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NUMERICAL STRESS INITIALIZATION IN GEOMECHANICS VIA THE FEM AND A TWO-STEP PROCEDURE

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Abstract.
The knowledge of the *in-situ* stress field in rock masses is crucial in different areas of geo-engineering, such as mining or civil underground excavations, hydrocarbon extraction, CO_2 storage, hydraulic fracture operations, etc. A method for the numerical generation of the *in-situ* stress state is described in this paper, which involves two steps: 1) an estimate of the stress state at each Gauss point is generated, and 2) global equilibrium is verified and re-balancing nodal forces are applied as needed. While the re-equilibration step is a closed procedure based only on statics, the first estimate of the stress state can be done in a variety of ways to incorporate all the information available. In this paper, the various options available are discussed and compared, and a new alternative procedure is presented which is based on the Airy stress function. The performance of the various procedures is illustrated with a real application example.

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1 INTRODUCTION

A rock mass or any geological material that is located at a certain depth is subjected to an *in situ* stress field. This stress field is the result not only of the geometry and weight of the geologic structure but also of a non-trivial geologic history. This history may include complex phenomena such as deposition, compaction, erosion or tectonic events.

Gravitational stresses are induced by the weight of the overburden, and often the vertical (or *z*) axis is a principal stress direction. Tectonic stresses may be the result of the tectonic movements at local or regional scale. The residual stresses are produced by strain energy locked-in in the rock from previous processes such as burial, lithification, denudation, heating and cooling. According to [4] a fraction of these residual stresses persist even after the rock is freed from boundary loads. When the *in-situ* stresses at a site are measured using any of the techniques available, the stress measure obtained includes the contributions from all those origins combined.

The knowledge of the stress field has been of great interest for both geotechnical/petroleum engineers and geologists. The first ones need a good estimate of the greenfield conditions, that are the unmodified initial stresses that exist within the rock mass, in the design of underground excavations for mining [2] or nuclear waste disposal [7], or hydraulic fracturing operations [6], while the geologist is usually more concerned about the processes that may have caused those stresses.

Together with an accurate description of the geologic structure, *in-situ* stresses existing in the rock mass constitute one of the most important factors for any rational and reliable analysis or design procedure, since initial stress may condition totally the response of the rock mass upon any actions considered in the analysis (loading, excavation, pore pressure variations, etc.).

2 GENERAL ASPECTS OF *IN-SITU* STRESS CALCULATIONS

In the context of a numerical analysis using the FEM, the initial conditions should be incorporated in the analysis following the generation of the FE mesh and prior to any further analysis. Ideally, this could be achieved by modelling the complete geological history in the rock mass. However, in the most common case that the complete geological history is not completely known, or that the realistic analysis of that history would be too complex or too expensive, the strategy is changed to simply trying to obtain the "current picture" of the stress state in the rock mass, which can be done using simplified procedures. To do this, one must consider carefully which conditions have to be rigorously enforced in this type of calculation, and which ones are actually not required.

In general terms, the set of governing equations commonly used for the formulation of the mechanical problem in small strains are:

The Equilibrium equation:

$$\sigma_{ij,j} + \rho g_i = 0. \tag{1}$$

The Compatibility equation:

$$\varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right). \tag{2}$$

The Material law:

$$\sigma_{ij} = f(\varepsilon_{kl}, \xi), \tag{3}$$

wherein ξ is one or more internal history variables, that may be present if the material is considered inelastic. If the material is elastic, the above equation takes the simplified form $\sigma_{ij} = E_{ijkl}\varepsilon_{kl}$ with no history variables.

The previous set of partial differential equations are valid over the domain Ω, and must be complemented with the appropriate boundary conditions over the boundary $\Gamma = \partial\Omega$, in the form of:

- *Dirichlet* boundary conditions with prescribed displacement (u_k) on the part Γ_u of the boundary where the displacements are known.
- *Newmann* boundary conditions with prescribed forces $(t_i = \sigma_{ij}n_j)$ on the remaining part Γ_σ of the boundary $(\Gamma = \Gamma_u \cup \Gamma_\sigma)$ where the external forces are known (including the part with zero stress, or "free boundary" condition).

