Averaged Similarities Generable by Single Attributes

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Abstract. In a theoretical setting, the Representation Theorem is used to generate T-indistinguishability operators (fuzzy similarity relations) from a given set of fuzzy criteria. In applied domains, though, other ways of generation are often used which involve quasi-arithmetic means. In this paper we study when one single fuzzy subset, obtained by averaging multiple fuzzy criteria, is able to generate the same T-indistinguishability operator obtained by all of them, either exactly or approximately.

Keywords. Indistinguishability operator, generator, quasi-arithmetic mean, Representation Theorem

Introduction

Indistinguishability operators with respect to a given t-norm T, or simply T-indistinguishability operators, are a class of fuzzy relations which are generally considered to be the natural fuzzification of classical equivalence relations. They are found under many names in the literature, depending on the author and on the chosen t-norm. $Similarity$ is perhaps the most common name applied to such fuzzy relations (Zadeh [7]), although it is sometimes associated with the particular t-norm $\min$. Other names are $Likeness$, $Fuzzy\ Equality$ or $Fuzzy\ Equivalence\ Relation$. We will use T-$indistinguishability\ operator$ (following Trillas, Valverde [6]), and also the term $similarity$ in an informal way.

Crisp equivalence relations are generally regarded as the mathematical construct for dealing with classifications. They are defined as those relations being reflexive, symmetric and transitive. If E is such a relation on a set X, for each element $x \in X$ we may consider all the elements $y \in Y$ that are related to x, that is, all $y \in Y$ such that $E(x, y) = 1$. We will refer to all these elements as *the class of x*. Here x acts as a prototype, and all the objects y in its class as its likes. As a result, the set X becomes partitioned into classes.

Often, equivalence relations are induced by attributes. For example, a given set X of plane closed polygonal lines becomes naturally partitioned into classes according to

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their number of sides. In addition to that, if the polygonal are real (drawn) lines, we may consider also color as an attribute, and the set becomes furtherly partitioned into, say, black and white lines. Each attribute is responsible for a partition of X and, therefore, for an equivalence relation. The final partition or equivalence relation is the intersection of the two, meaning with this that every two elements x and $y \in X$ are E-related if they have the same number of sides and the same color, but they are not if they fail to meet one of the two criteria, or neither of them. Formally, if E_s stands for *number of sides* and E_c means *color* then $E(x, y) = \min(E_s(x, y), E_c(x, y))$.

Attributes, though, may be of a graded nature. We may consider the attribute *perimeter* of a polygonal, whose range is the positive real numbers, or lines may be drawn in a variety of shades of gray which can be expressed as real numbers between 0 and 1. Attributes that take values on continuous universes are generally regarded as vague, and they are represented by fuzzy sets. Instead of considering a rectangle whose perimeter equals 5 as entirely different from another one of perimeter 5.15, and therefore belonging to two different classes, we will regard them as very similar objects whenever perimeter is the only attribute considered. They could share the same class, provided that classes are fuzzy sets and belonging to a class is a matter of degree.

The definition of T-indistinguishability operator axiomatically captures the intuitive idea of fuzzy equivalence relation.

Definition 0.1. *Let X be a universe and T a t-norm. A T-indistinguishability operator E on X is a fuzzy relation $E : X \times X \to [0, 1]$ satisfying, for all $x, y, z \in X$,*

1. $E(x, x) = 1$ *(Reflexivity)*
2. $E(x, y) = E(y, x)$ *(Symmetry)*
3. $T(E(x, y), E(y, z)) \leq E(x, z)$ *(T-Transitivity)*

A *t-norm* T is an operation on the unit interval which is associative, commutative and satisfies the boundary conditions $T(x, 0) = 0$ and $T(x, 1) = x$ for all $x, y \in [0, 1]$. It is generally accepted that t-norms are the AND connectives of Fuzzy Logic [2].

We will assume within this paper that the t-norm T is continuous and Archimedean [3]. Every continuous Archimedean t-norm is isomorphic to the sum of positive real numbers, bounded or unbounded, according to Ling's theorem [3]. The order reversing isomorphism $t : [0, 1] \to [0, +\infty]$ is called *an additive generator of T*, and $T(a, b) = t^{[-1]}(t(a) + t(b))$ for all $a, b \in [0, 1]$ where $t^{[-1]}$ is the pseudoinverse of t.

