

Master of Science in Advanced Mathematics and Mathematical Engineering

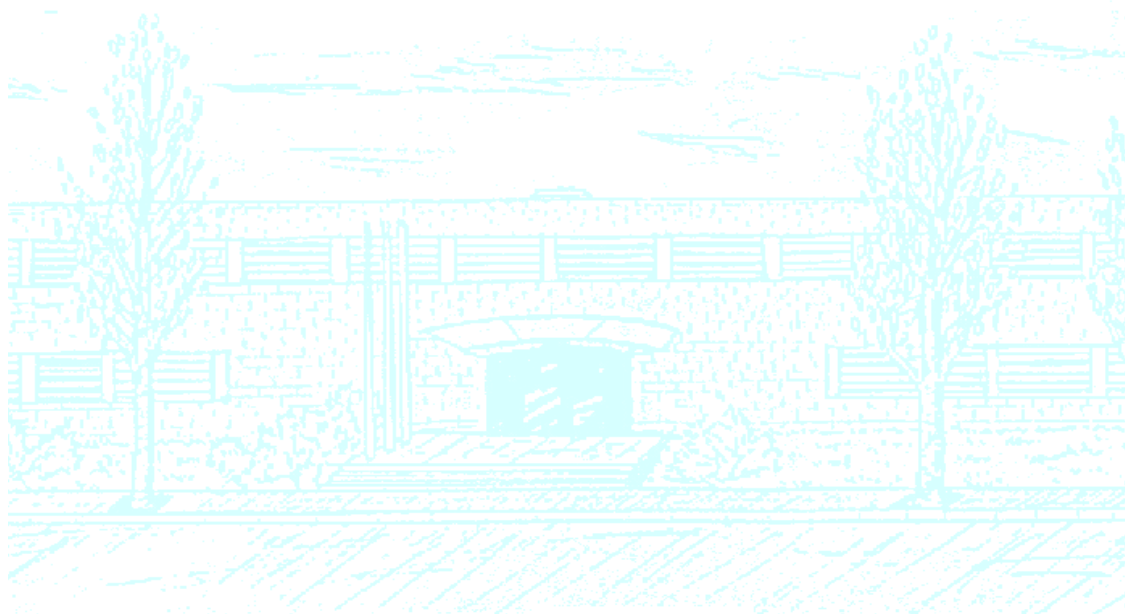
Title: Rainbow Matchings in Hypergraphs

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Rainbow Matchings in Hypergraphs

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A mis padres.

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Abstract

A rainbow matching in an edge colored hypergraph is a matching such that each pair of its edges have distinct colors. Brauldi, Ryser and Stein conjectured the existence of a partial $n - 1$ Latin transversal in an Latin square n -matrix. This problem can be translated to finding a rainbow matching in a complete bipartite graph. From this approach Aharoni and Berger introduced the problem of finding the minimum number of colors to ensure a t -rainbow matching in an r -partite r -uniform hypergraph.

The purpose of this master thesis is to give an overview of what has been done on this problems and apply the techniques given in this results for answering some new questions.

First of all, the results about rainbow matchings in bipartite graphs, one of them given by Erdős and Spencer using the Local Lovász Lemma.

Secondly, the study of matchings in r -partite r -uniform hypergraphs and the study of Ryser's conjecture, which is a generalization of Königs theorem. This conjecture has been proved only for the case $r = 3$ by Aharoni. This proof gives a relation between a matching in a 3-partite 3-uniform hypergraph with a rainbow matching in a bipartite graph.

Later, we continue with the study of rainbow matchings in r -partite r -uniform hypergraphs. In this part, we give the recent results in the area by Alon and by Glebov, Sudakov and Szabo.

Finally, using the techniques given by Erdős and Spencer with the Local Lovász Lemma, we answer some questions about rainbow matchings with edge colored complete r -partite r -uniform graphs, and about rainbow matchings in an edge colored bipartite graph with repeated edges, and the same question for a complete bipartite graph minus a matching. The three questions are motivated by the results described above.

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Introduction

Let us consider a certain firm, p workmen $X = \{x_1, x_2, \dots, x_p\}$ are available to fill p positions $Y = \{y_1, y_2, \dots, y_p\}$, each of the workmen being qualified for one or more of these jobs. Can each man be assigned to a position for which is qualified? If we draw a graph, with vertices $V = X \cup Y$ and edges $E = \{(x_i, y_j) : \text{if } x_i \text{ is qualified for the } y_j \text{ position}\}$, then the problem becomes one of matching in a bipartite graph with parts X and Y .

Suppose we have n men and n women and we wish to arrange n marriages. Let us supposed further that we wish to marry only men and women who are acquainted with each other. The Marriage Theorem states that this is possible if and only if for each k , $1 \leq k \leq n$, each set of k men collectively knows at least k women.

This Theorem was proved with one of the most famous theorems in bipartite matchings, the Theorem on Distinct Representatives by Philip Hall (1935). This Theorem was extended for hypergraphs by Aharoni and Haxell in 2003. A hypergraph is a generalization of a graph in which an edge may have more than two vertices.

Another famous theorem in bipartite matchings which is related with Hall's theorem is the König's Minimax Theorem, which states, in a bipartite graph, the cover number τ is equal to the matching number ν . Ryser wanted to extend this result and conjectured that, in an r -partite r -uniform hypergraph (an r -partite hypergraph where all its edges have

size r), we have $\tau \leq (r - 1)v$.

This result has been proven only for the case $r = 3$, by Aharoni, using the Hall's Theorem for hypergraphs in [4]. In this paper Aharoni gives a relation between a matching in 3-partite 3-uniform hypergraph with a "Rainbow matching" in bipartite graph, where a Rainbow matching can be described as a system of disjoint representatives of a family of matchings. A motivation for the study of rainbow matchings in bipartite graphs comes from Latin squares.

In the study of Latin squares, finding a (partial or full) Latin transversal is a subject of interest. Ryser conjectured that, for n odd, every $n \times n$ Latin square contains a full transversal. For even n this conjecture turns out false, so Brualdi and Stein independently conjectured that, in an $n \times n$ Latin square, there exists a partial transversal of size $n - 1$.

There have been several partial result about this conjecture. Erdős and Spencer proved using the probabilistic method with the Local Lovasz Lemma that if each symbol in an $n \times n$ Latin square appears at most $\frac{n-1}{4e}$ times then there is a full transversal.

In fact, it is known that every Latin square has a partial transversal of size $n - o(n)$, proved by Woolbright in [24] and independently Brower, de Vries and Wieringa [10] proved that for every Latin square there is a partial transversal of size $n - \sqrt{n}$. The best bound known was given by Hatami and Schor in [15], who proved that every Latin square contains a partial transversal of size $n - O(\log^2 n)$.

Now, if we take a complete bipartite graph such that one side of the partition represents the rows of an $n \times n$ Latin square, and the other side represents the columns. The edge (u, v) represents the cell $c_{u,v}$ of the Latin square. Finally, we color the edges of our bipartite graph in such a way that the edge (u, v) is colored by the symbol that is in the cell $c_{u,v}$. Then, by construction of the bipartite graph and definition of Latin square, each set of edges with the same symbol is a matching. So, if we get a rainbow matching of this family of matchings in our graph, then the Latin square will have a Latin transversal. Due to this and the fact that the conjecture for Latin squares have not been proven or disproven, finding rainbow matchings in bipartite graphs becomes a subject of interest.

Having in mind this duality, Aharoni and Berger made a generalization of the Latin square conjecture, which says that if a bipartite graph consisting of n matchings, each of at least size $n + 1$, then there is a rainbow matching with size n . We have to note that in this conjecture we are not considering disjoint matchings and their union may produce multigraphs rather than simple graphs.

Barát, Gyárfás and Sárközy [9], observed that the arguments that Woolbright, Brower, de Vries and Wierninga used to find a partial transversal of size $n - \sqrt{n}$ in a Latin square generalize to bipartite graphs. So, the best known bound at the moment is that a bipartite graph consisting of n matchings, each with at least n edges, has a rainbow matching of size $n - \sqrt{n}$.

There have been several ways to approximate to this conjecture, a natural one is by increasing the size of the matchings and getting a rainbow matching of size n . For the case that each matching has size at least $2n$. However, the conjecture becomes trivial. In this direction there have been some important improvements; Aharoni, Charbit and Howard in [3] proved that matchings of size $\frac{7n}{4}$ are sufficient to guarantee a rainbow matching of size n . Kotlar and Ziv [16] improved it with matchings of size $\frac{5n}{3}$. The best bound known for large size of the matchings was given by Clemens and Ehrenmüller [11] with matchings of size $\frac{3n}{2}$.

On the other hand, there have been other research lines on rainbow matchings, not only for bipartite graphs, but for r -partite hypergraphs. Aharoni and Berger [2] asked for the number of matchings of size n that guarantee a rainbow matching in an r -partite r -uniform graph. They defined $f(r, t)$ as the maximal size of a family of matchings of size t in an r -partite r -uniform hypergraph without a rainbow matching of size t . The case of a rainbow matching in a bipartite graph is $f(2, t)$. Aharoni and Berger in the same paper conjectured that for all $r > 1$ and t , one has $f(r, t) = 2^{r-1}(t - 1)$.

Aharoni and Berger proved their conjecture for the case of bipartite graphs, namely $f(2, t) = 2(t - 1)$. Later, Alon [6] disproved this conjecture for r large enough by showing that $f(r, 3) > 2.216^r$. He also proved with a probabilistic construction that for r large and all t , $f(r, t) > 2.71^r$.

Alon went further and defined $F(r, t)$ as the largest value of f such that there exists an r -uniform hypergraph (not necessarily r -partite) edge-colored with f colors, where each color is a matching of size t without a matching of size t . In addition, he proved that $f(r, t) \leq F(r, t) \leq \frac{t^{rt}(t-1)}{t!}$, where the upper bound is super exponential in t for fixed r . Glebov, Sudakov and Szabó [14] improved this upper bound to one which is polynomial in t . For r -partite r -uniform hypergraphs, they proved that $f(r, t) < (r + 1)^{r+1}(t - 1)t^{2r}$. For r -uniform hypergraphs in general, they obtain $F(r, t) < 8^{rt}$.

The aim of this work is to give an overview of rainbow matchings, giving the theorems that caught our attention, not only because of the results they state, but because of the techniques they introduce. Studying this theorems and techniques we tried to generalize some results using the Local Lemma.

In the first chapter, we give the basic notions of graph theory and define rainbow matchings in bipartite graphs. In the same chapter we define a Latin square and Latin transversal including the proof of Erdős and Spencer for Latin transversals in a large class of square matrices. We conclude there by giving the relation between a Latin square and an edge-colored bipartite graph, and between Latin transversals with a rainbow matching in an edge-colored bipartite graph. At the end of this chapter we include some results about Latin transversals in two particular cases.

In the second chapter we define hypergraphs, r -uniform hypergraphs, and r -partite r -uniform hypergraphs. We also introduce some notions of topology in order to give the proof of the Hall's theorem for hypergraphs given by Aharoni and Haxel. In addition, we include the proof by Aharoni of the Ryser's conjecture for the case $r = 3$. We conclude this chapter by giving the relation between a rainbow matching in a bipartite graph with a matching in an 3-partite 3-uniform hypergraph.

In the third chapter we define a rainbow matching in an r -partite r -uniform hypergraph. Here, we give the Aharoni-Berger's Theorem about $f(2, t)$. We continue with the results given by Alon mentioned above and conclude with the theorem given by Glebov, Sudakov and Szabó about

$F(r, t)$.

Finally, in the fourth chapter, we give some new results that generalize some of the results on rainbow matchings.

After studying Erdős-Spencer Theorem, we would like to know if we can do the same for hypergraphs. This problem is connected to the existence of rainbow matchings in hypergraphs, where now the constrain is on the size of each color class instead of the number of colors. Our first result is the case for 3-partite 3-uniform hypergraphs, which states:

Theorem 1. *Let G be a complete 3-partite 3-graph with n vertices in each part of the partition. For every edge coloring of G such that each color appears less than $k = \frac{(n-1)^2}{6e}$ times, there is a rainbow matching.*

From the proof of this Theorem we noticed that this result can be generalized for r -partite r -uniform hypergraphs. Another thing that is important to notice, is that this Theorem not only give a bound for the size of matchings in order to have a rainbow matching, but it does it for any coloring in its edges.

