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## MODELING THERMOELASTOPLASTICITY OF COMPOSITE MATERIALS

ABDALLA M. AHMED<sup>\*</sup> AND YASSER M. SHABANA<sup>\*†</sup>

<sup>\*</sup> Department of Materials Science and Engineering  
Egypt-Japan University of Science and Technology (E-JUST)  
P.O. Box 179, New Borg El-Arab City Postal Code 21934, Alexandria, Egypt  
e-mail: [abdalla.ahmed@ejust.edu.eg](mailto:abdalla.ahmed@ejust.edu.eg), [www.ejust.edu.eg](http://www.ejust.edu.eg)

<sup>†</sup> Mechanical Design Department  
Faculty of Engineering, El-Mataraia, Helwan University  
P.O. Box 11718, Cairo, Egypt  
e-mail: [yasser.shabana@gmail.com](mailto:yasser.shabana@gmail.com), [www.eng-mataraia.edu.eg](http://www.eng-mataraia.edu.eg)

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**Abstract.** The prediction of the thermoelastoplastic behaviors of composite materials and the corresponding constituents is essential and needs to be investigated from the theoretical and experimental aspects. One of the promising theories to predict the behaviors of a composite on the micro-scale using finite element analysis is the multi-continuum theory (MCT). It is also used for evaluating the constituent-averaged elastic stress and strain from the composite-averaged counterparts. In this research, the MCT is extended to handle the thermoelastoplastic behaviors of composites. A micromechanical model, which combines Eshelby and Mori-Tanaka models, is used to determine the effective composite properties using the constituents ones. These properties are used to propose incremental non-linear governing equations. Also, the thermoelastoplastic decomposition of the composite strain is carried out to determine the constituents stresses and strains. The current work is validated by comparing its results with some others in the literature and good agreement is obtained.

### 1 INTRODUCTION

The developments in performance and efficiency of aircrafts, nuclear power plants, and other sensitive industries rely on the improvements of materials that can operate under extreme mechanical and thermal loading conditions [1]. Accordingly, composite materials are frequently being used in severe thermo-mechanical loading environments. Thermal stresses mostly cause plastic yielding in many applications of different composite structures [2].

Generally, analysis of composite structures is complicated because of the presence of a variety of properties and the complex interaction of the microstructural level constituents. To overcome that, the constituent properties are homogenized to facilitate a single continuum analyses, and micromechanics can be used to obtain the composite properties. The relations between the macroscale composite properties and the microscale constituents counterparts are essential. Many analytical techniques of homogenization are based on the equivalent eigen-

strain method [3][4], which considers the problem of a single ellipsoidal inclusion embedded in an infinite elastic medium. Mori and Tanaka [5] used Eshelby solution to develop a method that takes into account the interactions among the inclusions. Various other investigators have persisted along these lines over the years. For example, Ju and Chen [6] proposed a micromechanical framework to investigate the effective properties of elastic composites with randomly dispersed ellipsoidal inclusions. One of the well-known homogenization techniques is the self-consistent method [7], which can be used for a random distribution of inclusions in an infinite medium that is assumed to have properties equal to the unknown properties sought. Therefore, an iterative procedure is used to obtain the overall moduli. Homogenization of composites with periodic microstructure has been accomplished by using various techniques including an extension of the Eshelby inclusion problem [3] ,[4], the Fourier series technique [8][9], and variational principles [10]. The periodic eigen-strain method was further developed to determine the overall relaxation moduli of linear viscoelastic composite materials [11] ,[12]. A particular case, the cell method for periodic composite considers a unit cell with a square inclusion [13]. The above mentioned analytical approaches predict approximate estimates of the exact solution of the micromechanics problem. These predictions should lie between the lower and upper bounds. Several variational principles were developed to evaluate bounds on the homogenized elastic properties of macroscopically isotropic heterogeneous materials [14]. Those bounds depend only on the volume fractions and the physical properties of the constituents. In general, these types of studies have shown that even for idealized microstructures it was difficult to quantify exact relations, except for few special cases, only bounding or approximate relations could be established. Nevertheless, many results have been obtained and some of the earlier works are among the most heavily cited in the field.

