

XIII International Conference on Computational Plasticity. Fundamentals and Applications
COMPLAS XIII
E. Oñate, D.R.J. Owen, D. Peric & M. Chiumenti (Eds)

ON EXPLOITING THE WEAK INVARIANCE OF MULTIPLICATIVE ELASTO-PLASTICITY FOR EFFICIENT NUMERICAL INTEGRATION

ALEXEY V. SHUTOV* †

* Novosibirsk State University
Pirogova 2, 630090 Novosibirsk, Russia

† Lavrentyev Institute of Hydrodynamics
Pr. Lavrentyeva 15, 630090 Novosibirsk, Russia
e-mail: alexey.v.shutov@gmail.com, web page: <http://sites.google.com/site/materialmodeling>

Key words: Multiplicative Elasto-plasticity, Weak Invariance, Kinematic Hardening, Implicit Time Stepping, Euler Backward

Abstract. The paper is devoted to the efficient and robust implementation of a certain finite-strain plasticity model, formulated within the popular multiplicative framework. The model was proposed by Shutov and Kreißig (2008), and it captures the nonlinear behavior of metallic materials, including the combined isotropic/kinematic hardening and viscosity. A new implicit time stepping procedure is suggested here, which can be used for the stress computation at each Gauß point of the finite element discretization.

The model of Shutov and Kreißig exhibits the so-called weak invariance of the solution under arbitrary isochoric change of the reference configuration. The presented algorithm benefits from this special structure of the constitutive equations: The weak invariance property is exploited for construction of the numerical integration procedure; the resulting procedure preserves the weak invariance. The inelastic incompressibility is exactly preserved as well to suppress the error accumulation; the algorithm is unconditionally stable and first-order accurate. In terms of accuracy, the proposed algorithm is comparable with the Euler Backward Method (EBM), but it is superior to EMB with respect to efficiency and robustness.

1 INTRODUCTION

In many practical metal forming applications it is necessary to estimate the residual stresses and the magnitude of spring back. Numerical computations accounting for these effects should employ models of finite strain plasticity with combined isotropic/kinematic hardening (cf., for example, [6, 2]). One of such models was constructed by Shutov and

Kreißig within the context of multiplicative plasticity/viscoplasticity [10]. The deformation-induced plastic anisotropy is captured by this model phenomenologically. The important feature of this model is that it is based on the double multiplicative split of the deformation gradient proposed by [7]. A similar model was also proposed in [16]. Lion's double multiplicative decomposition was adopted in different areas, including applications to shape memory alloys [3], coupled thermoplasticity [8], ratchetting [18], ductile damage [1, 17, 14]. One application of the model to microstructural simulations was discussed in [15].

For the FEM implementation, a numerical procedure is needed which computes the stresses locally at each integration point (Gauß point). For the considered model of Shutov and Kreißig, the numerical procedure is based on the integration of constitutive equations governing the inelastic flow of the material. Unfortunately, as is typical for metal plasticity, the underlying system of equations is stiff and implicit time stepping is needed to obtain a stable integration procedure. The implicit time discretization yields a system of nonlinear algebraic equations. The problem is aggravated by the fact that the general closed form solution for this system is unknown. Therefore, a local iterative procedure was implemented by many authors to solve a system of coupled nonlinear equations.¹ In the current study, *simple closed-form solutions are found* for elementary decoupled problems. The desired time stepping procedure for the coupled system of equations is obtained by a combination of these explicit solutions and by employing a sophisticated operator split. As a result, only one scalar equation has to be solved numerically with respect to the unknown inelastic strain increment. Clearly, such modification is the key to enhanced computational efficiency and robustness of the numerical integration.

The accuracy of the stress computation by the new algorithm is tested numerically. Concerning the accuracy, the new algorithm is comparable with the classical Euler Backward Method (EBM).

2 MATERIAL MODEL

2.1 System of constitutive equations

Let us shortly recall the viscoplastic material model proposed by Shutov and Kreißig [10]. For simplicity of the numerical implementation, we adopt here the Lagrangian formulation of the material model. Along with the well-known right Cauchy-Green tensor \mathbf{C} , we introduce tensor-valued internal variables \mathbf{C}_i , \mathbf{C}_{ii} . These variables are interpreted respectively as the inelastic right Cauchy-Green tensor and the inelastic right Cauchy-Green tensor of the substructure. By s and s_d we denote the inelastic arc length (Odqvist parameter) and its dissipative part, respectively.