In practice, this means that T-transitivity (definition 1.1.3) is simply a version of the *triangle inequality* for metrics, since $T(E(x, y), E(y, z)) \leq E(x, z)$ can be rewritten as $t(E(x, y)) + t(E(y, z)) \geq t(E(x, z))$ or, in a more convenient notation for the purposes of this paper,

$$t \circ E(x, y) + t \circ E(y, z) \geq t \circ E(x, z) .$$

Thus, the underlying semantics of T-indistinguishability operators is enhanced to include proximity in a metric sense in addition to fuzzy equivalence.

T-indistinguishability operators may also be induced by fuzzy attributes. These fuzzy attributes may be represented as fuzzy sets $h : X \to [0, 1]$, and then some procedure is needed to obtain the fuzzy relation E from the fuzzy sets h. Such procedure is provided by the Representation Theorem ([6])

Theorem 0.2. *Representation Theorem. Let E be a fuzzy relation on a set X and T a continuous t-norm. E is a T-indistinguishability operator if and only if there exists a family $H = (h_i)_{i\in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$E(x,y) = \inf_{i\in I} E_{h_i}(x,y).$$

We say that E is *generated* by H, or that H is a *generating family* of E.

Intuitively, H is a set of attributes relevant to the classification induced by E. Each attribute $h_i : X \to [0,1]$ is responsible for a singly generated T-indistinguishability E_{h_i} which is computed by

$$E_{h_i}(x,y) = E_T(x,y) = t^{[-1]}\left(|t \circ h_i(x) - t \circ h_i(y)|\right)$$

The metric interpretation becomes clear when we write the previous equation as

$$t \circ E_{h_i}(x,y) = |t \circ h_i(x) - t \circ h_i(y)|$$

since the right hand side is the real line distance between images of h_i via the isomorphism t.

The representation theorem is central to many theoretical developments in the field of fuzzy relations. Also, it provides a straight translation into the fuzzy framework of the crisp procedure described above to obtain an equivalence relation starting from a set of criteria. It first generates the equivalence relations for each attribute, and then combines all of them via AND, or MIN.

However, from an applied perspective this way of combining information is far from satisfactory. The notion of error is essential to applied domains, and the common way to deal with errors is by averaging information. If, for example, we perceive to different objects as somehow similar under a sequence of observations, we are not likely to thing of them as entirely different just because one particular observation indicates so. We may discard the conflicting piece of information or, more likely, we may aggregate all the evidence gathered throughout the sampling process by using some averaging operator.

Quasi-arithmetic means are a family of averaging operators which are widely used. Quasi-arithmetic means, or q-a means for short, may also be obtained from additive generators, in a very similar way to that of Archimedean t-norms.

Definition 0.3. [1] *The quasi-arithmetic mean M in* [0,1] *generated by a continuous monotonic map $t : [0,1] \to [-\infty, \infty]$ is defined for all $n \in \mathbb{N}$ and $x_1, ..., x_n \in [0,1]$ by*

$$M(x_1, ...x_n) = t^{-1}\left(\frac{t(x_1) + ... + t(x_n)}{n}\right).$$

M is continuous if and only if $Ran\, t \neq [-\infty, \infty]$.

Proposition 0.4. [5] *The map assigning to every continuous Archimedean t-norm T with additive generator t the quasi-arithmetic mean m_t generated by t is a canonical bijection between the set of continuous Archimedean t-norms and continuous quasi-arithmetic means with $t(1) \neq \pm\infty$.*

Similarly weighted quasi-arithmetic means can be defined in the following way.

