VARIATIONAL PRINCIPLES FOR MULTISYMPLECTIC SECOND-ORDER CLASSICAL FIELD THEORIES

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Abstract

We state a unified geometrical version of the variational principles for second-order classical field theories. The standard Lagrangian and Hamiltonian variational principles and the corresponding field equations are recovered from this unified framework.

Key words: *Second-order classical field theories; Variational principles; Unified, Lagrangian and Hamiltonian formalisms*

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Contents

1 Introduction

As stated in [9], the field equations of a classical field theory arising from a partial differential Hamiltonian system (in the sense of [9]) are locally variational, that is, they can be derived using a variational principle. In this work we use the geometric Lagrangian-Hamiltonian formulation for second-order classical field theories given in [4] to state the variational principles for this kind of theories from a geometric point of view, thus giving a different point of view and completing previous works on higher-order classical field theories [1, 8].

(All the manifolds are real, second countable and C^∞. The maps and the structures are assumed to be C^∞. Usual multi-index notation introduced in [6] is used).

2 Higher-order jet bundles

(See [6] for details). Let M be an orientable m-dimensional smooth manifold, and let $\eta \in \Omega^m(M)$ be a volume form for M. Let $E \xrightarrow{\pi} M$ be a bundle with $\dim E = m+n$. If $k \in \mathbb{N}$, the *kth-order jet bundle* of the projection π, $J^k\pi$, is the manifold of the k-jets of local sections $\phi \in \Gamma(\pi)$; that is, equivalence classes of local sections of π by the relation of equality on every partial derivative up to order k. A point in $J^k\pi$ is denoted by $j^k_x\phi$, where $x \in M$ and $\phi \in \Gamma(\pi)$ is a representative of the equivalence class. We have the following natural projections: if $r \leqslant k$,

$$\begin{array}{rcl} \pi^k_r \colon J^k\pi & \longrightarrow & J^r\pi \\ j^k_x\phi & \longmapsto & j^r_x\phi \end{array} \qquad \begin{array}{rcl} \pi^k \colon J^k\pi & \longrightarrow & E \\ j^k_x\phi & \longmapsto & \phi(x) \end{array} \qquad \begin{array}{rcl} \bar{\pi}^k \colon J^k\pi & \longrightarrow & M \\ j^k_x\phi & \longmapsto & x \end{array}$$

Observe that $\pi^s_r \circ \pi^k_s = \pi^k_r$, $\pi^k_0 = \pi^k$, $\pi^k_k = \mathrm{Id}_{J^k\pi}$, and $\bar{\pi}^k = \pi \circ \pi^k$.

If local coordinates in E adapted to the bundle structure are (x^i, u^α), $1 \leqslant i \leqslant m$, $1 \leqslant \alpha \leqslant n$, then local coordinates in $J^k\pi$ are denoted (x^i, u^α_I), with $0 \leqslant |I| \leqslant k$.

If $\psi \in \Gamma(\pi)$, we denote the *kth prolongation* of ϕ to $J^k\pi$ by $j^k\phi \in \Gamma(\bar{\pi}^k)$.

Definition 1 *A section* $\psi \in \Gamma(\bar{\pi}^k)$ *is* holonomic *if* $j^k(\pi^k \circ \psi) = \psi$*; that is,* ψ *is the* k*th prolongation of a section* $\phi = \pi^k \circ \psi \in \Gamma(\pi)$*.*