The motivation of the second comes from the fact that most of the examples for not having a rainbow matching are those who have matchings not edge disjoint. So, we would like to know what happen when we have a set of matchings which are not edge disjoint. So, we asked ourself, if we color a complete r -partite r -graph and each edge has multiplicity m , When can we ensure a rainbow matching?. It seems logic that the size of the colors can be greater than for a simple graph. We answered this question for $m = 2$.

Theorem 2. *Let G be a complete edge colored bipartite multigraph with n vertices in fact stable set such that each edge has multiplicity 2. Suppose that each color appears at most $k \leq \frac{n-1}{2e}$. Then G has a rainbow matching.*

Having this Theorem in mind and studying the proof, this result can be generalized for any m . Actually, we are convinced that we can do something else about complete colored bipartite multigraphs such that the edges have multiplicity at most m_1 and at least m_0 .

There are several results for edge colored complete bipartite graphs and multigraphs about having a rainbow matching, What if now we remove some edges?. We get a result for the case when we have a complete bipartite graph without a perfect matching.

Theorem 3. *Let G be $K_{n,n} - M$ be edge colored, where M is a perfect matching in $K_{n,n}$, and $n \geq 20$. If each color is of size less than $k = \frac{(n-3)(n-6)}{4e(n-1)} > 1$ then G has a rainbow matching.*

Since we used the Local Lóvasz Lemma for the proofs of all our new results, we always need some kind of symmetry in our graphs, from this fact, in the last case we would like to know how many matchings or edges we still can remove in order that this bipartite graph have the necessary symmetry and say something about having a rainbow matching in it.

Chapter 1

Rainbow matchings in bipartite graphs.

In graph theory the topic about matchings in Bipartite graphs has been well studied, but what about having a bunch of matchings in a Bipartite graph an getting a matching with one edge from each matching?

In this section we will give the basic notions and definitions about this topic which is called Rainbow Matchings.

1.1 Definitions

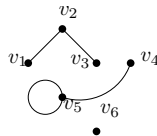


Figure 1.1: This is a graph with $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{(v_1, v_2), (v_2, v_3), (v_5, v_5), (v_5, v_4)\}$.

A *graph* is a pair of sets $G = (V, E)$, where $V = V(G)$ is a set of *vertices* and $E = E(G)$ is a multiset of *edges*, formed by pairs of vertices. Graphs loop-less and without multiple edges are called *simple*. In this work, unless explicit stated, when we talk about graphs, we will mean simple graphs. If we have an edge $a = (u, v)$ where u and v are vertices, we say that a is *incident* with u and v ; we also say that u and v are *adjacent* or *neighbors* and that a is *incident* to u and v . The *neighborhood* of $S \subseteq V$, denoted as $N_G(S)$, is the set of vertices adjacent to the vertices of S . The *degree* of u is the number of edges incident to u , denoted as $d_G(u) = |N_G(u)|$. If two edges in E have a common vertex, then we also say that they are *adjacent*. We define the maximum degree of G as the maximum degree among all its vertices and we denote it by $\Delta(G)$. We say that two edges are *adjacent* if they share a vertex.

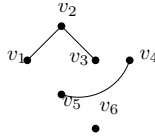


Figure 1.2: Example of a *simple* graph.

We say that G is *bipartite* if there exist a partition of V into two sets A and B called *parts*, such that every edge is incident to one vertex in A and with one in B .

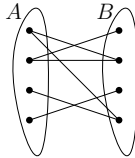


Figure 1.3: Example of a *bipartite* graph.

A *matching* of a graph G is a subset M of edges such that no pair of edges are adjacent. A *path* in G is a sequence of vertices u_1, u_2, \dots, u_n such that the edge (u_i, u_{i+1}) is in G for all $i \in \{1, \dots, n-1\}$.

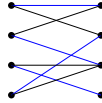


Figure 1.4: The set of blue edges is a matching of G .

A *proper edge coloring* (in this work we will simply say *edge coloring*) of a graph is an assignment of “colors” to the edges of the graph such that no two adjacent edges have the same color. If G is an edge colored graph then, by definition of matching and edge coloring, each color represents a matching of G . A *rainbow matching* of an edge-colored graph is a matching whose edges have distinct colors. In other words, each edge belongs to a different matching.

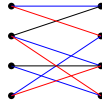


Figure 1.5: This is an edge colored graph, and each color represents a matching.

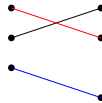


Figure 1.6: This is a rainbow matching of the above example.

1.2 Existence of rainbow matchings and Latin transversals.

Imagine that we have a collection of “bad” events A_1, \dots, A_m , we would like to know when there is some point in our probability space for which none of the “bad” events occurs. In other words, we need to prove that the probability of the event in which none of these “bad” events happen is positive.

So, in the case when the “bad” events are pairwise independent and the probability of each one is at most p then,

$$\mathbb{P}\left[\bigwedge_{i \in [m]} \overline{A_i}\right] \geq (1 - p)^m,$$

in which case is positive if $p < 1$.

But when there are some dependent events, the calculation can be difficult. Lovász gave a powerful tool that “extends” in some way the result above, this is the Lopsided Lovász Local Lemma.

Lemma 1. [7] [13] *Consider a set \mathcal{E} of (typically bad) events such that for each $A \in \mathcal{E}$, there is a set $D(A)$ of at most d other events, such that for all $S \subset \mathcal{E} - (A \cup D(A))$ we have that*

$$\mathbb{P}[A \mid \bigwedge_{A_j \in S} \overline{A_j}] \leq p.$$

If $ep(d + 1) \leq 1$ with $e \approx 2.718$, then with positive probability, none the events in \mathcal{E} occur.

In other words, if each event A_i is independent with at most $m - d$ events, $\mathbb{P}[A_j] \leq p$ and $ep(d + 1) \leq 1$, then $\mathbb{P}[\bigwedge_{j=1}^m \overline{A_j}] > 0$.

The next Theorem is an application of Local’s Lemma proved by Erdős and Spencer, which will be important for this work.

Let $A = (a_{ij})$ be an $n \times n$ matrix with integer entries. A permutation π is called a *Latin transversal* (of A) if the entries $a_{i\pi(i)}$ are all different, with $i \in \{1, \dots, n\}$.

Theorem 4. [13]

Suppose $k \leq (n - 1)/(4e)$ and suppose that no integer appears in more than k entries of A . Then A has a Latin transversal.

Proof. Denote by T the set of all ordered four-tuples (i, j, i', j') satisfying $i < i', j \neq j'$ and $a_{ij} = a_{i'j'}$. Let π be a random permutation of $\{1, 2, \dots, n\}$, choose it according to a uniform distribution among all the $n!$ possible permutations.

For each $(i, j, i', j') \in T$ we denote the event $A_{ijj'i'}$ so that $\pi(i) = j$ and $\pi(i') = j'$. We would like to show that the probability that none of these exists is positive. This way there is a Latin transversal in A .

Let us define the symmetric dependency graph $G = (T, E)$, such that (i, j, i', j') is adjacent to (p, q, p', q') if and only if $\{i, j\} \cap \{p, p'\} \neq \emptyset$ or $\{j, j'\} \cap \{q, q'\} \neq \emptyset$. For a given four-tuple (i, j, i', j') there are at most $4n$ choices of (s, t) with either $s \in \{i, i'\}$ or $t \in \{j, j'\}$, and for each of these choices each integer appears less than k times in A , hence $\Delta(G) < 4nk$.

Let S be any set of members of T that are non adjacent in G to (i, j, i', j') . By symmetry, we may assume that $i = j = 1$ and $i' = j' = 2$. Then no entry in $(p, q, p', q') \in S$ is 1 nor 2. We say that π is a *good permutation* if π satisfies that $\bigwedge_S \overline{A_{ppq'q'}}$.

Let S_{ij} denote the set of all good permutations π satisfying $\pi(1) = i$ and $\pi(2) = j$.

Claim 1) $|S_{12}| \leq |S_{ij}|$ for all $i \neq j$.

Indeed, first assume that $i, j > 2$. For each $\pi \in S_{12}$ we define the permutation π^* as follows. Suppose $\pi(x) = i$ and $\pi(y) = j$, then define $\pi^*(1) = i$, $\pi^*(2) = j$, $\pi^*(x) = 1$, $\pi^*(y) = 2$ and $\pi^*(t) = \pi(t)$ for all $t \neq 1, 2, y, x$. This clearly defines an injective function from S_{12} to S_{ij} . Analogous we can prove that $|S_{12}| \leq |S_{ij}|$ with $\{1, 2\} \cap \{i, j\} \neq \emptyset$, for instance, if $i = 1$ or $j = 2$ the function is the same as above, but if $i = 2$ or $j = 1$, without loss of generality assume that $j = 1$ and $\pi(x) = i$ with $x \neq 2$ (with $x = 2$) we define $\pi^*(1) = i$, $\pi^*(2) = 1$, $\pi^*(x) = 2$ ($\pi^*(1) = 2$, $\pi^*(2) = 1$) and $\pi^*(w) = \pi(w)$ for all $w \neq 1, 2, x$. We can do exactly the same for the case $i = 2$ and the function still injective. Therefore $|S_{12}| \leq |S_{ij}|$.

Hence,

$$\mathbb{P}[A_{1122} | \bigwedge_S \overline{A_{pp'q'}}] = \frac{|S_{12}|}{|S|} = \frac{|S_{12}|}{\sum_{i \neq j} |S_{ij}|} \leq \frac{|S_{12}|}{\sum_{i \neq j} |S_{12}|} = \frac{1}{n(n-1)} = \mathbb{P}[A_{1122}],$$

for all $i \neq j$.

By symmetry we get that

$$Pr[A_{ijj'j'} | \bigwedge_S \overline{A_{pp'q'}}] \leq \frac{1}{n(n-1)},$$

for all $i, j, i', j' \in T$.

Since $e4nk(\frac{1}{n(n-1)}) \leq 1$ and $\Delta(G) < 4nk$, the Lopsided Local's Lemma completes gives $\mathbb{P}[\bigwedge_T \overline{A_{ijj'j'}}] > 0$ and there is a transversal as claimed. \square

A matrix is called *Latin* if each symbol appears at most once in each row and column. A *partial transversal* in a Latin matrix $m \times n$ is a set of entries, each in a different row and in a different column, and each containing a different symbol. We call it *full transversal* or simply *transversal* if it is of size $\min(m, n)$.

Notice that a transversal is the same as we defined above but we did not ask for the matrix to be Latin.

The matrix $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ is a Latin matrix and its transversal is a latin transversal.

One can ask if all the Latin square have a full transversal just like Ryser [22] conjecture it but just for n odd.

Conjecture 1. *In a Latin square of order n , with n odd, there exists a full transversal.*

Can we conjecture the same for n even? Just notice that the 2×2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is latin. But does not have a latin transversal, but it has a partial transversal of length $n - 1 = 1$.

If we start to do examples of $n \times n$ matrices with n small, we can find out that it has at least a partial transversal of length $n - 1$. With this observation Brualdi and Stein independently gave a famous conjecture which makes Latin Transversal a topic of interest:

Conjecture 2. *In and $n \times n$ Latin matrix there exists a partial transversal of size $n - 1$.*

The conjecture is still open, although as we said before one can do small examples and find the transversal or the partial transversal. There have been several results about it, the best current was proved by Hatami and Schor [15] which says that every Latin square has a partial transversal of size $n - O(\log^2 n)$.

Now one can ask: “What is the relation of this with rainbow matchings?”, well the answer is simple:

We can build a complete bipartite graph that can represent a Latin matrix where the vertices of each part of the partition represents columns and rows respectively where each edge represents an entry and we colored such a way that each color represents one symbol of the Latin matrix. Notice that each color will represent a matching, so if we find a rainbow matching then the matrix will have a Latin transversal and viceversa.

In other words the Brualdi-Stein conjecture can be rewritten as:

Conjecture 3. *The complete bipartite graph $K_{n,n}$ with n edge disjoint matchings of size n has a rainbow matching of size $n - 1$.*

In addition, the Edös-Spencer theorem tells that a complete bipartite graph $K_{n,n}$ colored with edge disjoint matchings of size $k \leq (n - 1)/4e$, each has a rainbow matching.

1.3 Some results of Rainbow Matchings in graphs

There have been several cases solved of Ryser's conjecture for specific Latin squares. In this section we give two of them which caught our attention; one in Additive Latin squares and the other with random colorings in graphs.

Let G be an Abelian group (recall that an *Abelian* group is a commutative group) and two subsets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ of G , we construct the $k \times k$ matrix $L(A, B)$ in such a way that $L_{i,j} = a_i + b_j$. Notice that this matrix does not repeat entries in each row and column. In particular, if $A = B = G$ then $L(A, B)$ is a Latin square. Snevily conjectured in [23] that if G is a finite Abelian group with odd order, then $L(A, B)$ has a full transversal, which is the same state as Ryser's conjecture but for a specific Latin square.

