

# Distance Bound Smoothing under Orientation Constraints

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**Abstract**—Distance Bound Smoothing (DBS) is a basic operation originally developed in Computational Chemistry to determine point configurations that are within certain pairwise ranges of distances. This operation consists in the iterative application of filtering processes that reduce the given ranges using triangular and tetragonal inequalities.

Standard DBS has a limited range of applications because it does not take into account constraints on the orientations of simplices (triangles or tetrahedra, depending on the dimension of the problem). This paper discusses an extension of DBS that permits incorporating these constraints. This paves the way for the application of DBS techniques to a broad range of problems in Robotics.

## I. INTRODUCTION

Let us consider, for instance, the regional part of the wrist-partitioned 6R robot shown in Fig. 1(top), that is, the first three links and joints that permit locating the wrist center anywhere in the robot's workspace. Since the location of each of the three joint axes is determined by two points, the configuration of this regional part can be determined by seven points in  $\mathbb{R}^3$  [1]. Some of the distances between these points are constant regardless of the location of the wrist center. If, in addition, we fix the location of the wrist center with respect to the robot's base, the graph of distance constraints shown in Fig. 1(bottom) is derived. In this case, the graph is rigid in general. In other words, the unknown distances between the seven points can only have a discrete set of values compatible with the known distances. To obtain these values, a Distance Bound Smoothing (DBS) technique can be used [2]. In this example, such a technique would obtain, in general, eight different sets of compatible distances. Nevertheless, it is well-known that the inverse kinematics of a 3R robot can only have up to four solutions [3]. This apparent contradiction has a simple explanation: A standard DBS technique would not take into account the relative orientations of the tetrahedra defined by points  $\{P_1, P_2, P_3, P_4\}$  and  $\{P_3, P_4, P_5, P_6\}$ .

This paper presents a generalization of the standard DBS technique that permits incorporating orientation constraints thus making it applicable to solve problems such as inverse/direct kinematics of serial/parallel robots, mutual localization of robots in teams, sensor data fusion, and constraint-based robot programming, to name just a few.

We will focus our efforts on the 2D case where the problem can be formally stated as follows: Given lower and

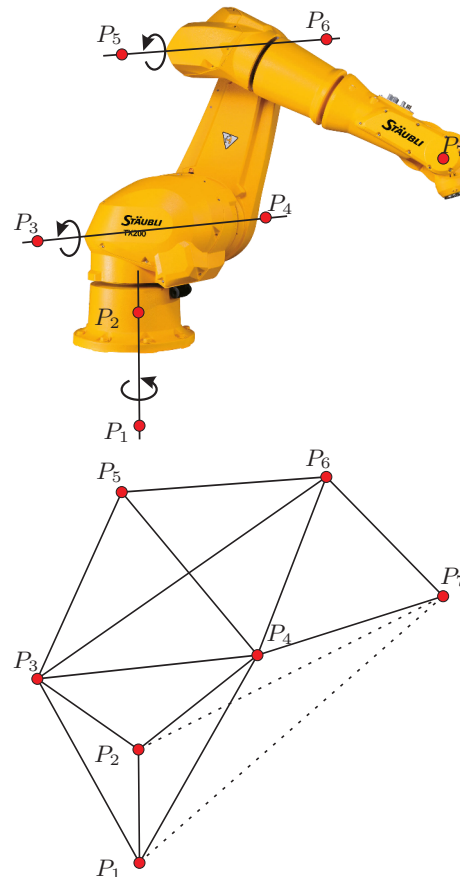


Fig. 1. Top: The configuration of the regional part of a wrist-partitioned 6R robot is determined by the location of seven points. Bottom: Associated distance graph where nodes stand for points and edges for known distances between the corresponding points. Solid lines represent constant distances, regardless of the location of the end-effector.

upper bounds on the pairwise distances between  $n$  points in  $\mathbb{R}^2$ , and a set of orientation constraints for subsets of three points, tighten the bounds ensuring that all distance ranges are mutually compatible and consistent with  $n$  points embedded on a plane and satisfying the orientation constraints. In Robotics, distances may be assumed to be exactly known (in which case the initial lower and upper bounds coincide), unknown (in which case the initial range goes from zero to infinity), or estimated using sensor information (in which case an initial finite range is known).