The free energy per unit mass is assumed to be given by $\psi = \psi_{el}(\mathbf{C}\mathbf{C}_i^{-1}) + \psi_{kin}(\mathbf{C}_i\mathbf{C}_{ii}^{-1}) + \psi_{iso}(s - s_d)$. Here, ψ_{el} is the energy storage due to elastic deformations, ψ_{kin} represents the

¹In [5], a semi-implicit procedure with only one scalar equation was developed assuming that elastic strains are small.

storage associated with the kinematic hardening, and ψ_{iso} is the energy storage related to the isotropic hardening. The functions ψ_{el} and ψ_{kin} are assumed to be isotropic.² Let $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{X}}$ denote respectively the 2nd Piola-Kirchhoff stress tensor and the total backstress tensor, both operating on the reference configuration. The backstress describes the translation of the yield surface in the stress space. The isotropic expansion of the yield surface is captured by the scalar quantity R . The rate-dependent overstress f is introduced to capture the viscous effects.

A local initial value problem is considered in this paper. Thus, the local deformation history $\mathbf{C}(t)$ is assumed to be given in the time interval $t \in [0, T]$. The corresponding stress response is governed by the following system of ordinary differential and algebraic equations

$$\dot{\mathbf{C}}_i = 2 \frac{\lambda_i}{\mathfrak{F}} (\mathbf{C}\tilde{\mathbf{T}} - \mathbf{C}_i\tilde{\mathbf{X}})^{\text{D}} \mathbf{C}_i, \quad \mathbf{C}_i|_{t=0} = \mathbf{C}_i^0, \quad \det \mathbf{C}_i^0 = 1, \quad \mathbf{C}_i^0 = (\mathbf{C}_i^0)^{\text{T}} > 0, \quad (1)$$

$$\dot{\mathbf{C}}_{\text{ii}} = 2\lambda_i \varkappa (\mathbf{C}_i\tilde{\mathbf{X}})^{\text{D}} \mathbf{C}_{\text{ii}}, \quad \mathbf{C}_{\text{ii}}|_{t=0} = \mathbf{C}_{\text{ii}}^0, \quad \det \mathbf{C}_{\text{ii}}^0 = 1, \quad \mathbf{C}_{\text{ii}}^0 = (\mathbf{C}_{\text{ii}}^0)^{\text{T}} > 0, \quad (2)$$

$$\dot{s} = \sqrt{\frac{2}{3}} \lambda_i, \quad \dot{s}_{\text{d}} = \frac{\beta}{\gamma} \dot{s} R, \quad s|_{t=0} = s^0, \quad s_{\text{d}}|_{t=0} = s_{\text{d}}^0, \quad (3)$$

$$\tilde{\mathbf{T}} = 2\rho_{\text{R}} \frac{\partial \psi_{\text{el}}(\mathbf{C}\mathbf{C}_i^{-1})}{\partial \mathbf{C}} \Big|_{\mathbf{C}_i = \text{const}}, \quad (4)$$

$$\tilde{\mathbf{X}} = 2\rho_{\text{R}} \frac{\partial \psi_{\text{kin}}(\mathbf{C}_i\mathbf{C}_{\text{ii}}^{-1})}{\partial \mathbf{C}_i} \Big|_{\mathbf{C}_{\text{ii}} = \text{const}}, \quad (5)$$

$$R = \rho_{\text{R}} \frac{\partial \psi_{\text{iso}}(s - s_{\text{d}})}{\partial s}, \quad (6)$$

$$\lambda_i = \frac{1}{\eta} \left\langle \frac{1}{f_0} f \right\rangle^m, \quad f = \mathfrak{F} - \sqrt{\frac{2}{3}} [K + R], \quad \mathfrak{F} = \sqrt{\text{tr}[(\mathbf{C}\tilde{\mathbf{T}} - \mathbf{C}_i\tilde{\mathbf{X}})^{\text{D}}]^2}. \quad (7)$$

Here, the material parameters $\rho_{\text{R}} > 0$, $\varkappa \geq 0$, $\beta \geq 0$, $\gamma \in \mathbb{R}$, $\eta \geq 0$, $m \geq 1$, $K > 0$, and the real-valued functions ψ_{el} , ψ_{kin} , ψ_{iso} are assumed to be known; $f_0 = 1$ MPa is used to obtain a dimensionless term in the bracket (f_0 is not a material parameter). The function $\lambda_i(t)$ is referred to as the inelastic multiplier; $\mathfrak{F}(t)$ is the norm of the driving force; the superposed dot denotes the material time derivative, $(\cdot)^{\text{D}}$ stands for the deviatoric part of a second-rank tensor; $\langle x \rangle := \max(x, 0)$ is the Macaulay bracket.

Assuming for the isotropic hardening

$$\rho_{\text{R}} \psi_{\text{iso}}(s - s_{\text{d}}) = \frac{\gamma}{2} (s - s_{\text{d}})^2 \quad (8)$$

we arrive at

$$R = \gamma (s - s_{\text{d}}). \quad (9)$$

²Some additional terms can be introduced for a more precise description of the energy storage [8].