Conventional modeling methods treat composite materials as homogeneous solids with uniform properties. The multi-continuum theory (MCT) incorporates the classical micromechanics-based strain decomposition technique, which was first developed by Hill [15], for evaluating the constituent-averaged stress and strain using the composite-averaged counterpart in a numerical algorithm. The behaviors of the constituents can be determined by assuming the constituents to be separate continua but linked together in the composite [16]. Garnich et al. [17] modified Hills relations to be valid for applications that include thermal loads.

In this article, further modification to the MCT is achieved so that the obtained relations of two constituents composite can predict the elastoplastic behavior of composite materials subjected to thermomechanical loads. The composite plasticity comes from the plastic behavior of the matrix material whereas the reinforcements behave elastically. In order to analyze the plastic behavior of a composite, incrementation of the constitutive equations is applied. In order to validate the proposed model, results are compared with some others in the literature and good agreement is obtained.

## **2 MODELING PROCEDURE**

### **2.1 Effective properties of composites**

In order to determine the effective mechanical and thermal properties of the composite, the constituents properties should be used. Also, one essential factor that affects the prediction of

these properties and composite behavior is the Eshelby tensor that is given by:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \quad (1)$$

For an isotropic spherical inclusion, where the aspect ratio equals one ( $\lambda = 1$ ), the components of Eshelby tensor are given as functions of the Poisson ratio of the matrix material ( $\nu_m$ ) as:

$$S_{11} = S_{22} = S_{33} = \frac{7 - 5\nu_m}{15(1 - \nu_m)} \quad (2)$$

$$S_{12} = S_{13} = S_{21} = S_{23} = S_{31} = S_{32} = \frac{5\nu_m - 1}{15(1 - \nu_m)} \quad (3)$$

$$S_{44} = S_{55} = S_{66} = \frac{4 - 5\nu_m}{15(1 - \nu_m)} \quad (4)$$

The composite mechanical and thermal properties are evaluated using the micromechanics equation [18] where the strain-stress relation, ( $d\varepsilon - d\sigma$ ), of the composite is expressed as:

$$d\varepsilon = (I + D_1)L_o^{-1}d\sigma - \{D_2 - \alpha_o(I + D_1)\}dT \quad (5)$$

where,

$$D_1 = \phi_f A^{-1} B \quad (6)$$

$$D_2 = \phi_f A^{-1} M_1 \quad (7)$$

$$A_1 = (C_f - C_m)^{-1} R_1 \quad (8)$$

$$B = -I \quad (9)$$

$$M_1 = -(C_f - C_m)^{-1} (C_f \alpha_1 - C_f \alpha_o) \quad (10)$$

$$R_1 = L_o + (C_f - C_m)S + (C_f - C_m)(I - S)\phi_f \quad (11)$$

$dT$  is the temperature difference;  $C$  and  $\alpha$  are the stiffness and thermal expansion coefficient and  $\phi_f$  is the reinforcement volume fraction. The subscripts  $o$  and  $I$  refer to the matrix and reinforcement materials, respectively.

Substituting Eqns. (6-11) into Eqn. (1) yield the compliance matrix and the thermal expansion coefficient of the composite ( $S_c$  and  $\alpha_c$ ) as:

$$S_c = \left( I + \phi_f \left( (C_f - C_m)^{-1} (C_m + (C_f - C_m)S + (C_f - C_m)(I - S)\phi_f) \right)^{-1} (-I) \right) C_m^{-1} \quad (12)$$

$$\alpha_c = - \left\{ \phi_f \left( (C_f - C_m)^{-1} (C_m + (C_f - C_m)S + (C_f - C_m)(I - S)\phi_f) \right)^{-1} \left( -(C_f - C_m)^{-1} (C_f \alpha_f - C_m \alpha_m) \right) - \alpha_m \left( I - \phi_f \left( (C_f - C_m)^{-1} (C_m + (C_f - C_m)S + (C_f - C_m)(I - S)\phi_f) \right)^{-1} \right) \right\} \quad (13)$$