In DBS, triangular and tetragonal inequalities are used to tighten the existing distance bounds [4, pp. 221-285]. Since an efficient algorithm exists for tightening bounds using triangular constraints, we assume that they are already

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satisfied by the initial ranges. Moreover, since tetragonal inequalities become equalities in planar problems, we focus our analysis on the application of the tetragonal equality under orientation constraints.

A planar set of four points defines six pairwise distances. In this paper we show that the derivative of one of this distances with respect to any other is monotone, provided that no subset of three points changes its orientation. Then, since the bounding of monotone functions is straightforward, DBS using tetragonal and orientation constraints can be reduced to the analysis of 14 patterns. Now, changes in orientations correspond alignments, which can also be analyzed characterizing their different monotone sections. The overall approach is simpler than the standard presentation of the method despite we are also including orientation constraints, which have only been considered in [5] for points on a line.

This paper is organized as follows. First of all, to make it as much self-contained as possible, Section II gives the needed mathematical background. Then, Section III discusses how to obtain tight bounds for the evaluation of functions analyzing their monotonicity and Section IV particularizes this analysis to the tetragonal equality. Section V applies the proposed method to the mutual localization of mobile robot teams. Finally, Section VI summarizes the presented work and points to directions deserving further attention.

## II. MATHEMATICAL BACKGROUND

### A. Cayley-Menger bideterminants

The Cayley-Menger bideterminant of two sequences of  $n$  points,  $(P_{i_1}, \dots, P_{i_n})$  and  $(P_{j_1}, \dots, P_{j_n})$  is defined as:

$$D(i_1, \dots, i_n; j_1, \dots, j_n) = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & s_{i_1, j_1} & \dots & s_{i_1, j_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & s_{i_n, j_1} & \dots & s_{i_n, j_n} \end{vmatrix}, \quad (1)$$

where  $s_{i,j}$  denotes the squared distance between  $P_i$  and  $P_j$ . This determinant plays a fundamental role in Distance Geometry, a term coined by Blumenthal in [6] which refers to the analytical study of Euclidean geometry in terms of invariants, without resorting to artificial coordinate systems.

Since in many cases of interest the two sequences of points are the same, it will be convenient to abbreviate  $D(i_1, \dots, i_n; i_1, \dots, i_n)$  by  $D(i_1, \dots, i_n)$ , which is simply called a Cayley-Menger determinant.

It can be shown that  $D(i_1, \dots, i_n; j_1, \dots, j_n)$  is  $-1^n 2^{n-1} ((n-1)!)^2$  times the product of the oriented hypervolumes of the simplices spanned by  $P_{i_1}, \dots, P_{i_n}$  and  $P_{j_1}, \dots, P_{j_n}$  in  $\mathbb{E}^{n-1}$  [7, pp. 12-129],[8]. Since we are going to constraint our analysis to  $\mathbb{E}^2$ , simplices will be triangles and  $-\frac{1}{16}D(i_1, i_2, i_3)$  will give the squared area of the triangle defined by  $P_{i_1}, P_{i_2}$ , and  $P_{i_3}$ .

### B. Conditions of embedability in $\mathbb{E}^2$

Let us assume that we have  $n > 3$  points in  $\mathbb{E}^2$  whose pairwise distances are constrained to lie within certain ranges

of validity. These ranges are mutually constrained and must clearly satisfy the following sets of algebraic conditions:

- For all sets of three points:

$$D(i_1, i_2, i_3) \leq 0. \quad (2)$$

- For all sets of four points:

$$D(i_1, i_2, i_3, i_4) = 0, \quad (3)$$

and

$$D(i_1, i_2, i_3; i_1, i_2, i_4) = \begin{cases} < 0 & \text{if } \sigma_{i_1, i_2, i_3} \sigma_{i_1, i_2, i_4} > 0 \\ \geq 0, & \text{otherwise} \end{cases} \quad (4)$$

where  $\sigma_{i,j,k}$  is defined as negative if points  $P_i, P_j$ , and  $P_k$  must be arranged clockwise and positive otherwise.