In the case of *in-situ* stress initialization, typically the entire set of equations is solved through the use of a FE code with certain values of the material parameters, loading and B.Cs. However, strictly speaking the only equations that *must* be satisfied by the *in situ* stress state are the equilibrium equations (1) together with the corresponding Neumann boundary condition on Γ_σ, and, regarding the Material laws (3), in the case that the material is non-linear with a limit stress condition (i.e. stress criterion or plastic stress threshold), this limit *cannot* be violated by the *in-situ* stress obtained. The remaining conditions, such as aspects of material laws other than the strength limits (e.g. material deformability, elastic moduli, etc.) or kinematic compatibility at continuum or boundary level, would not need to be strictly enforced. This in principle makes the system of equations incomplete, with many possible (actually infinite) solutions for any given geological geometry, densities, strength limits, etc. which would seem to make the problem ill-defined. However, in most cases the Engineer has some additional information about the *in situ* stress state coming from the field, either from generic regional information (world stress map [10]), of from specific filed information (*in situ* stress tests, etc.), and the strategy of analysis changes, form the classical direct problem, to a variety of partially inverse procedures by which one tries to combine solutions, or use convenient (fictitious) values of kinematic boundary conditions and/or selected material parameters [11], so that in the end the outcome in terms of stresses is at the same time:

a) strictly satisfying the required equilibrium+ strength conditions, and
b) providing the best possible fit to available *in-situ* information.

Note finally, that, at the end of the *in situ* stress calculations the values of all kinematic variables such as nodal displacements or material strains are typically discarded and reset to zero, which is in agreement with the fact that kinematic conditions and deformational parameters (which are needed to solve the problem via FEM codes) take only convenient values but in general have no intrinsic meaning.

Based on this general philosophy, the simplest procedure commonly used to numerically generate *in situ* stress states, consists of simply applying gravity loads on the domain considered, with the original elastic modulus and a fictitious Poisson ratio with value:

$$\nu = \frac{K_0}{1+K_0}, \tag{4}$$

In this way, if medium homogeneous, surface horizontal and fully laterally constrained conditions the desired K_0 is recovered. Note however that this method is limited to $K_0 < 1$, and that if conditions different than mentioned it will not in general lead to the desired horizontal-to-vertical stress ratio at all points of the domain. These are some of the reasons that motivate the development of more elaborated procedures.

In this paper, a two-step method based on a proposed estimate not necessarily in strict equilibrium, plus a re-equilibration step is described and demonstrated in the case of a real geological cross-section.

3 A GENERAL TWO-STEP PROCEDURE TO OBTAIN *IN-SITU* STRESSES

The procedure described consists of two steps:

1) A first step is determining a first "proposal" ($\boldsymbol{\sigma}^{prop}$) of *in-situ* stress state in the domain Ω that satisfies the following basic conditions locally:

   –vertical stress components are in equilibrium with gravity loads

   – horizontal stresses satisfy locally the pre-established K_0 ratio to vertical stresses,

   – any other condition that may be desirable to satisfy based on the knowledge of field conditions.

   This proposal will be as close to global equilibrium as possible, but it is not a requirement that global equilibrium be satisfied strictly.

2) A second step that equilibrates this initial guess, in case it was not in equilibrium, by evaluating first the unbalanced nodal forces and then applying those forces to the discretization. In the case of linear elastic materials, unbalanced forces are completely redistributed in a single calculation. However, if the material is non-linear, iterations may be needed in this second calculation.

4 GENERATION OF THE INITIAL STRESS PROPOSAL $\boldsymbol{\sigma}^{prop}$

To generate the initial proposal of stress state (step 1 of he general procedure above), there are various options, and two of them are outlined in this section.

4.1 Stress proposal ($\boldsymbol{\sigma}^{prop}$) based on horizontal stress ratio (K_0)

The vertical stress is obtained as a simple function of the depth of the corresponding Gauss point, and the horizontal stress is then obtained from the desired horizontal-to-vertical stress ratio K_0. Following this procedure, the initial stress proposal will exhibit vertical and horizontal principal directions, and the following values:

$$\sigma_{zz} = \sigma_V = \int \gamma dz \ ; \quad \sigma_{xx} = \sigma_h = K_0 \sigma_V; \quad \tau_{xz} = 0, \tag{5}$$

where γ is the specific weight of the material, and K_0 is the desired horizontal-to-vertical stress ratio.

