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Progress towards a unified approach to entanglement distribution

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Entanglement distribution is key to the success of secure communication schemes based on quantum mechanics, and there is a strong need for an ultimate architecture able to overcome the limitations of recent proposals such as those based on entanglement percolation or quantum repeaters. In this work we provide a broad theoretical background for the development of such technologies. In particular, we investigate the question of whether entanglement distribution is more efficient if some amount of entanglement—or some amount of correlations in general—is available prior to the transmission stage of the protocol. We show that in the presence of noise the answer to this question strongly depends on the type of noise and on the way the entanglement is quantified. On the one hand, subadditive entanglement measures do not show an advantage of preshared correlations if entanglement is established via combinations of single-qubit Pauli channels. On the other hand, based on the superadditivity conjecture of distillable entanglement, we provide evidence that this phenomenon occurs for this measure. These results strongly suggest that sending one half of some pure entangled state down a noisy channel is the best strategy for any subadditive entanglement quantifier, thus paving the way to a unified approach for entanglement distribution which does not depend on the nature of noise. We also provide general bounds for entanglement distribution involving quantum discord and present a counterintuitive phenomenon of the advantage of arbitrarily little entangled states over maximally entangled ones, which may also occur for quantum channels relevant in experiments.

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I. INTRODUCTION

Considered as a curiosity in the early days of quantum theory [1], entanglement has now been recognized as the essential ingredient for a growing number of applications in quantum technologies [2,3]. Among them we find, for example, the celebrated quantum cryptography [4] allowing for a provably secure communication between distant parties, and quantum teleportation [5], which offers the possibility of an intact transmission of a state of a particle over an arbitrarily long distance using preshared entanglement and classical communication. Entanglement is also necessary for quantum nonlocality, which is an even stronger resource for certain information-processing tasks, including the above-mentioned secure key distribution [4,6,7] and certified quantum randomness generation [8–10].

A common assumption behind entanglement-based protocols is that long-distance or at least medium-distance entanglement is available beforehand. Several remedies against this drawback have been recently proposed, with the most promising one being based on quantum repeaters [11] and entanglement distillation [12]. However, the necessity of powerful quantum memories appears as the main limiting factor in this proposal (cf. [13]). Another method is based on entanglement percolation [14], but it also suffers problems when considered in realistic situations in the presence of noise and decoherence [15].

The aim of the present work is to explore different realistic scenarios in which the long-distance entanglement can be distributed. The general framework for such a task we adopt here is the following (see Fig. 1). Two parties, Alice and Bob, initially share a three-particle quantum system. Two of the particles are with Alice, and the remaining one is in Bob's hands. In the most general situation we allow Alice and Bob to share some correlations established before the beginning of the protocol. The distribution of entanglement is then achieved with the aid of a quantum channel which is used to transmit one of Alice's particles to Bob.

A remarkable result with respect to such general entanglement distribution protocols has been obtained in [16]. There, it was shown that the process is even possible without sending entanglement directly: for successful entanglement distribution the exchanged particle does not need to be entangled with the rest of the system. This phenomenon has been termed "entanglement distribution with separable states," and its experimental verification has also been reported recently [17–20]. These results suggest that such a distribution procedure may be advantageous in the presence of noise: it could be possible to surpass the fragileness of entanglement by sending a separable particle.

Despite considerable attempts to understand this phenomenon [21–26], one of the most important questions remains unresolved: *Can noisy entanglement distribution with separable states provide an advantage when compared to sending one-half of the maximally entangled state through the same noisy channel?* Note that the answer for this question also has a direct importance for the theory and practice of quantum repeaters and quantum percolation where the intermediate-distance entanglement between the involved nodes must be established in some way. In this work we attack this problem by focusing on the following closely related questions:

(1) Given a noisy quantum channel, what is the maximal amount of entanglement that can be distributed with and without preshared correlations?

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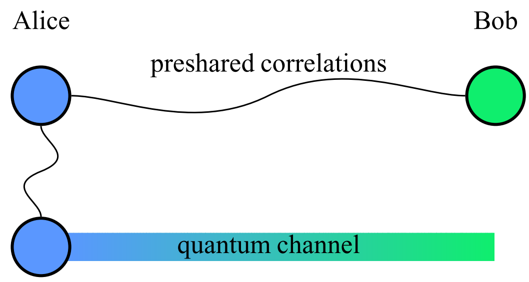

FIG. 1. (Color online) General framework for entanglement distribution. Alice is initially in possession of two particles, while one particle is in Bob's hands. Alice and Bob further have access to preshared correlations and an additional, possibly noisy quantum channel which is used for entanglement distribution.

(2) Are preshared correlations helpful for entanglement distribution via a given quantum channel?

Note that a negative answer to the second question also implies that entanglement distribution with separable states is not the best strategy in this situation.

As our study reveals, the answers to these questions depend on the way entanglement is quantified. In particular, we show that if the entanglement quantifier is subadditive (that is, its value for a tensor product of any two states is not greater than the sum of the values for the individual states), preshared correlations provide no advantage for single-qubit Pauli channels or any tensor product thereof. In this situation the best distribution strategy is to send one half of the maximally entangled state down the noisy channel. However, not all entanglement quantifiers are subadditive. In particular, it is conjectured that the distillable entanglement is superadditive [27]. Assuming this conjecture holds true, we show that preshared correlations can indeed provide an advantage for the distribution of distillable entanglement. Another surprising result is obtained for the logarithmic negativity: for this entanglement measure states with arbitrarily little entanglement can show better performance for entanglement distribution when compared to maximally entangled states. We further present bounds for noisy entanglement distribution given by quantum discord [28,29], thus significantly extending the results provided in [22,23] to the noisy scenario.

Moreover, the results presented in this work strongly suggest that a unified approach to entanglement distribution is indeed possible. In particular, based on our findings it is very reasonable to assume that preshared correlations do not provide an advantage for any subadditive entanglement quantifier, regardless of the type of noisy channel used for the distribution. If this assumption is correct, sending one half of some pure entangled state down a noisy channel will be the best strategy in this very general scenario. However, we also show that maximally entangled states are not necessarily optimal for this process.

This paper is organized as follows. In Sec. II we study noiseless entanglement distribution, while the scenario involving noise is considered in Sec. III. In Sec. IV we investigate the optimal entanglement distribution without preshared correlations, i.e., we consider the maximal amount of entanglement that can be distributed via a given noisy channel if Alice and Bob do not share any correlations initially. Finally, the possible advantage of preshared correlations for noisy entanglement distribution is discussed in Sec. V.

II. NOISELESS ENTANGLEMENT DISTRIBUTION

The starting point of this section is the general scenario for entanglement distribution considered in [22,23]; see also [29] for a detailed discussion. In particular, we assume that two parties, Alice and Bob, have access to a general tripartite quantum state $\rho = \rho^{ABC}$. We further assume, without loss of generality, that the entanglement distribution is realized by sending the particle C from Alice to Bob, and that during the entire process particles A and B are in possession of Alice and Bob, respectively. If the quantum channel used for the transmission of the particle C is noiseless, the amount of entanglement distributed in this process is quantified via the difference $E^{A|BC}(\rho) - E^{AC|B}(\rho)$ between the final amount of entanglement $E^{A|BC}(\rho)$ and the initial amount of entanglement $E^{AC|B}(\rho)$.

As it was shown in [16], entanglement distribution is also possible by sending a particle which is not entangled with the rest of the system, i.e., there exist states $\rho = \rho^{ABC}$ such that $E^{C|AB}(\rho) = 0$ and, at the same time, $E^{A|BC}(\rho) - E^{AC|B}(\rho) > 0$. This finding has triggered a debate about the type of correlations which are responsible for entanglement distribution. An important result in this context was provided in Refs. [22] and [23]. The amount of distributed entanglement cannot exceed the amount of quantum discord $\Delta^{C|AB}$ between the exchange particle C and the remaining system AB:

$$\Delta^{C|AB}(\rho) \geqslant E^{A|BC}(\rho) - E^{AC|B}(\rho). \tag{1}$$

At this point, it is also important to notice that in general quantum discord does not vanish on separable states. This inequality was shown to hold for all distance-based quantifiers of entanglement and discord [22]:

$$E^{X|Y}(\rho^{XY}) = \min_{\sigma^{XY}\in\mathcal{S}} D(\rho^{XY},\sigma^{XY}), \tag{2}$$

$$\Delta^{X|Y}(\rho^{XY}) = \min_{\{\Pi_i^X\}} D\left(\rho^{XY}, \sum_i \Pi_i^X \rho^{XY} \Pi_i^X\right). \tag{3}$$

Here, $\mathcal{S}$ is the set of bipartite separable states, $\{\Pi_i^X\}$ is a local von Neumann measurement on the subsystem X, and D can be any general distance which satisfies the following two properties [22]:

(1) D does not increase under quantum operations:

$$D(\Lambda[\rho],\Lambda[\sigma]) \leqslant D(\rho,\sigma) \tag{4}$$

for any quantum operation Λ and any pair of quantum states ρ and σ, and

(2) D satisfies the triangle inequality

$$D(\rho,\sigma) \leqslant D(\rho,\tau) + D(\tau,\sigma) \tag{5}$$

for any three quantum states ρ, σ, and τ.