The expansion of (2) leads to the triangular inequality involving the distances between  $P_{i_1}, P_{i_2}$ , and  $P_{i_3}$ . Moreover, the equality in (3) is nothing else than the tetragonal equality involving the six pairwise distances between  $P_{i_1}, P_{i_2}, P_{i_3}$ , and  $P_{i_4}$ . Finally, in (4) we assume that the relative orientations of all pairs of triangles sharing an edge are fixed, but the case where some orientations are undefined can be easily encompassed if necessary. Note, however, that the whole set of orientation constraints can not be fixed arbitrarily. Actually, it is possible to define a basis that determines all other orientations [9].

The above set of algebraic conditions involving all possible subsets of three and four points is highly redundant. Sets of necessary and sufficient conditions can be deduced from them [6]. Nevertheless, keeping a highly redundant set of conditions reduces the clustering effect in branch-and-prune schemes where only one necessary condition is applied at a time [10].

### C. The tetragonal equality

Let us consider a set of four points, say  $P_1, \dots, P_4$ . Then, the expansion of the tetragonal equality

$$D(1, 2, 3, 4) = 0, \quad (5)$$

yields a quadratic expression in any of the involved squared distances. For example, we can obtain a quadratic expression in terms of  $s_{3,4}$  whose root, after some algebraic manipulations, can be expressed as:

$$s_{3,4} = - \frac{D(1, 2, 4; 1, 2, 3)|_{s_{3,4}=0}}{D(1, 2)} + \frac{\sigma_{1,2,4} \sigma_{1,2,3} \sqrt{D(1, 2, 4) D(1, 2, 3)}}{D(1, 2)}. \quad (6)$$

A similar expression can be obtained for other squared distances by a simple permutation of indices. Alternatively, (6) can be written as

$$s_{3,4} = \frac{-D(1, 2, 4; 1, 2, 3)|_{s_{3,4}=0} + 16 A_3 A_4}{D(1, 2)}, \quad (7)$$

where  $A_i$  denotes the oriented area of the triangle defined by the ordered set  $\{P_1, P_2, P_3, P_4\} \setminus P_i$ , since  $4A_3 = \sigma_{1,2,4} \sqrt{-D(1, 2, 4)}$  and  $4A_4 = \sigma_{1,2,3} \sqrt{-D(1, 2, 3)}$ .

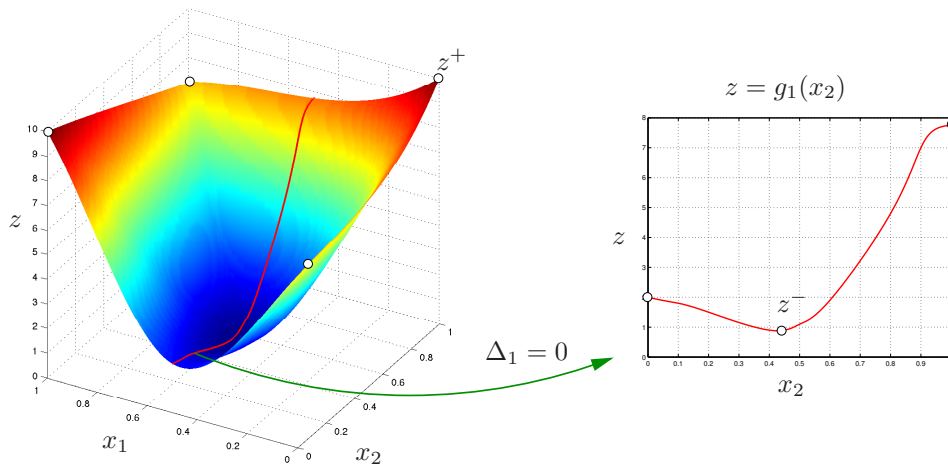


Fig. 2. Schematic illustration of how to determine the exact bounds of  $z = g(x_1, x_2)$  analyzing the monotonicity of  $g$ . In the figure, this function is monotone with respect to  $x_1$  and  $x_2$ , except when  $\Delta_1 = 0$  and when  $\Delta_2 = 0$ . These cases have to be recursively analyzed. For instance, for the first case,  $x_1$  is eliminated defining a new function  $z = g_1(x_2)$ , whose bounds can be readily determined.

