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Constraint algorithm for singular k-cosymplectic field theories

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The ideal is to reach proofs by comprehension rather than by computation.

—Bernhard Riemann

Abstract

The main objective of this thesis is to develop a constraint algorithm for singular k-cosymplectic field theories. We begin by reviewing the Hamiltonian and Lagrangian formalisms for autonomous and nonautonomous mechanics and field theory. Then we present the constraint algorithms for presymplectic and precosymplectic systems, which are geometric frameworks for singular autonomous and nonautonomous mechanical systems. We also review the constraint algorithm k-presymplectic systems, which are geometric models for singular autonomous field theories. Finally, the last part of the work is devoted to defining the concepts of k-precosymplectic manifold, proving the existence of Reeb vector fields in these manifolds, k-precosymplectic Hamiltonian system and to develop a constraint algorithm in order to find a submanifold where the existence of solutions of the field equations is assured.

Keywords: k-cosymplectic manifold, k-precosymplectic manifold, constraint algorithm, field theory, Hamiltonian formalism, Lagrangian formalism.

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Contents

Introduction

The main theories in modern physics can be formulated in geometric terms using geometric mechanics and geometric field theory. The natural framework for autonomous mechanical systems is symplectic geometry, whereas the nonautonomous counterpart can be nicely described using cosymplectic geometry. These two formulations admit generalizations to first order classical field theory using k-symplectic geometry for the autonomous case and k-cosymplectic geometry for the nonautonomous case. Moreover, this can be generalized even more by using multisymplectic geometry. Some references for these topics are [1] and [12].

Recently, singular systems are gaining importance because of their role in modern physics and control theory, both in mechanics and in classical field theory. In fact, some of the most important physical theories of the 20th century are singular: Maxwell's Theory of Electromagnetism, Einstein's Relativity and Classical String Theory are the most relevant examples but there are many more. One of the problems of singular systems is that in general, there are no global solutions to the system of differential equations which describe the system. This problem is usually solved by trying to find a submanifold N of our phase space manifold M where we can assure the existence of global solutions tangent to N. Usually this is done with what is known as constraint algorithms. These procedures add some constraints to the system at every step until we can ensure the existence of global solutions in the final constraint submanifold.

One of the firsts to find a constraint algorithm to solve the problem for the Hamiltonian formalism of singular mechanics was P. Dirac [13]. After this, many people tried to geometrize this algorithm and this was finally done by Gotay et al. [14] and [15]. These works dealt with the symplectic formulation of autonomous mechanics. This was later generalized by D. Chinea, M. de León et al. to nonautonomous systems [5], [7] and [8] in different ways using the Poincaré-Cartan 2-form available in the cosymplectic formulation of nonautonomous mechanics.

Some years later, these algorithms were generalized to the context of classical field theory by M. de León et al. to the multisymplectic formalism [6], [9] and by X. Gràcia et al. to the k-symplectic formulation of singular classical field theories [16].

The aim of this thesis is to complete the constraints algorithms stating a constraint algorithm for singular k-cosymplectic field theories. In order to do this, we define the concepts of k-precosymplectic manifold and k-precosymplectic Hamiltonian system and we develop a constraint algorithm generalizing the ones commented before to find an iterative process which will allow us, adding some constraints in every step, to find a final constraint submanifold where the existence of global solutions to the k-precosymplectic Hamiltonian system is assured.

The first chapter of this thesis is devoted to reviewing the geometric formulations of mechanics [1] and classical field theory [10], [11] and [12]. The first section presents the autonomous Hamiltonian and Lagrangian mechanics which use symplectic geometry to model the phase space of autonomous mechanical systems. We also introduce in this first section the presymplectic manifolds which model the phase space of singular autonomous mechanics. On the other hand, in the second section we present the nonautonomous counterparts of Hamiltonian and Lagrangian mechanics using cosymplectic geometry to model the phase space of nonautonomous mechanical systems. In the same way as in the previous section, precosymplectic manifolds model the phase space of singular nonautonomous mechanical systems. In the third and fourth sections we introduce the k-symplectic and k-cosymplectic formulations of classical field theory. In both cases we begin introducing the manifolds involved: k-symplectic and k-cosymplectic manifolds respectively, which model the phase spaces for autonomous and timedependent field theories. We also study the canonical models of these manifolds: $(T^1_k)^*Q$ and $\mathbb{R}^k \times (T^1_k)^*Q$. This presentation begins with the Hamiltonian formalism and at the end we also present the Lagrangian counterpart. In the third section of Chapter 1 we also introduce k-presymplectic manifolds, which model the phase space of singular k-presymplectic field theories.

In the second chapter, we review the constraint algorithms for singular mechanical systems. In the first section we study the singular autonomous mechanical systems and give a brief review of the algorithm by Gotay et al. [15]. In the second section we present with some detail the development of the constraint algorithm for nonautonomous singular mechanical systems developed by D. Chinea et al. [5].

In the third chapter we present the constraint algorithms for k-presymplectic and k-precosymplectic field theories. The algorithm for k-presymplectic field theory was developed by X. Gràcia, R. Martín and N. Román-Roy [16] as a generalization of the first algorithm by Gotay et al. [15]. Finally, the goal of this thesis is in Section 3.2., where we define the concept of a k-precosymplectic manifold and introduce the Darboux coordinates in these manifolds. We prove the existence of Reeb vector fields in k-precosymplectic manifolds and find a type of manifolds where we have these vector fields uniquely determined. With all this in mind, we develop a constraint algorithm to find a final constraint submanifold in singular k-precosymplectic systems.

All the manifolds and mappings appearing in this thesis are assumed to be smooth. Also, sum over crossed repeated indices is understood.

Chapter 1

Foundations

In this first chapter we present a review on Hamiltonian and Lagrangian mechanics. We begin with the symplectic approach for autonomous systems, while the cosymplectic approach is used to model nonautonomous systems. They generalize to the k-symplectic and k-cosymplectic approaches respectively. Some references on these topics are [1], [2] and [12].

1.1 Autonomous Hamiltonian and Lagrangian mechanics

In this first section we review the geometric formulation of autonomous Hamiltonian and Lagrangian mechanics. In order to do this, we will make wide use of symplectic geometry, and in particular of the canonical structures of the tangent bundle TQ and the cotangent bundle T^*Q of a smooth manifold Q.

1.1.1 Autonomous Hamiltonian mechanics

We begin the presentation of autonomous Hamiltonian mechanics by introducing our most important tool: symplectic manifolds. Throughout this subsection, M will be a finite-dimensional manifold.

Symplectic geometry

Definition 1.1.1. *Let $\omega \in \Omega^2(M)$, we say that ω is a* ***symplectic form*** *if it is closed ($\mathrm{d}\omega = 0$) and nondegenerate, that is, for every $p \in M$, $i_{X_p}\omega_p = 0$ if and only if $X_p = 0$. If the form ω is degenerate, we say that it is a* ***presymplectic form****. Let ω be a symplectic (resp. presymplectic) 2-form on M. Then the couple (M, ω) is called a* ***symplectic manifold*** *(resp. a* ***presymplectic manifold****).*

Notice that every symplectic manifold must be even dimensional and orientable (for instance ω^m is a volume form). The first important fact about symplectic manifolds is given by Darboux's Theorem, which basically says that every symplectic manifold is locally diffeomorphic to a cotangent bundle.

Theorem 1.1.2 (Darboux's Theorem). *Let (M, ω) be a symplectic manifold of dimension $2m$. Then, for every $x \in M$, there exists a local chart $(\mathcal{U}, q^i, p_i)_{i=1,\dots,m}$ such that $x \in \mathcal{U}$ and the local expression of the symplectic form ω in this chart is*

$$\omega|_{\mathcal{U}} = \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

These local charts are called ***symplectic charts*** *and its coordinates are called* ***canonical coordinates*** *or* ***Darboux coordinates*** *of the symplectic manifold M.*

Proof. The proof of this Theorem can be found in [17]. ■

For presymplectic manifolds we have a similar result:

Theorem 1.1.3 (Darboux's Theorem for presymplectic manifolds). *Let (M, ω) be a presymplectic manifold such that $\dim M = 2m + n$ and $\operatorname{rank} \omega = 2m$. Then, for every $x \in M$, there exists a chart $(\mathcal{U}, q^i, p_i, z^j)_{i=1,\dots,m, j=1,\dots n}$ such that $x \in \mathcal{U}$ and the presymplectic form expressed in this chart is written as*

$$\omega|_{\mathcal{U}} = \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

These charts are called ***presymplectic charts*** *and its coordinates are called* ***canonical coordinates*** *or* ***Darboux coordinates*** *of the presymplectic manifold M.*

Example 1.1.4. *Let Q be an m-dimensional manifold. Consider the canonical 2-form $\omega \in \Omega(T^*Q)$. Then, (T^*Q, ω) is a symplectic manifold.*

Given a presymplectic manifold (M,ω) we can consider the following morphism of $\mathcal{C}^\infty(M)$-modules

$$\begin{array}{rccc} \flat\colon & \mathfrak{X}(M) & \longrightarrow & \Omega^1(M) \\ & X & \longmapsto & i_X\omega \end{array}$$

Notice that if ω is a symplectic form, the morphism $\flat$ is also an isomorphism and its inverse is denoted by $\sharp = \flat^{-1}$.

Hamiltonian systems

Definition 1.1.5. *A* ***regular (resp. singular) symplectic (resp. presymplectic) Hamiltonian system*** *is a triple (M,ω,γ) where (M,ω) is a symplectic (resp. presymplectic) manifold and γ is a closed 1-form called the* ***Hamiltonian 1-form*** *of the system.*

Taking into account Poincaré's Lemma, for every $p \in M$, there exists an open neighbourhood $\mathcal{U}$ of p and a function $h \in \mathcal{C}^\infty(\mathcal{U})$ such that $\gamma|_{\mathcal{U}} = \mathrm{d}h$, called the **local Hamiltonian function** of the system. If the Hamiltonian 1-form γ is exact, then there exists a function $h \in \mathcal{C}^\infty(M)$ such that $\gamma = \mathrm{d}h$. In this case we say that h is the **global Hamiltonian function** of the system.

Given a symplectic Hamiltonian system (M,ω,γ), there exists a unique Hamiltonian vector field vector field $X_h \in \mathfrak{X}(M)$ such that

$$\flat(X_h) = i_{X_h}\omega = \gamma \tag{1.1}$$

or, equivalently, $X_h = \sharp(\gamma)$. Consider a canonical chart $(\mathcal{U}, q^i, p_i)$ and let X be an arbitrary vector field on M, whose local expression in this chart is

$$X|_{\mathcal{U}} = A^i\frac{\partial}{\partial q^i} + B_i\frac{\partial}{\partial p_i}.$$

Imposing that X must satisfy (1.1), which in coordinates looks like

$$i\left(A^i\frac{\partial}{\partial q^i} + B_i\frac{\partial}{\partial p_i}\right)(\mathrm{d}q^i\wedge\mathrm{d}p_i) = \frac{\partial h}{\partial q^i}\mathrm{d}q^i + \frac{\partial h}{\partial p_i}\mathrm{d}p_i,$$

we get that

$$\begin{cases} A^i = \dfrac{\partial h}{\partial p_i}, \\ B_i = -\dfrac{\partial h}{\partial q^i}. \end{cases}$$

Hence, the local expression of the Hamiltonian vector field in a canonical chart is

$$X_h = \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i}$$

In case the form ω is presymplectic, equation (1.1) may not have a solution defined on the whole manifold M, but just on some points of M. In this case we can use a constraint algorithm in order to find a submanifold $N \hookrightarrow M$ such that equation (1.1) has solutions in the submanifold N, if possible. We will see these algorithms later on in the second chapter of this work.

1.1.2 Autonomous Lagrangian mechanics

In this subsection we will give a brief presentation of the canonical structures in the tangent bundle of a manifold M. For more detail on this subject see [1] and [2].

Let Q be an m-dimensional manifold. Consider the bundle TQ as our phase space. Recall that in the tangent bundle we have the **vertical endomorphism** and the **Liouville vector field** whose coordinates in a natural chart of coordinates of TQ are

$$J = \frac{\partial}{\partial v^i} \otimes \mathrm{d}q^i$$
$$\Delta = v^i \frac{\partial}{\partial v^i}$$

where (q^i, v^i) are the canonical coordinates on TQ.

We want to make the tangent bundle TQ into a symplectic manifold. In order to do this we consider a **Lagrangian function** $\mathcal{L}\colon TQ \to \mathbb{R}$. Using this Lagrangian function we can construct the Poincaré-Cartan forms. First, consider the 1-form $\theta_{\mathcal{L}} \in \Omega^1(TQ)$ defined by

$$\theta_{\mathcal{L}} = \mathrm{d}\mathcal{L} \circ J.$$

Now we can define the Poincaré-Cartan 2-form $\omega_{\mathcal{L}} \in \Omega^2(TQ)$ as $\omega_{\mathcal{L}} = -\mathrm{d}\theta_{\mathcal{L}}$. Notice that the 2-form $\omega_{\mathcal{L}}$ is closed. The matrix of $\omega_{\mathcal{L}}$ is

$$\begin{pmatrix} \dfrac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i} - \dfrac{\partial^2 \mathcal{L}}{\partial q^i \partial v^j} & \dfrac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} \\ -\dfrac{\partial^2 \mathcal{L}}{\partial v^i \partial v^j} & 0 \end{pmatrix}.$$

It is important to note that the 2-form $\omega_{\mathcal{L}}$ is nondegenerate if and only if the matrix $\left(\dfrac{\partial^2\mathcal{L}}{\partial v^i \partial v^j}\right)$ is non singular. This motivates the following definition:

Definition 1.1.6. *A Lagrangian function $\mathcal{L}$ is said to be **regular** if the form $\omega_{\mathcal{L}}$ is nondegenerate.*

Proposition 1.1.7. *A Lagrangian function is regular if and only if the matrix $\left(\dfrac{\partial^2\mathcal{L}}{\partial v^i \partial v^j}\right)$ is non singular.*

If the Lagrangian is regular, then $\omega_{\mathcal{L}}$ is symplectic and thus we can consider the isomorphism

$$\begin{array}{rccl} \flat_{\mathcal{L}}\colon & \mathfrak{X}(TQ) & \longrightarrow & \Omega^1(TQ) \\ & X & \longmapsto & i_X\omega_{\mathcal{L}}. \end{array}$$

with inverse $\sharp_{\mathcal{L}} = \flat_{\mathcal{L}}^{-1}$.