The first result about this problem was given by Alon [5], although He did not solve it for all finite Abelian group with odd order, He proved it for Abelian groups with odd prime order, which is a close approach. This paper not only got close to the solution of this problem but gave a powerful method for solving combinatorial problems (this case in particular) with the polynomial method using the Combinatorial Nullstellensatz theorem given there. This technique was use later by Dasgupta, Károlyi, Szegedy and Serra [12] in order to prove the conjecture for all the cyclic groups of odd order, getting closer to the solution of this conjecture. In 2011 the conjecture was totally solved by Arsovski.

Theorem 5. [8]

Let G be a finite Abelian group of odd order. For any two subsets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ of G , the addition table $L = L(A, B)$ which has entries $L_{i,j} = a_i + b_j$, with $i, j \in \{1, \dots, k\}$, has a full transversal.

Another result about a Ryser's conjecture case is about random models for Latin squares.

As we said in the last section, a Latin square can be seen as an edge-colored complete Bipartite graph $K_{n,n}$ with n colors. Since there is not a good random model for proper n edge-coloring of $K_{n,n}$ (Latin squares), Perarnau and Serra, inspired by the configuration model used to produce random regular graphs, proposed in [20] one that gets close to it, in which the idea is to use all the colors the same number of times, but the edge-coloring is not necessarily of size n .

Let $C_u(n, s)$ be a Uniform random model such that each edge gets one of the $s = kn$ colors independently and uniformly at random, with $k \geq 10.93$.

We say that a coloring is *equitable* if each color class has the same size. The Regular random model, $C_r(n, s)$, we choose an edge-coloring uniformly at random among all the equitable edge-colorings.

We say that a property holds *with high probability (whp)* in $C_u(n, s)$ (or $C_r(n, s)$) if the probability that the property is satisfied by an edge-coloring chosen uniformly at random from $C_u(n, s)$ ($C_r(n, s)$), tends to one as $n \rightarrow +\infty$.

The result given by Serra and Perarnau in [20] is the following:

Theorem 6. [20]

Every edge-coloring of $K_{n,n}$ in the $C_u(n, s)$ model ($s \geq n$) contains a rainbow matching whp.

This result can be proven in an analogous for $C_r(n, s)$. There is no good model for random Latin squares, or equivalently, proper edge coloring of $K_{n,n}$. The proportion of the latter among all edge colorings of $K_{n,n}$ is exponentially small, so that Theorem 6 does not tell much about an asymptotic version of Ryser's conjecture. Nevertheless, it supplies a clue that the conjecture might be true, at least asymptotically.

Chapter 2

Matchings in hypergraphs.

2.1 Basic definitions.

In this section we will give the basic notions of hypergraphs and matching in order to start the theory.

A *hypergraph* is a set E of subsets, called edges, of some ground set V whose elements are called vertices. We say that a hypergraph is an *r -graph* or *r -uniform* if all its edges are of the same size r . Notice that in the case that G is a 2-graph, then G is simply a graph.

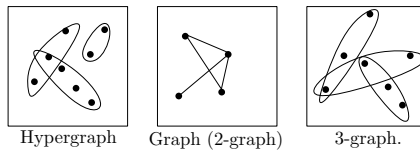


Figure 2.1: Examples of hypergraphs.

An r -graph is called r -partite if its vertex set $V(H)$ can be partitioned into r sets V_1, \dots, V_r (called the "sides" or "parts" of the hypergraph) in such a way that each edge meets each V_i in at most one vertex.

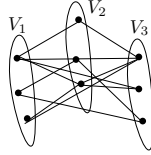


Figure 2.2: Example of 3-partite 3-graphs, here each triangle represents an edge.

A *matching* in a hypergraph is a set of disjoint edges.

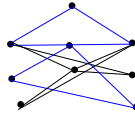


Figure 2.3: The blue edges represent one matching. Here $v(H) = 2$.

The *matching number*, $v(H)$, of a hypergraph is the maximal size of a matching in H . In the last example it is 2.

We will say that a set K of edges *pins* another set F of edges if every edge in F is met by some edge from K .

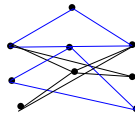


Figure 2.4: The set of blue edges pins the set of black ones.

The *matching width* of H , denoted by $mw(H)$, is the maximum over

all matchings M in H of the minimal size of a set of edges from H pinning M .

A *cover* of a hypergraph H is a subset of $V(H)$ meeting all edges of H . The *covering number*, $\tau(H)$, of H is the minimal size of a cover of H .

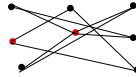


Figure 2.5: The red vertices represent a cover of this hypergraphs, in this case $\tau = 2$.

2.2 Hall's theorem for hypergraphs.

The aim of this section is the extension of Hall's theorem for hypergraphs, which gives a really nice proof that uses Sperner's lemma.

Let $\mathcal{A} = \{H_1, \dots, H_m\}$ be a family of hypergraphs. A *system of disjoint representatives* (abbreviated "SDR") for \mathcal{A} is a function $f : \mathcal{A} \rightarrow \cup_{i=1}^m H_i$ such that $f(H_i) \in E(H_i)$ for all i and $f(H_i) \cap f(H_j) = \emptyset$ whenever $i \neq j$. In other words, a SDR of a family of hypergraphs is a hypergraph such that each edge represent a unique hypergraph of \mathcal{A} and all its edges are pairwise disjoint.

In the figure 2.6 we can see an example of SDR.

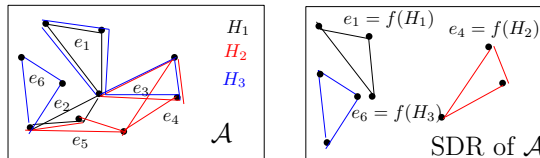


Figure 2.6: Each color represent different hypergraphs of the family \mathcal{A} .

Notice that this definition is an extension of the classic SDR of a family of sets, since a set can be seen as an r -graph with $r = 1$, i.e., the vertices are the elements of the set and an edge is also a vertex so, here the vertices (elements) of the SDR are different, but since each one is just a vertex, then they are pairwise disjoint. Indeed, a hypergraph is a set of edges, which are subsets of V , and since we are asking that the elements should be pairwise disjoint, then they must be different. Further more, if the family \mathcal{A} of hypergraphs are matchings of a certain hypergraph and each has a different color, then the SDR is a rainbow matching of the original hypergraph.

For the next definitions we need a notion of topology, but since we will use it only for the proof of one theorem we will not give proofs of the basic statements.

We will denote by Δ_n the n -dimensional simplex. We recall that a n -simplex is the n -dimensional polytope with $n + 1$ vertices. In other words, a simplex is the generalization of a triangle in every dimension, for example in dimension 3, Δ_3 is a tetrahedron (a pyramid where all its faces are triangles).

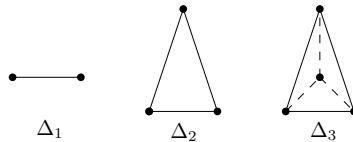


Figure 2.7: Example of simplexes in dimension 1, 2 and 3.

A k -face, or just *face*, of a n -dimensional polytope P are the elements of P with dimension $k \leq n$. For instance, the faces of Δ_2 are the vertices, the edges and the triangle itself. A *simplicial complex* is a set C of simplexes, satisfying the properties: (a) if $\sigma \in C$ then every face of σ is in C , and (b) the intersection of any two simplexes in C is a member of C .

The *support*, $\text{supp}(x)$, of a point x in Δ_k is the unique face of Δ_k containing x in its relative interior. For instance if x is a vertex of Δ_k then

the $supp(x)$ is x itself. More examples are illustrated in the Figure 2.8.

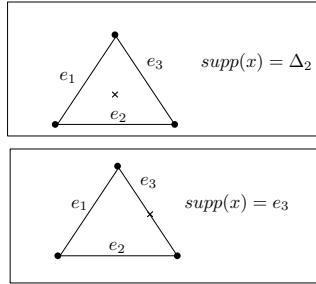


Figure 2.8: Example of $supp(x)$.

A *triangulation* of a topological space X is a simplicial complex satisfying the property that the relative interiors of its simplices partition the space. A triangulation T of Δ_k is called *hierarchic* if for any two points which are connected in the one-dimensional skeleton of T the support of one is contained in the support of the other one. For instance, Figure 2.9 shows a triangulation in which x_2 is connected to x_1, x_3 and x_7 ; the support of x_2 is the edge (x_1, x_3) , $supp(x_1) = x_1$, $supp(x_3) = x_3$ and $supp(x_7) = \Delta_2$, and we have that $supp(x_1) \subset supp(x_2)$, $supp(x_3) \subset supp(x_2)$ and $supp(x_2) \subset supp(x_7)$.

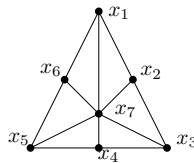


Figure 2.9: Example of a hierarchic triangulation of Δ_2 .

A triangulation T is called *economically hierarchic* if, for each point x of T , the neighbors of x in the one-dimensional skeleton of T on the boundary of $supp(x)$ form a simplex (possibly empty) in T .

The next Lemma about economically hierarchic triangulation in simplices was given by Aharoni and Haxell in [4]. Since the proof uses topology we will not prove it.

Lemma 2. [4] *The k -dimensional simplex Δ_k has an economically hierarchic triangulation.*

Sperner's lemma is a very famous result in topology. It states that every "Sperner coloring" of a triangulation of Δ_n contains a simplex with the $n + 1$ colors.

Lemma 3. *Let T be a triangulation of Δ_n , and let \mathcal{X} be a coloring of the points of T by $n + 1$ colors, which satisfies the following conditions:*

- (1) *Each vertex of Δ_n is colored by different color.*
- (2) *The points of T on a face τ of Δ_n are colored by the colors of the vertices of τ .*

Then there exist a simplex in the triangulation whose vertices receive all $n + 1$ colors.

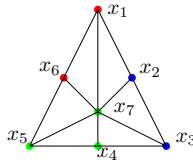


Figure 2.10: Example of a Sperner coloring in a triangulation of Δ_2 .

Sperner's lemma is of our interest since Aharoni and Haxell, in order to prove a generalization Hall's theorem for hypergraphs, had the clever idea to use Sperner's lemma, in such a way that they construct an auxiliary triangulation of a simplex Δ_{n-1} in which each point of the triangulation represent an edge of the hypergraph and the simplex with the n colors will represent a matching in the hypergraph. Hence, if the construction of a coloring in this triangulation satisfies the condition of the Sperner's

lemma then, the simplex in the triangulation whose vertices receive all n colors represents a rainbow matching (SDR).

Theorem 7. [4] *Let \mathcal{A} be a family of hypergraphs. If for every subfamily \mathcal{B} of \mathcal{A} there exists a matching $M_{\mathcal{B}}$ in $\cup\mathcal{B}$, which cannot be pinned by less than $|\mathcal{B}|$ disjoint edges from $\cup\mathcal{B}$, then there exists a SDR for \mathcal{A} .*

Proof. Let $|\mathcal{A}| = m$, and write $\mathcal{A} = (H_1, \dots, H_m)$. We assume that the members of \mathcal{A} are edge disjoint. One way to see this, is by adding to each edge a different element of the ground set (this cannot appear in any other edge) for each occurrence of the edge in a hypergraph H_i , i.e., we are “earmarking” the edge in such a way that we distinguish when it belongs to H_i . The matchings are the same but distinguishing if each edge belongs to a hypergraph or another, and they have to be pinned by the same number of edges since we do not remove vertices but add new ones in each edge. Hence this “earmarking” does not affect the assumption or the conclusion of the theorem.

Now, let T be a hierarchic triangulation of the simplex $\Delta_{|\mathcal{A}|-1} = \Delta_{m-1}$ which has m vertices. We will color the points of T with a Spencer coloring such that each color will represent an edge from $\cup\mathcal{A}$. This will be done by these two rules:

(a) At points of T lying in the interior of the face $F_{\mathcal{B}}$ of Δ_{m-1} spanned by the points $\{v_i : H_i \in \mathcal{B}\}$, only edges from $M_{\mathcal{B}}$ are placed.

(b) Any two adjacently placed edges are either identical or disjoint.

In order to construct and verify that (a) and (b) hold, we will proceed by induction on the dimension of the face:

($n = 1$) At each vertex v_i of Δ_{m-1} place an edge $e \in M_{\{H_i\}}$.