## 2.2 Thermoelastic multi-continuum decomposition

MCT starts with a continuum definition of stress at a point. It is clear that stresses differ from a point to another across the different phases of the composite. The homogenized value used to characterize the stress tensor at a point in a single continuum is derived by the volume average of all stresses in the region as:

$$\sigma = \frac{1}{V} \int_D \sigma(x) dV \quad (14)$$

where,  $D$  is the region representing the continuum point,  $V$  is the volume associated with the point in the averaging process, and  $x$  is a position vector locating a point in the domain  $D$ . So, the composite average (or homogenized) stress state  $\sigma_c$  can be expressed as:

$$\sigma_c = \frac{1}{V_c} \int_{D_c} \sigma(x,y,z) dV \quad (15)$$

The concept of a multicontinuum simply extends this concept to reflect coexisting materials within a continuum point. Consider a composite material with two clearly identifiable constituents. The reinforcement and matrix average stress states can be expressed as:

$$\sigma_f = \frac{1}{V_f} \int_{D_f} \sigma(x,y,z) dV \quad (16)$$

$$\sigma_m = \frac{1}{V_m} \int_{D_m} \sigma(x,y,z) dV \quad (17)$$

where,  $V_c = V_f + V_m$ , and  $D_c = D_f + D_m$

Combining equations (15-17) leads to:

$$\sigma_c = \phi_f \sigma_f + \phi_m \sigma_m \quad (18)$$

Similarly, the strain states of the composite, reinforcement and matrix have the same form as those of the above mentioned stress states.

Changing from direct tensor to contracted matrix notation, the volume-averaged linearized elastic constitutive relations for the composite and the constituents are given by:

$$\{\sigma_c\} = [C_c](\{\varepsilon_c\} - \{\varepsilon_{c0}\}) = [C_c](\{\varepsilon_c\} - \theta\{\alpha_c\}) \quad (19)$$

$$\{\sigma_f\} = [C_f](\{\varepsilon_f\} - \{\varepsilon_{f0}\}) = [C_f](\{\varepsilon_f\} - \theta\{\alpha_f\}) \quad (20)$$

$$\{\sigma_m\} = [C_m](\{\varepsilon_m\} - \{\varepsilon_{m0}\}) = [C_m](\{\varepsilon_m\} - \theta\{\alpha_m\}) \quad (21)$$

Substituting Eqns. (19-21) into Eqn. (18) yields

$$[C_c](\{\varepsilon_c\} - \theta\{\alpha_c\}) = \phi_f([C_f](\{\varepsilon_f\} - \theta\{\alpha_f\})) + \phi_m([C_m](\{\varepsilon_m\} - \theta\{\alpha_m\})) \quad (22)$$

Using Eqns. (18, 22) and apply some mathematical manipulations results in the reinforcement strain as a function of the composite strain including various properties as:

$$\{\varepsilon_f\} = (\phi_f[I] + \phi_m[A])^{-1} (\{\varepsilon_c\} - \theta\{a\}) \quad (23)$$

where,

$$[A] = \left[ -\frac{\phi_f}{\phi_m} ([C_c] - [C_m])^{-1} ([C_c] - [C_f]) \right] \quad (24)$$

$$\{a\} = \{ ([C_c] - [C_m])^{-1} ([C_c]\{\alpha_c\} - \phi_f[C_f]\{\alpha_f\} - \phi_m[C_m]\{\alpha_m\}) \} \quad (25)$$

Substituting Eqns. (19, 20) into Eqn. (23) yields the reinforcement stress

$$[C_f]^{-1}\{\sigma_f\} + \theta\{\alpha_f\} = (\phi_f[I] + \phi_m[A])^{-1} ([C_c]^{-1}\{\sigma_c\} + \theta\{\alpha_c\} - \theta\{a\}) \quad (26)$$

$$\{\sigma_f\} = [C_f](\phi_f[I] + \phi_m[A])^{-1} [C_c]^{-1}\{\sigma_c\} + \theta[C_f] \left( (\phi_f[I] + \phi_m[A])^{-1} (\{\alpha_c\} - \{a\}) - \{\alpha_f\} \right) \quad (27)$$