Next, let us assume that the remaining part of the free energy is governed by potentials of Neo-Hookean type

$$\rho_R \psi_{\text{el}}(\mathbf{C}\mathbf{C}_i^{-1}) = \frac{k}{2} (\ln \sqrt{\det \mathbf{C}\mathbf{C}_i^{-1}})^2 + \frac{\mu}{2} (\text{tr} \overline{\mathbf{C}\mathbf{C}_i^{-1}} - 3), \quad (10)$$

$$\rho_R \psi_{\text{kin}}(\mathbf{C}_i \mathbf{C}_{ii}^{-1}) = \frac{c}{4} (\text{tr} \overline{\mathbf{C}_i \mathbf{C}_{ii}^{-1}} - 3), \quad (11)$$

where $k > 0$, $\mu > 0$, $c > 0$ are material constants; the overline $\overline{(\cdot)}$ denotes the unimodular part of a tensor such that $\overline{\mathbf{A}} := (\det \mathbf{A})^{-1/3} \mathbf{A}$ for all \mathbf{A} . In this special case we have

$$\tilde{\mathbf{T}} = k \ln \sqrt{\det(\mathbf{C})} \mathbf{C}^{-1} + \mu \mathbf{C}^{-1} (\overline{\mathbf{C}\mathbf{C}_i^{-1}})^{\text{D}}, \quad (12)$$

$$\tilde{\mathbf{X}} = \frac{c}{2} \mathbf{C}_i^{-1} (\mathbf{C}_i \mathbf{C}_{ii}^{-1})^{\text{D}}, \quad (13)$$

and the evolution equations (1), (2) are reduced to

$$\dot{\mathbf{C}}_i = 2 \frac{\lambda_i}{\mathfrak{F}} \left(\mu (\overline{\mathbf{C}\mathbf{C}_i^{-1}})^{\text{D}} - \frac{c}{2} (\mathbf{C}_i \mathbf{C}_{ii}^{-1})^{\text{D}} \right) \mathbf{C}_i, \quad (14)$$

$$\dot{\mathbf{C}}_{ii} = \lambda_i \varkappa c (\mathbf{C}_i \mathbf{C}_{ii}^{-1})^{\text{D}} \mathbf{C}_{ii}. \quad (15)$$

2.2 Properties of the model

The exact solution of (1) – (7) exhibits the following *geometric property*

$$\mathbf{C}_i, \mathbf{C}_{ii} \in M, \quad \text{where } M := \{ \mathbf{B} \in \text{Sym} : \det \mathbf{B} = 1 \}. \quad (16)$$

Thus, we are dealing with a system of differential and algebraic equations on the manifold. The condition $\det(\mathbf{C}_i) = 1$ corresponds to the inelastic incompressibility, typically assumed for metallic materials; the condition $\det(\mathbf{C}_{ii}) = 1$ reflects the incompressibility on the substructural level. These incompressibility conditions should be exactly fulfilled by the numerical solution in order to suppress the error accumulation [11]. Moreover, the tensors \mathbf{C}_i and \mathbf{C}_{ii} are positive definite. From the physical standpoint, it is essential that they remain positive-definite within the numerical method, since they represent some metric tensors of Cauchy-Green type.

Another important aspect is the *weak invariance* of the considered constitutive equations. The essence of the weak invariance is as follows (cf. [9]): Let \mathbf{F}_0 be arbitrary second rank tensor such that $\det(\mathbf{F}_0) = 1$. If the prescribed loading program is replaced by a new program $\mathbf{F}^{\text{new}}(t) := \mathbf{F}(t) \mathbf{F}_0^{-1}$, $\mathbf{C}^{\text{new}}(t) := \mathbf{F}_0^{-\text{T}} \mathbf{C}(t) \mathbf{F}_0^{-1}$ and the initial conditions are transformed according to

$$\mathbf{C}_i^{\text{new}}|_{t=0} = \mathbf{F}_0^{-\text{T}} \mathbf{C}_i|_{t=0} \mathbf{F}_0^{-1}, \quad \mathbf{C}_{ii}^{\text{new}}|_{t=0} = \mathbf{F}_0^{-\text{T}} \mathbf{C}_{ii}|_{t=0} \mathbf{F}_0^{-1}, \quad (17)$$

then the same Cauchy stresses (true stresses) \mathbf{T} will be predicted by the model

$$\mathbf{T}^{\text{new}}(t) = \mathbf{T}(t) \text{ for all } t \in [0, T]. \quad (18)$$

The proof of (18) for the model of Shutov and Kreißig can be found in [13].