As it was further shown in [22,23], the results presented above also hold for the quantum relative entropy $S(\rho||\sigma) = \mathrm{Tr}[\rho \log \rho] - \mathrm{Tr}[\rho \log \sigma]$, despite the fact that the relative entropy in general does not satisfy the triangle inequality. The corresponding quantifiers of entanglement and discord in this case are known as the relative entropy of entanglement E_R and the relative entropy of discord Δ_R:

$$E_R^{X|Y}(\rho^{XY}) = \min_{\sigma^{XY}\in\mathcal{S}} S(\rho^{XY}||\sigma^{XY}), \tag{6}$$

$$\Delta_R^{X|Y}(\rho^{XY}) = \min_{\{\Pi_i^X\}} S\left(\rho^{XY}\middle\|\sum_i \Pi_i^X\rho^{XY}\Pi_i^X\right). \tag{7}$$

The relative entropy of entanglement E_R was originally introduced in [30,31]. By its relation to the relative entropy [32,33], it plays a fundamental role in quantum information theory. E_R is known to be an upper bound on the distillable entanglement E_d [34,35] and a lower bound on the entanglement of formation E_f [31]:

$$E_d \leqslant E_R \leqslant E_f. \tag{8}$$

The distillable entanglement E_d quantifies the maximal number of singlets that can be asymptotically obtained per copy of the given state via local operations and classical communication (LOCC) [12]. The entanglement of formation E_f is defined as [36]

$$E_f(\rho^{XY}) = \min\sum_i p_i E(|\psi_i\rangle^{XY}), \tag{9}$$

where the minimum is taken over all pure-state decompositions $\{p_i, |\psi_i\rangle^{XY}\}$ of the state ρ^{XY}, i.e., $\rho^{XY} = \sum_i p_i |\psi\rangle\langle\psi|^{XY}$, and $E(|\psi\rangle^{XY}) = S(\rho^X)$ is the von Neumann entropy of the reduced state.

The relative entropy of discord Δ_R was originally introduced in [37], where it was called "one-way information deficit" [38]. It quantifies the amount of information which cannot be localized by one-way classical communication between two parties.

A. Relation to distillable entanglement and entanglement cost

Equipped with these tools, we are now in a position to present the first results of this paper. In particular, we will provide a close connection between the relative entropy of discord Δ_R, the distillable entanglement E_d, and the entanglement cost E_c. The latter is defined as the minimal number of singlets per copy required for the asymptotic creation of a bipartite quantum state via LOCC [3], and can also be written as the regularized entanglement of formation [39]:

$$E_c(\rho) = \lim_{n\to\infty}\frac{1}{n}E_f(\rho^{\otimes n}). \tag{10}$$

The aforementioned relation between Δ_R, E_d, and E_c is provided in the following theorem.

Theorem 1. Given a tripartite state $\rho = \rho^{ABC}$, the following inequality holds:

$$\Delta_R^{C|AB}(\rho) \geqslant E_d^{A|BC}(\rho) - E_c^{AC|B}(\rho). \tag{11}$$

Proof. This inequality can be proven by noticing that the inequality (1) also holds for the regularized relative entropy of entanglement and discord:

$$\lim_{n\to\infty}\frac{\Delta_R^{C|AB}(\rho^{\otimes n})}{n} \geqslant \lim_{n\to\infty}\frac{E_R^{A|BC}(\rho^{\otimes n})}{n} - \lim_{n\to\infty}\frac{E_R^{AC|B}(\rho^{\otimes n})}{n}. \tag{12}$$

By applying Eq. (8) and using the fact that the distillable entanglement E_d does not change under regularization we arrive at the inequality

$$\lim_{n\to\infty}\frac{\Delta_R^{C|AB}(\rho^{\otimes n})}{n} \geqslant E_d^{A|BC}(\rho) - \lim_{n\to\infty}\frac{E_f^{AC|B}(\rho^{\otimes n})}{n}. \tag{13}$$

In the next step we recall that the entanglement cost is equal to the regularized entanglement of formation, see Eq. (10), and thus

$$\lim_{n\to\infty}\frac{\Delta_R^{C|AB}(\rho^{\otimes n})}{n} \geqslant E_d^{A|BC}(\rho) - E_c^{AC|B}(\rho). \tag{14}$$

Finally, the desired inequality (11) follows by observing that the relative entropy of discord does not increase under regularization: $\Delta_R(\rho) \geqslant \lim_{n\to\infty}\Delta_R(\rho^{\otimes n})/n$. ■

Notice that Eq. (11) has a clear operational interpretation: the relative entropy of discord is an upper bound on the number of singlets gained in the process of entanglement distribution in the asymptotic limit. This is because $E_d^{A|BC}(\rho)$ quantifies the number of singlets Alice and Bob can distill *after* performing the entanglement distribution, while $E_c^{AC|B}(\rho)$ quantifies the amount of singlets that Alice and Bob need to create the state $\rho = \rho^{ABC}$ *before* the entanglement distribution has been performed, both in the asymptotic limit. Moreover, as already mentioned in the proof of Theorem 1, this statement is also true for the regularized relative entropy of discord $\lim_{n\to\infty}\Delta_R(\rho^{\otimes n})/n$.

B. Relation to measures of NPT entanglement and distillability

The results presented above demonstrate that the relation between entanglement and discord in Eq. (1) is more general than anticipated by the original approach [22,23]. In the following we will go one step further by extending these results to general measures of NPT (nonpositive partial transpose) entanglement. In particular, we consider entanglement quantifiers of the form [3]

$$E_{\mathrm{PPT}}^{X|Y}(\rho^{XY}) = \min_{\sigma^{XY}\in\mathrm{PPT}} D(\rho^{XY},\sigma^{XY}), \tag{15}$$

where PPT is the set of states having positive partial transpose, and the distance D satisfies Eqs. (4) and (5). The amount of quantum discord is defined in the same way as in Eq. (3):

$$\Delta^{X|Y}(\rho^{XY}) = \min_{\{\Pi_i^X\}} D\left(\rho^{XY},\sum_i \Pi_i^X\rho^{XY}\Pi_i^X\right). \tag{16}$$

The following theorem shows that inequality (1) also applies to these measures of NPT entanglement.

Theorem 2. Given a tripartite state $\rho = \rho^{ABC}$, the following inequality holds:

$$\Delta^{C|AB}(\rho) \geqslant E_{\mathrm{PPT}}^{A|BC}(\rho) - E_{\mathrm{PPT}}^{AC|B}(\rho). \tag{17}$$

Proof. The proof goes along the lines of the one of Eq. (1), first presented in [22]. We start by introducing the state

$\sigma = \sigma^{ABC}$, which is PPT with respect to the bipartition $AC|B$, and, moreover, we assume that it is the closest PPT state to ρ: $E_{\mathrm{PPT}}^{AC|B}(\rho) = D(\rho,\sigma)$. We then define the states

$$\rho' = \sum_i \Pi_i^C \rho \Pi_i^C \tag{18}$$

and

$$\sigma' = \sum_i \Pi_i^C \sigma \Pi_i^C \tag{19}$$

to arise from ρ and σ via the local von Neumann measurement on C, minimizing the distance between ρ and ρ', i.e., $\Delta^{C|AB}(\rho) = D(\rho,\rho')$. Furthermore, we use the fact that the distance D satisfies the triangle inequality, and thus

$$D(\rho,\sigma') \leqslant D(\rho,\rho') + D(\rho',\sigma'). \tag{20}$$

Recalling that D does not increase under quantum operations, it follows that

$$D(\rho',\sigma') \leqslant D(\rho,\sigma), \tag{21}$$

and Eq. (20) becomes

$$D(\rho,\sigma') \leqslant \Delta^{C|AB}(\rho) + E_{\mathrm{PPT}}^{AC|B}(\rho). \tag{22}$$

In the final step we note that the state (19) is PPT with respect to all bipartitions, i.e., $E_{\mathrm{PPT}}^{AB|C}(\sigma') = E_{\mathrm{PPT}}^{AC|B}(\sigma') = E_{\mathrm{PPT}}^{A|BC}(\sigma') = 0$. The fact that $E_{\mathrm{PPT}}^{AB|C}(\sigma') = 0$ is obvious, since σ' arises by performing a local von Neumann measurement $\{\Pi_i^C\}$ on the state σ, and thus has the form of a quantum-classical state: $\sigma' = \sum_i \Pi_i^C \sigma \Pi_i^C = \sum_i p_i \sigma_i^{AB} \otimes |i\rangle\langle i|^C$. Moreover, by the very construction, the state σ is PPT with respect to the bipartition $AC|B$, and so is σ', meaning that $E_{\mathrm{PPT}}^{AC|B}(\sigma') = 0$. This, together with the fact that σ' is classical on the subsystem C, implies that it is also PPT with respect to the remaining bipartition $A|BC$: $E_{\mathrm{PPT}}^{A|BC}(\sigma') = 0$. This means that the distance between ρ and σ' is an upper bound on $E_{\mathrm{PPT}}^{A|BC}(\rho)$, which, when applied in Eq. (22), completes the proof. ■

The above theorem extends the range of applications of Eq. (1) to distance-based quantifiers of NPT entanglement. The same arguments can also be applied to measures of distillability defined as [3]

$$E_{\mathrm{ND}}^{X|Y}(\rho^{XY}) = \min_{\sigma^{XY}\in \mathrm{ND}} D(\rho^{XY},\sigma^{XY}). \tag{23}$$

Here, ND is the set of nondistillable states, and, as before, the distance D satisfies Eqs. (4) and (5). Using the same arguments as in the proof of Theorem 2, we see that Eq. (17) generalizes to these distillability measures:

$$\Delta^{C|AB}(\rho) \geqslant E_{\mathrm{ND}}^{A|BC}(\rho) - E_{\mathrm{ND}}^{AC|B}(\rho), \tag{24}$$

where the quantum discord $\Delta^{C|AB}$ is defined in the same way as in Eq. (16).

