i$ the nodal jump discontinuity values, $\boldsymbol{N'}_k(\boldsymbol{x})$ the interpolation functions, and $\varphi_i(x)$ the enrichment functions. Subscript $k = 1,2,3, \dots, n$ is for the traditional degrees of freedom with n representing the number of nodes, while subscript $i = 1,2,3, \dots, m$ stands for the additional degrees of freedom with m representing the number of enriched nodes.

This work is focused on the study of 2D geomechanical discontinuities, which in general have a known, pre-established location. For the sake of clarity, only linear quadrilateral elements are considered, although the formulation below may be extended to triangular or quadratic elements. Regarding the geometry of intersections between the quadrangle and the discontinuity, two main situations may be distinguished: two nodes on each side ("2-2" intersection), and one node plus three nodes ("1-3" intersection), see Figure 1:

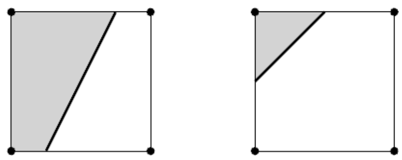

Figure 1. Basic intersection types: "2-2" (left) and a "1-3" (right).

In addition, different particular cases may be identified when the discontinuity coincides with a node or nodes (Figure 2) although these cases may be described as particular versions of the "2-2" or "1-3" intersections.

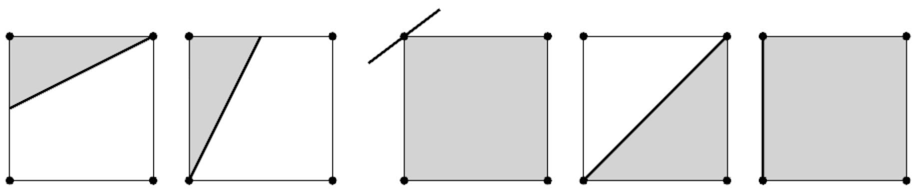

Figure 2. Special cases of intersection.

2.1 "Overhang" displacements $\boldsymbol{\delta}_i$ as additional variables.

Most XFEM formulations take as additional variables in Eq.(1) the displacement jumps $\boldsymbol{a}_i$. However, in this study the alternative choice proposed by Jirásek and Belytschko [1] is followed, in which the additional variables used are the "overhang" displacements across the discontinuity $\boldsymbol{\delta}_i$, with the physical meaning and the relation to the traditional displacement jumps $\boldsymbol{a}_i$ and total displacement values $\boldsymbol{u}_i$ as illustrated in Fig. 3.

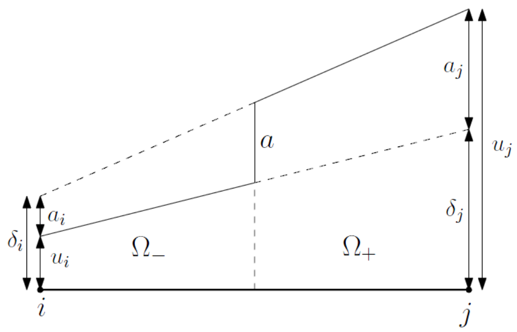

Figure 3. Schematic 1-D representation of real displacements ($\boldsymbol{u}_i$), fictitious displacements ($\boldsymbol{\delta}_i$) and jumps ($\boldsymbol{a}_i$).

The choice of δ_i instead of a_i has some advantages from the viewpoint of physical interpretation, and it also brings some theoretical advantages [3]. Mathematically, the relation between these variables is the following:

$$\begin{cases} \boldsymbol{\delta}_i = \boldsymbol{u}_i + \boldsymbol{a}_i \ \ in \ \ \Omega_- \\ \boldsymbol{\delta}_j = \boldsymbol{u}_j - \boldsymbol{a}_j \ \ in \ \ \Omega_+ \end{cases} \tag{2}$$

Considering separately the subdomains on each side of the discontinuity the previous relation leads to the corresponding interpolation expression and enrichment functions. On the negative side, this leads simply to:

$$u(x^-) = \sum_{i\in\Omega_-} N'_i(x)u_i + \sum_{j\in\Omega_+} N'_j(x)\delta_j \quad (3)$$

and on the positive side:

$$u(x^+) = \sum_{i\in\Omega_-} N'_i(x)\delta_i + \sum_{j\in\Omega_+} N'_j(x)u_j \quad (5)$$

that is, to the traditional FEM interpolation but using for each side of the discontinuity the appropriate combination of regular displacements $\boldsymbol{u}_i$ and overhang displacements $\boldsymbol{\delta}_j$. Finally, on the discontinuity itself the displacement jump may be defined from previous equations as the difference between real displacements on both sides of the discontinuity:

$$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{u}^+(x) - \boldsymbol{u}^-(x) \quad (7)$$

2.2 Other theoretical aspects and numerical implementation.

Based on the above equations, the full XFEM formulation, including stiffness matrices, force vectors, etc. has been developed according to the Principle of Virtual Work and other usual procedures in the literature. These aspects, together with some novel considerations on the so-called double interpolation assumption and equivalence to traditional double-node interface elements [5], are described can be found in [3].