4.2 Stress proposal (σ^{prop}) based on stress functions

This second method is a little more elaborated and is based on the use of the so called *stress functions* $\Phi(x,y,z)$, from which the components of the stress tensor will follow as derivatives:

$$\sigma_{ij} = F_{ij}\{\Phi\}, \tag{6}$$

where F_{ij} is a differential operator.

These functions are usually chosen in such a way that they automatically satisfy the equilibrium equations. In the particular case of a two-dimensional analysis in the domain *x,z*, this reduces to:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial z^2} - K_0 \gamma z + \sigma_{xx_0}$$

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial x^2} - \gamma z + \sigma_{zz_0} \tag{7}$$

$$\sigma_{xz} = \frac{\partial^2 \phi}{\partial x \partial z} + \sigma_{xz_0}$$

where Φ is known as the Airy stress function $\Phi(x,z)$ [1,9]. In our particular case, a third degree polynomial expression with constant coefficients is used as stress function:

$$\Phi(x,z) = \frac{a_1}{6}x^3 + \frac{a_2}{2}x^2 z + \frac{a_3}{2}xz^2 + \frac{a_4}{6}z^3 \tag{8}$$

Applying equations (7), the second derivatives of Φ lead to expressions for the stress components that are linear in *x,z* and involve the coefficients a_i, some of which may be fixed on the basis of simple geo-mechanical considerations (such as vertical stresses gradient being γ or horizontal stresses ratio K_0). To determine the remaining coefficients, a minimization procedure is established as described in the following.

The entire domain is decomposed into vertical strips, which are in turn subdivided by the sub-horizontal lines of the geological layers into trapezoidal subdomains (Fig. 1). Each subdomain is limited on the top and bottom by surfaces S1 and S2 which are assumed plane but not necessarily horizontal, and are subject to the following boundary conditions: 1) The normal stress $\bar{\sigma}^{(\alpha)}$ on the top surface S_1 is prescribed as a linear function of *x* and *z*. 2) The shear intensity $\bar{\tau}^{(\alpha)}$ on the top surface S_1 is also prescribed as a linear function of *x* and *z*. 3) The shear intensity $\bar{\tau}^{(\beta)}$on the bottom surface S_2 is linked to the amount of normal stress on the same surface.

According to the boundary conditions imposed on the top and bottom surfaces of the subdomain (Fig. 1), an objective function $G(X_i)$ that evaluates the square difference between the normal/shear stresses and the prescribed values can be established. Considering the derivatives of the objective function with respect to the coefficients X_i, then the stress state that the best fit of boundary conditions is defined by the parameters (X_i) that minimize the objective function:

$$\frac{\partial G}{\partial X_i}(X_i) = \frac{\partial}{\partial X_i}\int_{S_1}\left(\sigma_n{}^{(\alpha)} - \bar{\sigma}^{(\alpha)}\right)^2 dS_1 + \frac{\partial}{\partial X_i}\int_{S_1}\left(\tau^{(\alpha)} - \bar{\tau}^{(\alpha)}\right)^2 dS_1 + \\ + \frac{\partial}{\partial X_i}\int_{S_2}\left(\tau^{(\beta)} - \bar{\tau}^{(\beta)}\right)^2 dS_2, \quad (9)$$

wherein $\sigma_n{}^{(\alpha)}$, $\tau^{(\alpha)}$ are the normal and shear stresses of the proposed distribution on the top surface of the subdomain S_1, the $\tau^{(\beta)}$ is shear stress on the bottom surface of the subdomain S_2, the $\bar{\sigma}^{(\alpha)}$, $\bar{\tau}^{(\alpha)}$ are the normal and shear stress values to be prescribed on S_1, and finally, the $\bar{\tau}^{(\beta)}$ is the shear stress value to be prescribed on S_2.