(I.H.) Let now $1 \leq k \leq m$ and assume that we have placed edges at all points of T lying on all faces of dimension less than k of Δ_{m-1} such that (a) and (b) hold.

Let v be any point in the interior of a face $F_{\mathcal{B}}$ of dimension k , where \mathcal{B} is a subfamily of \mathcal{A} with size $k + 1$. Since T is economically hierarchic, the points on the boundary of $F_{\mathcal{B}}$ connected to v form a (possibly empty) simplex τ of size less than k . Let K be the set of edges placed at the

vertices of τ . By (I.H.) τ satisfies (b) and so K consists of disjoint edges, since the size of τ is at most $k - 1$ then $|K|$ is at most k . By the condition on $M_{\mathcal{B}}$, the set K does not pin it. So, there exists an edge $e \in M_{\mathcal{B}}$ which is not met by any of the edges in K . Place e at v , this clearly satisfies (a).

Let us see that this way of placing edges from $M_{\mathcal{B}}$ on the points of T in $F_{\mathcal{B}}$ satisfies (b).

Let u and v be two adjacent points. Since T is economically hierarchic, then $\text{supp}(v) \subseteq \text{supp}(u)$ or $\text{supp}(u) \subseteq \text{supp}(v)$. Without loss of generality assume that $\text{supp}(v) \subseteq \text{supp}(u)$.

If both u and v belong on the boundary of $F_{\mathcal{B}}$ then (b) follows from (I.H.), so let us assume that $\text{supp}(u) = F_{\mathcal{B}}$.

i) If $\text{supp}(v) = F_{\mathcal{B}}$, then the edges placed at u and v belong to the matching $M_{\mathcal{B}}$, then (b) follows.

ii) If v lies on the boundary of $F_{\mathcal{B}}$, then by the way we placed the points in the relative interior of $F_{\mathcal{B}}$, in particular u , u and v are disjoint and this yields (b).

Having placed edges at all points of T , we now color each point v in T by that color i for which the edge placed at v belongs to H_i .

By rule (a), this is a legal coloring for Sperner's lemma. Hence, by Sperner's lemma, there exists a simplex σ whose vertices are colored by all colors $1, \dots, m$.

Since the edges placed at the vertices of σ belong to distinct H_i 's, there are not identical, and hence, by (b), they are disjoint. Thus they form a SDR of \mathcal{A}

□

From this Theorem the next result is immediate.

Corollary 1. *Let \mathcal{A} be a family of n -graphs. If for every $\mathcal{B} \subseteq \mathcal{A}$ there exists a matching in $\cup \mathcal{B}$ of size greater than $n(|\mathcal{B}| - 1)$, then there exists a SDR for \mathcal{A} .*

By this corollary the condition is sufficient. If we add to this condition that $M_{\mathcal{B}}$ cannot be pinned by fewer than $|\mathcal{B}|$ edges from the matchings M_C where $C \subseteq \mathcal{B}$ makes then, we obtain a necessary and sufficient condition

for the existence of a SDR in a family \mathcal{A} . Why is this? Well, first of all, by Corollary 1 this still has a SDR. But, What if \mathcal{A} has a SDR? Then we simply take as $M_{\mathcal{B}}$ the edges of the SDR that represent the graphs in \mathcal{B} for all $\mathcal{B} \subseteq \mathcal{A}$, and this matchings satisfy that cannot be pinned by fewer than $|\mathcal{B}|$ edges from the matchings M_C where $C \subseteq \mathcal{B}$. This gives us a necessary condition for having a SDR.

Theorem 8. *\mathcal{A} has a SDR if and only if there exists an assignment of a matching $M_{\mathcal{B}} \subseteq \cup \mathcal{B}$ to each subfamily \mathcal{B} of \mathcal{A} , satisfying that $M_{\mathcal{B}}$ cannot be pinned by fewer than $|\mathcal{B}|$ edges from $\cup \{M_C : C \subseteq \mathcal{B}\}$.*

2.3 Matchings for 3-partite 3-graphs.

Recall that τ is the covering number of H and ν the matching number.

Notice that $\tau \geq \nu$ since, if there are ν disjoint edges then, we need at least one vertex for each edge in order to meet them.

If we have an r -uniform hypergraph, we can also notice that $\tau \leq r\nu$, since the union of the edges of a maximal matching forms a cover.

Thus, in particular for a bipartite graph we have that $\tau = \nu$ which is König's theorem.

Theorem 9. *For bipartite graphs $\tau = \nu$.*

Ryser's conjecture is a generalization of the Theorem above for r -uniform hypergraphs:

Conjecture 4. *In an r -partite r -graph, $\tau \leq (r - 1)\nu$.*

In case that the conjecture is true, then it is sharp. This is given by the truncated projective plane, i.e., a projective plane from which a vertex is deleted together with all edges incident with it (take a look to figure 2.11). Here the sides are the sets of vertices which together with the deleted vertex form an edge. The matching number for this hypergraph is 1, but τ is the number of the size of the sides (1 less than the size of the edges in the projective plane).

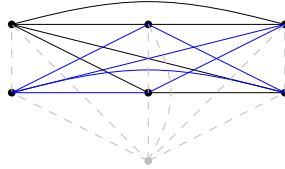


Figure 2.11: The gray part is the deleted edges and vertex, each color represent the edges containing a vertex in the first side.

In this section, we give the proof by Aharoni in 1999 [1] of the case $r = 3$ of Ryser's conjecture, and this also will give an approach to the latin squares conjecture. The proof makes use of Hall's Theorem discussed in the previous section.

Lemma 4. *In an r -graph H , $v(H) \leq r(mw(H))$.*

This Lemma comes from the fact that each matching in H is pinned by $mw(H)$ edges, containing together at most $r(mw(H))$ vertices.

We call *deficiency* $def(\mathcal{A})$ of a family of n graphs \mathcal{A} to the minimal natural number d such that $mw(\cup \mathcal{B}) \geq |\mathcal{B} - d|$ for every subfamily \mathcal{B} of \mathcal{A} .

Theorem 10. *Every family \mathcal{A} of hypergraphs has a partial SDR of size at least $|\mathcal{A}| - def(\mathcal{A})$.*

That is, there exist a subfamily \mathcal{D} of size at most $def(\mathcal{A})$ of \mathcal{A} such that $\mathcal{A} \setminus \mathcal{D}$ has a SDR.

Proof. Let $d = def(\mathcal{A})$ and let v_1, \dots, v_d be new vertices which do not belong to $V(\cup \mathcal{A})$. Add to each $H \in \mathcal{A}$ all singletons, namely replace it by the hypergraph $H' = H \cup \{\{v_1\}, \dots, \{v_d\}\}$ where these are the new edges: $E(H') = E(H) \cup \{\{v_1\}, \dots, \{v_d\}\}$.

Since a singleton can only be pinned by itself, the matching width increases by d for all subfamily of \mathcal{A} . Then we have that the family $\mathcal{A}' = \{H' : H \in \mathcal{A}\}$ satisfies that, for each subfamily \mathcal{B}' of \mathcal{A}' , $mw(\cup \mathcal{B}') \geq |\mathcal{B}'|$, since for each subfamily \mathcal{B} of \mathcal{A} $mw(\mathcal{B}) \geq |\mathcal{B}| - d$.

Thus, by Theorem 7, \mathcal{A}' has a SDR f' .

Let \mathcal{D} be the subfamily of \mathcal{A} such that if $H \in \mathcal{D}$, then $f'(H') = v_i$ for some $i \in \{1, \dots, d\}$. Notice that $|\mathcal{D}| \leq d$.

Let $f : \mathcal{A} \setminus \mathcal{D} \rightarrow \cup_{i=1}^m H_i$ such that $f(H) = f'(H')$, this function yields the desired partial SDR of \mathcal{A} . □

The next theorem proves Ryser's conjecture for $r = 3$.

Theorem 11. [1] *In a 3-partite 3-graph $\tau \leq 2v$.*

Proof. Let Γ be a 3-partite 3-graph with sides V_1, V_2, V_3 . We see one side (say, V_1) of Γ as a family \mathcal{A} of hypergraphs such that each vertex v in V_1 represents a hypergraph H_v built by the vertices adjacent to v in the sides V_2 and V_3 in such a way that if (v, u, w) is an edge in Γ then (u, w) is an edge in the hypergraph represented by v . Notice that the edges in the hypergraphs of \mathcal{A} are of size 2 (in the Figure 2.12 is illustrated an example of the H_v 's).

Let \mathcal{B} be a subfamily of \mathcal{A} at which the deficiency $d = \text{def}(\mathcal{A})$ is achieved, namely $mw(\cup \mathcal{B}) = |\mathcal{B}| - d$.

By Lemma 4 we have that $v(\cup \mathcal{B}) \leq 2mw(\cup \mathcal{B})$. Since the edges in $\cup \mathcal{B}$ form a bipartite graph, it follows by König's theorem that $v(\cup \mathcal{B}) = \tau(\cup \mathcal{B}) \leq 2mw(\cup \mathcal{B})$.

Let $\mathcal{C} = \mathcal{A} \setminus \mathcal{B}$, and write $|\mathcal{B}| = b$, $|\mathcal{C}| = c$. Then, by the above, $\tau(\cup \mathcal{B}) \leq 2mw(\cup \mathcal{B}) = 2(b - d)$. Let \mathcal{X} be the set of vertices in V_1 corresponding to elements of \mathcal{C} , together with vertices in a minimal cover of $\cup \mathcal{B}$.

Since \mathcal{X} covers all the edges containing V_1 , then it is a cover of Γ , and its size is at most $c + 2(b - d)$.

On the other hand, by Theorem 10, we have that $v(\Gamma) \geq |V_1| - d = c + b - d$.

Thus, $\tau(\Gamma) \leq |\mathcal{X}| \leq c + 2(b - d) \leq 2(c + b - d) \leq 2v(\Gamma)$. □

An important remark of this proof is to notice that if we get a rainbow matching in a bipartite graph we have a matching in a 3-partite 3-graph

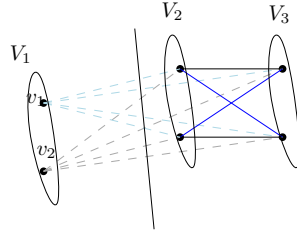


Figure 2.12: The black edges represent H_{v_2} and the blue ones represent H_{v_1} .

where each matching in the bipartite graph represents the adjacent vertices of a vertex in one of the parts of the 3-partite 3-graph or viceversa, if this 3-partite 3-graph has a t -matching, then the bipartite graph has a t -rainbow matching. In general, if a $(k - 1)$ -partite $(k - 1)$ -graph has a rainbow matching then the k -partite k -graph associated to this $(k - 1)$ -graph has a matching.

For instance, the latin transversal conjecture can be written as a conjecture for $K_{n,n}$ rainbow matchings (recall Conjecture 3), and by this remark we can write the conjecture as follow:

Conjecture 5. *Let H be a complete 3-partite 3-graph where each side has size n and each pair of edges intersect at most in one vertex. Then H has a matching of size $n - 1$*

Notice that in this hypergraph H each bipartite graph represented by a vertex in the side V_1 is a matching since each pair of edges in H intersect at most in one vertex, in this case v . Thus \mathcal{A} given as in the proof of Theorem 11 is a $K_{n,n}$ with n matchings of size n , and the matching of size $n - 1$ in H is a rainbow matching (SDR) of size $n - 1$ in \mathcal{A} .

In addition, if we take H as in the Conjecture 5 and use Theorem 11 we have that $n \leq 2v$. This because τ is the number of the size of the sides, which is n . So, we have that the maximum matching in H is at least $\frac{n}{2}$. This tell us that an $n \times n$ Latin square has at least a $\frac{n}{2}$ partial transversal.

Even if this result does not give new information in the context of Latin squares, it applies to a much more general class of bipartite colored graphs.

Chapter 3

Rainbow matchings in hypergraphs.

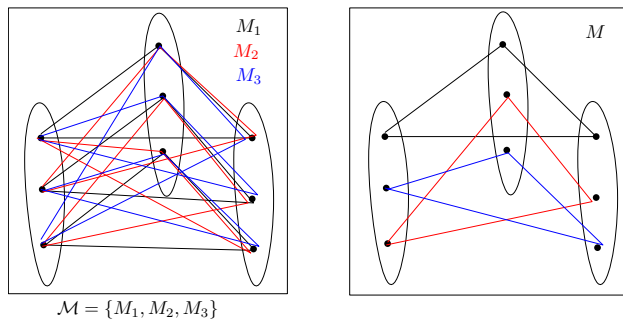


Figure 3.1: Each color represents one matching. Here M is a rainbow matching of \mathcal{M} .