Definition 0.5. *Let $\alpha_1, \alpha_2, ..., \alpha_n$ be positive numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. α_i are called weights. The weighted quasi-arithmetic mean $M^{\alpha_1,\alpha_2,...,\alpha_n}$ of $x_1, x_2, ..., x_n \in [0,1]$ with weights $\alpha_1, \alpha_2, ..., \alpha_n$ generated by a continuous strict monotonic map $t : [0,1] \to [-\infty, \infty]$ is*

$$M^{\alpha_1,\alpha_2,...,\alpha_n}(x_1, x_2, ..., x_n) = t^{-1}\left(\sum_{i=1}^{n} \alpha_i \cdot t(x_i)\right).$$

$M^{\alpha_1,\alpha_2,...,\alpha_n}$ *is continuous if and only if $Ran\ t \neq [-\infty, \infty]$.*

Proposition 0.6. [5] *The map assigning to every continuous Archimedean t-norm T with additive generator t the weighted quasi-arithmetic mean $M^{\alpha_1,\alpha_2,...,\alpha_n}$ generated by t is a canonical bijection between the set of continuous Archimedean t-norms and continuous weighted quasi-arithmetic means with weights $\alpha_1, \alpha_2, ..., \alpha_n$ and with $t(1) \neq \pm\infty$.*

For simplicity, we well write $M(\alpha_i, x_i)$ instead of $M^{\alpha_1,\alpha_2,...,\alpha_n}(x_1, x_2, ..., x_n)$.

1. q-a means of attributes and their relationship with q-a means of indistinguishabilities

In this section we deal with a family of fuzzy sets $H = (h_i)_{i\in I}$ which we assume to represent a set of attributes or criteria applicable to all $x \in X$. Examples of such attributes are perimeter, gray level, weight, suitability, smoothness etc. and, since they are obtained through empirical measuring or subjective assessment, they are bound to errors and uncertainty.

Each fuzzy set $h \in H$ allows for any pair of elements $x, y \in X$ to be regarded as similar up to a degree $E_h(x, y)$ and, since h is only an approximate instantiation of some theoretical graded variable, so is $E_h(x, y)$. Standard proceedings in such situations include averaging the empirically measured features or the subjectively assessed criteria in order to obtain a more reliable fuzzy set $\bar{h}$ and, therefore, a more accurate relation $E_{\bar{h}}(x, y)$.

Let M be a quasi-arithmetic mean with weights $(\alpha_i)_{i\in I}$ and additive generator t, the same additive generator as that of the Archimedean t-norm T.

In order to average the information via M there are two possible courses of action. We may first compute the quasi-arithmetic mean of all the fuzzy sets in the generating family H, $\bar{h} = M(\alpha_i, h_i)$ and use this single fuzzy set $\bar{h}$ to generate the indistinguishability operator $E_{\bar{h}}(x, y)$ afterwards. Or, we may start by generating a family of indistinguishability operators $(E_{h\,i})_{i\in I}$ and then averaging all the indistinguishabilities in the family as $\bar{E}_H = M(\alpha_i, E_{h\,i})$. We will show that the two procedures may throw different results, depending on how different are the orders induced by the fuzzy sets h on X.

Proposition 1.1. *$E_{\bar{h}}$ is an indistinguishability operator with respect to T.*

Proof. Obvious, since $E_{\bar{h}}$ is the T-indistinguishability generated by the fuzzy set $\bar{h}$. □

Proposition 1.2. [5][4] *$\bar{E}_H$ is an indistinguishability operator with respect to T.*

Proposition 1.3. [4] $\bar{E}_H \leq E_{\bar{h}}$

Each fuzzy set $h_i \in H$ induces a preorder $\leq_i$ on X as follows.

Definition 1.4. *$x \leq_i y$ if and only if i $h_i(x) \leq h_i(y)$ for all $x, y \in X$.*

Note that the induced preorders $\leq_i$ are $total$ preorders because $h_i : X \to [0,1]$ and $[0,1]$ is a totally ordered set.

Definition 1.5. *Two preorders $\leq_i$ and $\leq_j$ on X are compatible if and only if $x <_i y \Rightarrow x \leq_j y$ and $x <_j y \Rightarrow x \leq_i y$ for all $x, y \in X$*

Proposition 1.6. *$\bar{E}_H = E_{\bar{h}}$ if, and only if, $\leq_i$ and $\leq_j$ are compatible orders for all $i, j \in I$.*

Proof. The result is a consequence of the following simple lemma:

$\left|\sum_{i\in I} a_i\right| = \sum_{i\in I} |a_i|$ if, and only if, $a_i \geq 0$ for all $i \in I$, or else $a_i \leq 0$ for all $i \in I$.