Using Eqns. (18, 19, 21, 23) results in the matrix strain and stress as follows:

$$\{\varepsilon_m\} = \left( \frac{1 - \phi_f(\phi_f[I] + \phi_m[A])^{-1}}{\phi_m} \right) \{\varepsilon_c\} + \theta \left( \frac{\phi_f}{\phi_m} (\phi_f[I] + \phi_m[A])^{-1} \{a\} \right) \quad (28)$$

$$\begin{aligned} \{\sigma_m\} = & \left( [C_m][C_c]^{-1} \left( \frac{1 - \phi_f(\phi_f[I] + \phi_m[A])^{-1}}{\phi_m} \right) \right) \{\sigma_c\} \\ & + \theta \left( [C_m] \left( \left( \frac{1 - \phi_f(\phi_f[I] + \phi_m[A])^{-1}}{\phi_m} \right) \{\alpha_c\} + \left( \frac{\phi_f}{\phi_m} (\phi_f[I] + \phi_m[A])^{-1} \{a\} \right) - \{\alpha_m\} \right) \right) \end{aligned} \quad (29)$$

### 2.3 Thermoelastoplasticity

For elastic-plastic analysis, the equivalent stress-equivalent plastic strain relation is used to obtain the equivalent (von Mises) stress of the matrix that may written as:

$$\sigma_e = \sqrt{\frac{1}{2} ((\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2) + 3(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)} \quad (30)$$

It is assumed that the matrix material behaves elastoplastically according to the following relation:

$$\sigma_e = \sigma_o \left( 1 + \frac{\varepsilon_p}{\varepsilon_o} \right)^{0.1}, \quad \varepsilon_o = \frac{\sigma_o}{E_o}$$

The strain hardening parameter of the matrix can be evaluated as:

$$H' = \frac{1}{10} E_o \left( \frac{\sigma_o}{\sigma_e} \right)^9 \quad (31)$$

When the stress exceeds the yield limit and the matrix material behaves elastoplastically, the modulus of elasticity and Poisson ratio should be replaced by the equivalent ones ( $E'_o$  and  $\nu'_o$ ) as follows:

$$E'_o = \frac{E_o}{1 + (E_o/H')} \quad (32)$$

$$\nu'_o = \frac{\nu_o + (E_o/2H')}{1 + (E_o/H')} \quad (33)$$

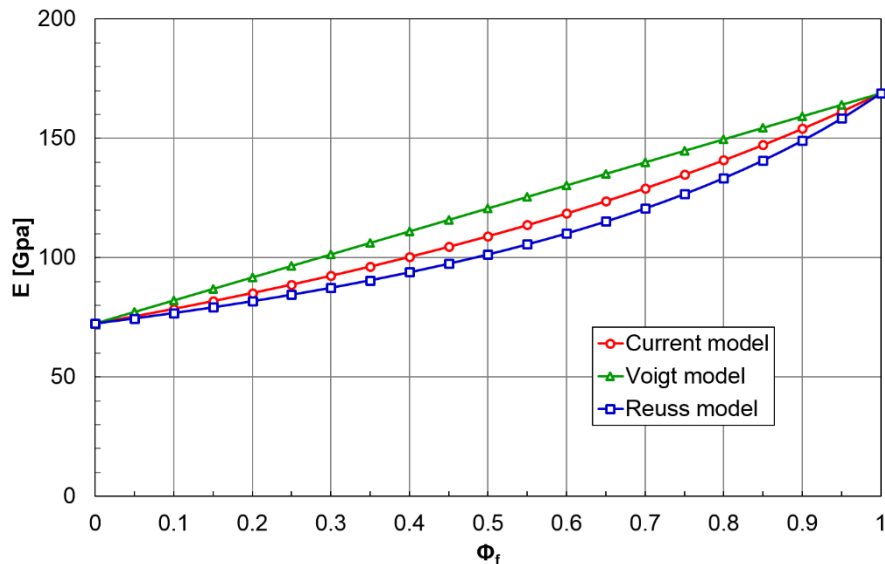
## 3 RESULTS AND DISCUSSION

The considered materials of the matrix and reinforcements are aluminum (Al) and partially stabilized zirconia (PSZ), respectively. Table 1 presents the thermal and mechanical properties of these materials [19].