Generally speaking, the weak invariance can be seen as a generalized symmetry property of the material [9]. Like any other symmetry, it should be exactly preserved by the numerical procedure. As will be shown in the following, the weak invariance requirement is helpful in constructing new efficient algorithms.

3 TIME STEPPING SCHEME

Consider a typical time step $t_n \mapsto t_{n+1}$ with $\Delta t := t_{n+1} - t_n > 0$. Assume that ${}^n\mathbf{C} := \mathbf{C}({}^n t)$ and ${}^{n+1}\mathbf{C} := \mathbf{C}({}^{n+1} t)$ are known and the internal variables $\mathbf{C}_i, \mathbf{C}_{ii}, s, s_d$ at t_n are given by ${}^n\mathbf{C}_i, {}^n\mathbf{C}_{ii}, {}^n s, {}^n s_d$. We need to update the internal variables and to compute the stress tensor ${}^{n+1}\mathbf{T}$. It is instructive to introduce the incremental inelastic parameter

$$\xi := \Delta t \, {}^{n+1}\lambda_i, \quad (19)$$

which is a non-dimensional quantity. For most practical applications we may assume that $\xi \leq 0.2$. A new implicit numerical procedure will be constructed in this section, which is based on closed-form solutions for decoupled problems. As a result, the original problem will be reduced to a simplified problem with only one scalar unknown ξ .

3.1 Update of s and s_d for known ξ

Integrating (3)₁ and (3)₂ within the time step, one easily obtains ${}^{n+1}R, {}^{n+1}s$ and ${}^{n+1}s_d$ as functions of ξ

$${}^{n+1}R(\xi) = \frac{{}^t R + \sqrt{2/3}\gamma\xi}{1 + \sqrt{2/3}\beta\xi}, \quad \text{where } {}^t R := \gamma({}^n s - {}^n s_d), \quad (20)$$

$${}^{n+1}s(\xi) = {}^n s + \sqrt{2/3} \xi, \quad {}^{n+1}s_d(\xi) = {}^n s_d + \frac{\beta}{\gamma} \sqrt{2/3} \xi \, {}^{n+1}R(\xi). \quad (21)$$

3.2 Update of \mathbf{C}_i for known ${}^{n+1}\mathbf{C}_{ii}, \xi$

In this subsection we assume that ${}^{n+1}\mathbf{C}_{ii}$ and ξ are known. We need to update \mathbf{C}_i by integrating the evolution equation (14). Toward that end, we rewrite (14) as follows

$$\dot{\mathbf{C}}_i = 2\frac{\lambda_i}{\mathfrak{F}}\left(\mu \overline{\mathbf{C}} - \frac{c}{2} \mathbf{C}_i \mathbf{C}_{ii}^{-1} \mathbf{C}_i\right) + \beta \mathbf{C}_i, \quad (22)$$

where

$$\beta = -\frac{2}{3}\frac{\lambda_i}{\mathfrak{F}}\left(\mu \operatorname{tr}(\overline{\mathbf{C}}\mathbf{C}_i^{-1}) - \frac{c}{2} \operatorname{tr}(\mathbf{C}_i \mathbf{C}_{ii}^{-1})\right) \in \mathbb{R}. \quad (23)$$

Next, we express the norm of the driving force as a function of ξ : Combining (7)₁ and (7)₂ we obtain

$$\mathfrak{F} = \mathfrak{F}_2(\xi) := f_0\left(\frac{\eta\xi}{\Delta t}\right)^{1/m} + \sqrt{2/3}(K + {}^{n+1}R(\xi)). \quad (24)$$

Applying the classical Euler backward method (EBM) to (22), and replacing \mathfrak{F} through $\mathfrak{F}_2(\xi)$ we obtain

$${}^{n+1}\mathbf{C}_i = {}^n\mathbf{C}_i + 2\frac{\xi}{\mathfrak{F}_2}\left(\mu {}^{n+1}\overline{\mathbf{C}} - \frac{c}{2} {}^{n+1}\mathbf{C}_i {}^{n+1}\mathbf{C}_{ii}^{-1} {}^{n+1}\mathbf{C}_i\right) + \beta\Delta t {}^{n+1}\mathbf{C}_i, \quad (25)$$

where the time-dependent scalar β is unknown.