Finally, the above results also hold for measures of NPT entanglement and distillability based on the relative entropy, although, as mentioned earlier, the latter does not satisfy the triangle inequality in general. The fact that Eqs. (17) and (24) still apply to these measures can be seen using the same arguments as in the proof of Theorem 2 by observing that for the relative entropy the inequality (20) becomes equality [22].

C. Relation to Schatten norms

The results presented so far hold for a very general class of quantifiers for entanglement and discord. In particular, we have seen that Eq. (1) applies for any entanglement measure E, which is defined via the minimal distance to the set of separable, nondistillable, or PPT states, if the amount of discord Δ is quantified as in Eq. (3). The corresponding distance only needs to satisfy two minimal requirements given in Eqs. (4) and (5): it should not increase under quantum operations and it should satisfy the triangle inequality. On the other hand, we have also seen that Eq. (1) can be still valid even if the distance violates one of these properties. This was demonstrated for the relative entropy which can violate the triangle inequality.

In the following we will show that Eq. (1) may also hold for distances violating Eq. (4), i.e., those that are not contractive under quantum operations. To this end we will consider the following distance:

$$D_p(\rho,\sigma) = ||\rho - \sigma||_p, \tag{25}$$

with $\|\cdot\|_p$ being the Schatten p norm of an operator M defined through

$$||M||_p = (\mathrm{Tr}[(M^\dagger M)^{p/2}])^{1/p}, \tag{26}$$

with $p \geqslant 1$. Clearly, D_1 coincides with the trace distance and thus does not increase under quantum operations [2]. However, contrary to what had been claimed in [40], already D_2 (so-called Hilbert-Schmidt distance) can increase under quantum operations as shown in [41]. The arguments from Ref. [41] can be further generalized to show this fact for any $p > 1$ (see also [42,43] for similar considerations).

Now let E_p be defined as

$$E_p^{X|Y}(\rho^{XY}) = \min_{\sigma^{XY}\in T} D_p(\rho^{XY},\sigma^{XY}), \tag{27}$$

with the minimization going over the set T, which here might denote either of the sets: separable, nondistillable, or PPT states. Let further Δ_p be defined by Eq. (3) with the distance taken to be D_p. The following theorem shows that Eq. (1) also holds in this situation.

Theorem 3. Given a tripartite state $\rho = \rho^{ABC}$, the following inequality holds:

$$\Delta_p^{C|AB}(\rho) \geqslant E_p^{A|BC}(\rho) - E_p^{AC|B}(\rho). \tag{28}$$

Proof. The proof follows exactly the lines of the proof of Theorem 2. The only thing which needs to be proved is the fact that although D_p may increase under general quantum operations, it does not for those operations that map the states ρ and σ to ρ' and σ' in Eqs. (18) and (19), respectively. For this purpose, we notice that such mapping is unital, i.e., $\sum_i \Pi_i^C \mathbb{1} \Pi_i^C = \mathbb{1}$, where $\mathbb{1} = \mathbb{1}^{ABC}$ is the identity operator, and it was shown in Ref. [42] that no unital map can increase the p norm for any $p \geqslant 1$. This implies that for $p \geqslant 1$, D_p does satisfy Eq. (21) for the states of interest, which completes the proof. ■

While quantifiers of discord based on Schatten norms have been considered only recently [43–54], entanglement quantifiers of this type were studied already more than a decade ago [40,41,55–57]. Despite this fact, it has been an open question if E_p is a proper entanglement measure, i.e., if it is nonincreasing under LOCC for $p > 1$ [41]. In what follows we will put this question to rest by showing that E_p can increase by simply discarding a part of the system. For this purpose, let ρ^{AB} be a quantum state such that $E_p^{A|B}(\rho^{AB}) > 0$. Then, consider its extension to a three-partite state defined as $\rho^{ABC} = \rho^{AB} \otimes \mathbb{1}^C/2$, where the particle C is a qubit. We will now show that the entanglement of ρ^{AB} is larger than the entanglement of ρ^{ABC}:

$$E_p^{A|B}(\rho^{AB}) > E_p^{A|BC}(\rho^{ABC}) \tag{29}$$

for all $p > 1$. To this end, observe that the amount of entanglement $E_p^{A|BC}(\rho^{ABC})$ is bounded from above by the distance $D_p(\rho^{ABC},\sigma^{ABC})$ for $\sigma^{ABC} = \sigma^{AB} \otimes \mathbb{1}^C/2$, where σ^{AB} is the closest separable state to ρ^{AB}. Moreover, notice that the distance between ρ^{ABC} and σ^{ABC} can also be expressed as [54]

$$D_p(\rho^{ABC},\sigma^{ABC}) = \left\| \frac{\mathbb{1}^C}{2} \right\|_p D_p(\rho^{AB},\sigma^{AB}). \tag{30}$$

Recalling that the state σ^{AB} was defined to be the closest separable state to ρ^{AB} and using the fact that $||\mathbb{1}^C/2||_p = 2^{1/p-1}$, one obtains

$$E_p^{A|BC}(\rho^{ABC}) \leqslant 2^{1/p-1} E_p^{A|B}(\rho^{AB}). \tag{31}$$

The inequality (29) follows by noting that for $p > 1$ the exponent $1/p - 1$ is negative, and thus $2^{1/p-1} < 1$ in this case.

Similar results with respect to quantum discord were also obtained recently [43,54]. In particular, it was pointed out in [43] that the geometric discord $\Delta_G^{X|Y} = (\Delta_2^{X|Y})^2$ can increase under local operations on any of the parties X or Y, while most quantifiers of discord known in the literature do not increase under quantum operations on the subsystem Y. This result was later extended to all measures of discord Δ_p for $p > 1$ [54]. On the one hand, this observation together with Eq. (29) provides strong constraints for the possible applications of entanglement and discord quantifiers based on Schatten norms. On the other hand, the close relation of E_2 to the problem of finding optimal entanglement witnesses [56,57] and the connection between E_p and Δ_p established in Theorem 3 demonstrate the use of these quantities for understanding the structure of entanglement from a geometric perspective.

III. NOISY ENTANGLEMENT DISTRIBUTION

In the scenario considered so far, Alice and Bob aimed at distributing entanglement by having access to a noiseless quantum channel. Since noise is unavoidable in any realistic experiment; we will now consider the more general situation in which the channel used for entanglement distribution is noisy. Similarly to the foregoing discussion, we assume that Alice and Bob have access to a tripartite initial state $\rho_i = \rho^{ABC}$, where Alice is initially in possession of the particles A and C, and Bob is in possession of the remaining particle B. If Alice uses a noisy channel Λ^C to send her particle C to Bob, they end up in the final state $\rho_f = \Lambda^C[\rho_i]$. The amount of entanglement distributed in this process is then given by $E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i)$.

Having introduced the concept of noisy entanglement distribution, we are now in a position to extend Eq. (1) to

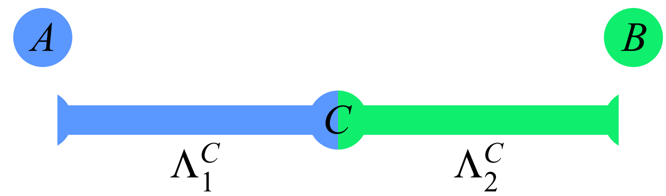

FIG. 2. (Color online) Decomposition of a noisy channel Λ^C in two channels Λ_1^C and Λ_2^C. For an initial state $\rho_i = \rho^{ABC}$ the final state after the application of the channel is given by $\rho_f = \Lambda^C[\rho_i] = \Lambda_2^C(\Lambda_1^C[\rho_i])$. The figure illustrates the intermediate state $\tilde{\rho} = \Lambda_1^C[\rho_i]$ after the application of Λ_1^C only. See main text for details.

this general scenario. In the following theorem we will show that noisy entanglement distribution is in general limited by the amount of discord in each of the states ρ_i and ρ_f.

Theorem 4. Given a quantum channel Λ^C and two states $\rho_i = \rho^{ABC}$ and $\rho_f = \Lambda^C[\rho_i]$, the following inequality holds:

$$\min\{\Delta^{C|AB}(\rho_i),\Delta^{C|AB}(\rho_f)\} \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i). \tag{32}$$

Here, E and Δ are any quantifiers of entanglement and discord which satisfy Eq. (1).

Proof. We first apply Eq. (1) to the state ρ_i, thus arriving at $\Delta^{C|AB}(\rho_i) \geqslant E^{A|BC}(\rho_i) - E^{AC|B}(\rho_i)$. Then the inequality $\Delta^{C|AB}(\rho_i) \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i)$ follows by recalling that entanglement does not increase under local noise, i.e., $E^{A|BC}(\rho_i) \geqslant E^{A|BC}(\rho_f)$. Using analogous reasoning one can also prove the inequality $\Delta^{C|AB}(\rho_f) \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i)$. Application of Eq. (1) to the state ρ_f gives us the inequality $\Delta^{C|AB}(\rho_f) \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_f)$. One then completes the proof by using $E^{AC|B}(\rho_f) \leqslant E^{AC|B}(\rho_i)$, which again follows from the fact that entanglement does not increase under local noise. As quantum discord can increase or decrease under local noise [44,58–60], the claim follows. ∎

A. Divisible channels

Let us consider a decomposition of the channel Λ^C into two channels Λ_1^C and Λ_2^C such that the successive application of these channels is equivalent to the application of Λ^C:

$$\rho_f = \Lambda^C[\rho_i] = \Lambda_2^C\big(\Lambda_1^C[\rho_i]\big). \tag{33}$$

(See also Fig. 2 for an illustration.) If such a decomposition is possible with nonunitary Λ_1^C and Λ_2^C, the channel Λ^C is called divisible [61]. By introducing an intermediate state

$$\tilde{\rho} = \Lambda_1^C[\rho_i], \tag{34}$$

we will now show that the amount of distributed entanglement is in general bounded above by the amount of discord in the state $\tilde{\rho}$:

$$\Delta^{C|AB}(\tilde{\rho}) \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i). \tag{35}$$

As in the foregoing discussion, we assume that E and Δ are quantifiers of entanglement and discord satisfying Eq. (1). Under this assumption, Eq. (35) can be proven using similar arguments as in the proof of Eq. (32). In particular, we can apply Eq. (1) to the intermediate state $\tilde{\rho}$, thus obtaining the inequality $\Delta^{C|AB}(\tilde{\rho}) \geqslant E^{A|BC}(\tilde{\rho}) - E^{AC|B}(\tilde{\rho})$. The proof of Eq. (35) is complete by making use of the fact that entanglement does not increase under local noise, leading to the inequalities $E^{A|BC}(\tilde{\rho}) \geqslant E^{A|BC}(\rho_f)$ and $E^{AC|B}(\tilde{\rho}) \leqslant E^{AC|B}(\rho_i)$.