**Table 1:** Mechanical and thermal properties of Al and PSZ [19]

Material	Young Modulus (GPa)	Poisson Ratio	Thermal expansion coefficient ( $10^{-6} 1/k$ )	Yield strength (MPa)
Al	72.4	0.349	23.1	276
PSZ	169	0.333	8.86	–

The properties of the composite are evaluated using the constituents counterparts. To validate the current model, the results of the modulus of elasticity are compared with those of Voigt (isostain) model [20] and Reuss (isostress) model [21]. The results are found to lie in between the upper and lower bounds, as seen in figure 1 and this is agreeable with the well-known information that particulate composites exhibit behavior that lie between the isostrain and isostress bounds. Consequently, the results are in good agreement with the two models. As seen, the modulus of elasticity increases with increasing  $\phi_f$  because the reinforcement are the main carrying load element.

**Figure 1:** Variation of the modulus of elasticity with the reinforcement volume fraction

The modulus of rigidity is compared to Voigt model [20] because it is applicable for evaluating the shear modulus. As seen in figure 2, the results are found to be close to those of Voigt model. Also, the modulus of rigidity increases with increasing  $\phi_f$ . Another property of the composite is the Poisson ratio that is compared to Reuss model [21] as shown in figure 3. It can be seen that the results are comparable to those of the reference model. Also, Poisson ratio decreases with increasing the  $\phi_f$ . For the thermal properties, figure 4 shows the thermal expansion coefficient (TEC) of the composite material compared to the upper and lower bounds that are predicted by the rule of mixtures and its inverse. The current results lie between the two bounds and decrease with increasing the  $\phi_f$ . Eventually, it can be concluded from figures 1-4 that the results of the mechanical properties, which represented by elastic and shear moduli and Poisson ratio, and the thermal properties, which represented by TEC, are all in good agreement with other models in the literature.

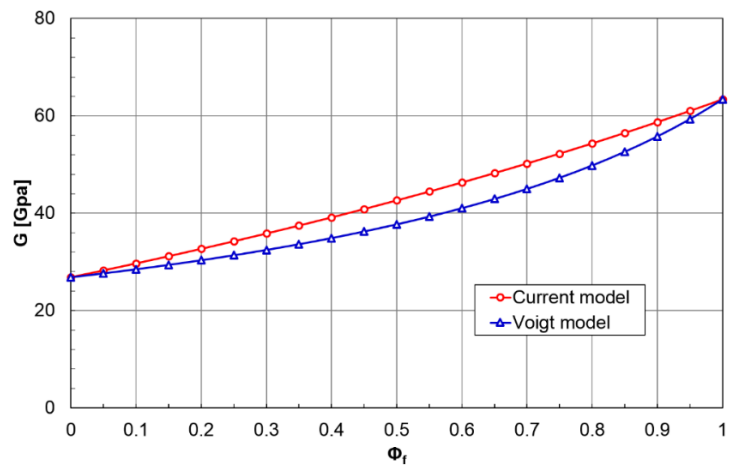


Figure 2: Variation of the modulus of rigidity with the reinforcement volume fraction