In the following we restrict ourselves to the case $k = 2$. According to [7], consider the subbundle of fiber-affine maps $J^1\bar{\pi}^1 \to \mathbb{R}$ which are constant on the fibers of the affine subbundle $(\bar{\pi}^1)^*(\Lambda^2 \mathrm{T}^* M) \otimes (\pi^1)^*(V\pi)$ of $J^1\bar{\pi}^1$ over $J^1\pi$. This subbundle is canonically diffeomorphic to the $\pi_{J^1\pi}$-transverse submanifold $J^2\pi^\dagger$ of $\Lambda_2^m(J^1\pi)$ defined locally by the constraints $p_\alpha^{ij} = p_\alpha^{ji}$, which fibers over $J^1\pi$ and M with projections $\pi_{J^1\pi}^\dagger \colon J^2\pi^\dagger \to J^1\pi$ and $\bar{\pi}_{J^1\pi}^\dagger \colon J^2\pi^\dagger \to M$, respectively. The submanifold $j_s \colon J^2\pi^\dagger \hookrightarrow \Lambda_2^m(J^1\pi)$ is the *extended* 2*-symmetric multimomentum bundle*.

All the canonical geometric structures in $\Lambda_2^m(J^1\pi)$ restrict to $J^2\pi^\dagger$. Denote $\Theta_1^s = j_s^*\Theta_1 \in \Omega^m(J^2\pi^\dagger)$ and $\Omega_1^s = j_s^*\Omega_1 \in \Omega^{m+1}(J^2\pi^\dagger)$ the pull-back of the Liouville forms in $\Lambda_2^m(J^1\pi)$, which we call the *symmetrized Liouville forms*.

Finally, let us consider the quotient bundle $J^2\pi^\ddagger = J^2\pi^\dagger / \Lambda_1^m(J^1\pi)$, which is called the *restricted* 2*-symmetric multimomentum bundle*. This bundle is endowed with a natural quotient map, $\mu \colon J^2\pi^\dagger \to J^2\pi^\ddagger$, and the natural projections $\pi_{J^1\pi}^\ddagger \colon J^2\pi^\ddagger \to J^1\pi$ and $\bar{\pi}_{J^1\pi}^\ddagger \colon J^2\pi^\ddagger \to M$. Observe that $\dim J^2\pi^\ddagger = \dim J^2\pi^\dagger - 1$.

3 Lagrangian-Hamiltonian unified formalism

(See [4] for details). Let $\pi \colon E \to M$ be the configuration bundle of a second-order field theory, where M is an orientable m-dimensional manifold with volume form $\eta \in \Omega^m(M)$, and $\dim E = m + n$. Let $\mathcal{L} \in \Omega^m(J^2\pi)$ be a second-order Lagrangian density for this field theory. The 2*-symmetric jet-multimomentum bundles* are

$$\mathcal{W} = J^3\pi \times_{J^1\pi} J^2\pi^\dagger \quad ; \quad \mathcal{W}_r = J^3\pi \times_{J^1\pi} J^2\pi^\ddagger \,.$$

These bundles are endowed with the canonical projections $\rho_1^r \colon \mathcal{W}_r \to J^3\pi$, $\rho_2 \colon \mathcal{W} \to J^2\pi^\dagger$, $\rho_2^r \colon \mathcal{W}_r \to J^2\pi^\ddagger$, and $\rho_M^r \colon \mathcal{W}_r \to M$. In addition, the natural quotient map $\mu \colon J^2\pi^\dagger \to J^2\pi^\ddagger$ induces a natural submersion $\mu_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{W}_r$.

Using the canonical structures in $\mathcal{W}$ and $\mathcal{W}_r$, we define a *Hamiltonian section* $\hat{h} \in \Gamma(\mu_{\mathcal{W}})$, which is specified by giving a local *Hamiltonian function* $\hat{H} \in C^\infty(\mathcal{W}_r)$. Then we define the forms $\Theta_r = (\rho_2 \circ \hat{h})^*\Theta \in \Omega^m(\mathcal{W}_r)$ and $\Omega_r = -\mathrm{d}\Theta_r \in \Omega^{m+1}(\mathcal{W}_r)$. Finally, $\psi \in \Gamma(\rho_M^r)$ is *holonomic* in $\mathcal{W}_r$ if $\rho_1^r \circ \psi \in \Gamma(\bar{\pi}^3)$ is holonomic in $J^3\pi$.