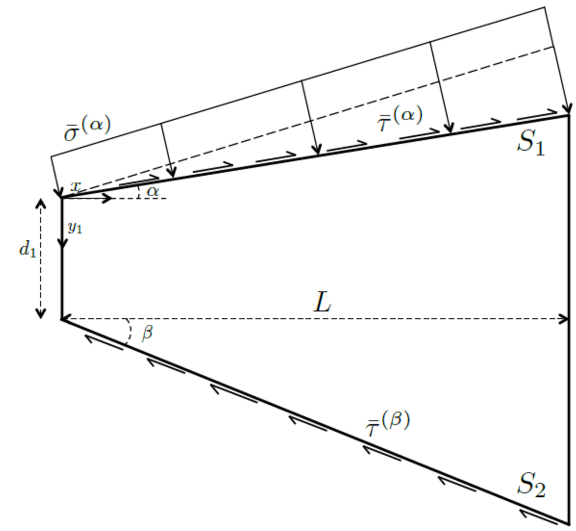

Figure 1 – General two-dimensional trapezoidal subdomain, with boundary surfaces S1 and S2. The conditions $\bar{\sigma}^{(\alpha)}$, $\bar{\tau}^{(\alpha)}$ and $\bar{\tau}^{(\beta)}$ are prescribed to these surfaces.

4 APPLICATION TO A GEOLOGICAL CROSS-SECTION

The example of application of the procedure described, consists of the geological 2D cross-section shown in Fig. 2 (upper diagram).

The geometry of the geological formation is very adequate for 2D analysis since all the cross-sections parallel to the one considered have a very similar geometry. Furthermore, the cross-section considered has the advantage of the availability of field information obtained using a variety of *in-situ* methods [3] from a wellbore. That information includes: Young's modulus (E), Poisson's ratio (ν), rock density (ρ), fluid pore pressure (P_f), and horizontal-to-vertical stress ratio (K_0). Table 1 shows these parameters for each layer of the domain (top to bottom).

In order to apply the procedure based on Airy stress functions, the domain is subdivided in a total of 22 vertical stripes, and each of these layers is in turn subdivided into a number of trapezoids by intersection with the 18 geological layers, as also shown in Fig.2 (up). Once the geometrical model is established, a FE mesh is generated as depicted in Fig. 2 (lower diagram), with a total of 2899 quadratic elements (2816 quadrangles and 83 triangles), and 8822 nodes.