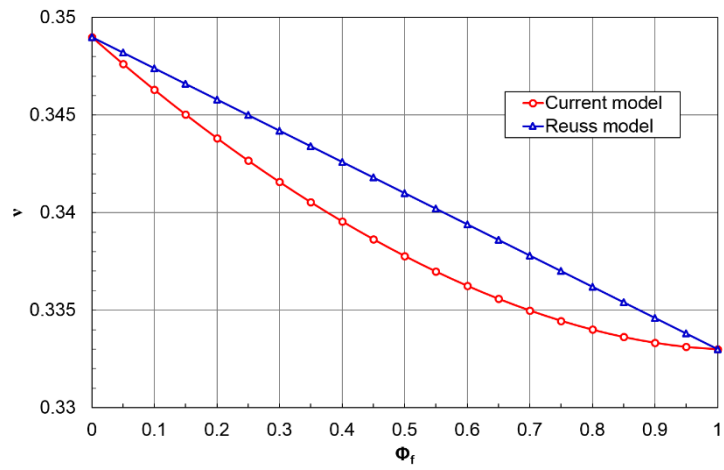


Figure 3: Variation of Poisson ratio with the reinforcement volume fraction

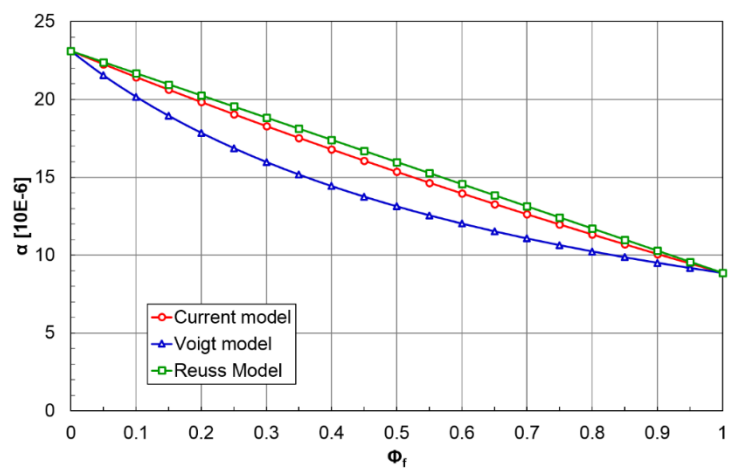
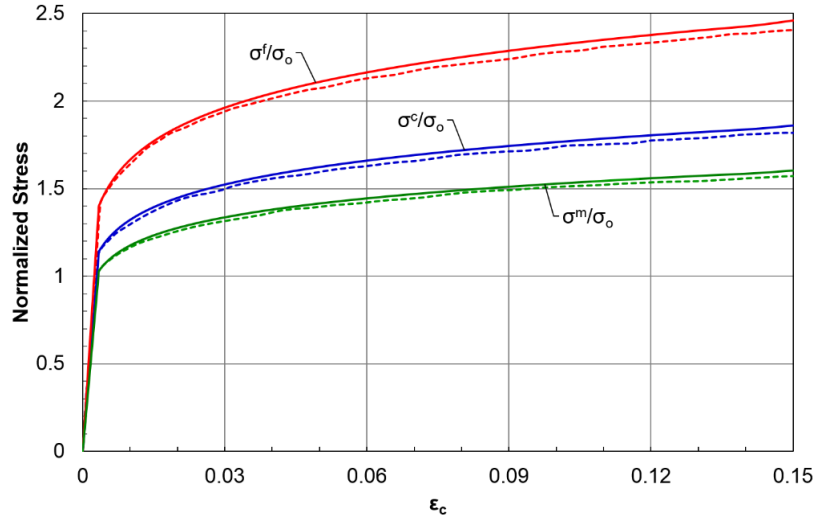


Figure 4: Variation of TEC with the reinforcement volume fraction.

Figure 5 shows the variation of the stresses of the composite, reinforcement and matrix normalized to the yield strength of the matrix with the composite strain. The reinforcement stress is the greatest while the matrix stress is the lowest. Therefore, the matrix has the least toughness. The results of the current model (solid lines) are compared with those of [18] (dashed lines) and it can be seen that good agreement is attained.



**Figure 5:** Comparison of the predicted composite, matrix and reinforcement stresses with another model.

Figure 6 shows the variation of the normalized composite stress with the composite strain considering  $\phi_f$  as a parameter. It can be seen that increasing  $\phi_f$  increases the composite toughness as the stress level is shifted up at the same strain leading to increase of the energy absorbed during deformation. The more curvature, the larger the area under the stress–strain curve, which is closely related to the energy absorbed during the deformation. Since the reinforcements are assumed to behave elastically and they are stiffer than the matrix, increasing  $\phi_f$  increases the stiffness of the composite and the yield strength of the composite as shown in the figure. Moreover, the strain hardening of the plastic part increases with  $\phi_f$  as the slope of the plastic part increases. Reinforcing phase imparts toughness to the composite. So, as increasing the  $\phi_f$ , the toughness of the composite increases. Figure 7 illustrates this clearly when plotting the normalized reinforcement stress with the composite strain. Figure 8 depicts the increase of normalized matrix stress with increasing both of the composite strain and  $\phi_f$ .