B. Markovian time evolution

Here we will see that the results presented in the previous section have a nice application in the scenario in which the particle C used for entanglement distribution is subject to a Markovian time evolution $\Lambda^C_{(t_2,t_1)}$. If we assume that the process starts with the initial state $\rho_i = \rho^{ABC}$ at the time $t = 0$, then for any time $t \geqslant 0$ the time-evolved state is given by

$$\rho_t = \Lambda^C_{(t,0)}[\rho_i]. \tag{36}$$

Denoting then by T the total time required for the process, the corresponding final state ρ_f can be written as

$$\rho_f = \rho_T = \Lambda^C_{(T,0)}[\rho_i]. \tag{37}$$

We are now in position to prove that the amount of entanglement distributed via a Markovian time evolution is bounded from above by the amount of discord in the time-evolved state ρ_t for any $T \geqslant t \geqslant 0$:

$$\Delta^{C|AB}(\rho_t) \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i). \tag{38}$$

Here, E and Δ are quantifiers of entanglement and discord satisfying Eq. (1). To prove the above statement, we use the fact that any Markovian time evolution $\Lambda^C_{(t_2,t_1)}$ obeys the composition law [62], that is,

$$\Lambda^C_{(t_2,t_1)}[\rho] = \Lambda^C_{(t_2,t)}\left[\Lambda^C_{(t,t_1)}[\rho]\right] \tag{39}$$

for any state ρ and all $t_2 \geqslant t \geqslant t_1 \geqslant 0$. This, together with Eqs. (36) and (37), leads us to the following expression for the final state:

$$\rho_f = \Lambda^C_{(T,0)}[\rho_i] = \Lambda^C_{(T,t)}[\rho_t], \tag{40}$$

for all $T \geqslant t \geqslant 0$. One then obtains Eq. (38) by applying Eq. (35) with $\tilde{\rho} = \rho_t$.

Let us notice that the inequality (38) also implies that the distribution of entanglement via a Markovian time evolution is bounded above by the minimal discord $\min_t \Delta^{C|AB}(\rho_t)$, minimized over all times t ranging between 0 and the duration of the total procedure T. On the other hand, any violation of Eq. (38) can also be regarded as a witness for the non-Markovianity of the underlying time evolution. These results support recent attempts to detect and quantify non-Markovianity via quantum entanglement [63] and quantum discord [47,64,65]. Noting that the inequality (38) is valid for a very general class of quantifiers for entanglement and discord, further investigation in this direction can lead to a better understanding of entanglement and discord in the context of detecting non-Markovianity.

IV. OPTIMAL ENTANGLEMENT DISTRIBUTION WITHOUT PRESHARED CORRELATIONS

In the foregoing discussion we considered noiseless and noisy entanglement distribution, and presented several tools for bounding the amount of entanglement distributed in this process. In this section we will apply them to the following problem: *How much entanglement can be distributed via a given quantum channel?*

Let us begin with the scenario in which Alice and Bob are not correlated initially, i.e., the initial and the final state are given by

$$\rho_i = \rho^{AC} \otimes \rho^B \tag{41}$$

and

$$\rho_f = \Lambda^C[\rho^{AC}] \otimes \rho^B, \tag{42}$$

respectively. We assume again that Alice is initially in possession of the particles A and C, while Bob holds the particle B. In the distribution process, the particle C is sent from Alice to Bob via the quantum channel Λ^C. Thus the initial entanglement between Alice and Bob is zero, and the amount of distributed entanglement is given by $E^{A|C}(\Lambda^C[\rho^{AC}])$.

In the following, we are interested in optimal entanglement distribution, i.e., we ask which initial states ρ^{AC} lead to the maximal final entanglement after the application of a quantum channel Λ^C. Clearly, if the quantum channel Λ^C is noiseless, the optimal distribution strategy is achieved if Alice prepares her particles A and C in the maximally entangled state,

$$|\phi^+\rangle^{AC} = \frac{1}{\sqrt{d_C}} \sum_{i=0}^{d_C-1} |ii\rangle^{AC}, \tag{43}$$

and sends the particle C to Bob.

Interestingly, as we will see below, this strategy is not always optimal if the quantum channel Λ^C is noisy. In passing, it is crucial to notice that all maximally entangled states show the same performance for entanglement distribution, i.e.,

$$E(\Lambda^C[|\phi_{\mathrm{me}}\rangle\langle\phi_{\mathrm{me}}|^{AC}]) = E(\Lambda^C[|\phi^+\rangle\langle\phi^+|^{AC}]) \tag{44}$$

is true for any maximally entangled state $|\phi_{\mathrm{me}}\rangle^{AC}$, any entanglement measure E, and any noisy channel Λ^C. This can be seen by first noting that any maximally entangled state $|\phi_{\mathrm{me}}\rangle^{AC}$ can be written as $|\phi_{\mathrm{me}}\rangle^{AC} = U_A|\phi^+\rangle^{AC}$, where U_A is a unitary acting on the subsystem A. Then, to get Eq. (44) one uses the facts that U_A commutes with Λ^C and that any entanglement quantifier E is invariant under local unitaries [3].

A. Relation to entanglement of formation

In this section we will show that maximally entangled states are optimal for entanglement distribution for all noisy channels if the exchanged particle C is a qubit and the amount of entanglement is quantified via the entanglement of formation E_f. We then have the following theorem.

Theorem 5. For any mixed state ρ^{AC} with $d_A \geqslant d_C = 2$ and any channel Λ^C, the following inequality holds:

$$E_f(\Lambda^C[|\phi^+\rangle\langle\phi^+|^{AC}]) \geqslant E_f(\Lambda^C[\rho^{AC}]). \tag{45}$$

Proof. We first recall that the entanglement of formation is a convex function of the state. This implies that for any mixed state ρ^{AC} there exists a pure state $|\psi\rangle^{AC}$ which shows at least the same performance for entanglement distribution:

$$E_f(\Lambda^C[|\psi\rangle\langle\psi|^{AC}]) \geqslant E_f(\Lambda^C[\rho^{AC}]). \tag{46}$$

To complete the proof we will show that the maximally entangled state has the best performance among all pure states, i.e.,

$$E_f(\Lambda^C[|\phi^+\rangle\langle\phi^+|^{AC}]) \geqslant E_f(\Lambda^C[|\psi\rangle\langle\psi|^{AC}]) \tag{47}$$

for any pure state $|\psi\rangle^{AC}$ with $d_A \geqslant 2$, $d_C = 2$, and any single-qubit channel Λ^C. At this point, it is important to note that the state $|\psi\rangle^{AC}$ is effectively a two-qubit state, even if the dimension of the subsystem A is larger than 2. This follows from the Schmidt decomposition of $|\psi\rangle^{AC}$, which due to the fact that the subsystem C is two-dimensional, is of the form $|\psi\rangle^{AC} = \lambda_0\,|00\rangle + \lambda_1\,|11\rangle$. The state $\Lambda^C[|\psi\rangle\,\langle\psi|^{AC}]$ can thus be regarded as a mixed state of two qubits. With this in mind, we can now use the fact that for all two-qubit states the entanglement of formation admits a simple formula: $E_f = g(\mathcal{C})$, where g is a nondecreasing function and $\mathcal{C}$ is the concurrence [66]. The final ingredient of our proof is the factorization law for concurrence (see Eq. (5) in [67]). Adapted to our notation it reads

$$\mathcal{C}(\Lambda^C[|\psi\rangle\langle\psi|^{AC}]) = \mathcal{C}(\Lambda^C[|\phi^+\rangle\langle\phi^+|^{AC}])\cdot\mathcal{C}(|\psi\rangle\langle\psi|^{AC}). \tag{48}$$

Since the concurrence is never larger than 1, we arrive at the following inequality:

$$\mathcal{C}(\Lambda^C[|\phi^+\rangle\langle\phi^+|^{AC}]) \geqslant \mathcal{C}(\Lambda^C[|\psi\rangle\langle\psi|^{AC}]). \tag{49}$$

Note that this inequality also holds if the concurrence $\mathcal{C}$ is replaced by the entanglement of formation E_f, since the latter is a nondecreasing function of the concurrence. This observation completes the proof [68]. ■

It is worth mentioning that the above result can be generalized to a larger class of entanglement measures, namely, to all those measures which for two qubits can be written as a nondecreasing function of concurrence, that is,

$$E = g(\mathcal{C}). \tag{50}$$

This can be seen by exploiting the same argumentation as before. Apart from the entanglement of formation, examples of such measures are the geometric measure of entanglement [69,70], the Bures measure of entanglement [30,31], and the Groverian measure of entanglement [71,72]. For two qubits all those measures reduce to a nondecreasing function of concurrence (see Fig. 4 in Ref. [73]).