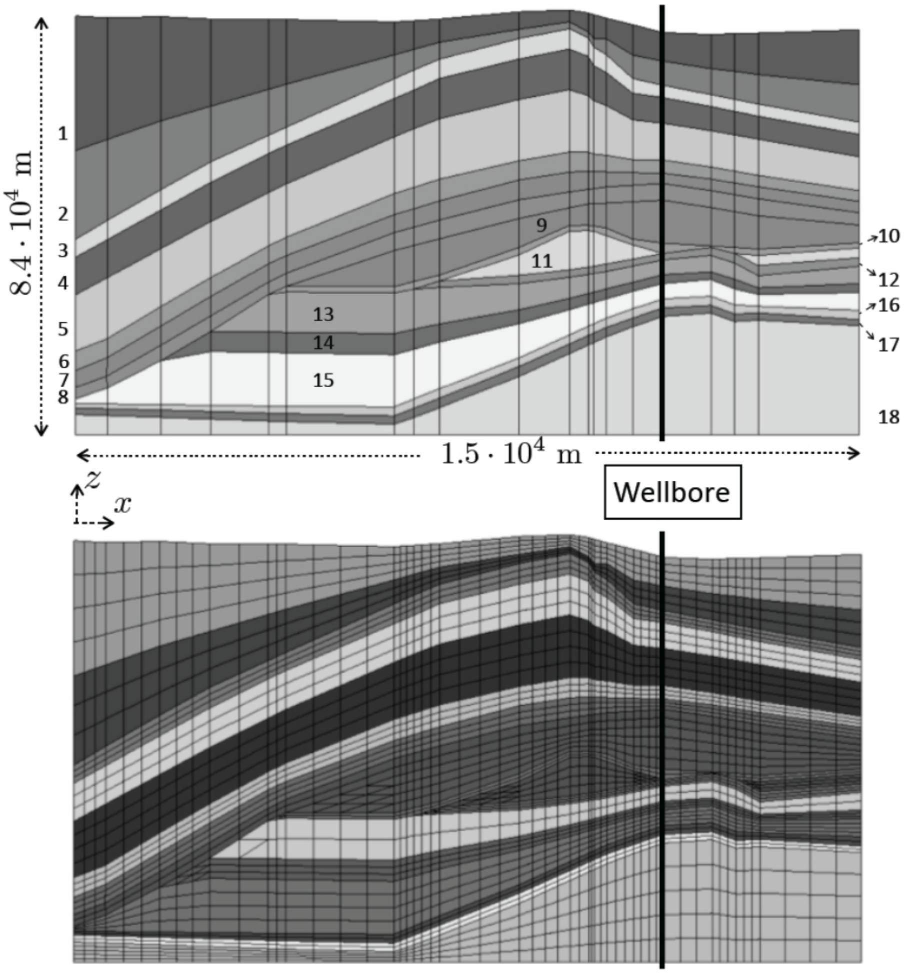

Figure 2 - Geomechanical model (above) and FE mesh (below) of the cross-section.

Rock properties assigned to each geomechanical unit are the ones detailed in Table 1. One aspect that has required some treatment for the comparison is the K_0 coefficient for each layer, because in the original information it was given as a rate between effective stresses, while in this study the FE analysis has been carried out is in terms of total stresses. Note also that, with values of horizontal-to-vertical stress ratios K_0 >1 (as shown in the table), the method of simply applying gravity with fictitious Poisson ratio would not be applicable.

Two calculations are presented in Fig. 3 in terms of the vertical and horizontal stress profiles along the wellbore. Both involve the two-step method described in Sect. 3, although they differ in the way the initial proposal (step 1) is generated:

a) The results in Fig. 3(a), correspond to an initial stress proposal consisting of calculating the vertical stress via application of gravity loads with the real material parameters of Table 1, and then replacing the horizontal stress with $\sigma_H = K_0 \sigma_V$.

b) In Fig. 3(b) the results correspond to the proposal of initial stresses is obtained with the Airy stress function procedure described in Sect. 3.

Table 1: Values of different parameters used in each layer in the geometry model

Layer	Young (GPa)	Specific weight	Poisson	K_0
1	13.79	24.13	0.224	2.21
2	27.23	25.21	0.23	2.00
3	22.75	25.11	0.227	1.31
4	19.99	24.33	0.22	1.09
5	31.03	25.51	0.235	1.18
6	44.82	25.90	0.235	1.36
7	37.92	24.82	0.24	1.21
8	46.88	25.41	0.25	1.31
9	17.24	24.13	0.233	1.13
10	42.75	25.02	0.25	1.18
11	49.64	25.80	0.253	1.21
12	46.88	25.80	0.22	1.24
13	39.99	25.02	0.244	1.20
14	43.78	25.11	0.245	1.23
15	37.92	25.21	0.24	1.25
16	51.71	25.90	0.25	1.24
17	31.03	25.21	0.245	1.29
18	42.75	25.02	0.25	1.18

As seen in the figure, in both cases the vertical stresses obtained in the calculations agree well with available values, especially for the upper-central layers down to 3500m, which are of main interest. In contrast, horizontal stresses exhibit a more significant difference between methods. For both strategies, the general trend is captured, but horizontal stress values obtained with strategy b) are closer to the in-situ measurements. Similarly to the vertical stresses case, the agreement between calculated and measured values is better in the upper part of the domain, from the surface to a depth of about 3000m.

5 CONCLUDING REMARKS

A methodology to generate an equilibrated initial stress state in geologic media and in a FE context has been described. This methodology constitutes one step forward with respect the simplest procedure of applying gravity loads with fictitious Poisson ratio, which is limited intrinsically to situations with K_0<1, and is also alternative to other existing procedures based on application of horizontal forces or displacements on the lateral (vertical) boundaries of the domain. The first step consists of proposing an initial stress state, not necessarily in equilibrium with the external loads, which is then re-equilibrated in the second step *via* FE calculation. The approaches to generate the initial stress state have been discussed, including a new strategy based on the subdivision in subdomains and Airy stress functions defined in each of those subdomains. This new strategy leads to a more realistic stress proposal, in which principal stresses do not need to be vertical and horizontal, but can be aligned with geological horizons, leading to lower unbalanced loads in the re-equilibration second step.