A possibility to tighten the range for  $s_{3,4}$  is to use interval arithmetics [11] to evaluate the right-hand of (7) and intersect the result with its current range. However, axiomatic interval arithmetic assumes that all variables are independent, including duplicate copies of the same variable. This typically leads to overestimations of the bounds. Simplifying mathematical expressions to reduce multiple copies of a variable can eliminate the overestimation, but is not always possible. An alternative for this is presented in next section.

### III. COMPUTING EXACT BOUNDS USING MONOTONICITY

If a function is monotone in a given domain, it is clear that its extrema are in the boundary of the domain. Moreover, if the domain is an axis-aligned box, as it is our case, we can appeal the following well-known proposition to find exact bounds without overestimation:

**Proposition 1** [12]. *Let  $x = (x_1, \dots, x_n)$  be a tuple of  $n$  real interval-valued variables defining an axis-aligned box such that  $x_i \in [x_i^-, x_i^+]$ , and let  $z = g(x) \in [z^-, z^+]$ . If  $g$  is continuous and locally monotonic with respect to each argument, then*

$$z^- = \min_{x \in \mathcal{H}} g(x),$$

and

$$z^+ = \max_{x \in \mathcal{H}} g(x),$$

where  $\mathcal{H}$  is the set of  $2^n$  vertices of the box defined by  $x$ .

If the derivatives of  $g$  with respect to  $x_1, \dots, x_n$  are known, then the vertices defining the extrema can be identified without the need of evaluating  $g$  in all vertices. Clearly, the maximum would correspond to the vertex where, for  $i = 1 \dots n$ ,  $x_i$  is set to  $x_i^-$  if  $\partial z / \partial x_i < 0$  and to  $x_i^+$  otherwise. The minimum would be in a vertex defined with the opposite criterion. Exploiting these properties, we can propose a general method to generate tight bounds for a given interval function by studying its monotonicity.

The analysis of the sign of  $\Delta_i = \partial z / \partial x_i$  reveals the monotonic areas for  $z$  as a function of  $x_i$ . Moreover, the

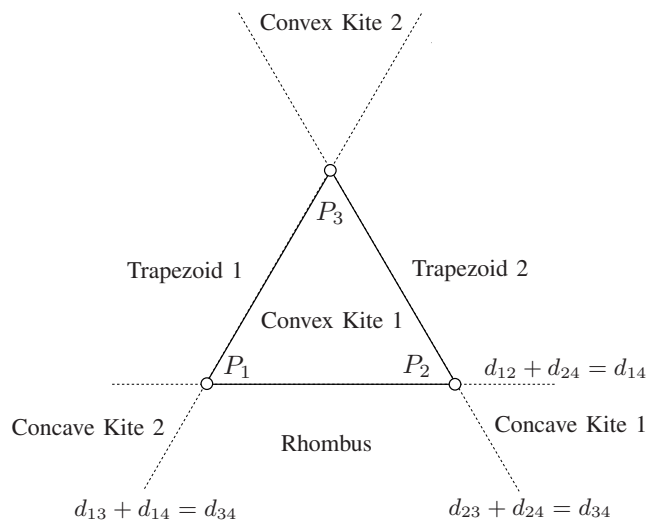


Fig. 3. Taking  $\triangle P_1 P_2 P_3$  as a reference, the plane is divided in seven regions. If  $P_4$  is bound to be in one of these regions,  $s_{3,4}$  is monotone with respect to the rest of squared distances. The boundaries separating the monotonic regions correspond to configurations where there is an alignment of three points.

boundaries between monotonic areas, i.e., the points where  $\Delta_i$  is 0, have to be analyzed separately since they may include the extrema of  $z$ . This can be done recursively, using the equation  $\Delta_i = 0$  to eliminate one variable from  $z = g(x)$  and define a new function  $z = g_1(\cdot)$ , whose monotonicity can be analyzed in the same way as that of  $g$ . One of these recursions is illustrated in Fig. 2.