The *Lagrangian-Hamiltonian problem for sections* associated with the system $(\mathcal{W}_r, \Omega_r)$ consists in finding holonomic sections $\psi \in \Gamma(\rho_M^r)$ satisfying

$$\psi^* \, i(X)\Omega_r = 0 \,, \quad \text{for every } X \in \mathfrak{X}(\mathcal{W}_r) \,. \tag{1}$$

Proposition 1 *A section* $\psi \in \Gamma(\rho_M^r)$ *solution to the equation* (1) *takes values in a* $n(m+m(m+1)/2)$*-codimensional submanifold* $j_{\mathcal{L}} \colon \mathcal{W}_{\mathcal{L}} \hookrightarrow \mathcal{W}_r$ *which is identified with the graph of a bundle map* $\mathcal{FL} \colon J^3\pi \to J^2\pi^\ddagger$ *over* $J^1\pi$ *defined locally by*

$$\mathcal{FL}^* p_\alpha^i = \frac{\partial \hat{L}}{\partial u_i^\alpha} - \sum_{j=1}^m \frac{1}{n(ij)} \frac{d}{dx^j} \left(\frac{\partial \hat{L}}{\partial u_{1_i+1_j}^\alpha} \right) \quad ; \quad \mathcal{FL}^* p_\alpha^I = \frac{\partial \hat{L}}{\partial u_I^\alpha} \,.$$

The map $\mathcal{FL}$ is the *restricted Legendre map* associated with $\mathcal{L}$, and it can be extended to a map $\widetilde{\mathcal{FL}} \colon J^3\pi \to J^2\pi^\dagger$, which is called the *extended Legendre map*.

4 Variational Principle for the unified formalism

If $\Gamma(\rho^r_M)$ is the set of sections of ρ^r_M, we consider the following functional (where the convergence of the integral is assumed)

$$\begin{array}{rcl} \mathbf{LH}\colon \Gamma(\rho^r_M) & \longrightarrow & \mathbb{R} \\ \psi & \longmapsto & \displaystyle\int_M \psi^*\Theta_r \end{array}$$

Definition 2 (Generalized Variational Principle) *The* Lagrangian-Hamiltonian variational problem *for the field theory* $(\mathcal{W}_r, \Omega_r)$ *is the search for the critical holonomic sections of the functional* **LH** *with respect to the variations of* ψ *given by* $\psi_t = \sigma_t \circ \psi$*, where* $\{\sigma_t\}$ *is a local one-parameter group of any compact-supported* ρ^r_M*-vertical vector field* Z *in* $\mathcal{W}_r$*, that is,*

$$\left.\frac{d}{dt}\right|_{t=0} \int_M \psi_t^*\Theta_r = 0\,.$$

Theorem 1 *A holonomic section* $\psi \in \Gamma(\rho^r_M)$ *is a solution to the Lagrangian-Hamiltonian variational problem if, and only if, it is a solution to equation* (1)*.*

(*Proof*) This proof follows the patterns in [2] (see also [3]). Let $Z \in \mathfrak{X}^{V(\rho^r_M)}(\mathcal{W}_r)$ be a compact-supported vector field, and $V \subset M$ an open set such that ∂V is a $(m-1)$-dimensional manifold and $\rho^r_M(\mathrm{supp}(Z)) \subset V$. Then,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \int_M \psi_t^*\Theta_r &= \left.\frac{d}{dt}\right|_{t=0} \int_V \psi_t^*\Theta_r = \left.\frac{d}{dt}\right|_{t=0} \int_V \psi^*\sigma_t^*\Theta_r = \int_V \psi^* \left(\lim_{t\to 0} \frac{\sigma_t^*\Theta_r - \Theta_r}{t}\right) \\ &= \int_V \psi^*\, \mathrm{L}(Z)\Theta_r = \int_V \psi^*(i(Z)\mathrm{d}\Theta_r + \mathrm{d}\, i(Z)\Theta_r) = \int_V \psi^*(-\, i(Z)\Omega_r + \mathrm{d}\, i(Z)\Theta_r) \\ &= -\int_V \psi^*\, i(Z)\Omega_r + \int_V \mathrm{d}(\psi^*\, i(Z)\Theta_r) = -\int_V \psi^*\, i(Z)\Omega_r + \int_{\partial V} \psi^*\, i(Z)\Theta_r \\ &= -\int_V \psi^*\, i(Z)\Omega_r\,, \end{aligned}$$