Unfortunately, the classical EBM violates the incompressibility condition $\det(\mathbf{C}_i) = 1$. In order to enforce this property, we may consider a modification of (25) as follows

$${}^{n+1}\mathbf{C}_i = {}^n\mathbf{C}_i + 2\frac{\xi}{\mathfrak{F}_2}\left(\mu {}^{n+1}\overline{\mathbf{C}} - \frac{c}{2} {}^{n+1}\mathbf{C}_i {}^{n+1}\mathbf{C}_{ii}^{-1} {}^{n+1}\mathbf{C}_i\right) + \beta\Delta t {}^{n+1}\mathbf{C}_i + \varepsilon\mathbf{P}, \quad (26)$$

where ε is a scalar and \mathbf{P} is a suitable second-rank tensor.³ The small scalar ε is determined such that $\det({}^{n+1}\mathbf{C}_i) = 1$. In some publications, the authors put $\mathbf{P} := \mathbf{1}$, but such choice of \mathbf{P} violates the weak invariance of the solution. In this study we put $\mathbf{P} := {}^{n+1}\mathbf{C}_i$. This approach is *compatible with the weak invariance*. Thus, we obtain from (26)

$$z {}^{n+1}\mathbf{C}_i = {}^n\mathbf{C}_i + 2\frac{\xi}{\mathfrak{F}_2}\left(\mu {}^{n+1}\overline{\mathbf{C}} - \frac{c}{2} {}^{n+1}\mathbf{C}_i {}^{n+1}\mathbf{C}_{ii}^{-1} {}^{n+1}\mathbf{C}_i\right), \quad (27)$$

where the unknown z is determined from the incompressibility condition $\det({}^{n+1}\mathbf{C}_i) = 1$. Next, introducing $\Phi := c {}^{n+1}\mathbf{C}_{ii}^{-1}$ we arrive at

$$z {}^{n+1}\mathbf{C}_i = {}^n\mathbf{C}_i + 2\frac{\xi}{\mathfrak{F}_2}\left(\mu {}^{n+1}\overline{\mathbf{C}} - \frac{1}{2} {}^{n+1}\mathbf{C}_i \Phi {}^{n+1}\mathbf{C}_i\right). \quad (28)$$

Further, multiplying both sides of (28) with $\Phi^{1/2}$ from left and right, we obtain

$$z \Phi^{1/2} {}^{n+1}\mathbf{C}_i \Phi^{1/2} = \Phi^{1/2} \left[{}^n\mathbf{C}_i + 2\frac{\xi}{\mathfrak{F}_2} \mu {}^{n+1}\overline{\mathbf{C}} \right] \Phi^{1/2} - \frac{\xi}{\mathfrak{F}_2} \Phi^{1/2} {}^{n+1}\mathbf{C}_i \Phi {}^{n+1}\mathbf{C}_i \Phi^{1/2}. \quad (29)$$

For what follows we introduce $\mathbf{Y} := \Phi^{1/2} {}^{n+1}\mathbf{C}_i \Phi^{1/2}$ and $\mathbf{A} := \Phi^{1/2} \left[{}^n\mathbf{C}_i + 2\frac{\xi}{\mathfrak{F}_2} \mu {}^{n+1}\overline{\mathbf{C}} \right] \Phi^{1/2}$; \mathbf{A} is known; \mathbf{Y} is unknown.

Recall that, for physical reasons, ${}^{n+1}\mathbf{C}_i$ must be positive definite. Therefore, we expect that \mathbf{Y} is positive definite as well. Substituting the abbreviations into (29), we obtain the quadratic equation with respect to \mathbf{Y} and its physically reasonable solution as follows

$$z \mathbf{Y} = \mathbf{A} - \frac{\xi}{\mathfrak{F}_2} \mathbf{Y}^2, \quad \mathbf{Y} > 0 \quad \Rightarrow \quad \mathbf{Y} = \frac{\mathfrak{F}_2}{2\xi} \left[-z\mathbf{1} + \sqrt{z^2\mathbf{1} + 4\frac{\xi}{\mathfrak{F}_2}\mathbf{A}} \right]. \quad (30)$$

³Another modification of EBM to enforce the incompressibility was presented in [4].

Recall that the unknown parameter z can be estimated using the incompressibility condition $\det({}^{n+1}\mathbf{C}_i) = 1$, which is equivalent to $\det(\mathbf{Y}) = \det(\Phi)$. Fortunately, we do not have to solve this equation exactly. Introducing $\mathbf{Y}_0 := \Phi^{1/2} {}^n\mathbf{C}_i \Phi^{1/2} = \mathbf{Y} + O(\xi)$ and taking the determinant of both sides of (30)₁, we have

$$z^3 \det(\Phi) = \det(\mathbf{A} - \frac{\xi}{\mathfrak{F}_2} \mathbf{Y}_0^2) + O(\xi^2), \quad z = \left[\det(\mathbf{A} - \frac{\xi}{\mathfrak{F}_2} \mathbf{Y}_0^2) / \det(\Phi) \right]^{1/3} + O(\xi^2). \quad (31)$$