Let H be a hypergraph and $\mathcal{M} = \{M_1, \dots, M_n\}$ be a collection of (possibly repeating and possibly with nonempty intersections) matchings,

and let M be a matching contained in $\cup \mathcal{M}$. If M is a system of representatives of \mathcal{M} , we say that M is a *rainbow matching* or that \mathcal{M} has a rainbow matching. If M is a partial SDR which does not represent all but only s of the matchings, we say that M is an *s -rainbow matching*.

In the study of Rainbow matchings, in order to approach to the latin square conjecture by other direction, one would like to answer the question: What size q of a collection \mathcal{M} of t -matchings (matchings of size t) guarantees the existence of an s -rainbow t -matching?

There are several works that give bounds for this question, in this chapter we give the ones that have made an impact for our study of rainbow matchings.

3.1 Aharoni-Berger conjecture.

Definition 1. Let r, s, t be numbers such that $s \leq t$. We write $f(r, s, t)$ for the maximal size of a family of t -matchings in an r -partite r -graph, possessing no s -rainbow t -matching. If $s = t$ we will simply write $f(r, t)$.

Aharoni and Berger in 2009 conjectured in [2], that:

Conjecture 6. $f(r, s, t) = 2^{r-1}(s-1)$ for all $r > 1$ and for all $s \leq t$.

In the same paper they proved the lower bound and for the case $r = 2$. For the next proofs, a family of matchings in an hypergraph (possibly with multiple edges) H can be seen as a coloring of the edges of H , where each color corresponds a matching.

Theorem 12. $f(r, s, t) \geq 2^{r-1}(s-1)$

Proof. We are going to denote the sides of the partition by V_0, \dots, V_{r-1} . For each function $p : [r-1] \rightarrow \{0, 1\}$ define a matching $M(p)$ of size t , whose i -th edge ($1 \leq i \leq t$) is $(u_0^i, \dots, u_{r-1}^i)$, where $u_0^i = i$ and for $j > 0$, $u_j^i = i + \sum_{k \leq j} p(k)$ modulo t . Figure 3.2 gives an example of a $M(p)$ for $r = 3$ and $t = 3$ and figure 3.3 gives the $M(p)$'s for $r = t = 3$.

Let \mathcal{M} consist of $s-1$ copies of each matching $M(p)$, $p \in \{0, 1\}^{[r-1]}$. In addition, each part of the partition is equal to $\mathbb{Z}/(t\mathbb{Z}) = \mathbb{Z}_t$. The figure 3.2 gives an example of a $M(p)$ for $r = 3$ and $t = 3$.

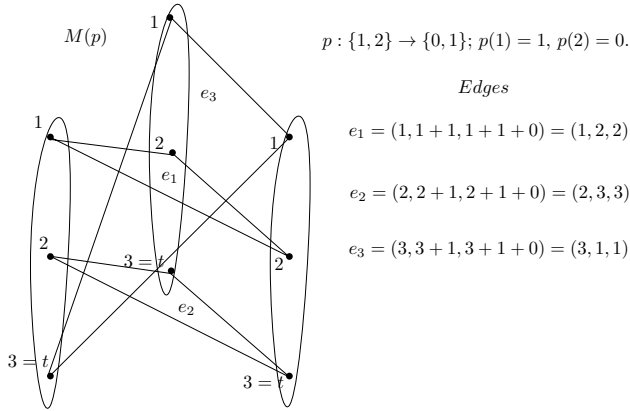


Figure 3.2: Example of $M(p)$.

Let M be a matching of size t contained in the union of the matchings $M(p)$. Since each side of the partition is \mathbb{Z}_t , M is perfect. We claim that it is equal to some $\mathcal{M} = \{M(p) | p \in \{0, 1\}^{[r-1]}\}$.

Let $e = (1, u_1, \dots, u_{r-1})$ be the edge in M whose first coordinate is 1, and let $f = (2, v_1, \dots, v_{r-1})$ be the edge in M whose first coordinate is 2. Suppose that e belongs to a copy of $M(p)$ and f to a copy of $M(q)$.

By contradiction, assume that $p \neq q$, and let j be the first index such that $p(j) \neq q(j)$.

Since M is a matching, $u_j \neq v_j$. Notice that $u_j + 1 = v_{j-1}$ and by definition of $M(p)$'s $u_j = u_{j-1} + p(j)$ and the same for v_j . If $p(j) > q(j)$, then $u_j = v_j$, which is a contradiction.

Thus $q(j) > p(j)$. Hence, u_{j+1} in the j -th side of the hypergraph cannot belong to any edge in M since M is a matching, but this contradicts the fact that M is perfect.

Doing the same with the next edges in M we show that all of them belong to the same $M(p)$. Since there are only $s - 1$ copies, then M is not a s -rainbow. Also, since there are 2^{r-1} functions p , we get that

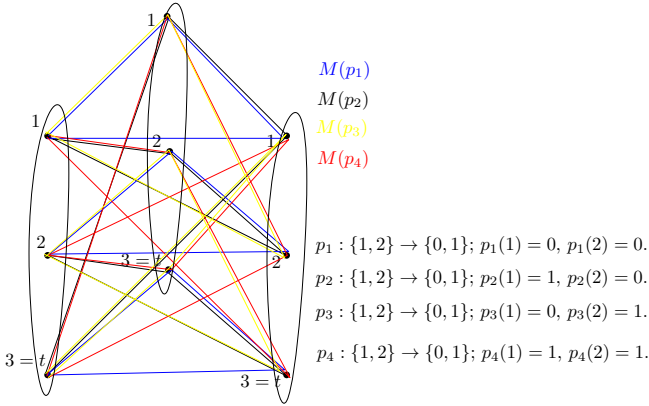


Figure 3.3: Example of all $M(p)$'s in a hypergraph with $r = t = 3$.

$$f(s, r, t) \geq 2^{r-1}(s - 1).$$

□

We should notice that the construction of \mathcal{M} in the proof of Theorem 12, that leads to the lower bound, contains repeated matchings. For the case that \mathcal{M} is a collection of edge disjoint matchings it is not clear which should be the lower bound, or if the multiplicities of every edge in $H = \bigcup_{M \in \mathcal{M}} M$ are bounded by a fixed constant, Which should be the lower bound?.

Theorem 13. $f(2, s, t) = 2(s - 1)$

Proof. By Theorem 12 we have that $f(r, s, t) \geq 2(s - 1)$, so we only need to prove that $f(r, s, t) \leq 2(s - 1)$.

Case 1: $s = t$

Let M_1, \dots, M_{2t-1} be a family of t -matchings in a bipartite graph with sides A and B . Let K be a k -rainbow k -matching of maximal size k .

We need to show that $k \geq t$. Assume by contradiction that $k < t$ and suppose without loss of generality that the edges of K are taken from the matchings $M_{2t-k}, M_{2t-k+1}, \dots, M_{2t-1}$.

Write $X_1 = A \cap V(K)$, so $|X_1| = |K| = k < t$. Since $|M_1| = t > k$ there exists some edge $e_1 = \{a_1, b_1\} \in M_1$ disjoint from X_1 . If e_1 is disjoint from $V(K)$, then adding it to K results in a $(k+1)$ -rainbow $(k+1)$ -matching, contradicting the assumption of maximality of k . Thus, we may assume that e_1 is incident with an edge $f_1 = \{b_1, c_1\} \in K$, write $X_2 = (X_1 \cup \{b_1\}) \setminus \{c_1\}$. Then $|X_2| = |X_1| = k$ (in the figure 3.4 is shown the construction of X_1 and X_2).

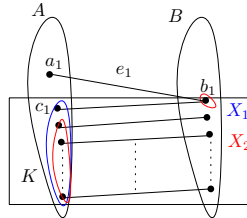


Figure 3.4: The set blue represents X_1 and the red one represents X_2 .

Since $|M_2| = t > k$ there exists an edge $e_2 = \{a_2, b_2\} \in M_2$ disjoint from X_2 . If $b_2 \notin V(K)$ (possibly with $a_2 = a_1$ or possibly $a_2 = c_1$), then there exists an alternating path which can be used to obtain a $(k+1)$ -rainbow $(k+1)$ -matching (see Figure 3.5). Thus we can assume that e_2 is incident with an edge $f_2 = \{b_2, c_2\} \in K$. Write now $X_3 = (X_2 \cup b_2) \setminus \{c_2\}$.

Continuing this way k steps, all edges of K must appear as f_i , and thus in the $k+1$ -st step the edge e_{k+1} does not meet $X_{k+1} = V(K) \cap B$. This yields an alternating path resulting in a $(k+1)$ -rainbow $(k+1)$ -matching, contradicting the maximality of k . Hence $k = t$ and $2(t-1) \geq f(2, t, t)$ (see Figure 3.6).

Case 2: $s < t$

Let M_1, \dots, M_{2s-1} be a family of t -matchings, K' be a k -rainbow t -matching, with maximal possible size k . Let ϕ the function that represents

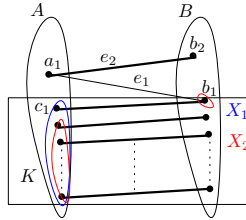


Figure 3.5: We get a larger matching if e_2 is disjoint from $V(K)$.

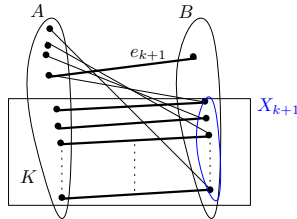


Figure 3.6: The bold edges represent a larger matching in the $k + 1$ -st step.

the SDR (recall the definition of SDR in Section 2.2). By contradiction assume that $k < s$, then there are at least s matchings M_i not represented in it, so assume $M_1, \dots, M_s \notin Im(\phi)$.

Let $K = K' \setminus \{e\}$ where e is an edge whose color appears more than once. We will make a similar process as for the case $t = s$, but instead of leaving each matching M_i after one edge e_i , we will continue choosing edges from M_i until all the edges in some M_j represented in K appear in " F_i ".

Let $e_1 = \{a_1, b_1\} \in M_1$ be disjoint from $X_1 = A \cap V(K)$, and let $f_1 = \{b_1, c_1\}$ be the edge in K meeting e_1 . Unless f_1 is the only one of its color in $(K, \phi|_K)$, we continue with M_1 .

Namely, we choose an edge $e_2 = \{a_2, b_2\} \in M_1$ disjoint from $X_2 = (X_1 \cup \{b_1\}) \setminus \{c_1\}$. If $b_2 \notin V(K)$, then we get a $(k + 1)$ -rainbow t -matching

contradicting the assumption of k .

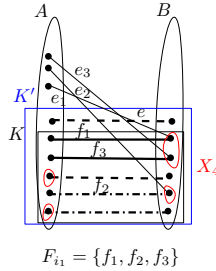


Figure 3.7: The bold, dotted and dash-dotted edges represent the matchings represented in K' . The edges f_1 and f_3 are all the edges that represent the bold matching in K .

Thus, we can assume that e_2 meets B in some edge $f_2 = \{b_2, c_2\} \in K$. We continue this way until the first time when the set $F_i = \{f_1, \dots, f_i\}$ satisfies $\phi^{-1}(j_i) \subseteq F_i$ for some j_i . When this happens, let $i = i_1$ (observe figure 3.7, we switch to M_2 , namely we find an edge $e_{i_1+1} = \{a_{i_1+1}, b_{i_1+1}\} \in M_2$ disjoint from X_{i_1+1} . Assuming by contradiction that $b_{i_1+1} \notin V(K)$, the matching obtained from K by applying the alternating path ending at b_{i_1+1} is a $(k+1)$ -rainbow t -matching. Thus we may assume that e_{i_1+1} meets some edge $f_{i_1+1} \in K$. We continue with M_2 until for some index $i_2 \neq i_1$ the set $F_{i_2} = \{f_1, \dots, f_{i_2}\}$ satisfies $F_{i_2} \supseteq \phi^{-1}(j_2)$ for some j_2 . We then switch to M_3 , and continue this process.

After k such switches, all colors j represented in $(K, \phi|_K)$ are exhausted, which means that at $k+1$ st step the edge e_{i_k+1} does not meet $X_{i_k+1} = B \cap V(K)$, which results in a $(k+1)$ -rainbow t -matching, contradicting the maximality of k . Therefore $k \geq s$.