3 NUMERICAL EXAMPLE

The example presented in this section shows the capability of the formulation to reproduce the correct stresses tractions along a radial XFEM discontinuity in a tunnel cross-section, and it also illustrates some effects of the mesh layout on the numerical results.

The example consists of a 6 meter radius tunnel cross-section, at the center of a 2D domain of 120x120m, with distributed loads of 8MPa applied on the top, 4MPa on the right side and prescribed movements (rollers) on bottom and left boundaries (Figure 4). The inclined discontinuity is assigned very high elastic stiffness values so that the overall behavior coincides with the classical Kirsch solution, with expressions:

$$\sigma_{rr} = \frac{P}{2}\left[(1+K)\left(1-\frac{a^2}{r^2}\right) - (1-K)\left(1-4\frac{a^2}{r^2}+\frac{3a^4}{r^4}\right)\cos 2\theta\right] \quad (13)$$

$$\sigma_{\theta\theta} = \frac{P}{2}\left[(1+K)\left(1+\frac{a^2}{r^2}\right) + (1-K)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right] \quad (14)$$

$$\sigma_{r\theta} = \frac{P}{2}\left[(1-K)\left(1+2\frac{a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta\right] \quad (15)$$

where P is the vertical load, K the horizontal-to-vertical remote stress ratio, a the tunnel radius, and r, θ the polar coordinates of any point with respect to the tunnel center.

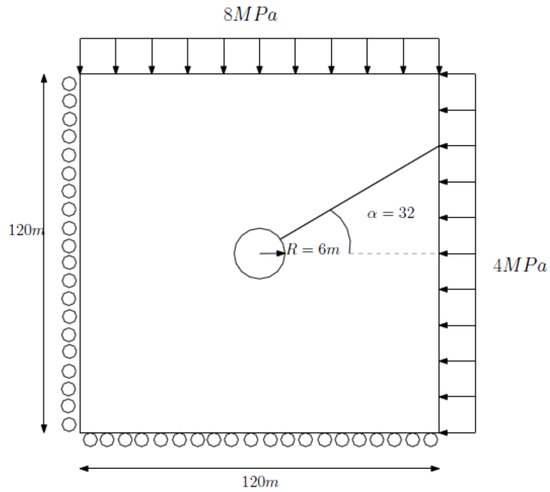

Figure 4. Square domain of 120x120m with a tunnel at the center.

The precise location of the discontinuity is defined by equation *y*=0.625*x*+22.5, and for the numerical analysis, this location is defined *via* level set. The elastic stiffness values for the discontinuity are $K_N = 10^8$ and $K_T = 10^8$ MPa/m. For the continuum, plane strain and linear elasticity are assumed, with E =1000MPa and ν = 0.3. The FE mesh is unstructured with 1029 quadrilateral elements, out of which 21 are XFEM-enriched (crossed by discontinuity), see Figure 5.

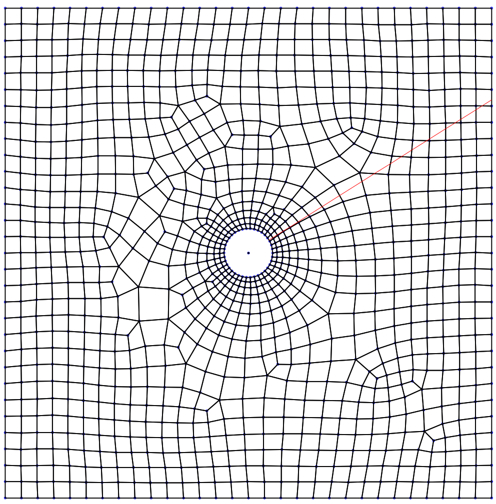

Figure 5. Unstructured squared mesh used for example 1, with a prefixed discontinuity (red line).

Figure 6 shows the normal and shear components of stress tractions transmitted across the discontinuity, that have been obtained from both the XFEM numerical calculation (in this case obtained directly as normal and shear stresses on the discontinuity itself, represented by dots in the figure), and from the Kirsch formulas (in this case projected on the discontinuity plane from the corresponding continuum stress components along the discontinuity location, solid line). As it is seen in the figure, XFEM results turn out very close to the analytical solution.