Given a Lagrangian function $\mathcal{L}$, we can define the **energy function** $E_{\mathcal{L}}$ as the function

$$E_{\mathcal{L}} = \Delta(\mathcal{L}) - \mathcal{L}.$$

With this last definition we can write the equation

$$\flat_{\mathcal{L}}(X_{\mathcal{L}}) = i_{X_{\mathcal{L}}}\omega_{\mathcal{L}} = \mathrm{d}E_{\mathcal{L}}. \tag{1.2}$$

It can be seen that if the Lagrangian $\mathcal{L}$ is regular, there exists a unique solution $X_{\mathcal{L}}$ of equation (1.2) and it is a second order differential equation, i.e., it satisfies the condition $JX_{\mathcal{L}} = \Delta$.

Proposition 1.1.8. *If $\mathcal{L}$ is regular, there exists a unique solution $X_{\mathcal{L}}$ of equation* (1.2) *and it is a second order differential equation, i.e., it satisfies the condition $JX_{\mathcal{L}} = \Delta$. Moreover, its integral curves are the solutions to the Euler-Lagrange equations*

$$\frac{\partial\mathcal{L}}{\partial q^i}\circ\alpha - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t \left(\frac{\partial\mathcal{L}}{\partial v^i}\circ\dot{\alpha}\right) = 0, \qquad 1 \leq i \leq m.$$

Hence, equation (1.2) is just a geometric version of the Euler-Lagrange equations.

1.2 Nonautonomous Hamiltonian and Lagrangian mechanics

In this section we will consider the case of time-dependent mechanics. Some references on this subject are [1], [4] and [5].

1.2.1 Nonautonomous Hamiltonian mechanics

As in the case of autonomous Hamiltonian systems, we will begin introducing the geometric tools needed in order to give an intrinsic description of the problem. We wil begin with a brief introduction to cosymplectic geometry.

Cosymplectic geometry

The main object of study of cosymplectic geometry are cosymplectic manifolds, which are somehow an odd-dimensional counterpart of symplectic manifolds. Some references on cosymplectic and precosymplectic geometry are [1] and [4].

Definition 1.2.1. *Let M be a smooth manifold of dimension $2m+1$ and $\omega \in \Omega^2(M)$, $\eta \in \Omega^1(M)$ be closed differential forms. If ω and η are such that $\operatorname{rank}\omega = 2r$ and $\omega^r \wedge \eta \neq 0$ we say that the triple (M,ω,η) is a* ***precosymplectic manifold*** *of rank $2r$. If, in addition, $\omega^m \wedge \eta \neq 0$, we say that (M,ω,η) is a* ***cosymplectic manifold****.*

Let (M,ω,η) be a cosymplectic manifold. From the definition it is clear that M is orientable as $\omega^m \wedge \eta$ is a volume form. Consider the following morphism between $\mathcal{C}^\infty(M)$-modules

$$\begin{array}{rccc} \flat\colon & \mathfrak{X}(M) & \longrightarrow & \Omega^1(M) \\ & X & \longmapsto & i_X\omega + (i_X\eta)\eta. \end{array} \tag{1.3}$$

It is clear that $\flat$ is an isomorphism and we will denote its inverse by $\sharp = \flat^{-1}$.

Remark 1.2.2. *Notice that if M is a precosymplectic manifold, this morphism $\flat$ is also defined but it is no longer an isomorphism.*

Now we define the **characteristic distribution** of ω as

$$\ker\omega = \{v \in TM \mid i_v\omega = 0\}.$$

Since $\operatorname{rank}\omega = 2m$, it is clear that $\ker\omega$ has dimension 1. Hence, there exists a unique vector field $\mathcal{R}$ called the **Reeb vector field** such that

$$i_{\mathcal{R}}\omega = 0, \quad i_{\mathcal{R}}\eta = 1.$$

Notice that the Reeb vector field is $\mathcal{R} = \sharp(\eta)$. As in the case of symplectic manifolds, we also have a Darboux's Theorem that ensures the existence of canonical charts.

Theorem 1.2.3 (Darboux's Theorem for cosymplectic manifolds)**.** *Let (M, ω, η) be a cosymplectic manifold. Then, for every $x \in M$ there exists a local chart $(\mathcal{U}, t, q^i, p_i)$, $i = 1, \dots, m$, with $x \in \mathcal{U}$ such that*

$$\eta|_{\mathcal{U}} = \mathrm{d}t, \qquad \omega|_{\mathcal{U}} = \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

Such a chart is called a Darboux or canonical chart.

Proof. The proof of this theorem can be found in [1]. ■

In Darboux coordinates, the Reeb vector field is expressed as $\mathcal{R} = \frac{\partial}{\partial t}$.

Example 1.2.4. *Let Q be an m-dimensional smooth manifold. Consider the manifold $\mathbb{R}\times T^*Q$ with coordinates (t, q^i, p_i) and the projection $\pi\colon \mathbb{R}\times T^*Q \to T^*Q$. The manifold $(\mathbb{R} \times T^*Q, \pi^*\omega, \mathrm{d}t)$ is a cosymplectic manifold.*

In case the manifold M is precosymplectic, we have the following Darboux's Theorem.

Theorem 1.2.5 (Darboux's Theorem for precosymplectic manifolds)**.** *If (M, ω, η) is a precosymplectic manifold of rank $2r$ and dimension $2m+1$, there exists a coordinate neighbourhood $\mathcal{U}$ at each point $x \in M$ with local coordinates (t, q^i, p_i, u^s), $1 \le i \le r, 1 \le s \le 2m - 2r$, such that*

$$\eta|_{\mathcal{U}} = \mathrm{d}t, \qquad \omega|_{\mathcal{U}} = \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

*Such a chart is called a **Darboux chart** or a **canonical chart**.*

Now let M be a precosymplectic manifold of rank $2r$. Then, there exists a vector field $\mathcal{R} \in \mathfrak{X}(M)$ such that

$$i_{\mathcal{R}}\omega = 0, \quad i_{\mathcal{R}}\eta = 1.$$

In fact, consider a partition of unity $(\mathcal{U}_\alpha, \varphi_\alpha)$ on M such that $(\mathcal{U}_\alpha, t_\alpha, q^i_\alpha, p^\alpha_i, u^s_\alpha)$ where $1 \leq i \leq r$, $1 \leq s \leq 2m - 2r$ is a Darboux chart on M. Then the local vector field

$$\mathcal{R}_\alpha = \frac{\partial}{\partial t_\alpha} \in \mathfrak{X}(\mathcal{U}_\alpha)$$

satisfies

$$i_{\mathcal{R}_\alpha}\omega = 0, \quad i_{\mathcal{R}_\alpha}\eta = 1.$$

Using the partition of unity, we can define global vector fields $\widetilde{\mathcal{R}}_\alpha$ as follows:

$$\widetilde{\mathcal{R}}_\alpha(x) = \begin{cases} \varphi_\alpha(x)\mathcal{R}_\alpha(x) & \text{if } x \in \mathcal{U}_\alpha, \\ 0 & \text{if } x \notin \mathcal{U}_\alpha. \end{cases}$$

Now we can construct a global vector field $\mathcal{R} = \sum_\alpha \widetilde{\mathcal{R}}_\alpha$ that satisfies

$$i_{\mathcal{R}}\omega = 0, \quad i_{\mathcal{R}}\eta = 1.$$

However, in the case of a precosymplectic manifold, the vector field $\mathcal{R}$ is not unique.

Hamiltonian systems

Definition 1.2.6. *Let (M, ω, η) be a cosymplectic manifold (resp. precosymplectic manifold) and $\gamma \in \Omega^1(M)$ be a closed form called the Hamiltonian 1-form. Then, $(M, \omega, \eta, \gamma)$ is called a* ***cosymplectic Hamiltonian system*** *(resp.* ***precosymplectic Hamiltonian system****).*

As in the autonomous case, in virtue of Poincaré's Lemma, we can put $\gamma|_{\mathcal{U}} = \mathrm{d}h$ for some $h \in \mathcal{C}^\infty(\mathcal{U})$ for every coordinate neighbourhood $\mathcal{U}$ of M. If, in addition, the 1-form γ is exact, we can put $\gamma = \mathrm{d}h$ for some $h \in \mathcal{C}^\infty(M)$.

Now given a cosymplectic Hamiltonian system $(M, \omega, \eta, \gamma)$, there exists a unique vector field $X_h \in \mathfrak{X}(M)$, called the **evolution vector field** that satisfies the equations

$$\begin{cases} \flat(X_h) = \gamma + (1 - \gamma(\mathcal{R}))\eta, \\ i_{X_h}\eta = 1, \end{cases} \tag{1.4}$$

which can also be written as

$$\begin{cases} i_{X_h}\omega = \gamma - \gamma(\mathcal{R})\eta, \\ i_{X_h}\eta = 1. \end{cases} \tag{1.5}$$

In order to write these equations in a more compact form, we can define the 2-form $\Omega \in \Omega^2(M)$ as

$$\Omega = \omega + \gamma \wedge \eta.$$

Using this form Ω, we can write equations (1.4) in the form

$$\begin{cases} i_{X_h}\Omega = 0 \\ i_{X_h}\eta = 1. \end{cases} \tag{1.6}$$

Consider a canonical chart $(\mathcal{U}, t, q^i, p_i)$ and let X be an arbitrary vector field on M, whose local expression in this chart is

$$X|_{\mathcal{U}} = A\frac{\partial}{\partial t}B^i\frac{\partial}{\partial q^i} + C_i\frac{\partial}{\partial p_i}.$$

Imposing that X must satisfy the system of equations (1.5), which in coordinates looks like

$$\begin{cases} i\left(A\frac{\partial}{\partial t} + B^i\frac{\partial}{\partial q^i} + C_i\frac{\partial}{\partial p_i}\right)(\mathrm{d}q^i \wedge \mathrm{d}p_i) = \frac{\partial h}{\partial q^i}\mathrm{d}q^i + \frac{\partial h}{\partial p_i}\mathrm{d}p_i, \\ i\left(A\frac{\partial}{\partial t} + B^i\frac{\partial}{\partial q^i} + C_i\frac{\partial}{\partial p_i}\right)(\mathrm{d}t) = 1, \end{cases}$$

we get that

$$\begin{cases} A = 1, \\ B^i = \frac{\partial h}{\partial p_i}, \\ C_i = -\frac{\partial h}{\partial q^i}. \end{cases}$$

Hence, the local expression of the evolution vector field X_h in a canonical chart is

$$X_h|_{\mathcal{U}} = \frac{\partial}{\partial t} + \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i}.$$

Remark 1.2.7. *Notice that if the Hamiltonian function h is time-independent, i.e., $\mathcal{R}(h) = \frac{\partial h}{\partial t} = 0$, we recover equations (1.1) of the autonomous case.*

1.2.2 Nonautonomous Lagrangian mechanics

In this subsection we generalize the autonomous Lagrangian systems to nonautonomous Lagrangian systems. We will briefly present the canonical structures of the manifold $\mathbb{R} \times TQ$. More information on this matter can be found in [1] and [12].

Consider an m-dimensional smooth manifold Q. In the autonomous case our phase space was the tangent bundle TQ. Hence, in the nonautonomous case we will use the product manifold $\mathbb{R} \times TQ$. We will denote by Δ the Liouville vector field on $\mathbb{R} \times TQ$ which has local expression on an adapted chart

$$\Delta = v^i \frac{\partial}{\partial v^i}.$$

We shall consider the extension of the vertical endomorphism J in an obvious way to $\mathbb{R} \times TQ$ which we denote by J and it has the same local expression $J = \frac{\partial}{\partial v^i} \otimes \mathrm{d}q^i$.