Neglecting the term $O(\xi^2)$, z is estimated by

$$z = \left[\det(\mathbf{A} - \frac{\xi}{\mathfrak{F}_2} \mathbf{Y}_0^2) / \det(\Phi) \right]^{1/3}. \quad (32)$$

Next, \mathbf{Y} is computed by (30)₃ and \mathbf{C}_i is updated through

$${}^{n+1}\mathbf{C}_i^* := \Phi^{-1/2} \mathbf{Y} \Phi^{-1/2}, \quad {}^{n+1}\mathbf{C}_i = \overline{{}^{n+1}\mathbf{C}_i^*}. \quad (33)$$

The correction step (33)₂ is needed to enforce the incompressibility, since z is not computed exactly. This step does not violate the weak invariance of the solution. The procedure, described in this subsection, yields ${}^{n+1}\mathbf{C}_i$ as a function of ${}^{n+1}\mathbf{C}_{ii}$ and ξ :

$${}^{n+1}\mathbf{C}_i = \mathfrak{C}_i({}^{n+1}\mathbf{C}_{ii}, \xi). \quad (34)$$

3.3 Update of \mathbf{C}_{ii} for known \mathbf{C}_i and ξ

In this subsection we update \mathbf{C}_{ii} using the evolution equation (15). Its EBM discretization yields

$${}^{n+1}\mathbf{C}_{ii}^* = {}^n\mathbf{C}_{ii} + \xi \mathcal{X} c ({}^{n+1}\mathbf{C}_i ({}^{n+1}\mathbf{C}_{ii}^*)^{-1})^D {}^{n+1}\mathbf{C}_{ii}^*. \quad (35)$$

Note that this equation has exactly the same structure as for the discretized finite-strain model of Maxwell fluid (cf. equation (25) in [12]). If the EBM solution ${}^{n+1}\mathbf{C}_{ii}^*$ is corrected to obtain ${}^{n+1}\mathbf{C}_{ii} = \overline{{}^{n+1}\mathbf{C}_{ii}^*}$, then a simple explicit update formula is available (cf. equation (29) in [12]):

$${}^{n+1}\mathbf{C}_{ii} = \overline{{}^n\mathbf{C}_{ii} + \xi \mathcal{X} c {}^{n+1}\mathbf{C}_i}. \quad (36)$$

Interestingly, this update formula for \mathbf{C}_{ii} can be derived by implementing the restriction of the weak invariance, as it was carried out in the previous subsection for \mathbf{C}_i .

3.4 Overall procedure with operator split

Multiplying (7)₁ with Δt , we arrive at the following incremental consistency condition for finding ξ

$$\xi \eta = \Delta t \left\langle \frac{1}{f_0} f \right\rangle^m, \quad \text{where } f = \tilde{f}({}^{n+1}\mathbf{C}_i, {}^{n+1}\mathbf{C}_{ii}, \xi). \quad (37)$$

Here, the overstress function $\tilde{f}({}^*\mathbf{C}_i, {}^*\mathbf{C}_{ii}, \xi)$ is determined from (7)₂ and (7)₃ as follows

$$\tilde{f}({}^*\mathbf{C}_i, {}^*\mathbf{C}_{ii}, \xi) := \mathfrak{F}_1({}^*\mathbf{C}_i, {}^*\mathbf{C}_{ii}) - \sqrt{\frac{2}{3}} [K + {}^{n+1}R(\xi)], \quad (38)$$

$$\mathfrak{F}_1(*\mathbf{C}_i, *\mathbf{C}_{ii}) := \sqrt{\text{tr}[(^{n+1}\mathbf{C} *\tilde{\mathbf{T}} - *\mathbf{C}_i *\tilde{\mathbf{X}})^D]^2}, \quad (39)$$

$$*\tilde{\mathbf{T}}(*\mathbf{C}_i) := k \ln \sqrt{\det(^{n+1}\mathbf{C})} {}^{n+1}\mathbf{C}^{-1} + \mu {}^{n+1}\mathbf{C}^{-1} (\overline{{}^{n+1}\mathbf{C}} *\mathbf{C}_i^{-1})^D, \quad (40)$$

$$*\tilde{\mathbf{X}}(*\mathbf{C}_i, *\mathbf{C}_{ii}) := \frac{c}{2} *\mathbf{C}_i^{-1} (*\mathbf{C}_i *\mathbf{C}_{ii}^{-1})^D. \quad (41)$$

The following procedure is implemented in this study:

1 Elastic predictor: Evaluate trial overstress as follows

$$\text{trial } f := \tilde{f}({}^n\mathbf{C}_i, {}^n\mathbf{C}_{ii}, 0). \quad (42)$$

If $\text{trial } f \leq 0$ then the current stress state remains in the elastic region. Put $\xi = 0$, ${}^{n+1}\mathbf{C}_i = {}^n\mathbf{C}_i$, ${}^{n+1}\mathbf{C}_{ii} = {}^n\mathbf{C}_{ii}$, ${}^{n+1}s = {}^n s$, ${}^{n+1}s_d = {}^n s_d$. The time step is complete. If $\text{trial } f > 0$ then proceed to the plastic corrector step.