□

3.2 Improving bounds.

After the work of Aharoni and Berger, Alon proved that their conjecture $f(r, s, t) \leq 2^{r-1}(s-1)$ is not true in [6]. In fact, using a probabilistic method he improves the lower bound for $f(r, t)$, showing that for all sufficiently large t and all r , $f(r, t) > 2.71828^{r-1}$.

Theorem 14. *For any real number $p \in (0, 1)$, $f(r, t) \geq pt^{r-1} - (t!)^{r-1}p^t$. Therefore, for every $\epsilon > 0$ and $t > t_0(\epsilon)$, $f(r, t) > (e - \epsilon)^{r-1}$.*

Proof. Let H be an r -partite r -graph such that the sides of the vertices are A_i , with $i = 1, \dots, r$ and $A_i = \mathbb{Z}/t\mathbb{Z} = \mathbb{Z}_t \forall i$. For each vector $s = (s_1, \dots, s_{r-1}, 0) \in \mathbb{Z}_t^r$ let M_s denote the matching consisting of the edges $(s_1 + i, \dots, s_{r-1} + i, i)$ with $0 \leq i < t$, $s_j + i$ module t .

Let \mathcal{M} be a collection of matchings obtained by choosing M_s randomly and independently with probability p .

Let $X(\mathcal{M}) = X$ be the random variable counting the number of matchings in \mathcal{M} and $Y = Y(\mathcal{M})$ counting the number of rainbow matchings in the union of \mathcal{M} .

Since X is a binomial random variable and all the possible matchings are represented by vectors in \mathbb{Z}_t^{r-1} , we have that $\mathbb{E}[X] = t^{r-1}p$.

Claim: $\mathbb{E}[Y] \leq (t!)^{r-1}p^t$.

Notice that there are at most $(t!)^{r-1}$ perfect matchings which correspond to the possible permutations of the vectors s . The probability that a possible rainbow matching lies in $\cup \mathcal{M}$ is p^t since each edge of this matching should belong to a different M_s . Therefore, $\mathbb{E}[Y] \leq (t!)^{r-1}p^t$. This proves the claim.

Hence, $\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] \geq t^{r-1}p - (t!)^{r-1}p^t$. Thus, there exist a choice \mathcal{M} such that $X(\mathcal{M}) - Y(\mathcal{M}) \geq pt^{r-1} - (t!)^{r-1}p^t$.

So, for each rainbow matching in \mathcal{M} , we can remove a matching that contributes on an edge of M . Therefore $f(r, t) \geq pt^{r-1} - (t!)^{r-1}p^t$.

Now, we can choose p in order to maximize the lower bound. We get that if $p = \left(\frac{1}{i[(t-1)!]^{r-1}}\right)^{\frac{1}{i-1}}$, then $f(r, t) \geq \left(\frac{1}{i[(t-1)!]^{r-1}}\right)^{\frac{1}{i-1}} t^{r-1} -$

$(\frac{1}{t^{[(t-1)!]^{r-1}}})(t!)^{r-1}$. Using Stirling's formula this tends to $e^{[(1+o(1))(r-1)] - 1}$.

For $r \geq (\frac{1}{2})\log(t)$ we have $f(r, t) \leq 2^{r-1}(t-1) > e^{r-1}$, thus $\forall \epsilon > 0$ there exists $t_0(\epsilon)$ such that for all $t > t_0(\epsilon)$, $f(r, t) > (e - \epsilon)^{r-1}$. □

Definition 2. Let $F(r, t)$ denote the maximum k for which there exists a collection of k matchings, each of size t , in some r -graph, such that there is no rainbow t -matching.

Since $F(r, t)$ is for every r -graph (not necessarily r -partite) it is trivial to see that $F(r, t) \geq f(r, t)$.

Moreover, notice that $F(r, t) \geq t$ for $t \geq 2$. For example, by taking an r -partite r -graph with a family of t matchings, where $t-1$ matchings are the same as $\{(1, 1, \dots, 1), (2, 2, \dots, 2), \dots, (t, t, \dots, t)\}$ and the last matching as the same but just changing the last two edges, $\{(1, 1, \dots, 1), \dots, (t-2, \dots, t-2), (t-1, t-1, \dots, t-1, t), (t, t, \dots, t, t-1)\}$. For the case $r = 2$ and $t = 3$, an example is illustrated in Figure 3.8

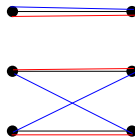


Figure 3.8: Example of $F(2, 3) \geq 3$.

Alon gave a new upper bound $f(r, t) \leq F(r, t) \leq \frac{t^{rt}(t-1)}{t!}$ in [6], but Glebov, Sudakov and Szabó [14] improved this upper bound with the next theorem.

Theorem 15. For every $r, t \geq 2$,

$$F(r, t) \leq \frac{2^{rt}}{\binom{r}{\lfloor \frac{t}{2} \rfloor}} (F(r, \lceil \frac{t}{2} \rceil) + \lfloor \frac{t}{2} \rfloor)$$

In particular, $F(r, t) < 8^{rt}$.

Proof. Let G be a (f, t) -colored (a collection of f t -matchings) r -graph with $f > \frac{2^{rt}}{\binom{t}{\lceil \frac{t}{2} \rceil}} (F(r, \lceil \frac{t}{2} \rceil) + \lfloor \frac{t}{2} \rfloor)$.

We color each vertex randomly, uniform and independently black and white. We say that a color class survives if all the vertices in exactly $\lfloor \frac{t}{2} \rfloor$ of its edges become black, and all its vertices in its $\lceil \frac{t}{2} \rceil$ edges become white.

The probability of one of its class survives is $\binom{t}{\lfloor \frac{t}{2} \rfloor} (\frac{1}{2})^{rt}$.

Therefore,

$$\mathbb{E}[X] = f \frac{\binom{t}{\lfloor \frac{t}{2} \rfloor}}{2^{rt}} > F(r, \lceil \frac{t}{2} \rceil) + \lfloor \frac{t}{2} \rfloor.$$

Then, there exist a coloring of its vertices such that $f' \geq f \frac{\binom{t}{\lfloor \frac{t}{2} \rfloor}}{2^{rt}} > F(r, \lceil \frac{t}{2} \rceil) + \lfloor \frac{t}{2} \rfloor$ of the f color classes survive.

Let G' be the graph resulting of the deletion of all the edges of the color classes that did not survive.

* G' consist of vertices black and white.

* G' has f' color classes such that every color class is a t -matching.

* Every t -matching has exactly $\lfloor \frac{t}{2} \rfloor$ edges with all its vertices colored black and $\lceil \frac{t}{2} \rceil$ colored white.

Let G_b be the graph of G' consisting only of the black vertices and the corresponding edges.

By construction G_b is a $(f', \lfloor \frac{t}{2} \rfloor)$ -colored graph, since $f' > F(r, \lfloor \frac{t}{2} \rfloor)$, then there exists a $\lfloor \frac{t}{2} \rfloor$ -rainbow matching M_b .

Now, let G_w be the subgraph of G' consisting only of the white vertices and its corresponding edges but deleting the color classes that corresponds to M_b .

Then G_w is a $(f' - \lfloor \frac{t}{2} \rfloor, \lceil \frac{t}{2} \rceil)$ -colored graph. Since $f' - \lfloor \frac{t}{2} \rfloor > F(r, \lceil \frac{t}{2} \rceil)$, then there exists a rainbow $\lceil \frac{t}{2} \rceil$ -matching in G_w , then the union of $M_b \cup M_w$ is a rainbow t -matching in G .

Therefore,

$$F(r, t) \leq \frac{2^{rt}}{\binom{t}{\lceil \frac{t}{2} \rceil}} (F(r, \lceil \frac{t}{2} \rceil) + \lfloor \frac{t}{2} \rfloor).$$

Using this result we will prove by induction on t that $F(r, t) < 8^{rt}$.

($t = 2$) Since $F(r, 1) = 0$ and by the result above we have that $F(r, 2) \leq \frac{2^{2r}}{2}(F(r, 1) + 1) = 2^{2r-1} < 8^{2r}$.

(I.H) Assume that $F(r, k) < 8^{rk}$ for all $2 \leq k \leq t - 1$.

Let us prove now that $F(r, t) < 8^{rt}$.

By the facts that $F(r, t) \geq t$ for every $r, t \geq 2$ and that $\binom{t}{\lceil \frac{t}{2} \rceil} \geq 2^{\lceil \frac{t}{2} \rceil}$ (this can be proved easily by induction).

Thus,

$$\begin{aligned} F(r, t) &\leq \frac{2^{rt}}{\binom{t}{\lceil \frac{t}{2} \rceil}} (F(r, \lceil \frac{t}{2} \rceil) + \lfloor \frac{t}{2} \rfloor) \\ &\leq \frac{2^{rt}}{2^{\lceil \frac{t}{2} \rceil}} (F(r, \lceil \frac{t}{2} \rceil) + F(r, \lfloor \frac{t}{2} \rfloor)) \\ &\leq \frac{2^{rt}}{2^{\lceil \frac{t}{2} \rceil}} (2F(r, \lceil \frac{t}{2} \rceil)) \end{aligned}$$

By the Induction Hypothesis,

$$\begin{aligned} F(r, t) &\leq \frac{2^{rt}}{2^{\lceil \frac{t}{2} \rceil}} (2F(r, \lceil \frac{t}{2} \rceil)) \\ &< \frac{2^{rt}}{2^{\lceil \frac{t}{2} \rceil}} (2(8^{r\lceil \frac{t}{2} \rceil})) = 2^{((r-1)\lceil \frac{t}{2} \rceil + r\lceil \frac{t}{2} \rceil + 1)} 8^{r\lceil \frac{t}{2} \rceil} \\ &\leq 2^{rt} 8^{r\lceil \frac{t}{2} \rceil} = 2^{3r\lceil \frac{t}{2} \rceil + rt} < 2^{3rt} = 8^{rt} \end{aligned}$$

□

Chapter 4

Rainbow n -matchings in an r -partite r -graphs.

As we have seen in the last chapters there are many ways that have been studied in order to get to the Ryser-Brauldi's conjecture. One of those was: If we have $K_{n,n}$ and we colored it with colors of size at most $\frac{n-1}{4e}$ then we know that the graph has a Rainbow n -matching. A good question now can be: "Can we do the same for complete r -partite r -graphs where each size has length n ?", or "what if instead of having simple graph $K_{n,n}$ each edge appears twice, Can we say the same about having a Rainbow n -matching?", or "If we have $K_{n,n}$ and we take off a matching of length n , What can we say about having a Rainbow n -matching?".

We will answer these questions in this chapter.

4.1 Rainbow matchings in complete r -partite r -graphs.

In the first chapter we have discussed when a square matrix has a Latin transversal. We have also seen that this can be translate into the existence

of rainbow matchings in complete properly edge colored bipartite graph.

Now, the immediate question could be if we can do the same with 3-partite 3-graphs and get something similar for this case. Well the answer is yes and the proof is very similar to the case of complete bipartite graphs.

Theorem 16. *Let G be a complete 3-partite 3-graph with n vertices in each part of the partition. For every edge coloring of G such that each color appears less than $k = \frac{(n-1)^2}{6e}$ times, there is a rainbow matching.*

Proof. Let π_1 and π_2 be random permutations on $\{1, \dots, n\}$ chosen uniformly from all the possible permutations (it is allowed that $\pi_1 = \pi_2$).

Define T as the set of 6-tuples (i, j, l, i', j', l') such that the edges $(i, j, l) \neq (i', j', l')$ with $i < i'$ and they have the same color. We define $f_{\pi_1, \pi_2} : \{1, \dots, n\} \rightarrow \{1, \dots, n\}^2$ such that $f_{\pi_1, \pi_2}(i) = (\pi_1(i), \pi_2(i))$ and, for each $(i, j, l, i', j', l') \in T$, denote $A_{i, (j, l), i', (j', l')}$ the event such that $f_{\pi_1, \pi_2}(i) = (j, l)$ and $f_{\pi_1, \pi_2}(i') = (j', l')$. In order to proof that G contains a rainbow matching we would like to prove that with positive probability none of these events happen.

Let us define the symmetric graph G' with vertices the 6-tuples in T we defined above and we say that two vertices (i, j, l, i', j', l') and (p, q, r, p', q', r') are adjacent if $\{i, i'\} \cap \{p, p'\} \neq \emptyset$ or $\{j, j'\} \cap \{q, q'\} \neq \emptyset$ or $\{l, l'\} \cap \{r, r'\} \neq \emptyset$.