as a consequence of Stoke's theorem and the assumptions made on the supports of the vertical vector fields. Thus, by the fundamental theorem of the variational calculus, we conclude

$$\left.\frac{d}{dt}\right|_{t=0} \int_M \psi_t^*\Theta_r = 0 \quad \Longleftrightarrow \quad \psi^*\, i(Z)\Omega_r = 0\,,$$

for every compact-supported $Z \in \mathfrak{X}^{V(\rho^r_M)}(\mathcal{W}_r)$. However, since the compact-supported vector fields generate locally the $C^\infty(\mathcal{W}_r)$-module of vector fields in $\mathcal{W}_r$, it follows that the last equality holds for every ρ^r_M-vertical vector field Z in $\mathcal{W}_r$. Now, for every $w \in \mathrm{Im}\psi$, we have a canonical splitting of the tangent space of $\mathcal{W}_r$ at w in a ρ^r_M-vertical subspace and a ρ^r_M-horizontal subspace,

$$\mathrm{T}_w\mathcal{W}_r = V_w(\rho^r_M) \oplus \mathrm{T}_w(\mathrm{Im}\psi)\,.$$

Thus, if $Y \in \mathfrak{X}(\mathcal{W}_r)$, then

$$Y_w = (Y_w - \mathrm{T}_w(\psi\circ\rho^r_M)(Y_w)) + \mathrm{T}_w(\psi\circ\rho^r_M)(Y_w) \equiv Y_w^V + Y_w^\psi\,,$$

with $Y_w^V \in V_w(\rho^r_M)$ and $Y_w^\psi \in \mathrm{T}_w(\mathrm{Im}\psi)$. Therefore

$$\psi^*\, i(Y)\Omega_r = \psi^*\, i(Y^V)\Omega_r + \psi^*\, i(Y^\psi)\Omega_r = \psi^*\, i(Y^\psi)\Omega_r\,,$$

since $\psi^* i(Y^V)\Omega_r = 0$, by the conclusion in the above paragraph. Now, as $Y^\psi_w \in \mathrm{T}_w(\mathrm{Im}\psi)$ for every $w \in \mathrm{Im}\psi$, then the vector field Y^ψ is tangent to $\mathrm{Im}\psi$, and hence there exists a vector field $X \in \mathfrak{X}(M)$ such that X is ψ-related with Y^ψ; that is, $\psi_* X = Y^\psi\big|_{\mathrm{Im}\psi}$. Then $\psi^* i(Y^\psi)\Omega_r = i(X)\psi^*\Omega_r$. However, as $\dim \mathrm{Im}\psi = \dim M = m$ and Ω_r is a $(m+1)$-form, we obtain that $\psi^* i(Y^\psi)\Omega_r = 0$. Hence, we conclude that $\psi^* i(Y)\Omega_r = 0$ for every $Y \in \mathfrak{X}(\mathcal{W}_r)$.

Taking into account the reasoning of the first paragraph, the converse is obvious since the condition $\psi^* i(Y)\Omega_r = 0$, for every $Y \in \mathfrak{X}(\mathcal{W}_r)$, holds, in particular, for every $Z \in \mathfrak{X}^{V(\rho^r_{\mathbb{R}})}(\mathcal{W}_r)$.