B. Relation to Pauli channel

We now show that for an important type of noise, the Pauli channel, the statement made in the previous section can be generalized to all entanglement measures. The action of the Pauli channel reads

$$\Lambda_{\mathrm{p}}^C[\rho^{AC}] = \sum_{i=0}^{3} p_i\sigma_i^C\rho^{AC}\sigma_i^C, \tag{51}$$

where the exchanged particle C is a qubit and σ_i are Pauli matrices with $\sigma_0 = \mathbb{1}$. We have the following:

Theorem 6. For any mixed state ρ^{AC} with $d_A \geqslant d_C = 2$ and any Pauli channel Λ_{p}^C the following inequality holds:

$$E\big(\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{AC}]\big) \geqslant E\big(\Lambda_{\mathrm{p}}^C[\rho^{AC}]\big) \tag{52}$$

for any entanglement measure E.

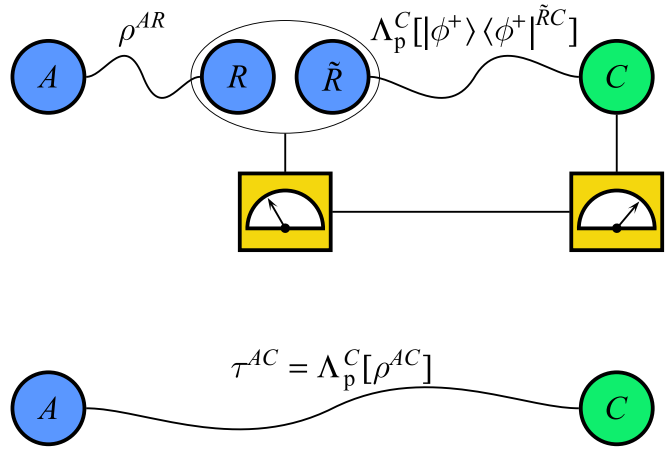

FIG. 3. (Color online) The state $\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\,\langle\phi^+|^{\tilde{R}C}]$ can be used to teleport the particle R of the state ρ^{AR} by performing a joint Bell measurement on R and $\tilde{R}$, and a conditional rotation on C (upper figure). This procedure leaves the subsystem AC in the final state $\tau^{AC} = \Lambda_{\mathrm{p}}^C[\rho^{AC}]$ (lower figure).

Proof. Let us start by introducing two additional particles R and $\tilde{R}$ with $d_R = d_{\tilde{R}} = 2$. We will now show that the state $\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}]$ can be used for teleportation in the following way: if two parties share the state $\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}]$ and apply the standard teleportation protocol [5] for teleporting the two-dimensional subsystem R of a total state ρ^{AR}, they will end up sharing the state $\Lambda_{\mathrm{p}}^C[\rho^{AC}]$. (See also [74] for similar considerations.) This can be seen explicitly by considering the essential steps of the standard teleportation protocol (see Fig. 3). In the first step, a joint Bell measurement is performed on the subsystems R and $\tilde{R}$. Depending on the outcome i of the measurement, the subsystem AC is found in one of the four states $\Lambda_{\mathrm{p}}^C[\sigma_i^C\rho^{AC}\sigma_i^C]$ with $0 \leqslant i \leqslant 3$. In the final step, a conditioned unitary rotation σ_i^C is applied on the subsystem C, leading to the final state

$$\tau^{AC} = \sigma_i^C\Lambda_{\mathrm{p}}^C\big[\sigma_i^C\rho^{AC}\sigma_i^C\big]\sigma_i^C. \tag{53}$$

At this point, it is crucial to note that the Pauli channel commutes with the Pauli matrices σ_i^C, i.e.,

$$\Lambda_{\mathrm{p}}^C\big[\sigma_i^C\rho^{AC}\sigma_i^C\big] = \sigma_i^C\Lambda_{\mathrm{p}}^C[\rho^{AC}]\sigma_i^C, \tag{54}$$

which can be seen by inspection using the anticommutation relation $\sigma_a\sigma_b = -\sigma_b\sigma_a$ for $1 \leqslant a,b \leqslant 3$. Using Eq. (54) we see that the final state τ^{AC} becomes independent from the outcome of the measurement i:

$$\tau^{AC} = \Lambda_{\mathrm{p}}^C[\rho^{AC}]. \tag{55}$$

Finally, note that all steps mentioned above can be performed by using local operations and classical communication (see Fig. 3). This implies that the final state τ^{AC} cannot have more entanglement than the state $\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}]$, regardless of the entanglement measure E used to quantify it. This completes the proof. ■

It should be stressed that the result presented in Theorem 6 can also be extended to the scenario in which the channel used for entanglement distribution is a tensor product of different single-qubit Pauli channels. As an example, consider a four-dimensional particle C consisting of two qubits C_1 and C_2. The channel $\Lambda_{\rm p}^C$ is now of the form $\Lambda_{\rm p}^C = \Lambda_{\rm p}^{C_1} \otimes \widetilde{\Lambda}_{\rm p}^{C_2}$, where $\Lambda_{\rm p}^{C_1}$ and $\widetilde{\Lambda}_{\rm p}^{C_2}$ are two (possibly different) Pauli channels. The action of this channel onto an arbitrary state $\rho^{AC} = \rho^{AC_1C_2}$ is given by

$$\Lambda_{\rm p}^C[\rho^{AC}] = \Lambda_{\rm p}^{C_1} \otimes \widetilde{\Lambda}_{\rm p}^{C_2}[\rho^{AC_1C_2}]. \tag{56}$$

Using similar lines of reasoning as in the proof of Theorem 6, we see that the best performance in this case is also achieved for the maximally entangled state, i.e., the inequality

$$E\big(\Lambda_{\rm p}^C[|\phi^+\rangle\langle\phi^+|^{AC}]\big) \geqslant E\big(\Lambda_{\rm p}^C[\rho^{AC}]\big) \tag{57}$$

holds for any state ρ^{AC} with $d_A \geqslant d_C = 4$ and the maximally entangled state $|\phi^+\rangle^{AC} = (1/2)\sum_{i=0}^{3}|ii\rangle^{AC}$. This statement is also true if the exchanged particle C consists of n qubits, and the channel $\Lambda_{\rm p}^C$ is a combination of n (possibly different) single-qubit Pauli channels. In this case, the best performance is achieved for the maximally entangled state (43) with $d_A \geqslant d_C = 2^n$.

Finally, we note that similar arguments can also be applied to a more general family of channels defined as follows:

$$\Lambda^C[\rho^{AC}] = \sum_i p_i U_i^C \rho^{AC} \big(U_i^C\big)^\dagger, \tag{58}$$

where the particles A and C can have arbitrary dimensions and U_i^C are unitary operators that act only on the particle C and have the following two properties:

(1) The unitaries $(U_i^C)^\dagger$ commute with the channel Λ^C, i.e.,

$$\Lambda^C\big[\big(U_i^C\big)^\dagger \rho^{AC} U_i^C\big] = \big(U_i^C\big)^\dagger \Lambda^C[\rho^{AC}] U_i^C. \tag{59}$$

(2) For the maximally entangled state $|\phi^+\rangle^{AC}$, the states

$$|\psi_i\rangle^{AC} = U_i^C\,|\phi^+\rangle^{AC} \tag{60}$$

form a complete orthonormal basis, i.e., $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ and $\sum_i |\psi_i\rangle\langle\psi_i| = \mathbb{1}$.

As we will show in the following, the maximally entangled state $|\phi^+\rangle^{AC}$ is optimal for entanglement distribution via a noisy channel given in Eq. (58). We will prove this statement by following the same reasoning as for Pauli channels (see also Fig. 3). In particular, we will show that the state $\Lambda^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}]$ can be used to teleport the particle R of dimension not larger than d_C, such that for any state ρ^{AR} the final state has the form $\tau^{AC} = \Lambda^C[\rho^{AC}]$. This can be proven by considering the state $\rho^{AR} \otimes \Lambda^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}]$, and applying a joint measurement on the particles R and $\tilde{R}$ in the basis $|\psi_i\rangle^{R\tilde{R}} = U_i^R\,|\phi^+\rangle^{R\tilde{R}}$. Conditioned on the measurement outcome i, the resulting postmeasurement state of the particles A and C is then given by $\Lambda^C[(U_i^C)^\dagger \rho^{AC} U_i^C]$. In the final step of the proof, we use Eq. (59) and apply conditional unitary rotations U_i^C, arriving at the desired final state $\tau^{AC} = \Lambda^C[\rho^{AC}]$. Using the same reasoning as for the Pauli channels, this proves the optimality of the maximally entangled state $|\phi^+\rangle^{AC}$ for the channels given in Eq. (58). Examples of such channels more general than the Pauli channels are the Weyl-covariant channels.

C. Relation to negativity and amplitude damping channel

All the results presented so far support the intuition that sending one-half of a maximally entangled state down a noisy quantum channel represents the optimal strategy if two parties wish to distribute entanglement between them. In particular, we have seen that this statement is true for all single-qubit channels if the figure of merit is the entanglement of formation, or any other entanglement measure which for two qubits reduces to a nondecreasing function of concurrence. Moreover, for Pauli channels we saw that this statement becomes completely general: in this case maximally entangled states are the optimal resource, regardless of the entanglement measure used.