Notice that the maximum degree of G' is at most $6n^2k$, since for each $(i, j, l, i', j', l') \in T$ there are at most $6n^2$ ways for choosing (p, q, r) such that $p \in \{i, i'\}$ or $q \in \{j, j'\}$ or $r \in \{l, l'\}$ and there are at most k different edges with the same color as (p, q, r) in G .

It is easy to see that $\mathbb{P}(A_{i, (j, l), i', (j', l')}) = \frac{1}{n^2(n-1)^2}$, since for each pair $i < i'$ there are n^2 choices for an edge with the first entry i . Since (i, j, l) and (i', j', l') should not be adjacent there are $(n-1)^2$ of possible edges that contain i' none adjacent with (i, j, l) , therefore $\mathbb{P}(A_{i, (j, l), i', (j', l')}) = \frac{1}{n^2(n-1)^2}$.

Let us show that G' satisfies the lopsidedness condition:

Let S denote the set of vertices non adjacent to (i, j, l, i', j', l') in G' . We will say that f_{π_1, π_2} is good if it satisfies $\bigwedge_S \overline{A_{p, (q, r), p', (q', r')}}.$ Let us

denote $S_{(i,j),(i',j')}$ the set of good f_{π_1, π_2} such that $f_{\pi_1, \pi_2}(1) = (i, j)$ and $f_{\pi_1, \pi_2}(2) = (i', j')$.

Let us show that $|S_{(1,1),(2,2)}| \leq |S_{(i,j),(i',j')}|$ for all $i \neq i'$ and $j \neq j'$.

Indeed, let us define the function $s : S_{(1,1),(2,2)} \rightarrow S_{(i,j),(i',j')}$ such that we map π_1 to some permutation π_1^* and π_2 to some permutation π_2^* so that we map $f_{\pi_1, \pi_2} \in S_{(1,1),(2,2)}$ to $f_{\pi_1^*, \pi_2^*} \in S_{(i,j),(i',j')}$. We define the new permutations as follows. For π_1 let x and y in $\{1, \dots, n\}$ such that $\pi_1(x) = i$ and $\pi_1(y) = i'$. Then we define $\pi_1^*(1) = i, \pi_1^*(2) = i', \pi_1^*(x) = 1$ and $\pi_1^*(y) = 2$ and $\pi_1^*(w) = \pi_1(w)$ for all $w \neq 1, 2, x, y$. Analogously, for π_2^* let x' and y' in $\{1, \dots, n\}$ such that $\pi_2(x') = j$ and $\pi_2(y') = j'$. We define $\pi_2^*(1) = j, \pi_2^*(2) = j', \pi_2^*(x') = 1$ and $\pi_2^*(y') = 2$ and $\pi_2^*(w) = \pi_2(w)$ for all $w \neq 1, 2, x'$ and y' . We can see that $f_{\pi_1^*, \pi_2^*} \in S_{(i,j),(i',j')}$ since the edges $(1, i, j), (2, i', j'), (x, 1, \pi_2^*(x)), (y, 2, \pi_2^*(y)), (x', \pi_1^*(x'), 1)$ and $(y', \pi_1^*(y'), 1)$ do not belong to S and it is clear that this function defines an injective mapping. Hence $|S_{(1,1),(2,2)}| \leq |S_{(i,j),(i',j')}|$.

It follows that

$$\mathbb{P}[A_{1,(1,1),2,(2,2)} | \bigwedge_S \overline{A_{p,(q,r),p',(q',r')}}] = \frac{|S_{(1,1),(2,2)}|}{\sum_{i \neq i', j \neq j'} |S_{(i,j),(i',j')}|},$$

for all $i \neq i'$ and $j \neq j'$. Hence, since there are $n^2(n-1)^2$ sets $S_{(i,j),(i',j')}$ with $i \neq i'$ and $j \neq j'$ and each one has cardinality at least $|S_{(1,1),(2,2)}|$, we have

$$\mathbb{P}[A_{1,(1,1),2,(2,2)} | \bigwedge_S \overline{A_{p,(q,r),p',(q',r')}}] \leq \frac{1}{n^2(n-1)^2}.$$

By symmetry of G' , the above inequality holds for any 6-tuple in T in the place of $A_{1,(1,1),2,(2,2)}$ proving the lopsidedependency of G' .

On the other hand, we have that $\Delta(G')\mathbb{P}[A_{i,(j,l),i',(j'l',r')}]e \leq 6n^2k(\frac{1}{n^2(n-1)^2})e = 1$. Therefore by the LLL (Lemma 1) G has a rainbow n -matching.

□

From Theorem 16 we easily get the following Corollary:

Corollary 2. *Let \mathcal{M} be a family of edge disjoint matchings whose union is the complete 3-partite 3-graph $K_{n,n,n}^{(3)}$. If each matching has size less than $k = \frac{(n-1)^2}{6e}$ then \mathcal{M} has a rainbow matching.*

In every proper edge coloring of $K_{n,n,n}^{(3)}$ every color appears at most n times. Therefore in contrast with the case of $K_{n,n}$, every proper edge coloring of $K_{n,n,n}^{(3)}$, with $n \geq 2$, has a rainbow matching. We note that the minimum number of edge disjoint matchings which decompose $K_{n,n,n}^{(3)}$, which corresponds to the edge chromatic number, is still unknown.

A second Corollary can be stated for a 4-partite 4-uniform hypergraph.

Corollary 3. *Let G be a 4-partite 4-uniform hypergraph on the vertex set $V_1 \cup V_2 \cup V_3 \cup V_4$, with $V_i = [n]$ for all $i \in \{1, 2, 3, 4\}$, such that for every $(i, j, k) \in [n]^3$ there is $l \in [n]$ such that the edge $(i, j, k, l) \in E(G)$. If the degree of each vertex in V_4 is less than $\frac{(n-1)^2}{6e}$ then G has a perfect matching.*

Using similar techniques Theorem 16 can be extended to r -partite r -uniform hypergraphs.

Theorem 17. *Each edge coloring of the complete r -partite r -uniform hyperegraph $K_{n,\dots,n}^{(r)}$ such that each color appears less than $\frac{(n-1)^r}{2re}$ times, contains a rainbow matching.*

4.2 Rainbow matchings in $K_{n,n}$ with multiple edges.

In the study of rainbow matchings in bipartite graphs there are several examples in which having matchings which are not edge-disjoint makes the existence of a rainbow matching far more difficult if we do not have enough edges and the multiplicity of the edges in our graph is big. But also, we

think that if we have an edge colored complete bipartite multigraph the probability for the existence of a rainbow matching should be bigger than for an edge colored complete bipartite simple graph. This way, if we mix in some way, results for rainbow matchings in an edge colored complete bipartite multigraph with other ones for rainbow matchings in an edge colored bipartite graph (not necessary complete) we think that we could enhance a result about rainbow matchings in an edge colored bipartite multigraph.

Again, Erdős and Spencer gave some conditions when a $K_{n,n}$ has a rainbow matching, what if now each edge appears twice. We give a condition for having a rainbow matching for these graphs and then maybe generalize this result.

Theorem 18. *Let G be a complete edge colored bipartite multigraph with n vertices in each stable set such that each edge has multiplicity two. Suppose that each color appears at most $k < \frac{n-1}{2e}$. Then G has a rainbow matching.*

Proof. Since all the edges are double then we will distinguish each edge by writing $(i, j)_l$ with $l = 0, 1$. Let $M_\pi^{(l_1, \dots, l_n)}$ be a matching such that π is a random permutation among all the possible $n!$ permutations, $l_i \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$, where the edges of the matching are $(i, \pi(i))_{l_i}$.

Denote by T the set of all ordered four-tuples $(i, j, i', j')^{l, l'}$ satisfying $i < i', j \neq j'$ and $c[(i, j)_l] = c[(i', j')_{l'}]$ where $c : E(G) \rightarrow \mathbb{N}$ in the given coloring of the multigraph (both edges have the same color). For each $(i, j, i', j')^{l, l'} \in T$, let $A_{(i,j),(i',j')}^{l, l'}$ denote the event that $(i, j)_l$ and $(i', j')_{l'}$ are in $M_\pi^{(l_1, \dots, l_n)}$. The existence of a rainbow matching is equivalent to the statement that with positive probability none of these events hold.

Define the symmetric graph D on the vertex set T by making $(i, j, i', j')^{l, l'}$ adjacent to $(p, q, p', q')^{l_p, l'_p}$ if and only if $\{i, i'\} \cap \{p, p'\} \neq \emptyset$ or/and $\{j, j'\} \cap \{q, q'\} \neq \emptyset$. The maximum degree of G is less than $8nk$ since there are at most $8n$ choices for each edge $(p, q)_{l_p}$ in G such that $p \in \{i, i'\}$ or $q \in \{j, j'\}$, and for each of these choices there are less than k choices such that $(p, q) \neq (p', q')$ with $c[(p, q)_{l_p}] = c[(p', q')_{l'_p}]$.

Notice that $\mathbb{P}[A_{(i,j),(i',j')}^{l,l'}] = \frac{1}{4n(n-1)}$. Indeed, since the probability of choosing (i, j) among all the permutations is $1/n$ and there are two ways of choosing this edge either $(i, j)_0$ or $(i, j)_1$ then probability of choosing $(i, j)_l$ is $1/2n$, so if we already have the edge $(ij)_l$ then the probability of choosing $(i', j')'_l$ is $1/(2(n-1))$. Therefore $\mathbb{P}[A_{ij,i'j'}^{l,l'}] = \frac{1}{4n(n-1)}$.

We will prove that $\mathbb{P}[A_{ij,i'j'}^{l,l'} \wedge_S \overline{A_{pq,p'q'}^{l,l'}}] \leq \frac{1}{4n(n-1)}$ for any $(i, j, i', j')^{l,l'} \in T$ and any set S of members of T that are nonadjacent in D to $(i, j, i', j')^{l,l'}$.

By symmetry we can assume that $i = j = 1$ and $i' = j' = 2$. We say that a matching $M_\pi^{(l_1, \dots, l_n)}$ is good if it satisfies $\wedge_S \overline{A_{pq,p'q'}^{l,l'}}$ and denote $S_{i,j}^{l_1, l_2}$ the set of all good matchings with the edges $(1, \pi(1) = i)_{l_1}$ and $(2, \pi(2) = j)_{l_2}$. We claim that $|S_{1,2}^{l_1, l_2}| \leq |S_{i,j}^{l,l'}|$ with l_1, l_2, l, l' modulo 2 and $i \neq j$. For each good matching $M_\pi^{(l_1, \dots, l_n)}$ we map it to a matching $M_{\pi^*}^{(l_1, l', l_3, \dots, l_n)}$. We define π^* as follows, if $\pi(x) = i$ and $\pi(y) = j$ then $\pi^*(x) = 1, \pi^*(y) = 2, \pi^*(1) = i, \pi^*(2) = j$, and for every $w \neq 1, 2, x, y$ $\pi^*(w) = \pi(w)$, then the matching $M_{\pi^*}^{(l, l', l_3, \dots, l_n)}$ will have the edges $(1, i)_l, (1, j)_{l'}$ and $(k, \pi^*(k))_{l_k}$ for all $k \in \{3, \dots, n\}$. This is a matching since π^* is a permutation of n , and it is good since $(1, i), (2, j), (x, 1)$ and $(y, 2)$ are adjacent to $(1, 1)$ and $(2, 2)$. This function is injective, therefore $|S_{1,2}^{l_1, l_2}| \leq |S_{i,j}^{l,l'}|$.

So we get that,

$$\mathbb{P}[A_{11,22}^{l_1, l_2} \wedge_S \overline{A_{pq,p'q'}^{l,l'}}] = \frac{|S_{1,2}^{l_1, l_2}|}{\sum_{i \neq j} \sum_{l=1}^2 \sum_{l'=1}^2 |S_{i,j}^{l,l'}|} \leq \frac{|S_{1,2}^{l_1, l_2}|}{\sum_{i \neq j} \sum_{l=1}^2 \sum_{l'=1}^2 |S_{1,2}^{l_1, l_2}|} = \frac{1}{4n(n-1)}$$

By symmetry the above inequality holds for each event $A_{ij,kr}^{l_1, l_2}$ and D is a lopsidedependency graph.

Since $e\Delta(D) \frac{1}{4n(n-1)} < e(8nk) \frac{1}{4n(n-1)} \leq 1$, then by LLLL G has a

4.3 Rainbow n -matchings in $K_{n,n}$ minus a perfect matching. 43

rainbow n -matching. □

The statement of Theorem 18 can be generalized in such a way that, if each edge appears exactly $m \geq 1$ times, and if each color appears less than $\frac{m(n-1)}{4e}$ times, then the complete bipartite multigraph has a rainbow matching.