Note that, in general, only $\left|\sum_{i\in I} a_i\right| \leq \sum_{i\in I} |a_i|$ holds.

Let t be the additive generator of both the quasi-arithmetic mean M and the Archimedean t-norm T. Let us take $a_i = \alpha_i(t \circ h_i(x)) - \alpha_i(t \circ h_i(y))$ for all $i \in I$.

Since t is monotonous, $\leq_i$ and $\leq_j$ being compatible is both a necessary and sufficient condition for all the a_i to have the same sign, which in turn is necessary and sufficient for (*) in the following equations:

$$
\begin{aligned}
&E_{\bar{h}}(x,y) = E_T(\bar{h}(x), \bar{h}(y)) \\
&= t^{[-1]} \circ \left(\left|t \circ \bar{h}(x) - t \circ \bar{h}(y)\right|\right) \\
&\underset{(*)}{=} t^{[-1]} \circ \left(\left|t \circ t^{[-1]}\left(\sum_{i\in I} \alpha_i t \circ h_i(x)\right) - t \circ t^{[-1]}\left(\sum_{i\in I} \alpha_i t \circ h_i(y)\right)\right|\right) \\
&= t^{[-1]} \circ \left(\left|\sum_{i\in I} \alpha_i t \circ h_i(x) - \sum_{i\in I} \alpha_i t \circ h_i(y)\right|\right) \\
&= t^{[-1]} \circ \left(\left|\sum_{i\in I} \alpha_i (t \circ h_i(x) - t \circ h_i(y))\right|\right) \\
&\underset{(**)}{=} t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i \left|t \circ h_i(x) - t \circ h_i(y)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t \circ t^{[-1]} \left|t \circ h_i(x) - t \circ h_i(y)\right|\right) \\
&= t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t \circ E_T\left(h_i(x), h_i(y)\right)\right) \\
&= t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t \circ E_{h_i}(x,y)\right) = M\left(\alpha_i, E_{h_i}(x,y)\right) \\
&= \overline{E}_H\left(x,y\right)
\end{aligned}
$$

□

Next, we are going to compute how different $E_{\bar{h}}(x,y)$ and $\overline{E}_H\left(x,y\right)$ are, provided that the orders induced by the fuzzy sets $h \in H$ do not generate compatible orders on X. The natural choice for measuring this difference or dissimilarity is E_T, so that we define:

$$C_H(x,y) = E_T(E_{\bar{h}}(x,y), \overline{E}_H(x,y))$$

for every pair $(x,y) \in X$.

$C_H(x,y) = 1$ if all the fuzzy sets h induce compatible orders on $\{x,y\}$, that is, if $h_i(x) \leq h_i(y)$ or either $h_i(x) \geq h_i(y)$ for all $h \in H$. When this does not happen $C_H(x,y)$ provides a measure of how compatible these orders are.

Given $(x,y) \in X$, we split the set $H = (h_i)_{i \in I}$ of all generators into two subsets, $I = P \cup N$, where $P = \{j \in I / h_j(x) \geq h_j(y)\}$ and $N = \{k \in I / h_k(x) < h_k(y)\}$. Note that both P or N may be empty, $P \cap N = \emptyset$, $P \cup N = I$ and $H = (h_j)_{j \in P} \cup (h_k)_{k \in N}$.

We may then split the sum $t \circ \overline{E}_H(x,y) = \sum_{i \in I} \alpha_i t \circ E_{h_i}(x,y)$ accordingly,

$$t \circ \overline{E}_H(x,y) = \sum_{j \in P} \alpha_j t \circ E_{h_j}(x,y) + \sum_{k \in N} \alpha_k t \circ E_{h_k}(x,y)$$

and rename

$$A(x,y) = \sum_{j \in P} \alpha_j t \circ E_{h_j}(x,y)$$

$$B(x,y) = \sum_{k \in N} \alpha_k t \circ E_{h_k}(x,y)$$

We are now in condition to compute the error made when we replace the T-indistinguishability $\overline{E}_H$ by $E_{\bar{h}}$, which is a lot simpler since it is generated by one single fuzzy set. Also, this error provides a measure of the compatibility C_H of the orders induced by H on (x,y).