The application of the procedure is demonstrated in the case of a real geological cross-section for which in-situ stress values along a wellbore are available exhibiting K_0>1. The results show that, while vertical stresses obtained are accurate enough and similar for both strategies, for the horizontal stresses the results obtained with the Airy stress functions fits better the *in-situ* stress measurements.

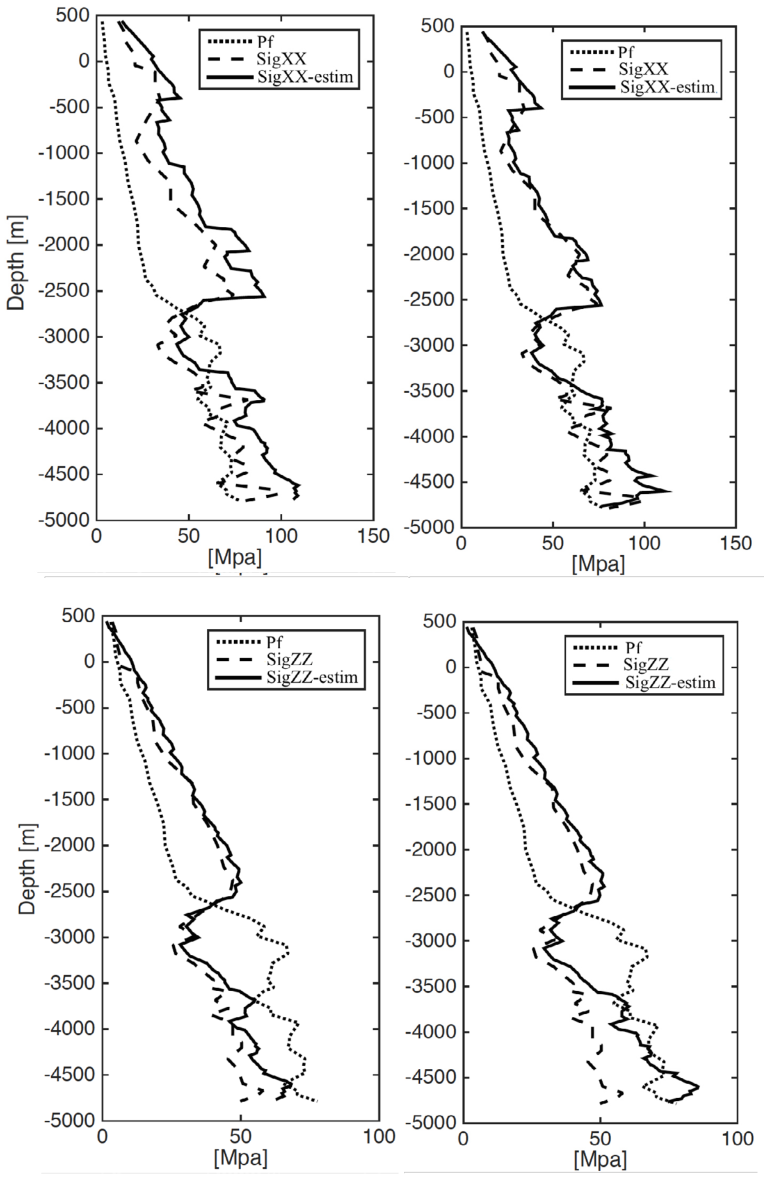

Figure 4 –Comparison with the in-situ values from the wellbore of: a) The horizontal stress profile calculated using the proposal stress based on horizontal stress ratio K_0 (top left); b) The horizontal stress profile obtained with the strategy based on Airy stress function (top right); c) The vertical stress profile calculated using the proposal stress based on horizontal stress ratio K_0. (bottom left); and d) The vertical stress profile obtained with strategy based on the Airy stress function (bottom right).

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