### IV. EXACT BOUNDS FOR THE TETRANGULAR EQUALITY

To apply the method in the previous section to the function in (7), we need to compute the derivatives of  $s_{3,4}$  with respect to  $s_{i,j}$ . Instead of computing these derivatives from (7) it is more convenient to obtain them from the linearization of (5) that can be shown to be

$$+A_1 A_2 \delta s_{1,2} - A_1 A_3 \delta s_{1,3} + A_1 A_4 \delta s_{1,4} + A_2 A_3 \delta s_{2,3} - A_2 A_4 \delta s_{2,4} + A_3 A_4 \delta s_{3,4} = 0. \quad (8)$$

TABLE I

THE PATTERNS GIVING THE VERTICES WHERE TO OBTAIN THE UPPER AND LOWER BOUNDS FOR  $d_{3,4}$  FOR THE DIFFERENT CHIROTOPE, I.E., ORIENTATIONS OF THE TRIANGLES  $A_1, A_2, A_3,$  AND  $A_4$ . SOLID AND DASHED LINES INDICATE DISTANCES AT THEIR LOWER AND UPPER LIMITS, RESPECTIVELY.

Chirotope	Lower Bound	Upper Bound
Rhombus (+, -, -, +) (-, +, +, -)		
Concave Kite 1 (-, -, -, +) (+, +, +, -)		
Concave Kite 2 (+, +, -, +) (-, -, +, -)		
Trapezoid 1 (+, +, +, +) (-, -, -, -)		
Trapezoid 2 (-, -, +, +) (+, +, -, -)		
Convex Kite 1 (+, -, +, +) (-, +, -, -)		
Convex Kite 2 (-, +, +, +) (+, -, -, -)		

Then, we have that

$$\frac{\partial s_{3,4}}{\partial s_{i,j}} = -1^{i+j} \frac{A_i A_j}{A_3 A_4}. \quad (9)$$

As long as the sign of the oriented areas of the triangles defined by  $P_1, \dots, P_4$  do not change,  $s_{3,4}$  is monotonic and, thus, the conditions of Proposition 1 holds. Therefore, in this case, we can readily identify the vertices providing tight bounds for  $s_{3,4}$ .

Fig. 3 shows a partition of the plane in regions where

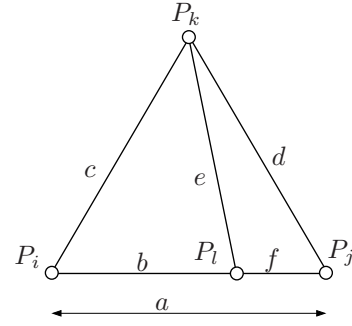


Fig. 4. The generic situation arising when three points are aligned. All the alignments in Fig. 3 can be reduced to this one.

the orientations of the triangles defined by  $P_1, \dots, P_4$  are constant, taking  $\triangle P_1 P_2 P_3$  as a reference. If  $P_4$  remains in one of these regions, the bounds for  $s_{3,4}$  can be readily determined. For instance, if  $P_4$  is in the *Rhombus* region then the patterns in the first row of Table I identify the vertices of the domain giving a tight range for  $s_{3,4}$ . Table I also contains the vertices for the rest of regions. This provides a simpler way to compute bounds on distances than using the non-linear optimization and Kuhn-Tucker conditions [7, p. 259].