2 Plastic corrector: The initial estimation for ${}^{n+1}\mathbf{C}_{ii}$ is obtained from ${}^n\mathbf{C}_{ii}$ by the same push-forward operation which brings $\overline{{}^n\mathbf{C}}$ to $\overline{{}^{n+1}\mathbf{C}}$. More precisely:

$$\mathbf{F}_0 := (\overline{{}^{n+1}\mathbf{C}})^{-1/2} (\overline{{}^n\mathbf{C}})^{1/2}, \quad \text{est } \mathbf{C}_{ii} := \mathbf{F}_0^{-T} {}^n\mathbf{C}_{ii} \mathbf{F}_0^{-1}. \quad (43)$$

Next, we estimate ξ by resolving the following incremental consistency condition

$$\xi \eta = \Delta t \left\langle \frac{1}{f_0} \tilde{f}(\mathfrak{C}_i(\text{est } \mathbf{C}_{ii}, \xi), \text{est } \mathbf{C}_{ii}, \xi) \right\rangle^m. \quad (44)$$

In other words, we perform a time step with a fixed $\mathbf{C}_{ii} \equiv \text{est } \mathbf{C}_{ii}$. Let $\text{est } \xi$ be the solution of (44), and $\text{est } \mathbf{C}_i := \mathfrak{C}_i(\text{est } \mathbf{C}_{ii}, \text{est } \xi)$. Now we can update \mathbf{C}_{ii} using the explicit update formula (36)

$${}^{n+1}\mathbf{C}_{ii} := \overline{{}^n\mathbf{C}_{ii} + \text{est } \xi \mathcal{Z} c \text{est } \mathbf{C}_i}. \quad (45)$$

Finally, we compute ξ by resolving the incremental consistency condition as follows

$$\xi \eta = \Delta t \left\langle \frac{1}{f_0} \tilde{f}(\mathfrak{C}_i({}^{n+1}\mathbf{C}_{ii}, \xi), {}^{n+1}\mathbf{C}_{ii}, \xi) \right\rangle^m. \quad (46)$$

This time we perform the time step with a fixed $\mathbf{C}_{ii} \equiv {}^{n+1}\mathbf{C}_{ii}$. Let ξ be the solution of (46). The variable \mathbf{C}_i is updated by ${}^{n+1}\mathbf{C}_i := \mathfrak{C}_i({}^{n+1}\mathbf{C}_{ii}, \xi)$. The variables s and s_d are updated by (21). The plastic corrector step is complete.

Note that instead of solving a nonlinear system of algebraic equations with respect to ${}^{n+1}\mathbf{C}_i$, ${}^{n+1}\mathbf{C}_{ii}$ and ξ as it was carried out in [10], only a scalar consistency equation has to be solved with respect to ξ : First, the consistency equations (44) has to be resolved, after that (46) is solved. Obviously, the resulting scheme is first order accurate. Moreover, the geometric property (16) is exactly satisfied and the positive definiteness of \mathbf{C}_i and \mathbf{C}_{ii} is guaranteed even for large time steps and strain increments. The numerical solution exhibits the same weak invariance property as the original continuum model.