Quite surprisingly, this intuition is generally not correct [75,76]. In particular, it was shown in Ref. [76] (see Sec. III therein) that maximally entangled states are not optimal for entanglement distribution if the amount of entanglement is quantified by the negativity [77,78], which is defined as $N(\rho^{AC}) = ||\rho^{T_A}||_1 - 1$, where T_A denotes the partial transposition over the system A and $||M||_1 = \mathrm{Tr}\sqrt{M^\dagger M}$ is the trace norm of M. In what follows we will recall this result, using, however, a slightly different entanglement monotone which is the logarithmic negativity given by

$$E_n(\rho^{AC}) = \log_2 ||\rho^{T_A}||_1 = \log_2[N(\rho^{AC}) + 1]. \tag{61}$$

We will also supplement the results of Ref. [76] by noting that for some quantum channels even arbitrarily little entangled states can outperform the maximally entangled state.

The effect of suboptimality of maximally entangled states was demonstrated for the single-qubit amplitude damping channel

$$\Lambda_{\rm ad}^C[\rho^{AC}] = K_1\rho^{AC}K_1^\dagger + K_2\rho^{AC}K_2^\dagger, \tag{62}$$

with Kraus operators $K_1 = |0\rangle\langle 0|^C + \sqrt{1-\gamma}\,|1\rangle\langle 1|^C$ and $K_2 = \sqrt{\gamma}\,|0\rangle\langle 1|^C$, and the damping parameter $0 \leqslant \gamma \leqslant 1$. If the initial state is chosen as

$$|\alpha\rangle^{AC} = \sqrt{1-\alpha}\,|00\rangle^{AC} + \sqrt{\alpha}\,|11\rangle^{AC} \tag{63}$$

with the real parameter $0 \leqslant \alpha \leqslant 1$, it is straightforward to verify that for $\tilde{\alpha} = (1-\gamma)/2$ the state $|\tilde{\alpha}\rangle^{AC}$ shows the same performance as the maximally entangled state, that is,

$$E_n\big(\Lambda_{\rm ad}^C[|\tilde{\alpha}\rangle\langle\tilde{\alpha}|^{AC}]\big) = E_n\big(\Lambda_{\rm ad}^C[|\phi^+\rangle\langle\phi^+|^{AC}]\big). \tag{64}$$

This is illustrated in Fig. 4, where the parameter space of α and γ is shown. The dashed line for $\alpha = \tilde{\alpha} = (1-\gamma)/2$ divides the parameter space into two parts. For $\alpha \leqslant (1-\gamma)/2$ (lower-left triangle in Fig. 4), the state $|\alpha\rangle^{AC}$ shows no advantage when compared to the maximally entangled state, i.e.,

$$E_n\big(\Lambda_{\rm ad}^C[|\alpha\rangle\langle\alpha|^{AC}]\big) \leqslant E_n\big(\Lambda_{\rm ad}^C[|\phi^+\rangle\langle\phi^+|^{AC}]\big). \tag{65}$$

However, for $(1-\gamma)/2 < \alpha < 1/2$ (upper right triangle in Fig. 4) the corresponding state $|\alpha\rangle^{AC}$ always outperforms

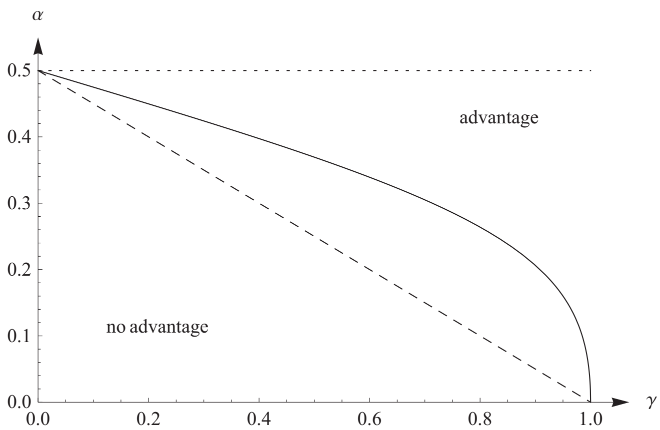

FIG. 4. The plot shows the relevant parameter regions of the damping parameter γ and the parameter α which enters the initial state $|\alpha\rangle^{AC}$ [see Eq. (63)]. Dashed line $\alpha = (1-\gamma)/2$ separates the parameter space in two parts. For $(1-\gamma)/2 < \alpha < 1/2$ all states $|\alpha\rangle^{AC}$ outperform the maximally entangled state. The solid line shows the value of $\alpha_{\max}$ which leads to the maximal logarithmic negativity for a given damping parameter γ. The dotted line shows $\alpha = 1/2$.

the maximally entangled state for the damping parameter $0 < \gamma < 1$:

$$E_n\left(\Lambda_{\mathrm{ad}}^C[|\alpha\rangle\langle\alpha|^{AC}]\right) > E_n\left(\Lambda_{\mathrm{ad}}^C[|\phi^+\rangle\langle\phi^+|^{AC}]\right). \tag{66}$$

For a given damping parameter γ we can further maximize the logarithmic negativity of the state $\Lambda_{\mathrm{ad}}^C[|\alpha\rangle\langle\alpha|^{AC}]$ with respect to the parameter α. Direct algebra shows that the maximum is achieved for

$$\alpha_{\max} = \frac{1}{\frac{\gamma}{\sqrt{1-\gamma}} + 2} \tag{67}$$

(see the solid line in Fig. 4). The corresponding quantity $E_n(\Lambda_{\mathrm{ad}}^C[|\alpha_{\max}\rangle\langle\alpha_{\max}|^{AC}])$ is shown in Fig. 5 as a function of the damping parameter γ (solid line). There we also show the logarithmic negativity $E_n(\Lambda_{\mathrm{ad}}^C[|\phi^+\rangle\langle\phi^+|^{AC}])$ for the maximally entangled state (dashed line).

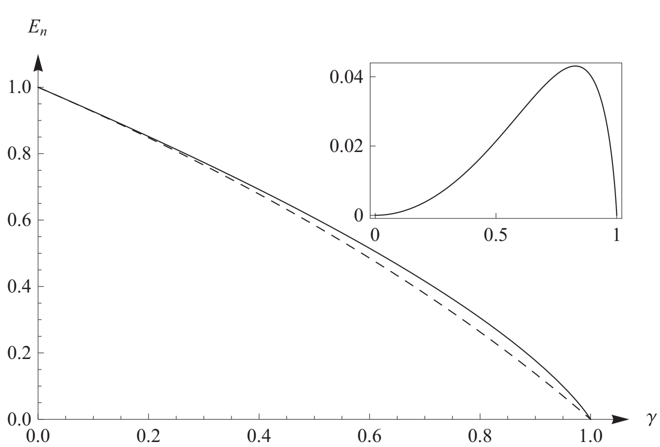

FIG. 5. The solid line shows the logarithmic negativity E_n of the state $\Lambda_{\mathrm{ad}}^C[|\alpha_{\max}\rangle\langle\alpha_{\max}|^{AC}]$ as a function of the damping parameter γ. Then the dashed line is the corresponding logarithmic negativity of $\Lambda_{\mathrm{ad}}^C[|\phi^+\rangle\langle\phi^+|^{AC}]$. The inset shows the difference $E_n(\Lambda_{\mathrm{ad}}^C[|\alpha_{\max}\rangle\langle\alpha_{\max}|^{AC}]) - E_n(\Lambda_{\mathrm{ad}}^C[|\phi^+\rangle\langle\phi^+|^{AC}])$. The states $|\alpha_{\max}\rangle^{AC}$ outperform the maximally entangled state $|\phi^+\rangle^{AC}$ in the whole region $0 < \gamma < 1$.

More interestingly, however, it turns out that for the logarithmic negativity, maximally entangled states can be outperformed even by states with arbitrarily little entanglement, which we prove in the following theorem.

Theorem 7. For any $\varepsilon > 0$ there exists a state ρ_ε^{AC} with logarithmic negativity at most ε and a channel Λ_ε^C such that

$$E_n\left(\Lambda_\varepsilon^C[\rho_\varepsilon^{AC}]\right) > E_n\left(\Lambda_\varepsilon^C[|\phi^+\rangle\langle\phi^+|^{AC}]\right). \tag{68}$$

Proof. We show this result for the amplitude damping channel Λ_{ad}^C given in Eq. (62) and the pure state $|\alpha\rangle^{AC}$ in Eq. (63). From the fact that the state $|\alpha\rangle^{AC}$ is separable for $\alpha = 0$ and maximally entangled for $\alpha = 1/2$, it follows that for any $\varepsilon > 0$ there exists $\alpha_\varepsilon \in (0,1/2)$ such that the logarithmic negativity of the state $|\alpha_\varepsilon\rangle^{AC}$ is nonzero and at most ε, i.e.,

$$0 < E_n(|\alpha_\varepsilon\rangle^{AC}) \leqslant \varepsilon. \tag{69}$$

To complete the proof it is enough to show that for any $\varepsilon > 0$ there exists an amplitude damping channel Λ_{ad}^C with the damping parameter γ_ε such that

$$E_n\left(\Lambda_{\mathrm{ad}}^C[|\alpha_\varepsilon\rangle\langle\alpha_\varepsilon|^{AC}]\right) > E_n\left(\Lambda_{\mathrm{ad}}^C[|\phi^+\rangle\langle\phi^+|^{AC}\right]. \tag{70}$$

The existence of such a channel follows directly from the arguments presented above. Precisely, by virtue of the inequality (66) we immediately see that Eq. (70) is true for any damping parameter γ_ε chosen such that $1 - 2\alpha_\varepsilon < \gamma_\varepsilon < 1$. ∎

We have then shown that, in some scenarios, states with very little entanglement are a better resource for noisy entanglement distribution when compared to maximally entangled states if the logarithmic negativity E_n is used to quantify entanglement. It is worth mentioning that this entanglement measure is closely related to the PPT entanglement cost, i.e., the entanglement cost under quantum operations preserving the positivity of the partial transpose. Precisely, E_n is always a lower bound on the PPT entanglement cost [79], while for all two-qubit states both quantities coincide [80]. For this reason the logarithmic negativity is equivalent to the PPT entanglement cost within the framework presented in this section, and all statements made for one quantity are also valid for the other.