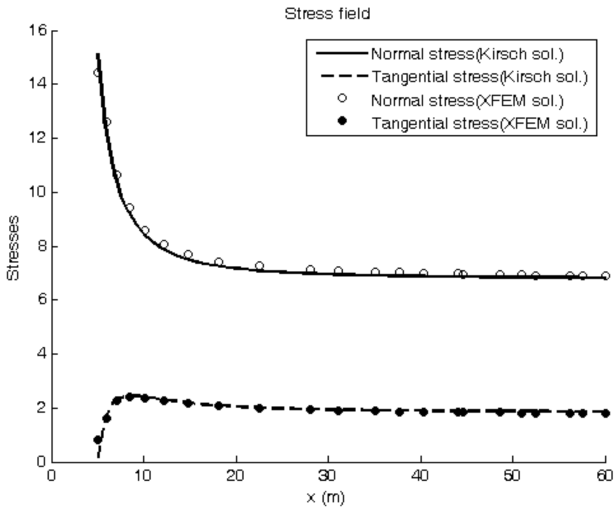

Figure 6. Stress field calculated on the discontinuity. Lines represent represents the analytical solution of Kirsch and dots are XFEM's solution.

The second set of results correspond to the same geometry and boundary conditions, but using another unstructured mesh defined by 920 quadrilateral elements, of which 25 are XFEM enriched, Figure 7.

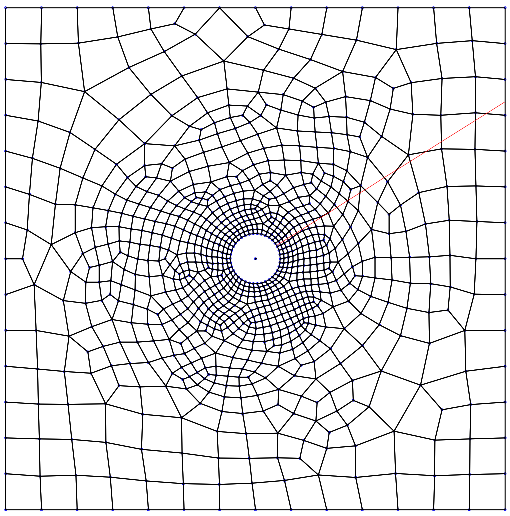

Figure 7. Unstructured square mesh used for example 2, with a prefixed discontinuity (red line).

The results should be similar, and in fact they mostly are; however, this second mesh leads to some stress oscillations, as we can see on Figure 8, especially in the tangential component.

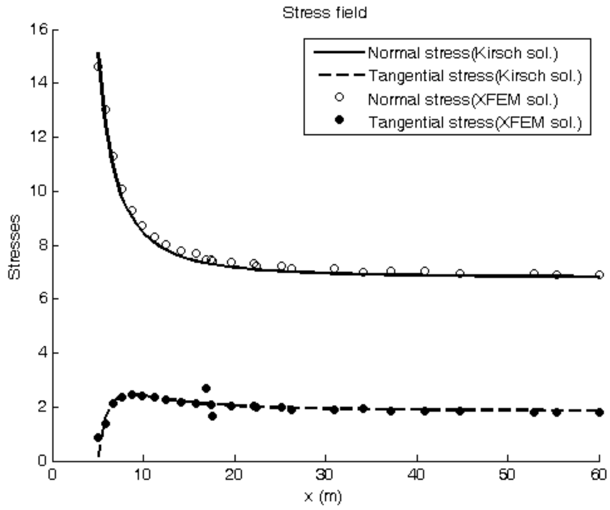

Figure 8. Stress components along the discontinuity. Lines represent the analytical solution of Kirsch and dots are XFEM solution with mesh 2.

The discontinuity oscillations happen to take place in the area where it intersects the FE lines at a skew angle and very near a node (see Figure 9 for a close-up of that part of the mesh). This leads to the suspicion that the oscillations are related to this type of intersection, and to verify that a few more calculations have been run after changing slightly the position of the nodes in that area, which seem to confirm the assumption.