Definition 1.2.8. *A vector field $X \in \mathfrak{X}(\mathbb{R} \times TQ)$ is a **second order partial differential equation** if*

$$i_X \eta = 1, \qquad J(X) = \Delta.$$

Consider a time-dependent Lagrangian function $\mathcal{L} \colon \mathbb{R} \times TQ \to \mathbb{R}$. Using the vertical endomorphism J we can construct the forms $\theta_{\mathcal{L}}$, $\omega_{\mathcal{L}}$ as

$$\theta_{\mathcal{L}} = \mathrm{d}\mathcal{L} \circ J \qquad \text{and} \qquad \omega_{\mathcal{L}} = -\mathrm{d}\theta_{\mathcal{L}}.$$

With these forms we can write the equations

$$\begin{cases} i_X \Omega_{\mathcal{L}} = 0, \\ i_X \mathrm{d}t = 1, \end{cases} \tag{1.7}$$

where $\Omega_{\mathcal{L}} = \omega_{\mathcal{L}} + \mathrm{d}E_{\mathcal{L}} \wedge \mathrm{d}t$ is the **Poincaré-Cartan 2-form**, or equivalently,

$$\begin{cases} i_X \omega_{\mathcal{L}} = \mathrm{d}E_{\mathcal{L}} + \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t, \\ i_X \mathrm{d}t = 1. \end{cases}$$

We say the Lagrangian is **regular** if the matrix $(\partial^2 \mathcal{L} / \partial v^i \partial v^j)$ is nonsingular. If this is the case, $(\Omega_{\mathcal{L}}, \mathrm{d}t)$ is a cosymplectic structure on $\mathbb{R} \times TQ$ and hence equations (1.7) have a unique solution X.

Proposition 1.2.9. *Let L be a time-dependent Lagrangian on $\mathbb{R} \times TQ$ and $X \in \mathfrak{X}(\mathbb{R} \times TQ)$ the vector field solution of* (1.7)*. Then, X is a second order partial differential equation and its integral curves α are solutions of*

$$\frac{\partial \mathcal{L}}{\partial q^i} \circ \alpha - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial v^i} \circ \alpha\right) = 0,$$

*which are called the **Euler-Lagrange equations** for the time-dependent Lagrangian $\mathcal{L}$.*

However, we are interested on those Lagrangians which are not regular. We call these Lagrangians **singular Lagrangians**. In this case we cannot assure uniqueness nor even existence of global solutions for the system. However, sometimes we can find a submanifold $N \hookrightarrow M$ where we have existence (but not necessarily uniqueness!) of global solutions. This is what we will be dealing with in the next chapter.

1.3 k-symplectic formulation of classical field theories

The k-symplectic formulation of classical field theories is the simplest way of generalizing the symplectic formulation of classical mechanics. In this section we will see how to extend this formalism to the cases of Hamiltonian and Lagrangian field theories. It was introduced by A. Awane in 1992 [3]. See also [12] for details.

1.3.1 k-symplectic geometry

This first subsection is devoted to introducing k-symplectic manifolds, which are a natural generalization of symplectic manifolds.

Definition 1.3.1. *Let M be a smooth manifold. Suppose* $\dim M = m(k+1)$*. Let $\omega^1, \ldots, \omega^k$ be a family of closed 2-forms on M and let V be an integrable distribution of dimension mk such that*

(1) $\omega^\alpha|_{V \times V} = 0$*, for* $1 \leq \alpha \leq k$*,*

(2) $\bigcap_{\alpha=1}^k \ker \omega^\alpha = \{0\}$.

In this case, $(M, \omega^1, \ldots, \omega^k, V)$ *is a* k***-symplectic manifold***.

On the other hand, in the degenerate case, we have the following definition:

Definition 1.3.2. *A* k***-presymplectic manifold*** *is a family* $(M, \omega^1, \ldots, \omega^k)$ *where* M *is a manifold of dimension* $m(k+1)$ *and the* ω^α *are closed 2-forms on* M.

Remark 1.3.3. *In the case* $k = 1$*, a 1-symplectic (resp. 1-presymplectic) manifold* (M, ω^1) *is a symplectic (resp. presymplectic) manifold*

In the case of k-symplectic manifolds we have also a Darboux's Theorem that assures us the existence of some particular local coordinates.

Theorem 1.3.4 (Darboux's Theorem for k-symplectic manifolds)**.** *Let* $(M, \omega^1, \ldots, \omega^k, V)$ *be a* k*-symplectic manifold. For every* $x \in M$ *we can find a local chart* $(\mathcal{U}, q^i, p_i^\alpha)$*,* $1 \leq i \leq m$*,* $1 \leq \alpha \leq k$ *such that* $x \in \mathcal{U}$ *and*

$$\omega^\alpha = \mathrm{d}q^i \wedge \mathrm{d}p_i^\alpha$$

for every $1 \leq \alpha \leq k$*, and*

$$V = \left\langle \frac{\partial}{\partial p_i^\alpha}, 1 \leq i \leq m, 1 \leq \alpha \leq k \right\rangle.$$

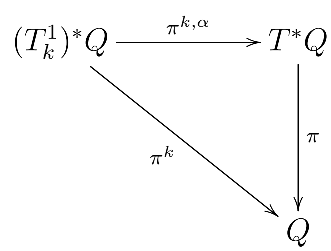
Proof. The proof of this theorem can be found in [3]. ■

It is an open problem to find necessary and sufficient conditions to a k-presymplectic manifold in order to assure the existence of Darboux coordinates.

Example 1.3.5. *The main example of* k*-symplectic manifold is the so-called* ***cotangent bundle of*** k^1***-covelocities*** *of an* m*-dimensional manifold* Q*:*

$$(T_k^1)^*Q = T^*Q \oplus_Q \stackrel{k}{\cdots} \oplus_Q T^*Q.$$

We can endow $(T_k^1)^*Q$ *with a* k*-symplectic structure using the natural projections*



by defining the forms $\theta^1, \dots, \theta^k \in \Omega^1((T^1_k)^*Q)$ *as*

$$\theta^\alpha = (\pi^{k,\alpha})^*\theta,$$

and the 2-forms $\omega^1, \dots, \omega^k$ *as*

$$\omega^\alpha = -\mathrm{d}\theta^\alpha,$$

or equivalently as $\omega^\alpha = (\pi^{k,\alpha})^*\omega$*, where* θ, ω *are respectively the Liouville 1-form and 2-form of the cotangent bundle* T^*Q*. It is an easy exercise to check that taking* $V = \ker(\pi^k)_*$*, we have that* $((T^1_k)^*Q, \omega^1, \dots, \omega^k, V)$ *is a* k*-symplectic manifold.*

1.3.2 k-vector fields and integral sections

In this subsection we introduce the concept of k-vector fields and we shall discuss what integrability means in the case of k-vector fields. These notions will be of great importance when talking about k-symplectic and k-cosymplectic field theories.

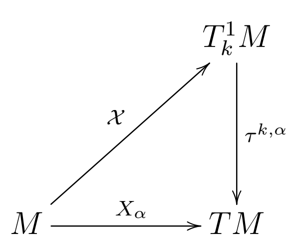
Let M be an m-dimensional smooth manifold. Consider its tangent bundle $\tau\colon TM \to M$. Now consider the **tangent bundle of k^1-velocities** defined as the Whitney sum

$$T^1_k M = TM \oplus_M \overset{k}{\cdots} \oplus_M TM.$$

with the canonical projection $\tau^k\colon T^1_k M \to M$.

Definition 1.3.6. *A* k***-vector field*** $\mathcal{X}$ *on* M *is a section of the projection* τ^k*. We will denote by* $\mathfrak{X}^k(M)$ *the set of all* k*-vector fields on* M*.*

Notice that using the diagram



we can decompose every k-vector field $\mathcal{X}$ as $\mathcal{X} = (X_1, \dots, X_k)$ where $X_\alpha \in \mathfrak{X}(M)$.

Definition 1.3.7. *Let $\mathcal{X} = (X_1, \ldots, X_k)$ be a k-vector field on a smooth manifold M. An **integral section** of $\mathcal{X}$ passing through $p \in M$ is a map $\varphi\colon \mathcal{U} \subset \mathbb{R}^k \to M$, such that $0 \in \mathcal{U}$ and such that*

(1) $\varphi(0) = p$,

(2) $\varphi_*(x)\left(\frac{\partial}{\partial x^\alpha}\Big|_x\right) = X_\alpha(\varphi(x))$

for every $x \in \mathcal{U}$ and for all $1 \leq \alpha \leq k$.

1.3.3 k-symplectic Hamiltonian formalism

Definition 1.3.8. *Let $(M, \omega^1, \ldots, \omega^k, V)$ be a k-symplectic manifold and let $\gamma \in \Omega^1(M)$ be a closed 1-form which will be called the Hamiltonian 1-form. Then, we say that $(M, \omega^1, \ldots, \omega^k, V, \gamma)$ is a k-**symplectic Hamiltonian system**.*

Given a k-symplectic manifold M, we can define the $\mathcal{C}^\infty(M)$-module morphism

$$\begin{array}{rccc} \flat\colon & \mathfrak{X}^k(M) & \longrightarrow & \Omega^1(M) \\ & \mathcal{X} & \longmapsto & i_{X_\alpha}\omega^\alpha \end{array} \tag{1.8}$$

where $\mathcal{X} = (X_1, \ldots, X_k)$. It can be checked that this morphism $\flat$ is surjective.

Consider the equation

$$\flat(\mathcal{X}) = i_{X_\alpha}\omega^\alpha = \gamma. \tag{1.9}$$

Consider an arbitrary k-vector field $\mathcal{X} = (X_\alpha) \in \mathfrak{X}^k(M)$, which in a canonical chart is expressed as

$$X_\alpha = (A_\alpha)^i \frac{\partial}{\partial q^i} + (B_\alpha)_i^\beta \frac{\partial}{\partial p_i^\beta}, \qquad 1 \leq \alpha \leq k.$$

Imposing equation (1.9), we get the conditions

$$\begin{cases} \dfrac{\partial h}{\partial p_i^\alpha} = (A_\alpha)^i, \\ \dfrac{\partial h}{\partial q^i} = -\sum_{\beta=1}^k (B_\beta)_i^\beta, \end{cases}$$

with $1 \leq i \leq m$ and $1 \leq \alpha \leq k$. We will denote by $\mathfrak{X}_h^k(M)$ the set of k-vector fields of M which are solution of equation (1.9).

Remark 1.3.9. *Notice that in the k-symplectic case, the existence of solutions of* (1.9) *is guaranteed, although in general we do not have uniqueness of solution. On the other hand, in the k-presymplectic case, we do not even have assured the existence of solutions. Later on, we will show an algorithm that allows us to find (if possible) a submanifold $N \hookrightarrow M$ where we can at least assure the existence of solutions.*

1.3.4 k-symplectic Lagrangian formalism

Consider the manifold $T^1_k Q$. Extending the case of the tangent bundle, we have a canonical k-tangent structure in $T^1_k Q$ given by a family $(J^1, \ldots, J^k)$ of (1,1)-tensor fields on $T^1_k Q$. The local expression of J^α in a canonical chart of $T^1_k Q$ is

$$J^\alpha|_{\mathcal{U}} = \frac{\partial}{\partial v^i_\alpha} \otimes \mathrm{d}q^i.$$

We can also construct a Liouville vector field Δ which has local expression

$$\Delta = \sum_{i=1}^{m} \sum_{\alpha=1}^{k} v^i_\alpha \frac{\partial}{\partial v^i_\alpha}.$$

Given a Lagrangian function $\mathcal{L}\colon T^1_k Q \to \mathbb{R}$, in a similar way as in Lagrangian mechanics, the k-tangent structure allows us to define k 1-forms $\theta^1_{\mathcal{L}}, \ldots, \theta^k_{\mathcal{L}}$ as

$$\theta^\alpha_{\mathcal{L}} = \mathrm{d}\mathcal{L} \circ J^\alpha.$$

Using these 1-forms, we can define a family $\omega^1_{\mathcal{L}}, \ldots, \omega^k_{\mathcal{L}}$ of presymplectic forms on $T^1_k Q$ by

$$\omega^\alpha_{\mathcal{L}} = -\mathrm{d}\theta^\alpha_{\mathcal{L}}.$$

Definition 1.3.10. *We say that a Lagrangian function $\mathcal{L}\colon T^1_k Q \to \mathbb{R}$ is regular if $(\omega^1_{\mathcal{L}}, \ldots, \omega^k_{\mathcal{L}}, V)$ is a k-symplectic structure on $T^1_k Q$ where*

$$V = \ker(\tau^k)_* = \left\langle \frac{\partial}{\partial v^1_1}, \ldots, \frac{\partial}{\partial v^m_k} \right\rangle.$$

Proposition 1.3.11. *A Lagrangian function $\mathcal{L}$ on $T^1_k Q$ is regular if and only if the matrix*

$$\left(\frac{\partial^2 \mathcal{L}}{\partial v^i_\alpha \partial v^j_\beta} \right)$$

is regular.

In the same way that we did in the Hamiltonian case, given a Lagrangian function $\mathcal{L}\colon T^1_kQ \to \mathbb{R}$, we consider the manifold T^1_kQ endowed with the 2-forms $\omega^1_{\mathcal{L}}, \dots, \omega^k_{\mathcal{L}}$, which in the case $\mathcal{L}$ is regular give T^1_kQ the structure of a k-symplectic manifold. Using the function $E_{\mathcal{L}} = \Delta(\mathcal{L}) - \mathcal{L}$, we can write the equation

$$i_{X_\alpha}\omega^\alpha_{\mathcal{L}} = \mathrm{d}E_{\mathcal{L}}. \tag{1.10}$$

We denote by $\mathfrak{X}^k_{\mathcal{L}}(T^1_kQ)$ the set of solutions of equation (1.10).