If orientation constraints have not to be considered, not all patterns need to be used: the upper bound is the maximum of the upper bounds for the *Rhombus* and the two *Concave Kite* cases, where  $P_4$  is below the line supporting  $\overline{P_1 P_2}$ . The converse applies for the lower bound, which is the minimum of the two *Convex Kite* and the two *Trapezoid* cases. Observe that the resulting seven patterns are by ones used in [7].

Some presentations of DBS oversimplify the problem by reducing it to the application of these seven patterns [13]. Nevertheless, as already proved in [4], this is only valid when no three points can be aligned within the allowed distance ranges. If this occurs, the boundaries separating the monotonic areas must be recursively analyzed.

With the appropriate permutation of indices, all possible alignments follow the pattern shown in Fig. 4. In this situation,  $f$  is replaced by  $a - b$  and, as proved in [14], (5) reduces to the Stewart's theorem [15, p. 6], i.e.,

$$-b a^2 + a b^2 + (a - b) c^2 + b d^2 - a e^2 = 0, \quad (10)$$

whose linear approximation is

$$(b^2 + c^2 - e^2 - 2 a b) \delta a - (a^2 + c^2 - d^2 - 2 a b) \delta b + 2 (a - b) c \delta c + 2 b d \delta d - 2 a e \delta e = 0.$$

The coefficients of this linear approximation are monotonic except when  $\overline{P_k P_l}$  or  $\overline{P_j P_k}$  form a right angle with  $\overline{P_i P_j}$ . Again, these two changes in the monotonicity need to be analyzed recursively.

When segment  $\overline{P_k P_l}$  is orthogonal to  $\overline{P_i P_j}$  we have that  $c^2 = b^2 + e^2$  and  $d^2 = (a - b)^2 + e^2$ , which can be subtracted to eliminate  $e$  leading to

$$c^2 - d^2 + a^2 - 2 a b = 0, \quad (11)$$

whose linear approximation is

$$2(a-b)\delta a - 2a\delta b + 2c\delta c - 2d\delta d = 0, \quad (12)$$

where all coefficients have constant sign. If  $e$  is the variable of interest so that it can not be eliminated from our analysis,  $c^2 = b^2 + e^2$  can be used to eliminate  $c$  from (10). In this case, the resulting coefficients also have constant sign.

When  $\overline{P_j P_k}$  is orthogonal to  $\overline{P_i P_j}$ , a similar analysis also leads to a linear approximations with constant sign coefficients.

Finally, if not only three points are aligned, but the four of them, all distances are linearly related. Thus, the derivatives are constant and the bounds for any given distance can be readily determined.

This completes the monotonicity analysis to obtain tight bounds for any distance in (5).

## V. EXAMPLE: POSITIONING OF ROBOT TEAMS

Networked mobile robots interact over a signal exchange network for its coordinated operation and behavior. Such systems have found many applications in diverse areas of science and engineering such as rescue operations [16], distributed arrays of sensors [17], or networked vehicles [18].

The robots teams can exploit collaboration to maintain global positioning as they move through space. Each robot is usually equipped with an ultrasonic module to estimate its distance to the teammates [19]. Moreover, an on-board camera provides information about the relative bearing of the robot with respect to the rest of the team. Thus, the global positioning of the team must fulfill a set of distance and orientation constraints as the ones given in Section II-B.

Since the distance measurements are noisy, or simply missing due to the limited range of the ultrasonic sensors, maximum-likelihood estimators that determine the most likely position of all the robots have been used in the past [20]. Nevertheless, it is difficult to give realistic probability density functions of the sensor readings due to the complexity of the physical process on which the estimated distance is based. Instead, if we simply assume that errors in measurements are bounded, it is possible to apply a DBS process to obtain tight bounds, as exemplified next.