Another good question of this type, can be: What happen if $K_{n,n}$ have multiple edges and each edge appears at least $m_0 \geq 1$ and at most $m_0 \leq m_1$ times?. As a first approach it is easy to see that if the matchings are of size less than $\frac{m_0(n-1)}{4e}$ then the graph has a rainbow matching. We would like to prove that the bound on the repetition of colors should be between $\frac{m_0(n-1)}{4e}$ and $\frac{m_1(n-1)}{4e}$.

4.3 Rainbow n -matchings in $K_{n,n}$ minus a perfect matching.

As we said in the last Section we would like to mix results about rainbow matchings in an edge colored bipartite graph with results about rainbow matchings in an edge colored bipartite multigraph.

There are already several results about rainbow matchings in an edge colored complete bipartite graph but, What about rainbow matchings in an edge colored bipartite graph without all the possible edges?. In this direction, we studied the case when we have an edge colored complete bipartite graph without a perfect matching.

Theorem 19. *Let G be $K_{n,n} - M$ be edge-colored, where M is a perfect matching in $K_{n,n}$ and $n \geq 20$. If each color is of size less than $k = \frac{(n-3)(n-6)}{4e(n-1)} > 1$ then G has a rainbow matching.*

Proof. Without loss of generality we can assume that M is the identity. Let $A_{x,y}^{u,v}$ for every $x \neq y, u \neq v, x$ and $v \neq y$, be the event that $c[(x, u)] = c[(y, v)]$ and $(x, u), (y, v)$ are both in a matching chosen uniformly at random and have the same color. As in the above cases we define

a graph be making $A_{x,y}^{u,v}$ adjacent to $A_{x',y'}^{u',v'}$ if the four edges involved of not form a matching. Let $D(A_{x,y}^{u,v})$ denote the neighbors of $A_{x,y}^{u,v}$ in this graph. Notice that $D(A_{x,y}^{u,v})$ is less than $d = 4(n-1)k$ since at least one of $\{(p, l), (q, r)\}$ is incident to (x, u) or (y, v) and there are less than k edges in the same matching.

Let $\mathcal{E} = \{A_{x,y}^{u,v} : x \neq y, i \neq x \text{ and } j \neq y, i\}$, if we prove that with positive probability none of the events in \mathcal{E} occur then G has a rainbow matching.

Let $\mathcal{T} \subseteq \mathcal{E} - (A_{x,y}^{u,v} \cup D(A_{x,y}^{u,v}))$ and $p = \frac{1}{(n-3)(n-6)}$. If we show that

$$\mathbb{P}[A_{x,y}^{u,v} \cap \overline{A_{p,q}^{l,r}}] \leq p, \quad (4.1)$$

using Lemma 1 we can conclude since $e(d+1)p \leq 1$

Here are three cases to keep in mind, which are:

- 1) $x = v$ and $y = u$,
- 2) $y = u$ and $x \neq v$, and
- 3) $x \neq v$ and $y \neq u$.

Without loss of generality since the graph is symmetric, we can assume that $x = 1$ and $y = 2$. We will prove (4.1) for each of the three cases.

Case $v = 1, u = 2$)

Assume that $v = 1, u = 2$ we would like to compute the conditional probability (4.1) of $A_{1,2}^{2,1}$. For this and the rest of cases, we say that a permutation π is *good* if satisfies $\bigwedge_{A \in \mathcal{T}} \overline{A}$. We define $S_{(u_1, \dots, u_n)}^{(v_1, \dots, v_n)}$ the set of all good permutations such that $\pi(u_i) = v_i$ for all $i \in \{1, \dots, n\}$, in these cases we will use at most $n = 4$. We denote by S the set of all good permutations.

We have that, for every pair x, y such that $x \neq y$ and $x, y \notin \{1, 2\}$, $|S_{1,2}^{2,1}| = \sum_{i \neq 1, 2, x} \sum_{j \neq 1, 2, y, i} |S_{1,2,x,y}^{2,1,i,j}|$. As before we can notice that $|S_{1,2,x,y}^{2,1,i,j}| \leq |S_{1,2,x,y}^{i,j,2,1}|$ by mapping the permutation $\pi \in S_{1,2,x,y}^{2,1,i,j}$ to another permutation

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$\pi^* \in S_{1,2,x,y}^{i,j,2,1}$, such that $\pi^*(1) = i, \pi^*(2) = j, \pi^*(x) = 2, \pi^*(y) = 1$. This is always well defined and injective for all $x, y \notin \{1, 2\}$. So,

$$\begin{aligned} (n-2)(n-3)|S_{1,2}^{2,1}| &= \sum_{x \neq 1,2} \sum_{y \neq x,1,2} \sum_{i \neq 1,2,x} \sum_{j \neq 1,2,y,i} |S_{1,2,x,y}^{2,1,i,j}| \\ &\leq \sum_{x \neq 1,2} \sum_{y \neq x,1,2} \sum_{i \neq 1,2,x} \sum_{j \neq 1,2,y,i} |S_{1,2,x,y}^{i,j,1,2}| \\ &\leq |S| \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}[A_{1,2}^{2,1} \bigwedge_{A \in \mathcal{T}} \overline{A}] &\leq \frac{|S_{1,2}^{2,1}|}{|S|} \\ &\leq \frac{1}{(n-2)(n-3)} \\ &\leq \frac{1}{(n-3)(n-6)} = p. \end{aligned}$$

Case $u = 2$ and $1 \neq v$)

Without loss of generality assume that $v = 3$, we would like to compute the conditional probability (4.1) of $A_{1,2}^{2,3}$.

Notice that for every $x \neq y$ and $x, y \notin \{1, 2\}$ we have that $|S_{1,2}^{2,3}| = \sum_{i \neq 2,3,x} \sum_{j \neq 2,3,y,i} |S_{1,2,x,y}^{2,3,i,j}|$.

We claim that $|S_{1,2,x,y}^{2,3,i,j}| \leq |S_{1,2,x,y}^{i,j,2,3}|$ if $y \neq 3$ and $i \neq 1$. Indeed, since we can send every permutation $\pi \in S_{1,2,x,y}^{2,3,i,j}$ to a permutation $\pi^* \in S_{1,2,x,y}^{i,j,2,3}$ in such a way that $\pi^*(1) = i, \pi^*(2) = j, \pi^*(x) = 2, \pi^*(y) = 3$ and $\pi^*(w) = \pi(w)$ for all $w \neq 1, 2, x, y$, this function is injective and it is trivial to see that π^* is good since $(1, i), (2, j), (x, 2), (y, 3)$ are adjacent to $(1, 2), (2, 3)$, but this is not true if either $y = 3, i = 1$.

Also notice that $|S_{1,2}^{2,3}| = \sum_{x \neq 1,2} |S_{1,2,x}^{2,3,1}|$ and for all $y \neq 1, 2, x$ we have that $|S_{1,2,x}^{2,3,1}| = \sum_{j \neq 1,2,3,y} |S_{1,2,x,y}^{2,3,1,j}|$.
So that,

$$\begin{aligned}
(n-2)(n-4)|S_{1,2}^{2,3}| &= \sum_{x \neq 1,2} \sum_{y \neq 1,2,3,x} \sum_{i \neq 2,3,x} \sum_{j \neq 2,3,y,i} |S_{1,2,x,y}^{2,3,i,j}| \\
&\leq \sum_{x \neq 1,2} \sum_{y \neq 1,2,3,x} \left(\sum_{j \neq 1,2,3,y} |S_{1,2,x,y}^{2,3,1,j}| + \sum_{i \neq 1,2,3,x} \sum_{j \neq 2,3,y,i} |S_{1,2,x,y}^{i,j,2,3}| \right) \\
&\leq (n-2)|S_{1,2}^{2,3}| + \sum_{i \neq 1,2,3} \sum_{j \neq 2,3,i} \sum_{x \neq 1,2,i} \sum_{y \neq 1,2,3,j} |S_{1,2,x,y}^{i,j,2,3}| \\
&\leq (n-2)|S_{1,2}^{2,3}| + |S|.
\end{aligned}$$

Thus,

$$(n-2)(n-5)|S_{1,2}^{2,3}| \leq |S|.$$

Therefore,

$$\mathbb{P}[A_{1,2}^{2,3} \bigwedge_{A \in \mathcal{T}} \bar{A}] \leq \frac{|S_{1,2}^{2,3}|}{|S|} \leq \frac{1}{(n-2)(n-5)} \leq p.$$

Case $v \neq 1, u \neq 2$

Without loss of generality assume that $v = 4, u = 3$, we would like to compute the conditional probability (4.1) of $A_{1,2}^{3,4}$.

Notice that for every $x \neq y$ and $x, y \notin \{1, 2\}$ we have that $|S_{1,2}^{2,3}| = \sum_{i \neq 3,4,x} \sum_{j \neq 3,4,y,i} |S_{1,2,x,y}^{3,4,i,j}|$.

We claim that $|S_{1,2,x,y}^{3,4,i,j}| \leq |S_{1,2,x,y}^{i,j,3,4}|$ if $x \neq 3, y \neq 4, i \neq 1$ and $j \neq 2$.
Indeed, since we can send every permutation $\pi \in S_{1,2,x,y}^{3,4,i,j}$ to a permutation

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$\pi^* \in S_{1,2,x,y}^{i,j,3,4}$ in such a way that $\pi^*(1) = i, \pi^*(2) = j, \pi^*(x) = 3, \pi^*(y) = 4$ and $\pi^*(w) = \pi(w)$ for all $w \neq 1, 2, x, y$, it is easy to see that this function is injective and that π^* is good since $(1, i), (2, j), (x, 3), (y, 4)$ are adjacent to $(1, 3), (2, 4)$, but this is not true if either $x = 3, y = 4, i = 1$ and $j = 2$.

Also notice that $|S_{1,2}^{3,4}| = \sum_{x \neq 1,2} |S_{1,2,x}^{3,4,1}|$ and for all $y \neq 1, 2, x$ we have that $|S_{1,2,x}^{3,4,1}| = \sum_{j \neq 1,3,4,y} |S_{1,2,x,y}^{3,4,1,j}|$ the same with $|S_{1,2}^{3,4}| = \sum_{y \neq 1,2} |S_{1,2,y}^{3,4,2}|$ and $|S_{1,2,y}^{3,4,2}| = \sum_{i \neq 2,3,4,x} |S_{1,2,x,y}^{3,4,i,2}|$ for all $x \neq 1, 2, y$.

So that,

$$\begin{aligned}
 (n-3)(n-4)|S_{1,2}^{3,4}| &= \sum_{x \neq 1,2,3} \sum_{y \neq 1,2,4,x} \sum_{i \neq 3,4,x} \sum_{j \neq 3,4,y,i} |S_{1,2,x,y}^{3,4,i,j}| \\
 &\leq \sum_{x \neq 1,2,3} \sum_{y \neq 1,2,4,x} \left(\sum_{i \neq 2,3,4,x} |S_{1,2,x,y}^{3,4,i,2}| + \sum_{j \neq 1,3,4,y} |S_{1,2,x,y}^{3,4,1,j}| + \sum_{i \neq 1,3,4,x} \sum_{j \neq 2,3,4,y,i} |S_{1,2,x,y}^{i,j,3,4}| \right) \\
 &\leq 2(n-3)|S_{1,2}^{3,4}| + \sum_{i \neq 1,3,4} \sum_{j \neq 2,3,4,i} \sum_{x \neq 1,2,3,i} \sum_{y \neq 1,2,4,j} |S_{1,2,x,y}^{i,j,3,4}| \\
 &\leq 2(n-3)|S_{1,2}^{3,4}| + |S|.
 \end{aligned}$$

Thus,

$$(n-3)(n-6)|S_{1,2}^{3,4}| \leq |S|.$$

Therefore,

$$\mathbb{P}[A_{1,2}^{3,4} \bigwedge_{A \in \mathcal{T}} \overline{A}] \leq \frac{|S_{1,2}^{3,4}|}{|S|} \leq \frac{1}{(n-3)(n-6)} = p.$$

By symmetry of our graph we have that (4.1) holds for each $A_{x,y}^{u,v}$. Therefore by LLL (Lemma 1) $K_{n,n} - M$ has a rainbow matching. \square

Having proved these Theorems, an interesting path to follow could be finding how many matchings can we take off and still the kind of symmetry in our graph in order to use this method again, or mixing $K_{n,n} - M$ with multiple edges and say something about having a rainbow matching since these are most of the time the problematic cases for finding the rainbow matching.

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