Theorem 1.3.12. *Let $\mathcal{L}\colon T^1_kQ \to \mathbb{R}$ be a Lagrangian function and $\mathcal{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k_h(T^1_kQ)$. Then,*

(1) If $\mathcal{L}$ is regular, then we have that $J^\alpha(X_\alpha) = \Delta_\alpha$ for every $1 \leq \alpha \leq k$. In other words, $\mathcal{X}$ is a second order partial differential equation. Moreover, if $\varphi\colon \mathbb{R}^k \to T^1_kQ$ is an integral section of $\mathcal{X}$, then the map $\phi = \tau^k \circ \psi\colon \mathbb{R}^k \to Q$ is a solution of equations

$$\left.\frac{\partial \mathcal{L}}{\partial q^i}\right|_{\phi^{(1)}(x)} - \sum_{\alpha=1}^{k} \left.\frac{\partial}{\partial x^\alpha}\right|_{x} \left(\left.\frac{\partial \mathcal{L}}{\partial v^i_\alpha}\right|_{\phi^{(1)}(x)} \right) = 0, \tag{1.11}$$

where $\phi^{(1)}$ denotes the first prolongation of $\phi\colon \mathbb{R}^k \to Q$.

(2) If $\mathcal{X} = (X_1, \dots, X_k)$ is integrable and $\phi^{(1)}\colon \mathbb{R}^k \to T^1_kQ$ is an integral section of $\mathcal{X}$ then $\phi\colon \mathbb{R}^k \to Q$ is a solution to equations (1.11)*.*

The differential equations (1.11) are the **Euler-Lagrange Field Equations** for the Lagrangian system $(T^1_kQ, \mathcal{L})$.

Remark 1.3.13. *Notice that if we put $k = 1$, equation* (1.10) *becomes the Euler-Lagrange equation of Lagrangian mechanics. An important difference between the case $k > 1$ and the case $k = 1$, is that in the case $k > 1$ we cannot assure the uniqueness of solutions.*

1.4 k-cosymplectic formulation of classical field theories

In this section we present the nonautonomous counterpart of k-symplectic field theory. k-cosymplectic field theory generalizes k-symplectic field theory in the same way as cosymplectic mechanics generalize symplectic mechanics. Some references on k-cosymplectic geometry and k-cosymplectic field theory are [10], [11] and [12].

1.4.1 k-cosymplectic geometry

Definition 1.4.1. *Let M be a manifold of dimension $k(m+1)+m$. A* ***k-cosymplectic structure*** *on M is a family $(\eta^\alpha, \omega^\alpha, V; 1 \leq \alpha \leq k)$, where each η^α is a closed 1-form, each ω^α is a closed 2-form and V is an integrable mk-dimensional distribution on M satisfying*

(1) $\eta^1 \wedge \cdots \wedge \eta^k \neq 0$, $\eta^\alpha|_V = 0$, $\omega^\alpha|_{V\times V} = 0$,

(2) $\left(\bigcap_{\alpha=1}^k \ker \eta^\alpha\right) \cap \left(\bigcap_{\alpha=1}^k \ker \omega^\alpha\right) = \{0\}$, $\dim \left(\bigcap_{\alpha=1}^k \ker \omega^\alpha\right) = k$.

Then, $(M, \eta^\alpha, \omega^\alpha, V)$ is said to be a ***k-cosymplectic manifold****.*

In particular, if $k = 1$, then $\dim M = 2m+1$ and (η^1, ω^1) is a cosymplectic structure on M.

Definition 1.4.2. *Let $(M, \eta^\alpha, \omega^\alpha, V)$ be a k-cosymplectic manifold. Then there exists a family of k vector fields $\{\mathcal{R}_\alpha\}$ which are called* ***Reeb vector fields****, characterized by the following conditions*

$$i_{\mathcal{R}_\alpha}\eta^\beta = \delta_\alpha^\beta, \qquad i_{\mathcal{R}_\alpha}\omega^\beta = 0.$$

Theorem 1.4.3 (Darboux Theorem for k-cosymplectic manifolds)**.** *Let $(M, \eta^\alpha, \omega^\alpha, V)$ be a k-cosymplectic manifold. Then around each point of M there exist local coordinates $(x^\alpha, q^i, p_i^\alpha)$ with $1 \leq \alpha \leq k, 1 \leq i \leq n$ such that*

$$\eta^\alpha = \mathrm{d}x^\alpha, \qquad \omega^\alpha = \mathrm{d}q^i \wedge \mathrm{d}p_i^\alpha, \qquad V = \left\langle \frac{\partial}{\partial p_i^1}, \ldots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1,\ldots,n}.$$

Proof. The proof of this theorem can be found in [10]. ■

These are called **Darboux** or **canonical coordinates** of the k-cosymplectic manifold M. Given a k-cosymplectic manifold $(M, \eta^\alpha, \omega^\alpha, V)$, we can define two vector bundle morphisms

$$\begin{array}{rccl} \widetilde{\flat}: & TM & \longrightarrow & (T_k^1)^*M \\ & X & \longmapsto & (i_{X_1}\omega^1 + (i_{X_1}\eta^1)\eta^1, \ldots, i_{X_k}\omega^k + (i_{X_k}\eta^1)\eta^k) \end{array}$$

and

$$\begin{array}{rccl} \flat: & T_k^1 M & \longrightarrow & T^*M \\ & \mathcal{X} & \longmapsto & i_{X_\alpha}\omega^\alpha + (i_{X_\alpha}\eta^\alpha)\eta^\alpha \end{array}$$

Remark 1.4.4. *Notice that* $\flat = \operatorname{tr}(\widetilde{\flat})$, *and hence in the case* $k = 1$ *we have that* $\flat = \widetilde{\flat}$ *which is the* $\flat$ *morphism defined for cosymplectic manifolds.*

Example 1.4.5. *Let* $(N, \varpi^\alpha, \mathcal{V})$ *be an arbitrary* k*-symplectic manifold. Then, denoting by*

$$\pi_{\mathbb{R}^k} \colon \mathbb{R}^k \times N \longrightarrow \mathbb{R}^k, \qquad \pi_N \colon \mathbb{R}^k \times N \longrightarrow N$$

the canonical projections, we consider the differential forms

$$\eta^\alpha = \pi^*_{\mathbb{R}^k}(\mathrm{d}x^k), \qquad \omega^\alpha = \pi^*_N \varpi^\alpha,$$

and the distribution $\mathcal{V}$ *in* N *defines a distribution* V *in* $M = \mathbb{R}^k \times N$ *in a natural way. All conditions given in Definition 1.4.1 are verified, and hence* $M = \mathbb{R}^k \times N$ *endowed with the* k*-cosymplectic structure* $(\eta^\alpha, \omega^\alpha, V)$ *is a* k*-cosymplectic manifold.*

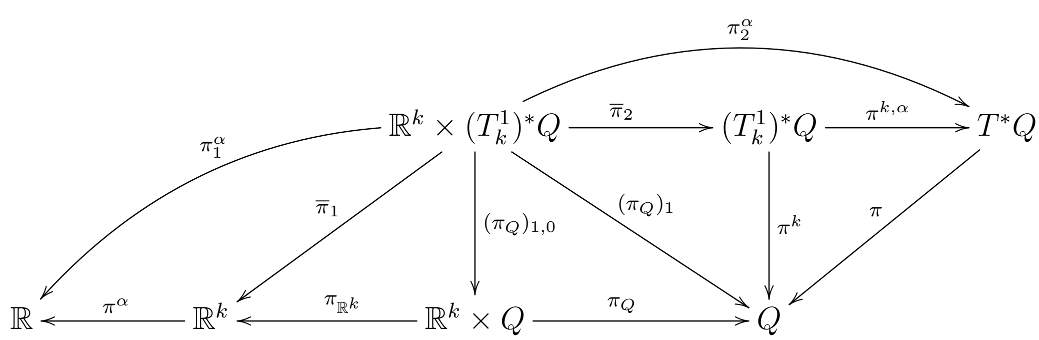
The canonical model of a k-cosymplectic manifold is the so called **stable cotangent bundle of k^1-covelocities** of an n-dimensional manifold Q

$$\mathbb{R}^k \times (T^1_k)^*Q$$

where $(T^1_k)^*Q$ is the Whitney sum of k copies of the cotangent bundle of Q, i.e. $(T^1_k)^*Q = T^*Q \oplus_Q \overset{(k)}{\cdots} \oplus_Q T^*Q$.

Thus, the elements of $\mathbb{R}^k \times (T^1_k)^*Q$ are of the form $(x, \nu_{1_q}, \dots, \nu_{k_q})$ where $x \in \mathbb{R}^k$, $q \in Q$ and $\nu_{\alpha_q} \in T^*_q Q$ where $1 \le \alpha \le k$.

In the following diagram we collect the projections we will use from now on:



If (q^i) with $1 \le i \le n$, is a local coordinate system defined on an open set $U \subset Q$, the induced **local coordinates** $(x^\alpha, q^i, p^\alpha_i)$, $1 \le i \le n$, $1 \le \alpha \le k$

on $\mathbb{R}^k \times (T^1_k)^*U = \big((\pi_Q)_1\big)^{-1}(U)$ are given by

$$\begin{aligned} x^\alpha(x,\nu_{1_q},\dots,\nu_{k_q}) &= x^\alpha(x) = x^\alpha, \\ q^i(x,\nu_{1_q},\dots,\nu_{k_q}) &= q^i(q), \\ p^\alpha_i(x,\nu_{1_q},\dots,\nu_{k_q}) &= \nu_{\alpha_k}\left(\frac{\partial}{\partial q^i}\bigg|_q\right). \end{aligned}$$

Thus, $\mathbb{R}^k \times (T^1_k)^*Q$ is endowed with a k-cosymplectic structure and thus it is a k-cosymplectic manifold of dimension $k + n(k+1)$, and the manifold $\mathbb{R}^k \times (T^1_k)^*Q$ with the projection $(\pi_Q)_1$ has the structure of a vector bundle over Q.

On $\mathbb{R}^k \times (T^1_k)^*Q$ we can define a family of canonical forms as follows

$$\eta^\alpha = (\pi^\alpha_1)^*\mathrm{d}x, \qquad \Theta^\alpha = (\pi^\alpha_2)^*\theta \quad \text{and} \quad \omega^\alpha = (\pi^\alpha_2)^*\omega,$$

with $1 \leq \alpha \leq k$, being $\pi^\alpha_1 \colon \mathbb{R}^k \times (T^1_k)^*Q \to \mathbb{R}$ and $\pi^\alpha_2 \colon \mathbb{R}^k \times (T^1_k)^*Q \to T^*Q$ the projections defined by

$$\pi^\alpha_1(x,\nu_{1_q},\dots,\nu_{k_q}) = x^\alpha, \qquad \pi^\alpha_2(x,\nu_{1_q},\dots,\nu_{k_q}) = \nu_{\alpha_q}$$

and θ and ω are the canonical Liouville and symplectic forms on T^*Q, respectively. Let us observe that, since $\omega = -\mathrm{d}\theta$, then $\omega^\alpha = -\mathrm{d}\theta^\alpha$.

If we consider a local coordinate system $(x^\alpha, q^i, p^\alpha_i)$ on $\mathbb{R}^k \times (T^1_k)^*Q$, the **canonical forms** η^α, θ^α and ω^α have the following local expressions:

$$\eta^\alpha = \mathrm{d}x^\alpha, \qquad \theta^\alpha = p^\alpha_i \mathrm{d}q^i, \qquad \omega^\alpha = \mathrm{d}q^i \wedge \mathrm{d}p^\alpha_i.$$

Moreover, let $V = \ker\big((\pi_Q)_{1,0}\big)_*$. Then, it is easy to see that in local coordinates the forms η^α and ω^α, with $1 \leq \alpha \leq k$, are closed and the following relations hold:

(1) $\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^k \neq 0$, $\mathrm{d}x^\alpha|_V = 0$, $\omega^\alpha|_{V\times V} = 0$,

(2) $\left(\bigcap_{\alpha=1}^k \ker \mathrm{d}x^\alpha\right) \cap \left(\bigcap_{\alpha=1}^k \ker \omega^k\right) = \{0\}$, $\dim\left(\bigcap_{\alpha=1}^k \ker \omega^k\right) = k$.

Remark 1.4.6. *Notice that the canonical forms on $(T^1_k)^*Q$ and $\mathbb{R}^k \times (T^1_k)^*Q$ are related by $(\overline{\pi}_2)^*$.*

1.4.2 k-cosymplectic Hamiltonian formalism

Definition 1.4.7. *Consider a k-cosymplectic $(M, \omega^\alpha, \eta^\alpha, V)$ and let $\gamma \in \Omega^1(M)$ be a closed 1-form on M, which will be called the Hamiltonian 1-form. The family $(M, \omega^\alpha, \eta^\alpha, V, \gamma)$ is a k-**cosymplectic Hamiltonian system**.*

Let $(M, \omega^\alpha, \eta^\alpha, V, \gamma)$ be a k-cosymplectic Hamiltonian system. We say that a k-vector field $\mathcal{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ is called a **k-cosymplectic Hamiltonian k-vector field** if it is solution of the system of equations

$$\begin{cases} i_{X_\alpha}\omega^\alpha = \gamma - \gamma(\mathcal{R}_\alpha)\eta^\alpha \\ i_{X_\beta}\eta^\alpha = \delta^\alpha_\beta. \end{cases} \tag{1.12}$$

We denote this fact by $\mathcal{X} \in \mathfrak{X}^k_h(M)$. Notice that, if we put $k = 1$, we recover equation (1.5). Using the $\flat$ morphism defined in the previous section, we can write equations (1.12) as

$$\begin{cases} \flat(\mathcal{X}) = \gamma + (1 - \gamma(\mathcal{R}_\alpha))\eta^\alpha \\ i_{X_\beta}\eta^\alpha = \delta^\alpha_\beta. \end{cases} \tag{1.13}$$

Consider an arbitrary k-vector field $\mathcal{X} = (X_\alpha) \in \mathfrak{X}^k(M)$, which in a canonical chart is expressed as

$$X_\alpha = (A_\alpha)_\beta \frac{\partial}{\partial t^\beta} + (B_\alpha)^i \frac{\partial}{\partial q^i} + (C_\alpha)^\beta_i \frac{\partial}{\partial p^\beta_i}, \qquad 1 \le \alpha \le k.$$

Imposing equation (1.12), we get the conditions

$$\begin{cases} (A_\alpha)_\beta = \delta^\beta_\alpha, \\ \dfrac{\partial h}{\partial p^\alpha_i} = (B_\alpha)^i, \\ \dfrac{\partial h}{\partial q^i} = -\sum_{\beta=1}^k (C_\beta)^\beta_i, \end{cases} \tag{1.14}$$

where $1 \le i \le m$ and $1 \le \alpha \le k$.