Consider the team of six mobile robots shown in Fig. 5. The maximum range of the ultrasonic sensors of set to 4 meters and the error in the estimation of distances is  $\pm 1$  cm. Moreover, the orientations of the triangles given by the robot's cameras are those in the figure. For instance,  $\triangle R_1 R_2 R_3$  is negative and  $\triangle R_2 R_3 R_4$  is positive. In these conditions, the sensors provide the matrix of distance ranges between the robots  $\mathbf{D}_1$  shown in Fig. 6. For the non-measured distances, we can only infer that the corresponding robots are further than the maximum span of the ultrasonic sensor. After applying the triangular inequalities, these undefined ranges get bounded, as shown in matrix  $\mathbf{D}_2$ . Then, the tetragonal equality can be used to further reduce the ranges. If we use it without considering the orientation constraints, i.e., we apply one iteration of the standard DBS method [4], we obtain the ranges in matrix  $\mathbf{D}_3$ . Due to the ambiguities

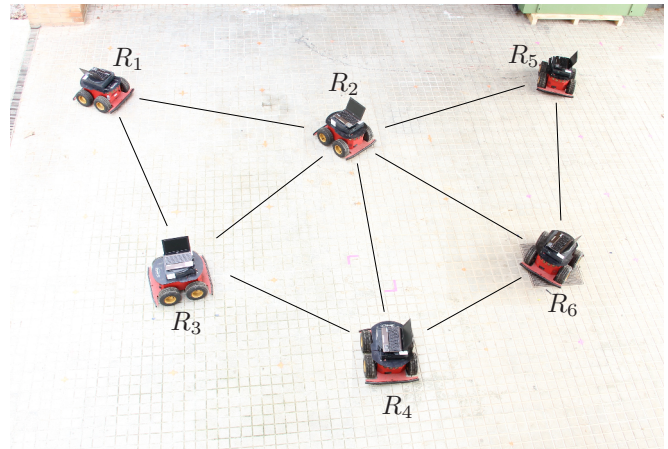


Fig. 5. A robot formation. Each robot is equipped with an ultrasound sensor to measure the distances to nearby teammates. The lines in the figure represent the distances actually measured. The orientation of the triangles is given by cameras mounted on the robots.

inherent to the used distance formulation, this process only produces a marginal reduction of some of the ranges. In contrast, applying one iteration of the method introduced in this paper, matrix  $\mathbf{D}_4$  is obtained, where some of the ranges are significantly narrower than the ones obtained with the standard DBS method. Finally, the ranges can be reduced even more applying a second iteration of the DBS with the orientation constraints. The final ranges are displayed in matrix  $\mathbf{D}_5$ . From this point no further reduction is possible. The number of iterations necessary to produce the tightest possible distance ranges is an open issue, but is probably related with the radius of the graph of constraints.

The plot on the right side of Fig. 6, shows a graphical representation of the tightening of the ranges as we apply the different filtering processes.

## VI. CONCLUSIONS

This paper presented an extension of DBS than can take into account orientation constraints in the reduction of the distance ranges. This extension is based on the monotonicity analysis of the tetragonal equality, which offers a geometric interpretation of the DBS process contrary the existing approach based on non-linear optimization [7].

Traditionally, standard DBS ignores orientation constraints or, at most, uses post-processing steps to discard the solutions whose orientations differ from the desired ones. Clearly, the integration of these constraints in the DBS process is more efficient as it permits focusing the efforts in obtaining the valid solutions.

The extension of the proposed method to 3D requires to study the monotonicity of the pentagonal equality. So, in principle the same method introduced in this paper could be used, just with additional levels of recursion. Finally, the proposed monotonicity analysis to obtain tight bounds seems particularly well-suited for geometric problems. Thus, it is potentially applicable to problems beyond DBS.