Proposition 1.7. $C_H(x,y) = t^{[-1]}(\min((2A(x,y)), 2B(x,y))$

Proof. We will show that

$$C_H(x,y) = t^{[-1]} \circ (A(x,y) + B(x,y) - |A(x,y) - B(x,y)|)$$
$$= \begin{cases} t^{[-1]}(2B) & \text{if } A \geq B \\ t^{[-1]}(2A) & \text{if } A < B \end{cases}$$

From this, the result follows immediately.

$$C_H(x,y) = E_T\left(\underset{H}{\bar{E}}(x,y), E_{\bar{h}}(x,y)\right)$$

$$\underset{(*)}{=} \overrightarrow{T}\left(E_{\bar{h}}(x,y) \,|\bar{E}_H(x,y)\right)$$

$$= t^{[-1]} \circ \left(t \circ \bar{E}_H(x,y) - t \circ E_{\bar{h}}(x,y)\right)$$

$$= t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t\left(E_i(x,y)\right) - \left|t \circ \bar{h}(x) - t \circ \bar{h}(y)\right|\right)$$

$$= t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t\left(E_i(x,y)\right) - \left|\sum_{i\in I} \alpha_i t \circ h_i(x) - \sum_{i\in I} \alpha_i t \circ h_i(y)\right|\right)$$

$$= t^{[-1]} \circ \left(\sum_{i\in I} \alpha_i t\left(E_i(x,y)\right) - \left|\sum_{i\in I} \alpha_i \left(t \circ h_i(x) - t \circ h_i(y)\right)\right|\right)$$

$$= t^{[-1]} \circ \left(\sum_{j\in P} \alpha_j t\left(E_j(x,y)\right) + \sum_{k\in N} \alpha_k t\left(E_k(x,y)\right)\right.$$

$$\left. - \left|\sum_{j\in P} \alpha_j \left(t \circ h_j(x) - t \circ h_j(y)\right) - \sum_{k\in N} \alpha_k \left(t \circ h_k(y) - t \circ h_k(x)\right)\right|\right)$$

$$= t^{[-1]} \circ \left(\sum_{j\in P} \alpha_j t\left(E_j(x,y)\right) + \sum_{k\in N} \alpha_k t\left(E_k(x,y)\right)\right.$$

$$- \left|\sum_{j\in P} \alpha_j t \circ t^{[-1]} \circ \left(t \circ h_j(x) - t \circ h_j(y)\right)\right.$$

$$\left.\left. - \sum_{k\in N} \alpha_k t \circ t^{[-1]} \circ \left(t \circ h_k(y) - t \circ h_k(x)\right)\right|\right)$$

$$= t^{[-1]} \circ \left(\sum_{j\in P} \alpha_j t\left(E_j(x,y)\right) + \sum_{k\in N} \alpha_k t\left(E_k(x,y)\right)\right.$$

$$\left. - \left|\sum_{j\in P} \alpha_j t\left(E_j(x,y)\right) - \sum_{k\in N} \alpha_k t\left(E_k(y,x)\right)\right|\right)$$

$$= t^{[-1]} \circ \left(A(x,y) + B(x,y) - |A(x,y) - B(x,y)|\right)$$

□

2. Conclusions and Future Works

The operator C_H that we have introduced provides a measure of how coincident the orders induced by a set of attributes H are on a given pair (x, y).

C_H also indicates whether the set H may be replaced by a single attribute $\bar{h}$ when the T-indistinguishability E is the result of averaging T-indistinguishabilities E_h, instead of taking infima (Representation Theorem).

Future work should address mostly three issues.

- First, the study of the theoretical properties of C_H.
- Second, finding ways of extending $C_H(x, y)$ over the whole set $X \times X$.
- And finally, the definition of an index of complexity or dimension of $E = \bar{E}_H = M(\alpha_i, E_{hi})$ with respect to the set H. We have already seen that the whole set H may be replaced by a single fuzzy set $\bar{h}$ when $C_H(x, y) = 1$ for all $x, y \in X$. This corresponds to the case of dimension 1, or minimum complexity. In general, the dimension will be the minimum number of fuzzy sets h needed to obtain $E = \bar{E}_H$ when $C_H(x, y) < 1$ for some (x, y).

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