These equations always have a global solution that can be defined by pasting together local sections using an adequate partition of unity. However, the solution, in general, will not be unique. On the other hand, in the singular case, we will not even be able to assure existence of global solutions, but we will be able to find (if possible!) a submanifold $N \hookrightarrow M$ where we will have existence of solutions.

1.4.3 k-cosymplectic Lagrangian formalism

Consider the phase space $\mathbb{R}^k \times T^1_kQ$. We can trivially extend the canonical structure $\{J^\alpha\}_\alpha$ from T^1_kQ to $\mathbb{R}^k \times T^1_kQ$, denoting these new tensor fields also by J^α. Their local expression is

$$J^\alpha = \frac{\partial}{\partial v^i_\alpha} \otimes \mathrm{d}q^i.$$

In the same fashion, we can extend the Liouville vector fields $\Delta, \Delta_1, \ldots, \Delta_k$ from T^1_kQ to $\mathbb{R}^k \times T^1_kQ$, and they have the same local expression. Using these Liouville vector fields, we can define

Definition 1.4.8. *A k-vector field $\mathcal{X} \in \mathfrak{X}^k(\mathbb{R}^k \times T^1_kQ)$ is a **second order partial differential equation** (SOPDE) if*

(1) $J^\alpha(X_\alpha) = \Delta_\alpha$ for every $1 \leq \alpha \leq k$,

(2) $i_{X_\beta}\eta^\alpha = \delta^\alpha_\beta$ for every $1 \leq \alpha, \beta \leq k$.

In a very similar way as we did in the previous chapter for the k-symplectic approach, we can define a family of 1-forms $\theta^1_{\mathcal{L}}, \ldots, \theta^k_{\mathcal{L}} \in \Omega^1(\mathbb{R}^k \times T^1_kQ)$ from a Lagrangian function $\mathcal{L}\colon \mathbb{R}^k \times T^1_kQ \to \mathbb{R}$ by

$$\theta^\alpha_{\mathcal{L}} = \mathrm{d}\mathcal{L} \circ J^\alpha,$$

and from these 1-forms we can define the so-called **Poincaré-Cartan** 2-forms

$$\omega^\alpha_{\mathcal{L}} = -\mathrm{d}\theta^\alpha_{\mathcal{L}}.$$

Definition 1.4.9. *Let $\mathcal{L}$ a Lagrangian function on $\mathbb{R}^k \times T^1_kQ$. We say $\mathcal{L}$ is regular if and only if $(\mathrm{d}x^\alpha, \omega^\alpha_{\mathcal{L}}, V)$ is a k-cosymplectic structure on $\mathbb{R}^k \times T^1_kQ$, where*

$$V = \ker((\pi_{\mathbb{R}^k})_{1,0})_*.$$

Definition 1.4.10. *We say that a k-vector field $\mathcal{X}$ of $\mathbb{R}^k \times T^1_kQ$ is a **k-cosymplectic Lagrangian k-vector field** if it is a solution of equations*

$$\begin{cases} i_{X_\alpha}\omega^\alpha_{\mathcal{L}} = \mathrm{d}E_{\mathcal{L}} + \frac{\partial \mathcal{L}}{\partial x^\alpha}\mathrm{d}x^\alpha, \\ i_{X_\beta}\mathrm{d}x^\alpha = \delta^\alpha_\beta, \end{cases} \tag{1.15}$$

where $E_{\mathcal{L}} = \Delta(\mathcal{L}) - \mathcal{L}$. We denote by $\mathfrak{X}^k_{\mathcal{L}}(\mathbb{R}^k \times T^1_kQ)$ the set of all k-cosymplectic Lagrangian k-vector fields.

Equations (1.15) are called k**-cosymplectic Lagrangian equations**.

Notice that if $\mathcal{L}$ is regular, then $(\mathrm{d}x^\alpha, \omega_{\mathcal{L}}^\alpha, V)$ is a k-cosymplectic structure on $\mathbb{R}^k \times T_k^1Q$. We denote by $\mathcal{R}_\alpha^{\mathcal{L}}$ the corresponding Reeb vector fields. Hence, if we write the k-cosymplectic Hamilton equations for the system $(\mathbb{R}^k \times T_k^1Q, \mathrm{d}x^\alpha, \omega_{\mathcal{L}}^\alpha, \mathcal{L})$ we get

$$\begin{cases} i_{X_\alpha}\omega_{\mathcal{L}}^\alpha = \mathrm{d}E_{\mathcal{L}} - \mathcal{R}_\alpha^{\mathcal{L}}(E_{\mathcal{L}})\mathrm{d}x^\alpha, \\ i_{X_\beta}\mathrm{d}x^\alpha = \delta_\beta^\alpha, \end{cases} \tag{1.16}$$

which are equivalent to (1.15).

Remark 1.4.11. *If we consider the case $k = 1$, we can see that equations* (1.15) *become*

$$\begin{cases} i_X\omega_{\mathcal{L}} = \mathrm{d}E_{\mathcal{L}} + \frac{\partial\mathcal{L}}{\partial t}\mathrm{d}t, \\ i_X\mathrm{d}t = 1, \end{cases}$$

which are equivalent to the dynamical equations

$$\begin{cases} i_X\Omega_{\mathcal{L}} = 0, \\ i_X\mathrm{d}t = 1, \end{cases}$$

where $\Omega_{\mathcal{L}} = \omega_{\mathcal{L}} + \mathrm{d}E_{\mathcal{L}} \wedge \mathrm{d}t$ is the Poincaré-Cartan 2-form.

Chapter 2

Constraint algorithms for singular mechanics

This second chapter is devoted to the studying of constraint algorithms for singular mechanical systems. It contains the constraint algorithms for symplectic mechanics and cosymplectic mechanics.

2.1 A constraint algorithm for presymplectic mechanics

In this section we will review the Gotay-Nester-Hinds algorithm for singular symplectic mechanics [14] and [15]. Consider a $2m$-dimensional manifold M. Endow M with a presymplectic form $\omega \in \Omega^2(M)$. Consider a Hamiltonian 1-form $\gamma \in \Omega^1(M)$ (recall that γ must be closed). Hence, (M, ω, γ) is a presymplectic Hamiltonian system.

If the form ω is symplectic (nondegenerate), the morphism $\flat$ is an isomorphism and hence the equation

$$\flat(X) = i_X \omega = \gamma \tag{2.1}$$

has a unique solution $X = \sharp(\gamma) = \flat^{-1}(\gamma)$.

However, if the form ω is not degenerate, we do not have existence nor uniqueness of solutions to equation (2.1) on M. In this case, we want to find (if possible) a maximal submanifold $N \hookrightarrow M$ such that equation (2.1) has global solutions $X \in \mathfrak{X}(N)$ (not necessarily unique!). In order to do this, we can proceed in an algorithmic way adding some constraints in every iteration to obtain a chain of submanifolds

$$\cdots \hookrightarrow M_j \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow M,$$

where

$$M_j = \{p \in M_{j-1} \mid \exists X_p \in T_pM_{j-1} \text{ such that } i_{X_p}\omega_p = \gamma_p\}$$

The first constraint submanifold M_1 is the set of all the points in M where we have solutions of equation (2.1). However, these solutions are not necessarily tangent to this new submanifold M_1. Hence, we restrict ourselves to the submanifold $M_2 \hookrightarrow M_1$ made by those points where those solutions are tangent to M_1. Again, these solutions are not necessarily tangent to M_2. Hence, we iterate this procedure until we find (if it exists!) a submanifold $M_l = M_{l-1}$ such that $\dim M_l > 0$. If $\dim M_l = 0$, in this case, M_l is a set of discrete points and the solutions have no interest. However, if $\dim M_l > 0$, we call M_l the **final constraint submanifold** and we have solutions $X \in \mathfrak{X}(M_l)$ of (2.1) which are tangent to the final constraint submanifold M_l.

Proposition 2.1.1. *The constraint submanifolds can be characterized by*

$$M_j = \{p \in M_{j-1} \mid i_{Y_p}\gamma_p = 0,\ \forall Y_p \in \ker\omega \cap TM_{j-1}\}.$$

This proposition allows us to compute the constraint submanifolds using the **j-ary constraint functions** $i_{Y_p}\gamma_p$ where $Y_p \in \ker\omega \cap TM_{j-1}$

Notice that we can apply this algorithm not only to singular Hamiltonian systems, but also,for instance, we can apply it to every singular Lagrangian system $(TQ, \omega_{\mathcal{L}}, \mathcal{L})$ if the Lagrangian is singular (if it is regular we do not need the algorithm, because we already have existence and uniqueness of solution).

2.2 A constraint algorithm for precosymplectic mechanics

The aim of this section is to present a constraint algorithm to solve nonautonomous singular mechanical systems. The algorithm we present was developed by D. Chinea et al. in 1994 [5]. There are alternative presentations of this algorithm which are equivalent in [7] and [8].

2.2.1 Description of the algorithm

The purpose of this section is, given a singular cosymplectic Hamiltonian or Lagrangian system, to find a submanifold $N \hookrightarrow M$ such that we can assure the existence of global solutions of the Hamilton or Euler-Lagrange equations. Suppose for instance that we have a singular nonautonomous Lagrangian $\mathcal{L}\colon \mathbb{R} \times TQ \to \mathbb{R}$. This was studied by D. Chinea et al. in 1994 [5]. In this case, $(\Omega_{\mathcal{L}}, \mathrm{d}t)$ is not a cosymplectic structure on $\mathbb{R} \times TQ$, but in general it is not even precosymplectic. This situation leads us to define the following model.

We suppose that there exist closed forms $\Omega \in \Omega^2(M)$, $\eta \in \Omega^1(M)$ such that

$$\Omega^r \wedge \eta \neq 0, \quad \Omega^{r+1} \wedge \eta = 0, \quad \Omega^{r+2} = 0.$$

In such a case, we can deduce that $2r \leq \operatorname{rank} \Omega \leq 2r + 2$. Under these assumptions, we have the following result:

Proposition 2.2.1. *For every $p \in M$, $\operatorname{rank} \Omega_p = 2r$ if and only if there exists a tangent vector $v \in T_pM$ such that*

$$\begin{cases} i_v \Omega_p = 0, \\ i_v \eta_p = 1. \end{cases}$$

Proof. This proof can be found in [5]. ∎

Consider now the system of equations

$$\begin{cases} i_X \Omega = 0, \\ i_X \eta = 1, \end{cases} \tag{2.2}$$

where $X \in \mathfrak{X}(M)$. Now, taking into account Proposition 2.2.1, we can deduce that equations (2.2) have solution at a point $p \in M$ if and only if $\operatorname{rank} \Omega_p = 2r$. It is natural then to take as our first constraint manifold

$$M_1 = \{p \in M \mid \operatorname{rank} \Omega_p = 2r\}.$$

At this point we need to suppose that M_1 is a submanifold of M. In this case, we denote by j_1 the natural embedding $M_1 \overset{j_1}{\hookrightarrow} M$. Now it is clear that $(j_1^*\Omega)^{r+1} = 0$, but it is not true in general that $(j_1^*\Omega)^r \wedge j_1^*\eta \neq 0$ and then $(j_1^*\Omega, j_1^*\eta)$ is not always a precosymplectic structure on M_1. The only thing that is clear is that $\operatorname{rank}(j_1^*\Omega) \leq 2r$.

Consider now the morphism $\flat$ defined in (1.3):

$$\begin{array}{rccl} \flat\colon & \mathfrak{X}(M) & \longrightarrow & \Omega^1(M) \\ & X & \longmapsto & i_X\Omega + (i_X\eta)\eta. \end{array}$$

From Proposition 2.2.1 we can get

Proposition 2.2.2. *Given $p \in M_1$, the system of equations* (2.2) *have a solution $v \in T_pM_1$ if and only if $\eta_p \in \flat(T_pM_1)$.*

We have that $\operatorname{rank}\Omega = 2r$ at every point of M_1. Hence, there exists a vector field X on M such that X is a solution of (2.2) on M_1. However, the vector field X must be a vector field on M_1, i.e., for every $p \in M_1$, X_p must be in T_pM_1 and not only on T_pM. Hence, we restrict ourselves to the submanifold M_2 defined as

$$M_2 = \{p \in M_1 \,|\, \eta_p \in \flat(T_pM_1)\}.$$

We denote by j_2 the corresponding embedding $M_2 \overset{j_2}{\hookrightarrow} M_1$. In this way we obtain a vector field Y on M_1 solution of (2.2) on M_2 but, again, this vector field is not necessarily tangent to M_2. Hence, we must iterate this procedure and we obtain a sequence of submanifolds

$$\cdots \hookrightarrow M_j \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow M,$$

where each constraint submanifold is defined by

$$M_j = \{p \in M_{j-1} \,|\, \eta_p \in \flat(T_pM_{j-1})\}, \quad j \geq 2.$$

We call this submanifold M_j the j**-ary constraint manifold**.