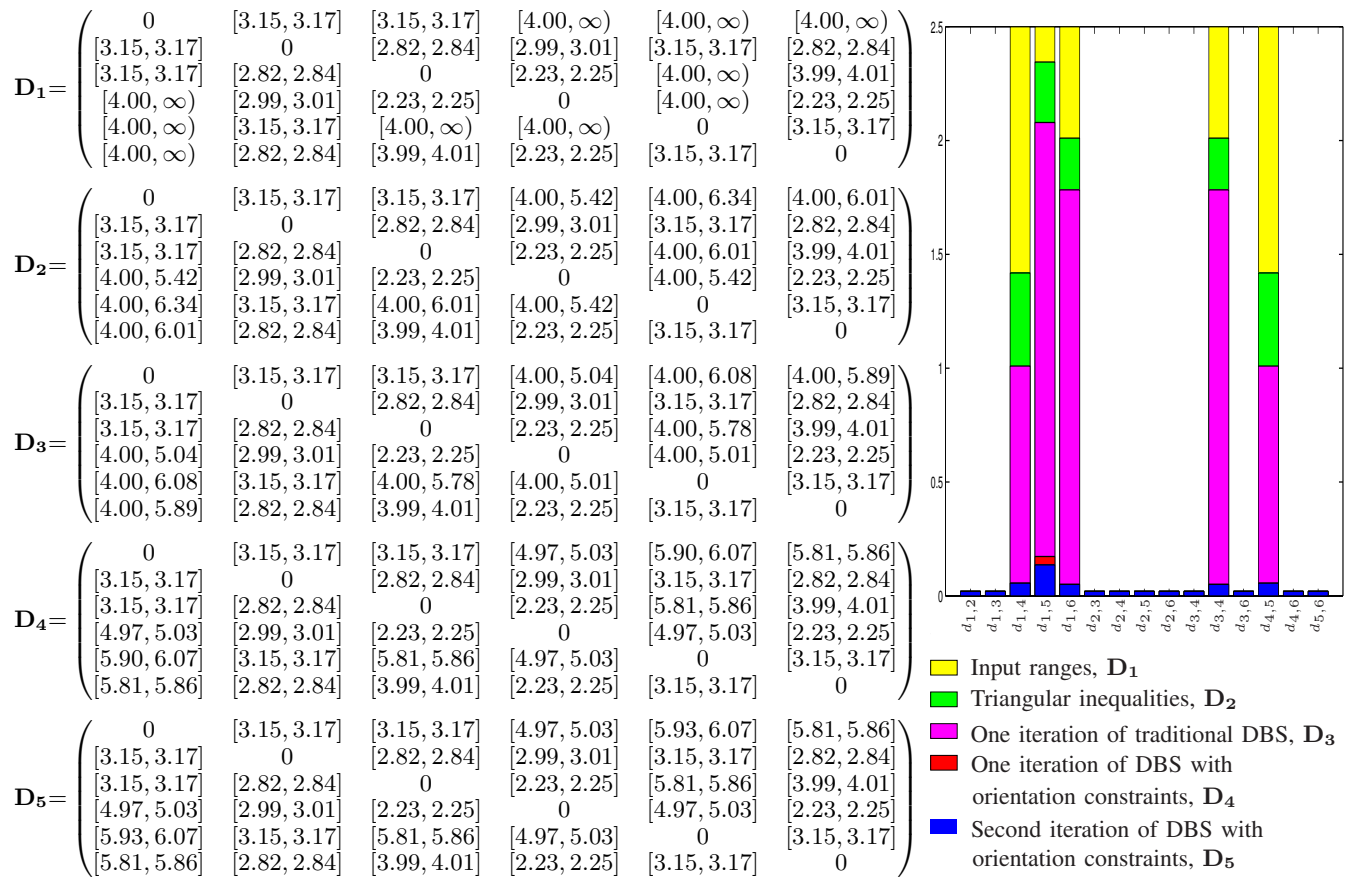


Fig. 6.  $D_1$  stands for the matrix of input ranges as provided by the ultrasonic sensors;  $D_2$ , for the matrix of ranges after imposing triangular inequalities;  $D_3$ , the matrix of ranges resulting from applying one iteration of the standard DBS algorithm;  $D_4$ , for the matrix of ranges resulting from applying one iteration of the proposed algorithm that takes into account orientation constraints; and  $D_5$ , the matrix of ranges after applying two iterations of the proposed algorithm. No further improvements are obtained with more iterations. The plot on the right shows the progressive reduction of the size of the intervals for the distances in the problem as we apply the different filters.

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