Moreover, we point out that the result presented in Theorem 7 can also be extended to the multicopy scenario, where Alice and Bob have access to many copies of a quantum channel Λ^C. The aim of the process in this case is to distribute the maximal logarithmic negativity per copy of the channel. The aforementioned results together with additivity of the logarithmic negativity [78] imply that for amplitude damping noise maximally entangled states can be outperformed by states with arbitrary little entanglement also in this scenario.

Let us finally mention that in Ref. [76] the authors show that the maximally entangled states are optimal for entanglement distribution if the single-qubit channel used to transmit the particle is unital and negativity is used as the entanglement measure.

V. OPTIMAL ENTANGLEMENT DISTRIBUTION WITH PRESHARED CORRELATIONS

In the foregoing discussion on optimal entanglement distribution, we assumed that initially Alice and Bob do not share any correlations, i.e., the initial state is fully product, see Eq. (41). Here, we will relax this assumption and allow for more general initial quantum states

$$\rho_i = \rho^{ABC}. \tag{71}$$

The main question we want answer in this section can be formulated as follows: *Are preshared correlations useful for entanglement distribution?*

The answer to this question is negative for any convex entanglement measure E if Alice and Bob initially share a separable state:

$$\rho_i = \sum_k p_k \rho_k^{AC} \otimes \rho_k^B. \tag{72}$$

In this case the initial entanglement is zero, and the amount of distributed entanglement is thus given by $E^{A|BC}(\rho_f) = E^{A|BC}(\sum_k p_k \Lambda^C[\rho_k^{AC}] \otimes \rho_k^B)$. For any convex entanglement quantifier E these arguments imply that any separable initial state ρ_i given in Eq. (72) can be outperformed by some pure state $|\psi\rangle^{AC}$:

$$E^{A|C}(\Lambda^C[|\psi\rangle\langle\psi|^{AC}]) \geqslant E^{A|BC}(\Lambda^C[\rho_i]). \tag{73}$$

This proves that preshared correlations are not useful for entanglement distribution for any convex entanglement measure if the preshared state is separable.

A. Subadditive entanglement measures

We will now consider subadditive entanglement measures, which are the ones that satisfy the following inequality:

$$E^{A_1A_2|B_1B_2}(\rho^{A_1B_1} \otimes \sigma^{A_2B_2}) \leqslant E^{A_1|B_1}(\rho^{A_1B_1}) + E^{A_2|B_2}(\sigma^{A_2B_2}) \tag{74}$$

for any two states $\rho^{A_1B_1}$ and $\sigma^{A_2B_2}$. Well-known examples of such measures are the entanglement of formation, the relative entropy of entanglement, and the logarithmic negativity, which in fact is additive, i.e., it satisfies Eq. (74) with equality. Moreover, we will also consider single-qubit Pauli channels Λ_{p}^C which were already introduced in Sec. IV B. As it is proven in the following theorem, for this type of noise preshared correlations are not useful if the corresponding entanglement quantifier is subadditive.

Theorem 8. Given a single-qubit Pauli channel Λ_{p}^C and two states $\rho_i = \rho^{ABC}$ and $\rho_f = \Lambda_{\mathrm{p}}^C[\rho_i]$, the following inequality holds for any subadditive entanglement measure E:

$$E^{A|C}\left(\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{AC}]\right) \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i). \tag{75}$$

Proof. The proof goes along the same lines as that of Theorem 6. We denote the initial state by $\rho_i = \rho^{ABR}$, where Alice is in possession of the particle A and a qubit R, and Bob is in possession of the remaining particle B. Additionally, Alice and Bob have access to the qubits $\tilde{R}$ and C of the state $\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}]$. By applying the standard teleportation protocol [5] to teleport the qubit R from Alice to Bob and using the state $\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}]$ as the resource, we see that Alice and Bob end up in the final state $\rho_f = \Lambda_{\mathrm{p}}^C[\rho^{ABC}]$. Using the fact that all steps in the standard teleportation protocol can be performed by LOCC and that the amount of entanglement cannot increase in this process, it follows that the final entanglement $E^{A|BC}(\rho_f)$ is bounded from above by the amount of entanglement in the total initial state: $E^{A|BC}(\rho_f) \leqslant E^{AR\tilde{R}|BC}(\rho^{ABR} \otimes \Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}])$. Finally, for a subadditive entanglement quantifier we can apply Eq. (74), which gives us $E^{A|BC}(\rho_f) \leqslant E^{AR|B}(\rho^{ABR}) + E^{\tilde{R}|C}(\Lambda_{\mathrm{p}}^C[|\phi^+\rangle\langle\phi^+|^{\tilde{R}C}])$. To complete the proof it is enough to notice that the state ρ^{ABR} is equivalent to the initial state ρ_i. ■

From the theorem it also follows that sending one-half of a maximally entangled state down the noisy Pauli channel is the optimal strategy. Let us notice that similarly as in Sec. IV B, the above result can further be generalized to the scenario in which the exchanged particle C consists of n qubits, and the channel Λ_{p}^C is a tensor product of n (possibly different) single-qubit Pauli channels. For any subadditive entanglement quantifier, preshared correlations do not provide any advantage also in this scenario, and the best performance is achieved for the maximally entangled state.

Note that these arguments also cover the situation where the channel used for entanglement distribution is noiseless. On the other hand, if the measure of entanglement is *not* subadditive, preshared correlations can indeed be helpful even in the noiseless scenario. This can be seen by considering the second power of the entanglement of formation: $E = E_f^2$. Note that E is a proper entanglement quantifier, i.e., it is nonincreasing under LOCC and zero only on separable states. If Alice and Bob have access to a noiseless single-qubit channel and do not share any initial correlations, the optimal strategy for Alice is to prepare locally two qubits A and C in the maximally entangled state $|\phi^+\rangle^{AC}$, and to send the qubit C to Bob. The amount of entanglement distributed in this way is given by $E^{A|C}(|\phi^+\rangle^{AC}) = 1$. However, Alice and Bob can achieve a better performance if they initially share the state $|\psi\rangle = |\psi\rangle^{ABC} = (|000\rangle + |101\rangle + |210\rangle + |311\rangle)/2$. In this case the amount of distributed entanglement is given by $E^{A|BC}(|\psi\rangle) - E^{AC|B}(|\psi\rangle) = 3$.

B. Distillable entanglement

The results presented so far can also be extended to the distillable entanglement E_d, which was conjectured to be superadditive in [27], i.e., it violates the inequality (74) for some states. Based on this conjecture, we will now show that preshared correlations can be useful for the distribution of distillable entanglement. In particular, we will consider *entanglement binding channels* Λ_{eb}^C, i.e., channels that destroy the distillable entanglement in any initial state ρ^{AC}: $E_d^{A|C}(\Lambda_{\mathrm{eb}}^C[\rho^{AC}]) = 0$. This implies that this type of channel cannot be used for the distribution of distillable entanglement if Alice and Bob do not share any correlations initially. However, provided the superadditivity conjecture is true, one can show that entanglement binding channels can still be used for entanglement distribution if preshared correlations are available.

Conjecture 9. There exists a state $\rho_i = \rho^{ABC}$ and an entanglement binding channel $\Lambda_{\rm eb}^C$ for which the following inequality holds:

$$E_d^{A|BC}\big(\Lambda_{\rm eb}^C[\rho_i]\big) > E_d^{AC|B}(\rho_i). \tag{76}$$

In the following we will prove this conjecture, assuming the validity of the superadditivity conjecture for distillable entanglement. To this end, we consider two bound entangled states $\rho_{\rm be}^{X_1Y_1}$ and $\sigma_{\rm be}^{X_2Y_2}$ for which

$$E_d^{X_1X_2|Y_1Y_2}\big(\rho_{\rm be}^{X_1Y_1} \otimes \sigma_{\rm be}^{X_2Y_2}\big) > 0, \tag{77}$$

and assume that initially Alice and Bob share one of them, say $\rho_{\rm be}$. In the next step, Alice and Bob use an entanglement binding channel to establish the additional state $\sigma_{\rm be}$ between them. The existence of such an entanglement binding channel is assured by results provided in [81]. After this procedure Alice and Bob share the conjectured distillable state $\rho_{\rm be} \otimes \sigma_{\rm be}$. As a consequence, entanglement binding channels can be used for entanglement distribution in the presence of preshared correlations under the assumption that the distillable entanglement is superadditive.