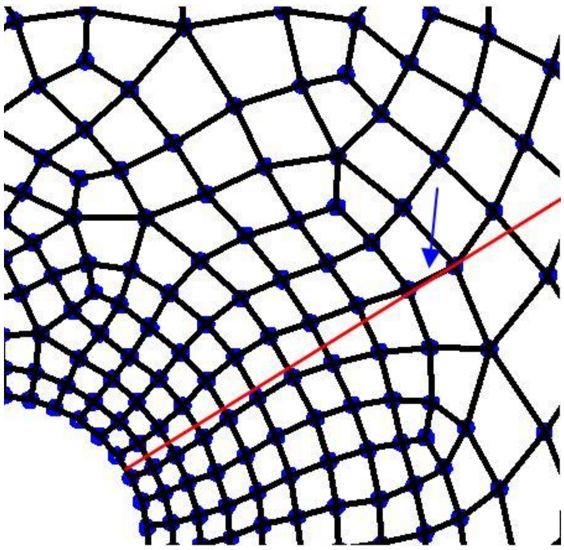

Figure 9. Close up of the area with oscillatory results.

As a remedy, in the literature one can find suggestions to shifting the position of the node of interest in that area slightly away from the discontinuity. However this requires establishing the precise strategy including by how much and in which direction this movement should take place. In this study, the opposite strategy has been explored, that is, moving these nodes *onto* the discontinuity itself. This alternative strategy requires fewer parameters, although the implementation may be a little longer because it requires to identify and treat separately the additional configurations shown in Fig. 2. By applying this strategy to this example, only the one node indicated in Fifg.9 is moved onto the discontinuity, and the stress oscillation disappears from the results as shown in Fig.10.

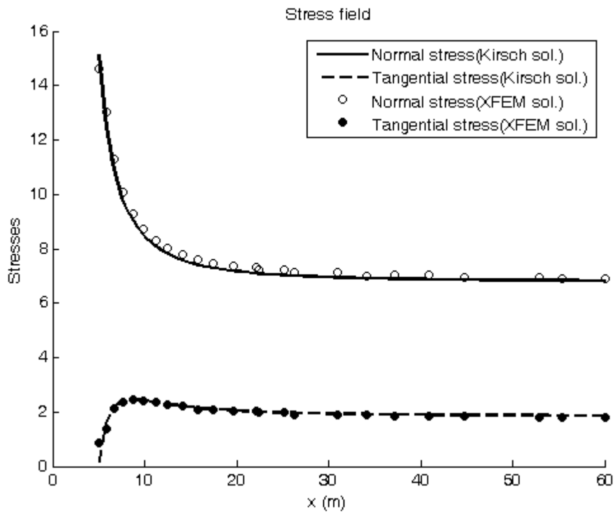

Figure 10. Stress field calculated on the discontinuity. Lines represent the analytical solution of Kirsch and dots are XFEM's solution.

5 CONCLUDING REMARKS

The application of XFEM to represent Geomechanical discontinuities has been outlined in this paper, with emphasis on basic aspects of the formulation and on numerical oscillations that may take place depending on the mesh layout. An example of application to recover the stresses along a discontinuity line emanating from a tunnel cross-section has been presented together with the comparison to an analytical solution. The formulation in terms of the “overhang” displacement variables on the other side of the discontinuity (instead of more traditional displacement jump variables) has some advantages in the mechanical interpretation and clearer physical meaning of the boundary conditions. A strategy to deal with the oscillations due to the discontinuity crossing the mesh lines very near some nodes, which is based on moving the nodes onto the discontinuity itself (rather than moving them away) has been also demonstrated successfully.

ACKNOWLEDGMENTS

The work was partially supported by research grants BIA2012-36898 from MEC (Madrid),which includes FEDER funds from the European Union, and 2014SGR-1523 from Generalitat de Catalunya (Barcelona). The first author acknowledges MICINN (Madrid) for her FPI doctoral fellowship.

REFERENCES

[1] Jirásek, M. and Belytschko, T. (2002). Computational resolution of strong discontinuities. CCM V Fifth World Congress on Computational Mechanics, July 7–12, 2002, Vienna, Austria. Eds.: H.A. Mang, F.G. Rammerstorfer, J. Eberhardsteiner

[2] Moës, N. and Belytschko, T. (2002). Extended finite element method for cohesive crack growth. Engineering Fracture Mechanics, (69):813-833.

[3] Crusat, L. , Garolera, D. and Carol, I (2015). XFEM with double interpolation and equivalence to zero-thickness interface elements. Submitted.

[4] Sanders, J. D., Dolbow, J. E., & Laursen, T. A. (2009). On methods for stabilizing constraints over enriched interfaces in elasticity. International Journal for Numerical Methods in Engineering, 78(9), 1009-1036.

[5] A.Gens, I.Carol, and E.E.Alonso (1988). An interface element formulation for the analysis of soil-reinforcement interaction. Computers and Geotechnics, (7):133-151.