■

5 Lagrangian variational problem

Consider the submanifold $j_\mathcal{L} \colon \mathcal{W}_\mathcal{L} \hookrightarrow \mathcal{W}_r$. Since $\mathcal{W}_\mathcal{L}$ is the graph of the restricted Legendre map, the map $\rho^\mathcal{L}_1 = \rho^r_1 \circ j_\mathcal{L} \colon \mathcal{W}_\mathcal{L} \to J^3\pi$ is a diffeomorphism. Then we can define the *Poincaré-Cartan m-form* as $\Theta_\mathcal{L} = (j_\mathcal{L} \circ (\rho^\mathcal{L}_1)^{-1})^*\Theta_r \in \Omega^m(J^3\pi)$. This form coincides with the usual Poincaré-Cartan m-form derived in [5, 7].

Given the Lagrangian field theory $(J^3\pi, \Omega_\mathcal{L})$, consider the following functional

$$\begin{array}{rccl} \mathbf{L} \colon \Gamma(\pi) & \longrightarrow & \mathbb{R} \\ \phi & \longmapsto & \displaystyle\int_M (j^3\phi)^*\Theta_\mathcal{L} \end{array}$$

Definition 3 (Generalized Hamilton Variational Principle) *The* Lagrangian variational problem *(or* Hamilton variational problem*) for the second-order Lagrangian field theory* $(J^3\pi, \Omega_\mathcal{L})$ *is the search for the critical sections of the functional* $\mathbf{L}$ *with respect to the variations of* ϕ *given by* $\phi_t = \sigma_t \circ \phi$*, where* $\{\sigma_t\}$ *is a local one-parameter group of any compact-supported* $Z \in \mathfrak{X}^{V(\pi)}(E)$*; that is,*

$$\frac{d}{dt}\bigg|_{t=0} \int_M (j^3\phi_t)^*\Theta_\mathcal{L} = 0\,.$$

Theorem 2 *Let* $\psi \in \Gamma(\rho^r_M)$ *be a holonomic section which is critical for the functional* $\mathbf{LH}$*. Then,* $\phi = \pi^3 \circ \rho^r_1 \circ \psi \in \Gamma(\pi)$ *is critical for the functional* $\mathbf{L}$*.*

Conversely, if $\phi \in \Gamma(\pi)$ *is a critical section for the functional* $\mathbf{L}$*, then the section* $\psi = j_\mathcal{L} \circ (\rho^\mathcal{L}_1)^{-1} \circ j^3\phi \in \Gamma(\rho^r_M)$ *is holonomic and it is critical for the functional* $\mathbf{LH}$*.*

(*Proof*) The proof follows the same patterns as in Theorem 1. The same reasoning also proves the converse.

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6 Hamiltonian variational problem

Let $\widetilde{\mathcal{P}} = \mathrm{Im}(\widetilde{\mathcal{FL}}) \overset{\tilde{\jmath}}{\hookrightarrow} J^2\pi^\dagger$ and $\mathcal{P} = \mathrm{Im}(\mathcal{FL}) \overset{\jmath}{\hookrightarrow} J^2\pi^\ddagger$ the image of the extended and restricted Legendre maps, respectively; $\bar{\pi}_\mathcal{P} \colon \mathcal{P} \to M$ the natural projection, and $\mathcal{FL}_o \colon J^3\pi \to \mathcal{P}$ the map defined by $\mathcal{FL} = \jmath \circ \mathcal{FL}_o$.

A Lagrangian density $\mathcal{L} \in \Omega^m(J^2\pi)$ is *almost-regular* if (i) $\mathcal{P}$ is a closed submanifold of $J^2\pi^\ddagger$, (ii) $\mathcal{FL}$ is a submersion onto its image, and (iii) for every $j^3_x\phi \in J^3\pi$, the fibers $\mathcal{FL}^{-1}(\mathcal{FL}(j^3_x\phi))$ are connected submanifolds of $J^3\pi$.