This procedure can end up in three different ways. It may happen that at some point, $M_k = \emptyset$. In this case, the system of equations (2.2) has no solution. It may also happen that we get a submanifold such that $\dim M_k = 0$, in this case the manifold M_k consists of isolated points and the dynamics on this case have no interest. Finally, the interesting case is when there exists $l \geq 1$ such that $M_l = M_{l+1}$ and hence $M_l = M_{l+i}$ for every $i > 0$. In this situation we have a vector field X on M_l such that

$$\begin{cases} i_X\Omega = 0, \\ i_X\eta = 1 \end{cases}$$

on M_l. In this case we call M_l the **final constraint manifold**.

Summing up, we have constructed an iterative procedure that theoretically gives us in favorable cases a final constraint submanifold. However, in this way it cannot even be computed. Hence we will devote the next subsection to give an alternative description of the algorithm in such a way that it will be computable. By computable, we mean that it will allow us to explicitly obtain the constraints at each step of the algorithm.

2.2.2 Alternative description of the algorithm

Consider a precosymplectic manifold (M, ω, η) of rank $2r$. Hence we have that the 2-form ω has constant rank $2r$ and $\omega^r \wedge \eta \neq 0$. Given a 1-form $\gamma \in \Omega^1(M)$, we consider the 2-form

$$\Omega = \omega + \gamma \wedge \eta.$$

This 1-form γ takes the role of the Hamiltonian 1-form in the Hamiltonian formulation while it represents $\mathrm{d}E_{\mathcal{L}}$ in the Lagrangian counterpart. We will suppose that Ω is closed. This is not asking too much because if ω, η and γ are closed, so is Ω. ω and η are closed because they are a precosymplectic structure on M. On the other hand, both in the Hamiltonian and in the Lagrangian cases, the form γ will be exact and hence closed. Notice that since $\omega^r \neq 0$ and $\omega^{r+1} = 0$, we have that

$$\Omega^r \wedge \eta \neq 0, \quad \Omega^{r+1} \wedge \eta = 0, \quad \Omega^{r+2} = 0.$$

In addition, we suppose there exists a vector field $\mathcal{R} \in \mathfrak{X}(M)$ such that

$$\begin{cases} i_{\mathcal{R}}\omega = 0, \\ i_{\mathcal{R}}\eta = 1. \end{cases}$$

Under all these hypothesis, we have

Proposition 2.2.3. *Let $v \in T_pM$. Then we have that*

$$\begin{cases} i_v\Omega_p = 0, \\ i_v\eta_p = 1 \end{cases}$$

if and only if

$$\begin{cases} i_v\omega_p = \gamma_p - \gamma_p(\mathcal{R}_p)\eta_p, \\ i_v\eta_p = 1. \end{cases}$$

Proof. The proof of this result can be found in [5]. ■

We have that $\operatorname{rank}\omega = 2r$. We can construct a basis $\mathcal{B} = \{e^1, \ldots, e^r, e^{r+1}, \ldots, e^{2r}, \eta_p, u^1, \ldots,$ of T_p^*M such that in this basis,

$$\omega_p = \sum_{i=1}^{r} e^i \wedge e^{r+i}.$$

Remark 2.2.4. *Without loss of generality, we can assume that $u^\alpha(\mathcal{R}_p) = 0$, because if $u^\alpha(\mathcal{R}_p) \neq 0$, it is enough to take $\bar{u}^\alpha = u^\alpha - u^\alpha(\mathcal{R}_p)\eta_p$.*

Hence, the corresponding dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \{e_1, \ldots, e_r, e_{r+1} \ldots, e_{2r}, \mathcal{R}_p, u_1, \ldots, u_{2n-2r}\}$ With all this in mind it is easy to prove

Proposition 2.2.5. *There exists a tangent vector $v \in T_pM$ satisfying*

$$\begin{cases} i_v\omega_p = \gamma_p - \gamma_p(\mathcal{R}_p)\eta_p, \\ i_v\eta_p = 1, \end{cases}$$

if and only if

$$\gamma_p \in (\ker\omega_p \cap \ker\eta_p)^0.$$

Proof. The proof of this result can be found in [5]. ■

Taking into account Proposition 2.2.1, 2.2.3, 2.2.5 and the definition of M_1 given before, we have the following corollary:

Corollary 2.2.6. *The first constraint manifold can also be characterized by*

$$M_1 = \{p \in M \mid \gamma_p \in (\ker\omega_p \cap \ker\eta_p)^0\}.$$

The conditions

$$\langle \ker\omega_p \cap \ker\eta_p, \gamma_p \rangle = 0,$$

are called **primary constraints**. The functions $i_Z\gamma$, where $Z \in \ker\omega_p \cap \ker\eta_p$, are also called primary constraints.

Now let $p \in M_1$. Consider the anihilator of $\flat(T_pM_1)$,

$$(\flat(T_pM_1))^0 = \{v \in T_pM \mid \langle v, \flat(T_pM_1)\rangle = 0\}.$$

If $u \in T_pM_1$, $v \in (\flat(T_pM_1))^0$, we have that

$$(\flat(u))(v) = (i_u\Omega_p + \eta_p(u)\eta_p)(v) = (-i_v\Omega_p + \eta_p(v)\eta_p)(v) = 0,$$

and hence,

$$(\flat(T_pM_1))^0 = \{v \in T_pM \mid \langle T_pM_1, -i_v\Omega_p + \eta_p(v)\eta_p\rangle = 0\}.$$

And now, defining the **cosymplectic orthogonal** of T_pM_1 as

$$(T_pM_1)^\perp = \{v \in T_pM \mid \langle T_pM_1, -i_v\Omega_p + \eta_p(v)\eta_p\rangle = 0\},$$

we can deduce that

$$\flat(T_pM_1) = ((T_pM_1)^\perp)^0.$$

With all this, we have already proved

Proposition 2.2.7. *The constraint submanifold M_2 can be characterized as*

$$M_2 = \{p \in M_1 \mid \eta_p \in ((T_pM_1)^\perp)^0\} = \{p \in M_1 \mid \langle (T_pM_1)^\perp, \eta_p\rangle = 0\}.$$

We will call the conditions $\langle (T_pM_1)^\perp, \eta_p\rangle = 0$ **secondary constraints**. The functions $i_v\eta$, where $v \in (T_pM_1)^\perp$ will also be called secondary constraints. Now, iterating this procedure we obtain the following.

Proposition 2.2.8. *The $(j+1)$-ary constraint manifold can be characterized as*

$$M_{j+1} = \{p \in M_j \mid \eta_p \in ((T_pM_j)^\perp)^0\} = \{p \in M_j \mid \langle (T_pM_j)^\perp, \eta_p\rangle = 0\},$$

where

$$(T_pM_j)^\perp = \{v \in T_pM \mid \langle T_pM_j, -i_v\Omega_p + \eta_p(v)\eta_p\rangle = 0\},$$

for every $j \geq 2$.

We call the conditions $\langle (T_pM_j)^\perp, \eta_p\rangle = 0$ **$(j+1)$-ary constraints**. The functions $i_v\eta$, where $v \in (T_pM_j)^\perp$, are also called $(j+1)$-ary constraints. We are going to close this subsection giving an alternative proof of the existence of solutions in the case the algorithm ends on a final constraint submanifold M_k.

Proposition 2.2.9. *Suppose that the constraint algorithm ends in a submanifold M_k, then the system of equations*

$$\begin{cases} i_X\Omega = 0, \\ i_X\eta = 1 \end{cases} \tag{2.3}$$

possesses solutions tangent to M_k if and only if

$$\langle (TM_k)^\perp, \eta\rangle = 0.$$

Proof. First we suppose that there exists a vector field $X \in \mathfrak{X}(M_k)$ such that the system of equations (2.3) holds on M_k. Now, given $v \in (T_pM_k)^\perp$,

$$0 = (-i_v\Omega + \eta(v)\eta)(X_p) = \eta(v).$$

Conversely, let us suppose that $\langle (TM_k)^\perp, \eta\rangle = 0$. Hence, $\eta|_{M_k} \in ((TM_k)^\perp)^0 = \flat(TM_k)$. Then, there exists a vector field $X \in \mathfrak{X}(M_k)$ such that $\flat(X) = i_X\Omega + \eta(X)\eta = \eta$. Hence, we have that $i_X\Omega = 0$ and $i_X\eta = 1$, since $\operatorname{rank}\Omega = 2r$ on M_k. ■

It is important to point out that this previous result holds for any submanifold $N \hookrightarrow M$. Actually, given a submanifold N of M, there exists a solution of the system (2.3) on N if and only if

$$\langle (TN)^\perp, \eta\rangle = 0,$$

where

$$(TN)^\perp = \{v \in TM \mid \langle TN, -i_v\Omega + (i_v\eta)\eta\rangle = 0\}.$$

Taking into account what we just pointed out, we could ask ourselves if the final constraint submanifold M_k obtained with the constraint algorithm is maximal. This is what states the following theorem.

Theorem 2.2.10 (Maximality of the final constraint manifold)**.** *Given a submanifold $N \overset{j}{\hookrightarrow} M$ and a vector field $X \in \mathfrak{X}(N)$ solution of* (2.3) *on N, then we have that $j(N) \subseteq M_k$.*

Proof. The proof of this result can be found on [5]. ■

2.2.3 Constraint algorithm for degenerate time-dependent systems

Now that we have the general algorithm for a precosymplectic manifold, it is time to apply it in the particular cases of singular nonautonomous Hamiltonian and Lagrangian systems.

Hamiltonian systems

Consider a presymplectic manifold (M, ω) of rank $2r$. Now we can take the product manifold $\mathbb{R} \times M$ with the nonautonomous Hamiltonian function

$h\colon \mathbb{R}\times M \to \mathbb{R}$. With all this we can construct the precosymplectic Hamiltonian system $(\mathbb{R}\times M, \Omega, \mathrm{d}t, h)$ where $\Omega = \omega + \mathrm{d}h\wedge \mathrm{d}t$, because it is easy to check that

$$\Omega^r \wedge \mathrm{d}t \neq 0, \quad \Omega^{r+1}\wedge \mathrm{d}t = 0, \quad \Omega^{r+2} = 0,$$

and thus we have that $2r \leq \operatorname{rank}\Omega \leq 2r+2$. Moreover we have that

$$\begin{cases} i_{\frac{\partial}{\partial t}}\omega = 0, \\ i_{\frac{\partial}{\partial t}}\mathrm{d}t = 1. \end{cases}$$

In this situation, we can apply the algorithm described before in order to obtain a sequence of constraints submanifolds

$$\cdots \hookrightarrow M_3 \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow \mathbb{R}\times M,$$

defined by

$$M_1 = \{p\in \mathbb{R}\times M \,|\, \langle \ker\omega_p \cap \ker \mathrm{d}_p t, \mathrm{d}_p h\rangle = 0\},$$
$$M_{j+1} = \{p \in M_j \,|\, \langle (T_pM_j)^\perp, \mathrm{d}_p t\rangle = 0\},$$

where

$$(T_pM_j)^\perp = \{v\in T_p(\mathbb{R}\times M) \,|\, \langle T_pM_j, -i_v\Omega_p + (i_v\mathrm{d}_pt)\mathrm{d}_pt\rangle = 0\}, \qquad j\geq 1.$$

Lagrangian systems

Now let us consider a degenerate nonautonomous Lagrangian $\mathcal{L}\colon \mathbb{R}\times TQ \to \mathbb{R}$ and suppose that $(\omega_{\mathcal{L}}, \mathrm{d}t)$ is a precosymplectic structure on $\mathbb{R}\times TQ$. Moreover $\frac{\partial}{\partial t} \in \mathfrak{X}(\mathbb{R}\times TQ)$ is a global vector field satisfying

$$\begin{cases} i_{\frac{\partial}{\partial t}}\omega_{\mathcal{L}} = 0, \\ i_{\frac{\partial}{\partial t}}\mathrm{d}t = 1. \end{cases}$$

In this case, we can apply the constraint algorithm described before to obtain a sequence of submanifolds

$$\cdots \hookrightarrow P_3 \hookrightarrow P_2 \hookrightarrow P_1 \hookrightarrow \mathbb{R}\times TQ,$$

defined as

$$P_1 = \{p\in \mathbb{R}\times TQ \,|\, \langle \ker\omega_{\mathcal{L}p}\cap \ker \mathrm{d}_pt, \mathrm{d}_pE_{\mathcal{L}}\rangle = 0\},$$

$$P_{j+1} = \{p \in P_j \,|\, \langle (T_pP_j)^\perp, \mathrm{d}_pt\rangle = 0\},$$

where

$$(T_pP_j)^\perp = \{v \in T_p(\mathbb{R}\times TQ) \,|\, \langle T_pP_j, -i_v\Omega_{\mathcal{L}p} + (i_v\mathrm{d}_pt)\mathrm{d}_pt\rangle = 0\}, \qquad j \geq 1.$$

Chapter 3

Constraint algorithms for singular field theories

In this chapter we present the algorithm developed by X. Gràcia en al. in 1994 to find in favorable cases a final constraint submanifold for k-presymplectic field theories. We also develop an analogous algorithm for the nonautonomous counterpart: k-precosymplectic field theory.