#### 4 CONCLUSIONS

Extending and modifying the multi-continuum theory to be valid for elastoplastic behaviors of composites with different shapes of the reinforcements and subjected to thermomechanical loading conditions is achieved. The plasticity of the composite is originated from the matrix plasticity while the reinforcements behave elastically. The effective properties of the composite are extracted as functions of the constituents counterparts. Results of the different mechanical and thermal properties and the stress-strain relations of the composite, reinforcement and matrix are compared with those of other models in the literature and good agreement is obtained.



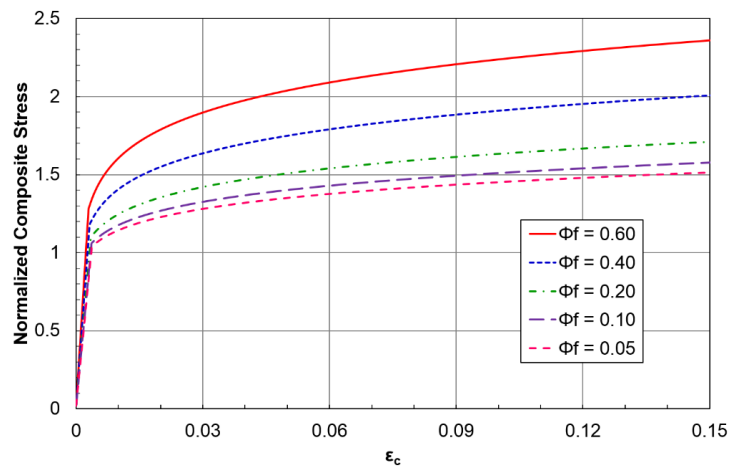


Figure 6: Composites normalized stress-strain curves at different reinforcement volume fraction.

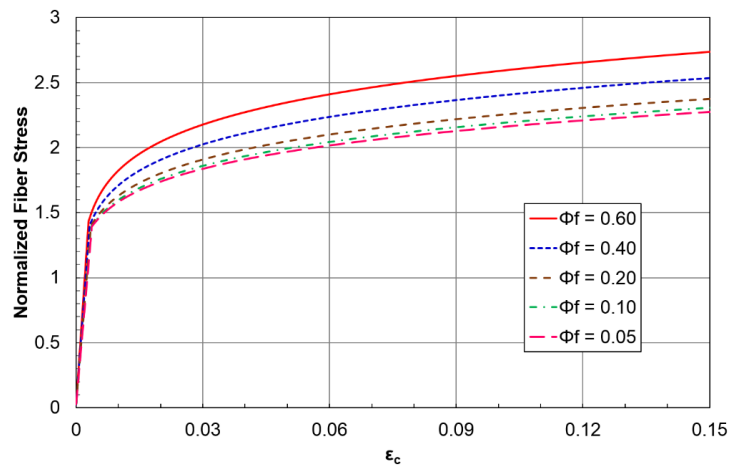


Figure 7: Reinforcement normalized stress with composites strain at different reinforcement volume fraction.

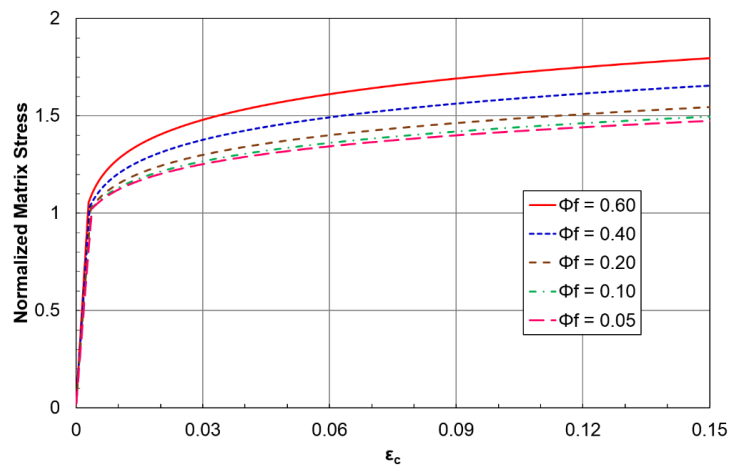


Figure 8: Matrix normalized stress with composites strain at different reinforcement volume fraction.

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