The Hamiltonian section $\hat{h} \in \Gamma(\mu_{\mathcal{W}})$ induces a Hamiltonian section $h \in \Gamma(\mu)$ defined by $\rho_2 \circ \hat{h} = h \circ \rho_2^r$. Then, we define the *Hamilton-Cartan m-form* in $\mathcal{P}$ as $\Theta_h = (h \circ \jmath)^* \Theta_1^s \in \Omega^m(\mathcal{P})$. Observe that $\mathcal{FL}_o^* \Theta_h = \Theta_{\mathcal{L}}$.

In what follows, we consider that the Lagrangian density $\mathcal{L} \in \Omega^m(J^2\pi)$ is, at least, almost-regular. Given the Hamiltonian field theory $(\mathcal{P}, \Omega_h)$, let $\Gamma(\bar{\pi}_{\mathcal{P}})$ be the set of sections of $\bar{\pi}_{\mathcal{P}}$. Consider the following functional

$$\begin{array}{rcl} \mathbf{H}\colon \Gamma(\bar{\pi}_{\mathcal{P}}) & \longrightarrow & \mathbb{R} \\ \psi_h & \longmapsto & \displaystyle\int_M \psi_h^* \Theta_{\mathcal{P}} \end{array}$$

Definition 4 (Generalized Hamilton-Jacobi Variational Principle) *The* Hamiltonian variational problem *(or* Hamilton-Jacobi variational problem*) for the second-order Hamiltonian field theory $(\mathcal{P}, \Omega_h)$ is the search for the critical sections of the functional* $\mathbf{H}$ *with respect to the variations of ψ_h given by $(\psi_h)_t = \sigma_t \circ \psi_h$, where $\{\sigma_t\}$ is a local one-parameter group of any compact-supported $Z \in \mathfrak{X}^{V(\bar{\pi}_{\mathcal{P}})}(\mathcal{P})$,*

$$\left.\frac{d}{dt}\right|_{t=0} \int_M (\psi_h)_t^* \Theta_h = 0\,.$$

Theorem 3 *Let $\psi \in \Gamma(\rho_M^r)$ be a critical section of the functional* $\mathbf{LH}$*. Then, the section $\psi_h = \mathcal{FL}_o \circ \rho_1^r \circ \psi \in \Gamma(\bar{\pi}_{\mathcal{P}})$ is a critical section of the functional* $\mathbf{H}$*.*

Conversely, if $\psi_h \in \Gamma(\bar{\pi}_{\mathcal{P}})$ is a critical section of the functional $\mathbf{H}$*, then the section $\psi = j_{\mathcal{L}} \circ (\rho_1^{\mathcal{L}})^{-1} \circ \gamma \circ \psi_h \in \Gamma(\rho_M^r)$ is a critical section of the functional* $\mathbf{LH}$*, where $\gamma \in \Gamma_{\mathcal{P}}(\mathcal{FL}_o)$ is a local section of $\mathcal{FL}_o$.*

(*Proof*) The proof follows the same patterns as in Theorem 1. The same reasoning also proves the converse, bearing in mind that $\gamma \in \Gamma_{\mathcal{P}}(\mathcal{FL}_o)$ is a local section.

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7 The higher-order case

As stated in [4], this formulation fails when we try to generalize it to a classical field theory of order greater or equal than 3. The main obstruction to do so is the relation among the multimomentum coordinates used to define the submanifold $J^2\pi^\dagger$, $p_\alpha^{ij} = p_\alpha^{ji}$ for every $1 \leqslant i, j \leqslant m$ and every $1 \leqslant \alpha \leqslant n$. Although this "symmetry" relation on the multimomentum coordinates can indeed be generalized to higher-order field theories, it only holds for the highest-order multimomenta. That is, this relation on the multimomenta is not invariant under change of coordinates for lower orders, and hence we do not obtain a submanifold of $\Lambda_2^m(J^{k-1}\pi)$.

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