3.1 A constraint algorithm for k-presymplectic field theories

The algorithm described in Section 2.1 can be generalized to k-presymplectic field theories [16]. In this section we will focus on the techniques used to do it, as these are the same techniques we will use to generalize the algorithms in [5], [7] and [8] to the case of k-precosymplectic field theories.

Definition 3.1.1. *A* ***k-presymplectic Hamiltonian system*** *is a family* $(M, \omega^\alpha, \gamma)$ *where* (M, ω^α) *is a k-presymplectic manifold and* γ *is a closed* 1*-form called the* ***Hamiltonian 1-form****.*

Consider a k-presymplectic Hamiltonian system $(M, \omega^\alpha, \gamma)$. We consider the Hamilton equation

$$\flat(\mathcal{X}) = i_{X_\alpha}\omega^\alpha = \gamma, \tag{3.1}$$

where $\mathcal{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$. If the system was k-symplectic, the morphism $\flat \colon \mathfrak{X}^k(M) \to \Omega^1(M)$ defined in (1.8) would be surjective and hence we would have existence (not uniqueness) of global solutions of equation (3.1). However, this is not always the case, and we do not have the existence of global solutions assured. We will try to find a submanifold $N \hookrightarrow M$ where we can assure the existence of global solutions.

Given a k-presymplectic Hamiltonian system $(M, \omega^\alpha, \gamma)$ we want to find a submanifold $N \hookrightarrow M$ and k-vector fields $\mathcal{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$ such that equation (3.1) holds on N and such that $\mathcal{X}$ is tangent to N (which is the same as asking $X_1, \ldots, X_k$ to be tangent to N).

Before we begin the description of the algorithm we need to introduce some concepts. Given a submanifold $N \subseteq M$ we can extend the natural embedding $j \colon N \hookrightarrow M$ to $T^k_1 j \colon T^1_k N \to T^1_k M$ in a natural way. We will denote its image as $\underline{T^1_k N} = T^1_k j(T^1_k N) \subseteq T^1_k M$.

Using the morphism $\flat$ previously defined, we denote by $(TN)^\perp$ the **annihilator** of the image of $\underline{T^1_k N}$ by $\flat$:

$$(TN)^\perp = \left(\flat(\underline{T^1_k N})\right)^0 = \left\{ X_p \in T_p M \mid \text{ for all } (Y_{p1}, \ldots, Y_{pk}) \in \underline{T^1_k N}\,,\ \langle i(Y_{p_\alpha})\omega^\alpha_p, X_p \rangle = 0 \right\}.$$

We say that $(TN)^\perp$ is the **k-presymplectic orthogonal complement** of $\underline{T^1_k N}$ in $T^1_k M$. It is easy to check that in the case $N = M$,

$$(TM)^\perp = \bigcap_{\alpha=1}^{k} \ker \omega^\alpha.$$

The following theorem will allow us to use the algorithm to do computations in particular cases.

Theorem 3.1.2. *Let $N \hookrightarrow M$ be a submanifold. Then, the following conditions are equivalent:*

(1) there exists a k-vector field $\mathcal{X} = (X_1, \ldots, X_k)$, tangent to N, such that

$$i_{X_\alpha}\omega^\alpha = \gamma.$$

(2) for every $p \in N$ and every $Y_p \in (T_p N)^\perp$, we have that

$$i_{Y_p}\gamma_p = 0.$$

Proof. The proof of this result can be found in [16]. ∎

Now we are ready to use the preceeding result to describe an algorithmic procedure which will give us a sequence of subsets,

$$\cdots \hookrightarrow M_j \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow M,$$

which we will suppose to be submanifolds. The first submanifold, $M_1 \subset M$, is defined as

$$M_1 = \{p \in M \,|\, \exists \mathcal{X}_p = (X_1, \dots, X_k)_p \in T^1_k M \text{ such that } i(X_{\alpha p})\omega^\alpha_p = \gamma_p\}.$$

So we restrict ourselves to the submanifold M_1 where there exist k-vector fields satisfying equation (3.1). However, in general, these k-vector fields solution of (3.1) will not be tangent to M_1. In this case, we need to consider the submanifold

$$M_2 = \{p \in M_1 \,|\, \exists \mathcal{X}_p = (X_1, \dots, X_k)_p \in T^1_k M_1 \text{ such that } i(X_{\alpha p})\omega^\alpha_p = \gamma_p\}.$$

Again, the solutions may not be tangent to M_2, so we need to iterate this procedure. By doing this, we obtain a sequence of submanifolds,

$$\cdots \hookrightarrow M_j \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow M,$$

which we will call **constraint submanifolds**. As in the presymplectic case discussed in the preceeding chapter, this procedure may end in a **final constraint submanifold** M_l with $\dim M_l > 0$ (this is the interesting case) or may end in $\dim M_l = 0$ or $M_l = \emptyset$.

Now, taking into account Theorem 3.1.2, we see that each constraint submanifold can also be defined as

$$M_j = \{p \in M_{j-1} \,|\, i(Y_p)\gamma_p = 0 \text{ for every } Y_p \in (T_p M_{j-1})^\perp\},$$

where $M_0 = M$.

We will denote by $\mathfrak{X}(M_j)^\perp$ the set of vector fields Y in M such that $Y_p \in (T_p M_j)^\perp$. We can obtain constraint functions $\{f_\mu\}$ defining each M_j from a local basis $\{Z_1, \dots, Z_r\}$ of $\mathfrak{X}(M_{j-1})^\perp$ by setting $f_\mu = i_{Z_\mu}\gamma$. With all this in mind, we can proceed to describe the k**-presymplectic constraint algorithm**:

(1) Obtain a local basis $\{Z_1, \dots, Z_r\}$ of the submodule $\bigcap_{\alpha=1}^k \ker \omega^\alpha$.

(2) Use Theorem 3.1.2 to obtain a set of independent constraints $f_\mu = i_{Z_\mu}\gamma$ defining the submanifold $M_1 \hookrightarrow M$.

(3) Compute solutions $\mathcal{X} = (X_1, \dots, X_k)$ of (3.1) on M_1.

(4) Impose the tangency condition of $X_1, \dots, X_k$ on the constraints f_μ.

(5) Iterate last item until no new constraints appear.

Notice that it is Theorem 3.1.2 what allows us to do actual computations with concrete cases.

Remark 3.1.3. *In particular, this algorithm works for singular Hamiltonian systems of the type* $((T^1_k)^*Q, \omega^1, \dots, \omega^k, h)$ *and also for singular Lagrangian systems* $(T^1_k Q, \omega^1_{\mathcal{L}}, \dots, \omega^k_{\mathcal{L}}, \mathcal{L})$.

In particular, if we put $k = 1$ in this algorithm, we recover the Gotay-Nester-Hinds algorithm for presymplectic manifolds.

3.2 A constraint algorithm for k-precosymplectic field theories

This section is devoted to the study of singular k-precosymplectic field theories and, in particular, to the development of a constraint algorithm in order to obtain a submanifold where we can assure the existence of solutions of the problem. We begin defining the k-precosymplectic manifolds as a generalization of k-presymplectic manifolds and introducing a particular kind of coordinates in them (although the precise necessary and sufficient conditions to ensure their existence is still unknown). We also prove the existence of Reeb vector fields in k-presymplectic manifolds and find a particular type of k-precosymplectic manifolds where they are uniquely determined.

With all this in mind, we proceed to develop a constraint algorithm which will allow us to find (when possible!) a submanifold $N \hookrightarrow M$ where we can assure the existence of global solutions tangent to N to our problem.

3.2.1 k-precosymplectic manifolds

We begin by introducing our main object of study: k-precosymplectic manifolds, which are a generalization of k-cosymplectic manifolds described in Section 1.4.

Definition 3.2.1. *A k-**precosymplectic manifold** is a family $(M, \omega^\alpha, \eta^\alpha, V)$ where M is a manifold of dimension $k + m(k+1)$, $\omega^\alpha \in \Omega^2(M)$ and $\eta^\alpha \in \Omega^1(M)$ where $\alpha = 1, \dots, k$ are closed and V is an integrable mk-dimensional distribution on M such that*

(1) $\eta^1 \wedge \cdots \wedge \eta^k \neq 0$,

(2) $\eta^\alpha|_V = 0$, $\omega^\alpha|_{V\times V} = 0$ *for every* $\alpha = 1, \dots, k$.

Example 3.2.2. *As in the regular case, consider a k-presymplectic manifold (P, ϖ^α). Then, the product manifold $\mathbb{R}^k \times P$ is a k-precosymplectic manifold taking $\eta^\alpha = \tau^* \mathrm{d}t^\alpha$ where t^α are the canonical coordinates in $\mathbb{R}^k$ and τ is the canonical projection $\mathbb{R}^k \times P \xrightarrow{\tau} \mathbb{R}^k$ and $\omega^\alpha = \pi^*\varpi^\alpha$ where π is the canonical projection $\mathbb{R}^k \times P \xrightarrow{\pi} P$. In the description of the algorithm, we will ask our manifolds to be of this type in order to have the problem well defined.*

Now we are going to describe a particular kind of local coordinates, that we will call **Darboux coordinates**. We begin by setting $\operatorname{rank}\omega^\alpha = 2r_\alpha$. Notice that in the regular case, $r_\alpha = m$ for every $\alpha = 1, \dots, k$, however this is not true in the singular case.

Given $p \in M$, a Darboux chart is an open neighbourhood $\mathcal{U} \subset M$ of p and a set of coordinates $\{t^\alpha, q^i, p^\alpha_{i_\alpha}; u^j\}$ where

(1) $\alpha = 1, \dots, k$,

(2) $I_\alpha \subset \{1, \dots, m\}$ such that $|I_\alpha| = r_\alpha$,

(3) $i \in \bigcup_\alpha I_\alpha$,

(4) $i_\alpha \in I_\alpha$,

(5) $j = 1, \dots, d$, where $d = m(k+1) - \left|\bigcup_\alpha I_\alpha\right| - \sum_\alpha |I_\alpha|$.

such that in these coordinates,

$$\omega^\alpha|_{\mathcal{U}} = \sum_{i\in I_\alpha} \mathrm{d}q^i \wedge \mathrm{d}p^\alpha_i \qquad \text{and} \qquad \eta^\alpha|_{\mathcal{U}} = \mathrm{d}t^\alpha.$$

In Definition 3.2.1 we have imposed the condition of the existence of a distribution V because it is precisely the existence of this distribution what assures the existence of Darboux coordinates in the regular case. However, it is an open problem to characterize the necessary and sufficient conditions for these coordinates to exist. Hence, we will assume the existence of Darboux coordinates around every point.

Recall that the Hamilton equations for a k-cosymplectic Hamiltonian system are

$$\begin{cases} i_{X_\alpha}\omega^\alpha = \gamma - \gamma(\mathcal{R}_\alpha)\eta^\alpha, \\ i_{X_\beta}\eta^\alpha = \delta^\alpha_\beta, \end{cases} \tag{3.2}$$

where the $\mathcal{R}_\alpha$ are the Reeb vector fields which in the regular case where uniquely determined. Now we are going to prove that in the singular case we can also asure the existence of Reeb vector fields, although they will not be unique.

Proposition 3.2.3. *Given a k-precosymplectic manifold $(M, \omega^\alpha, \eta^\alpha, V)$ with Darboux charts, there exists a family $Y_1, \dots, Y_k \in \mathfrak{X}(M)$ of vector fields satisfying*

$$\begin{cases} i_{Y_\alpha}\omega^\beta = 0, \\ i_{Y_\alpha}\eta^\beta = \delta^\beta_\alpha. \end{cases}$$

Proof. Consider a partition of unity $\{(\mathcal{U}_b, \psi_b)\}_{b\in\Lambda}$ on M such that on every $\mathcal{U}_b$ we have Darboux coordinates $\{t^\alpha_b, q^i_b, p^\alpha_{i_\alpha,b}; u^j_b\}$. Consider now the local vector fields $Y^b_\alpha = \frac{\partial}{\partial t^\alpha_b}$. These vector fields satisfy

$$\begin{cases} i_{Y^b_\alpha}\omega^\beta = 0, \\ i_{Y^b_\alpha}\eta^\beta = \delta^\beta_\alpha \end{cases}$$

on U_b. Using these vector fields, we can define global vector fields

$$\widetilde{Y}^b_\alpha(p) = \begin{cases} \psi_b(p)Y^b_\alpha(p), & \text{if } p \in \mathcal{U}_b, \\ 0 & \text{if } p \notin \mathcal{U}_b. \end{cases}$$

With these global vector fields we can construct global vector fields $Y_\alpha = \sum_b \widetilde{Y}^b_\alpha$ which satisfies

$$\begin{cases} i_{Y_\alpha}\omega^\beta = 0, \\ i_{Y_\alpha}\eta^\beta = \delta^\beta_\alpha, \end{cases}$$

for every $\alpha, \beta = 1, \dots, k$. ■

However, these vector fields are not necessarily unique, and hence the system of equations (3.2) is not uniquely determined, so we need to impose some extra condition on M in order to uniquely determine the Reeb vector fields. We will restrict ourselves to the situation where the k-precosymplectic manifold M is of the type $\mathbb{R}^k \times P$ where P is a k-presymplectic. Now, if we ask the Reeb vectors fields to be vertical with respect to the projection $\mathbb{R}^k \times P \xrightarrow{\tau} \mathbb{R}^k$, we can say that we have a uniquely determined family of **Reeb vector fields** $\mathcal{R}_1, \dots, \mathcal{R}_k$, and hence the system of equations (3.2) is well defined. An equivalent way of obtaining the same family of Reeb vector fields is taking the vector fields $\frac{\partial}{\partial t^\alpha}$ on the base space $\mathbb{R}^k$ and lifting them to $\mathbb{R}^k \times P$ with the trivial connection $\nabla = \mathrm{d}t^\alpha \otimes \frac{\partial}{\partial t^\alpha}$.