C. Distance-based entanglement measures

In the last part of this section we consider distance-based entanglement quantifiers E, as defined in Eq. (2). We have the following (without loss of generality we assume that $d_A \geqslant d_C$):

Theorem 10. For any noisy channel Λ^C there exists a pure state $|\psi\rangle = |\psi\rangle^{AC}$ such that the following inequality holds for any two states $\rho_i = \rho^{ABC}$ and $\rho_f = \Lambda^C[\rho_i]$:

$$\Delta^{C|A}(\Lambda^C[|\psi\rangle\langle\psi|]) \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i). \tag{78}$$

Proof. From Theorem 4 it follows that the amount of distributed entanglement is bounded above by the amount of discord between the exchanged particle C and the remaining system AB in the final state $\rho_f = \Lambda^C[\rho_i]$:

$$\Delta^{C|AB}(\Lambda^C[\rho_i]) \geqslant E^{A|BC}(\rho_f) - E^{AC|B}(\rho_i). \tag{79}$$

Then, to obtain Eq. (78) from Eq. (79), it suffices to show that for any channel Λ^C there exists a pure state $|\psi\rangle = |\psi\rangle^{AC}$ such that the following inequality,

$$\Delta^{C|A}(\Lambda^C[|\psi\rangle\langle\psi|]) \geqslant \Delta^{C|AB}(\Lambda^C[\rho_i]), \tag{80}$$

holds for any initial state $\rho_i = \rho^{ABC}$. For this purpose, let us first denote by $|\phi\rangle = |\phi\rangle^{ABCR}$ the purification of ρ_i, i.e., $\rho_i = \mathrm{Tr}_R[|\phi\rangle\langle\phi|]$. Then, we recall that all distance-based quantifiers of discord $\Delta^{X|Y}$ do not increase under quantum operations on the subsystem Y if the corresponding distance does not increase under quantum operations [82]. This implies that the inequality $\Delta^{C|ABR}(\Lambda^C[|\phi\rangle\langle\phi|]) \geqslant \Delta^{C|AB}(\Lambda^C[\rho_i])$ is satisfied. The proof of Eq. (80) is complete by recalling that $d_A \geqslant d_C$, and thus there must exist a pure state $|\psi\rangle = |\psi\rangle^{AC}$ such that $\Delta^{C|A}(\Lambda^C[|\psi\rangle\langle\psi|]) \geqslant \Delta^{C|ABR}(\Lambda^C[|\phi\rangle\langle\phi|])$ is true for any state $|\phi\rangle = |\phi\rangle^{ABCR}$. ■

Let us now apply the above result to the single-qubit phase damping channel $\Lambda_{\rm pd}^C$, which is a special case of a Pauli channel and is defined as follows:

$$\Lambda_{\rm pd}^C[\rho_i] = (1-p)\cdot\rho_i + p\cdot\sigma_z^C\rho_i\sigma_z^C, \tag{81}$$

with an initial state $\rho_i = \rho^{ABC}$ and the damping parameter p ranging from 0 to 1/2. While $p = 0$ corresponds to the noiseless scenario, full phase damping is achieved for $p = 1/2$. Using Theorem 8, it follows that for this type of noise maximally entangled states are optimal for entanglement distribution if the quantifier of entanglement is subadditive. In particular, this is true for the relative entropy of entanglement E_R defined in Eq. (6). As we will see in the following, for this entanglement measure the bound provided in Theorem 10 turns out to be tight for any single-qubit phase damping channel:

$$\begin{aligned}\Delta_R^{C|A}\big(\Lambda_{\rm pd}^C[|\phi^+\rangle\langle\phi^+|^{AC}]\big) &= E_R^{A|C}\big(\Lambda_{\rm pd}^C[|\phi^+\rangle\langle\phi^+|^{AC}]\big)\\ &\geqslant E_R^{A|BC}(\rho_f) - E_R^{AC|B}(\rho_i).\end{aligned} \tag{82}$$

Here, Δ_R is the relative entropy of discord defined in Eq. (7), $\rho_i = \rho^{ABC}$ is an arbitrary initial state with $d_A \geqslant d_C = 2$, and $\rho_f = \Lambda_{\rm pd}^C[\rho_i]$ is the final state after the application of the noisy channel.

To prove Eq. (82), let us notice that the following chain of inequalities holds:

$$\begin{aligned}S\left(\rho^{XY}\middle\|\sum_i \Pi_i^X\rho^{XY}\Pi_i^X\right) &\geqslant \Delta_R^{X|Y}(\rho^{XY}) \geqslant E_R^{X|Y}(\rho^{XY})\\ &\geqslant S(\rho^X) - S(\rho^{XY}),\end{aligned} \tag{83}$$

where $\{\Pi_i^X\}$ is a local von Neumann measurement on the particle X, and the last inequality was proven in [83]. If we now choose the projectors $\Pi_i^C = |i\rangle\langle i|^C$, it can be verified that for the state $\sigma^{AC} = \Lambda_{\rm pd}^C[|\phi^+\rangle\langle\phi^+|^{AC}]$ the upper and the lower bounds in Eq. (83) coincide: $S(\sigma^{AC}\|\sum_i\Pi_i^C\sigma^{AC}\Pi_i^C) = S(\sigma^C) - S(\sigma^{AC})$. Together with Theorem 8, this completes the proof of Eq. (82). In particular, this also shows that the bound provided in Theorem 10 is tight for single-qubit phase damping channels, since for this type of noise the amount of distributed entanglement is bounded above by $\Delta_R^{C|A}(\Lambda_{\rm pd}^C[|\phi^+\rangle\langle\phi^+|^{AC}])$, and this bound is also reachable according to Eq. (82).

VI. CONCLUSIONS AND OUTLOOK

A concise summary of our results is presented in Table I, where we list several entanglement quantifiers and types of noisy channels considered in this work, show the corresponding optimal state, and discuss the advantage of preshared correlations. In two of the cases it remains unclear if preshared correlations provide an advantage for entanglement distribution. We leave this question open for future research.

The results presented in this work can be regarded as a major step towards a unified approach to entanglement distribution. In particular, it can be seen from the first row in Table I that preshared correlations do not provide an advantage for any subadditive entanglement quantifier if entanglement is distributed via a combination of single-qubit Pauli channels. In this context, it is tempting to assume that these results extend to arbitrary noisy channels, and thus preshared correlations do not provide an advantage for any subadditive entanglement measure and any type of noise. Sending one-half of a pure entangled state down a noisy channel would then be the optimal strategy for any subadditive entanglement measure.

TABLE I. Overview of the entanglement quantifiers and types of noisy channels considered in this paper. The table shows also the optimal state for entanglement distribution without preshared correlations for the corresponding entanglement measure and quantum channel. As can be seen from the fourth column, in some situations preshared correlations show an advantage for entanglement distribution. The last column shows the section in this article where the corresponding result has been obtained.

Entanglement measure	Type of noisy channel	Optimal states (without preshared correlations)	Advantage of preshared correlations	Section
Subadditive entanglement measures	Single-qubit Pauli channel or any combination thereof	Maximally entangled states	No advantage	V A
Entanglement measures which are not subadditive	Noiseless channel	Maximally entangled states	Some of these measures show advantage even in the noiseless scenario	V A
Distillable entanglement	Entanglement binding channels	Without preshared correlations, no entanglement distribution possible	Conjectured advantage, based on the superadditivity conjecture of distillable entanglement	V B
Measures which for two qubits reduce to a nondecreasing function of concurrence	Single-qubit noise	Maximally entangled states	Unknown	IV A
Logarithmic negativity	Single-qubit amplitude damping channel	Maximally entangled states are not always optimal	Unknown	IV C

While we cannot prove this conjecture in full generality at this point, our results strongly support this statement. In particular, the advantage of preshared correlations was only found for measures which are not subadditive and for distillable entanglement, which is conjectured to be superadditive.

Regarding entanglement distribution with separable states, our results show that this strategy is not reasonable for any subadditive entanglement measure if a combination of single-qubit Pauli channels is used for the process. On the other hand, this result does not rule out the superiority of separable states for other types of noise. In this direction we have found, supplementing the results of Ref. [76], that states with arbitrarily little entanglement can outperform maximally entangled states for amplitude damping noise, if entanglement is quantified via the logarithmic negativity. These counterintuitive results also imply that a closer investigation of entanglement distribution with separable states is necessary, since, contrary to recent claims made, e.g., in [17,24], maximally entangled states are not necessarily the best resource to benchmark this process.

The results of this paper can also be seen as the first step toward similar considerations in quantum many-body systems. Note that over the last decade entanglement has proven to be extremely useful to characterize properties of many-body systems and the nature of quantum phase transitions [84]. For instance, in the ground states and low-energy states of quantum spin models the following properties hold (see [85,86] for a review):

(1) The two-body reduced density matrix typically exhibits entanglement for short separations of the spins only, even at criticality; still, entanglement measures show signatures of quantum phase transitions [87,88].

(2) One can concentrate entanglement between the chosen two spins by optimized measurements on the rest of the system, obtaining in this way the so-called *localizable entanglement* [89]; the corresponding entanglement length diverges when the standard correlation length diverges, i.e., at standard quantum phase transitions.

(3) For noncritical systems, ground states and low-energy states exhibit area laws: the von Neumann or Rényi entropy of the reduced density matrix of a block of size I scales as the size of the boundary of the block, ∂I; at criticality logarithmic divergence occurs frequently [90] (see also [91,92] for a review).

(4) Ground states and low-energy states can be efficiently described by matrix product states, or more generally tensor network states (cf. [93]).

(5) Topological order for gapped systems in one and two dimensions exhibits itself frequently in the properties of the *entanglement spectrum*, i.e., the spectrum of the logarithm of the reduced density matrix of a block I [94], and in the appearance of the *topological entropy*, i.e., negative constant correction to the area laws in 2D [95,96].

All the above results indicate the importance of few-body entanglement in the low-energy physics of many-body systems (cf. [97–99]). It is to be expected that few-body entanglement will also play a role in characterizing the out-of-equilibrium dynamics of quantum many-body systems [100].

Note that the scheme of entanglement distribution discussed in this paper, at least in the noiseless case, can be considered in the context of *the real transfer of particle* C to Bob, or as the *change of partition from* $AC|B$ *to* $A|BC$. In this sense, one can view our results as a characterization of entanglement in three-body reduced density matrix in a many-body system. Generalizations including noisy transfer are possible, for instance, by coupling C locally to a reservoir or an ancillary particle. It would also be interesting to consider the entanglement distribution scheme with many recipients (Bobs). Finally, asking analogous questions for Bell nonlocality or steering seems to be a fascinating perspective that would also lead to a better understanding of these phenomena.

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