3.2.2 The constraint algorithm

Now we are ready to tackle the description of the constraint algorithm. We consider the k-precosymplectic manifold $M = \mathbb{R}^k \times P$, where P is a k-presymplectic manifold and suppose it has Darboux charts around every $p \in M$. We now that this implies that we have uniquely determined Reeb vector fields $\mathcal{R}_1, \dots, \mathcal{R}_k$.

Definition 3.2.4. *A k-**precosymplectic Hamiltonian system** is a family $(M, \omega^\alpha, \eta^\alpha, V, \gamma)$ where $(M, \omega^\alpha, \eta^\alpha, V)$ is a k-precosymplectic manifold of the type $\mathbb{R}^k \times P$ and γ is a closed 1-form on M called the **Hamiltonian 1-form**.*

The solutions of a k-presymplectic Hamiltonian system $(M, \omega^\alpha, \eta^\alpha, V, \gamma)$ are the integral sections of the k-vector fields $\mathcal{X} = (X_\alpha) \in \mathfrak{X}^k(M)$ solution of the system of differential equations

$$\begin{cases} i_{X_\alpha}\omega^\alpha = \gamma - \gamma(\mathcal{R}_\alpha)\eta^\alpha, \\ i_{X_\alpha}\eta^\beta = \delta^\beta_\alpha, \end{cases}$$

Remark 3.2.5. *Notice that in the case $k = 1$, we recover the case of singular nonautonomous mechanics studied in Section 2.2. In that case, we made wide use of the Poincaré-Cartan 2-form in the development of the constraint algorithm. However, in the k-precosymplectic case we do not have Poincaré-Cartan 2-forms and we will have to use other tools.*

We want to find a submanifold $N \hookrightarrow M$ such that the system of equations (3.2) has global solutions on N tangent to N. In order to find this submanifold (if it exists!) we construct a constraint algorithm which provides us a sequence of submanifolds

$$\cdots \hookrightarrow M_j \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow M$$

which in favorable cases will end in a final constraint submanifold.

The following Theorem will be the core of our algorithm and will give us a way to compute the constraints at every step of the algorithm.

Theorem 3.2.6. *Let $(M, \omega^\alpha, \eta^\alpha, V, \gamma)$ be a k-precosymplectic Hamiltonian system. Consider a submanifold $C \hookrightarrow M$ and a k-vector field $\mathcal{X}\colon C \to (T^1_k)_C M$ such that $\mathcal{X}_p \in (T^1_k)_p C$ for every $p \in C$. The following two conditions are equivalent:*

(1) There exists a k vector field $\mathcal{X} = (X_\alpha)\colon C \to (T^1_k)_C M$ tangent to C such that the system of equations

$$\begin{cases} i_{X_\alpha}\omega^\alpha = \gamma - \gamma(\mathcal{R}_\alpha)\eta^\alpha, \\ i_{X_\alpha}\eta^\beta = \delta^\beta_\alpha, \end{cases} \tag{3.3}$$

holds on C.

(2) For every $p \in C$, there exists $\mathcal{Z}_p = (Z_\alpha)_p \in (T^1_k)_p C$ such that $i_{Z_{\alpha p}}\eta^\beta_p = \delta^\beta_\alpha$ and $\sum_\alpha \eta^\alpha_p + \widetilde{\gamma}_p = \flat(\mathcal{Z}_p)$ where $\widetilde{\gamma}_p = \gamma_p - \gamma_p(\mathcal{R}_{\alpha p})\eta^\alpha_p$.

Proof. Take $\mathcal{Z}_p = \mathcal{X}_p \in (T^1_k)_p C$. It is clear that $i_{Z_{\alpha p}}\eta^\beta_p = \delta^\beta_\alpha$ for every $p \in C$. On the other hand,

$$\flat(\mathcal{Z}_p) = i_{Z_{\alpha p}}\omega^\alpha_p + (i_{Z_{\alpha p}}\eta^\alpha_p)\eta^\alpha_p = \widetilde{\gamma}_p + \sum_\alpha \eta^\alpha_p.$$

Conversely, suppose that for every $p \in C$, there exists $\mathcal{Z}_p \in (T^1_k)_p C$ such that $i_{Z_{\alpha p}}\eta^\beta_p = \delta^\beta_\alpha$ and $\flat(\mathcal{Z}_p) = \widetilde{\gamma}_p + \sum_\alpha \eta^\alpha_p$. Let $p \in C$. We take a Darboux chart $(\mathcal{U}, \{t^\alpha, q^i, p^\alpha_{i_\alpha}; u^j\})$ around p and hence

$$\eta^\alpha = \mathrm{d}t^\alpha,$$
$$\omega^\alpha = \sum_{i \in I_\alpha} \mathrm{d}q^i \wedge \mathrm{d}p^\alpha_i,$$
$$\gamma = \frac{\partial h}{\partial q^i}\mathrm{d}q^i + \frac{\partial h}{\partial p^\alpha_{i_\alpha}}\mathrm{d}p^\alpha_{i_\alpha} + \frac{\partial h}{\partial t^\alpha}\mathrm{d}t^\alpha + \frac{\partial h}{\partial u^j}\mathrm{d}u^j.$$

In this coordinates, $\widetilde{\gamma} = \gamma - \gamma(\mathcal{R}_\alpha)\eta^\alpha$ is

$$\widetilde{\gamma} = \frac{\partial h}{\partial q^i}\mathrm{d}q^i + \frac{\partial h}{\partial p^\alpha_{i_\alpha}}\mathrm{d}p^\alpha_{i_\alpha} + \frac{\partial h}{\partial u^j}\mathrm{d}u^j.$$

In the following, we will omit the point p everywhere in order to simplify the notation. We write our k-vector $\mathcal{Z}$ in coordinates:

$$Z_\alpha = A^\beta_\alpha \frac{\partial}{\partial t^\beta} + B^i_\alpha \frac{\partial}{\partial q^i} + C^\beta_{\alpha, i_\beta} \frac{\partial}{\partial p^\beta_{i_\beta}} + D^j_\alpha \frac{\partial}{\partial u^j}.$$

Now let us compute its image by the morphism $\flat$:

$$\begin{aligned}
\flat(\mathcal{Z}) &= \sum_\alpha i_{Z_\alpha}\omega^\alpha + (i_{Z_\alpha}\eta^\alpha)\eta^\alpha \\
&= \sum_\alpha \sum_{i\in I_\alpha} i_{Z_\alpha}(\mathrm{d}q^i \wedge \mathrm{d}p^\alpha_i) + \sum_\alpha (i_{Z_\alpha}\mathrm{d}t^\alpha)\mathrm{d}t^\alpha \\
&= \sum_\alpha \sum_{i\in I_\alpha} (i_{Z_\alpha}\mathrm{d}q^i)\cdot \mathrm{d}p^\alpha_i - \sum_\alpha \sum_{i\in I_\alpha} \mathrm{d}q^i \cdot (i_{Z_\alpha}\mathrm{d}p^i_\alpha) + \sum_\alpha (i_{Z_\alpha}\mathrm{d}t^\alpha)\mathrm{d}t^\alpha \\
&= \sum_\alpha \sum_{i\in I_\alpha} B^i_\alpha \mathrm{d}p^\alpha_i - \sum_\alpha \sum_{i\in I_\alpha} C^\alpha_i \mathrm{d}q^i + \sum_\alpha A^\alpha_\alpha \mathrm{d}t^\alpha.
\end{aligned}$$

Comparing this expression with

$$\sum_\alpha \eta^\alpha + \widetilde{\gamma} = \sum_\alpha \mathrm{d}t^\alpha + \frac{\partial h}{\partial q^i}\mathrm{d}q^i + \frac{\partial h}{\partial p^\alpha_{i_\alpha}}\mathrm{d}p^\alpha_{i_\alpha} + \frac{\partial h}{\partial u^j}\mathrm{d}u^j,$$

we get the following conditions on Z:

$$A^\alpha_\alpha = 1, \qquad \frac{\partial h}{\partial u^j} = 0, \qquad \frac{\partial h}{\partial q^i} = -\sum_{\substack{\alpha \text{ such} \\ \text{that } i\in I_\alpha}} C^\alpha_{\alpha,i}, \qquad \frac{\partial h}{\partial p^\alpha_{i_\alpha}} = B^{i_\alpha}_\alpha.$$

Moreover, we know by hypothesis that $A^\beta_\alpha = \delta^\beta_\alpha$. The second condition is a compatibility condition of the Hamilton equations in the k-precosymplectic case. It can be stated as follows: the Hamiltonian function must not depend on the gauge variables. The second pair of conditions together with the condition $A^\beta_\alpha = \delta^\beta_\alpha$ are equivalent to the system of equations (3.3) when written in coordinates (see equation (1.14)). This finnishes the proof. ■

Now using the previous Theorem, we are able to give a description of the constraint algorithm. First of all, we need to restrict orselves to those points such that $\gamma\left(\frac{\partial}{\partial u^j}\right) = 0\ \forall j$, because it is a compatibility condition of the system. The j-ary constraint submanifold $M_j \subset M_{j-1}$ is defined as

$$M_j = \Big\{ p \in M_{j-1} \,|\, \exists \mathcal{Z} = (Z_\alpha) \in (T^1_k) M_{j-1} \text{ such that} \\ \flat(\mathcal{Z}) = \widetilde{\gamma} + \sum_\alpha \eta^\alpha \text{ and } i_{Z_\alpha}\eta^\beta = \delta^\beta_\alpha \Big\},$$

where $M_0 = M$.

Definition 3.2.7. *Let $C \hookrightarrow M$ be a submanifold of a k-precosymplectic manifold M. The k**-precosymplectic orthogonal complement** of C is*

$$TC^\perp = \Big(\flat\left((T^1_k) C \cap D_C \right) \Big)^0$$

where D_C is the set of all k-vectors $\mathcal{Z}_p = (Z_\alpha)_p$ on C such that $i_{Z_{\alpha p}}\eta^\beta_p = \delta^\beta_\alpha$.

With this definition and Theorem 3.2.6 we can give an alternative characterization of the constraints submanifolds:

$$M_j = \Big\{ p \in M_{j-1} \,|\, \widetilde{\gamma} + \sum_\alpha \eta^\alpha \in ((TC)^\perp)^0 \Big\},$$

Although this allows us to effectively compute the constraints at every step of the algorithm, an alternative and equivalent way to compute the constraint submanifolds of the k**-precosymplectic constraint algorithm**, which is much more operational, is the following:

(1) Obtain a local basis $\{Z_1, \dots, Z_r\}$ of $(TM)^\perp$.

(2) Use Theorem 3.2.6 to obtain a set of independent constraint functions $f_\mu = i_{Z_\mu}(\widetilde{\gamma} + \sum_\alpha \eta^\alpha)$ defining the submanifold $M_1 \hookrightarrow M$.

(3) Compute solutions $\mathcal{X} = (X_\alpha)$ of (3.2).

(4) Impose the tangency condition of $X_1, \dots, X_k$ on M_1.

(5) Iterate item (4) until no new constraints appear.

If this iterative procedure ends in a submanifold M_l with nonzero dimension, then we can assure the existence of global solutions to equation (3.2) on this submanifold M_l.

Remark 3.2.8. *As in the k-presymplectic case, this algorithms also works for singular Hamiltonian field theories of the type* $(\mathbb{R}^k \times (T^1_k)^*Q, \omega^\alpha, \eta^\alpha, h)$ *and also for singular Lagrangian field theories* $(\mathbb{R}^k \times T^1_k Q, \omega^\alpha_{\mathcal{L}}, \mathrm{d}x^\alpha, \mathcal{L})$.

Notice that we can treat k-presymplectic (autonomous) field theories as k-precosymplectic (nonautonomous) field theories. In this case, we do not have the 1-forms η^α and we recover the k-presymplectic algorithm described in Section 3.1.

Conclusions and outlook

In this thesis we have given a brief review of autonomous and nonautonomous mechanics and field theory, focusing on the singular cases. We have reviewed the constraint algorithms to solve the equations of motion of presymplectic and precosymplectic mechanical systems and the field equations of k-presymplectic field theories.

We have also defined the concepts of k-precosymplectic manifold and proved the existence of global Reeb vector fields in these manifolds. We have also defined the notion of k-precosymplectic Hamiltonian system. Finally we have developed a constraint algorithm for singular k-precosymplectic field theories in order to find a submanifold of the phase bundle where there are solutions to the field equations. These algorithms can be applied to the Hamiltonian and Lagrangian formalisms of timedependent singular field theories.

However, there is still a lot of work to do in this area. It is still an open problem to find necessary and sufficient conditions to assure the existence of some kind of Darboux coordinates in both k-presymplectic and k-precosymplectic manifolds. We also want to work out some examples of singular k-cosymplectic field theories to put to work the algorithm we have developed.

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