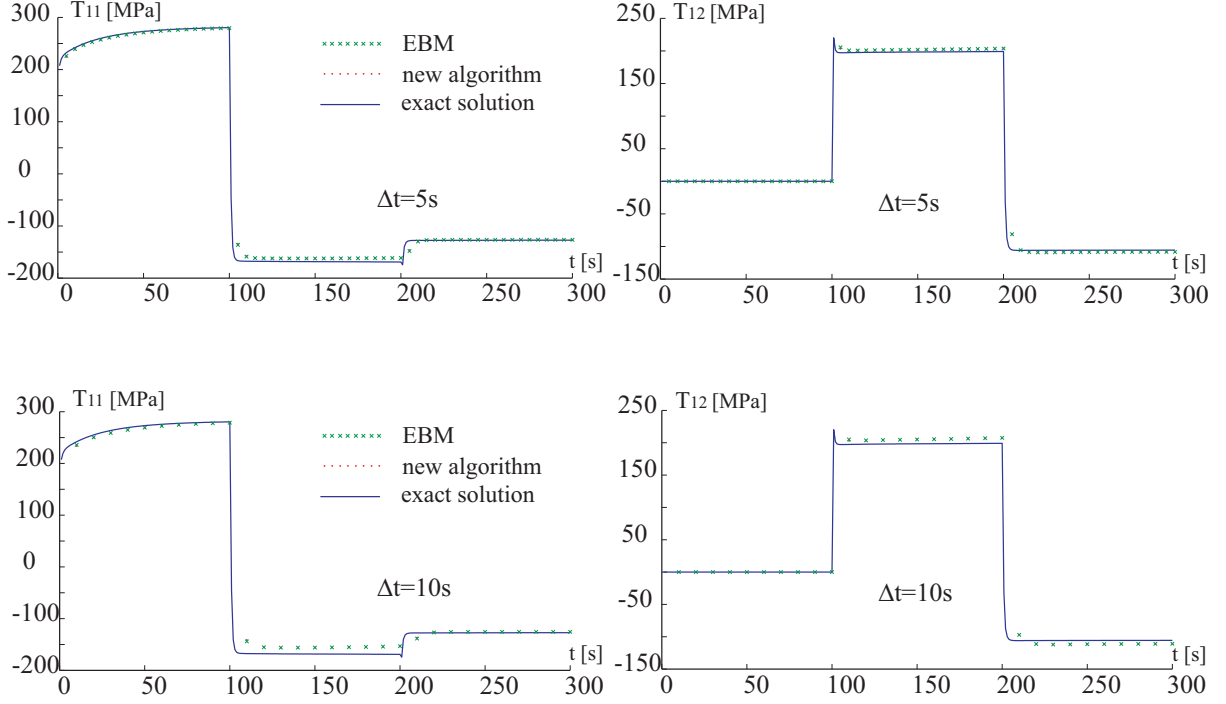


Figure 1: Simulated stress response for the load path (47). Top: results for constant time step size $\Delta t = 5\text{s}$. Bottom: results for $\Delta t = 10\text{s}$.

4 NUMERICAL TEST

Let us test the accuracy of the new algorithm numerically. We will compare this algorithm with the Euler backward method (EBM) with subsequent correction of incompressibility. The set of material parameters used for the simulation is taken from paper [10], which qualitatively corresponds to an aluminum alloy. The parameters are summarized in Table 1.

Table 1: Set of material parameters

k [MPa]	μ [MPa]	c [MPa]	γ [MPa]	
73500	28200	3500	460	
K [MPa]	m [-]	η [s]	\varkappa [MPa ⁻¹]	β [-]
270	3.6	$2 \cdot 10^6$	0.028	5

To test the accuracy of the stress computation we simulate the material response under strain controlled loading. The local loading program in the time interval $t \in [0, 300]$ is given by

$$\mathbf{F}(t) = \overline{\mathbf{F}'(t)}, \quad (47)$$

where $\mathbf{F}'(t)$ is a piecewise linear function of time t

$$\mathbf{F}'(t) := \begin{cases} (1 - t/100)\mathbf{F}_1 + (t/100)\mathbf{F}_2 & \text{if } t \in [0, 100] \\ (2 - t/100)\mathbf{F}_2 + (t/100 - 1)\mathbf{F}_3 & \text{if } t \in (100, 200] \\ (3 - t/100)\mathbf{F}_3 + (t/100 - 2)\mathbf{F}_4 & \text{if } t \in (200, 300] \end{cases} .$$

Here, the key points of the loading path are given by

$$\mathbf{F}_1 := \mathbf{1}, \quad \mathbf{F}_2 := \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \mathbf{F}_3 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{F}_4 := \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} .$$

Moreover, we consider the initial conditions as follows

$$\mathbf{C}_i|_{t=0} = \mathbf{1}, \quad \mathbf{C}_{ii}|_{t=0} = \mathbf{1}, \quad s|_{t=0} = 0, \quad s_d|_{t=0} = 0. \quad (48)$$

The numerical solution obtained with extremely small time step ($\Delta t = 0.005\text{s}$) will be seen as the exact solution. The components of the Cauchy stress tensor \mathbf{T} are plotted in figure 1 for different time step sizes Δt . The numerical test reveals that the new method and the EBM with incompressibility correction have a similar integration error even for big time steps $\Delta t = 10\text{ s}$. Both algorithms are first order accurate and they are *comparable in accuracy*, but the computational effort for the new algorithm is much smaller than for the EBM.

5 CONCLUSIONS

A new highly efficient time stepping algorithm is presented. The derivation of the algorithm is based on the property of the weak invariance of the solution. Instead of solving 13 scalar equations with respect to 13 unknowns, a single incremental consistency condition has to be resolved with respect to the inelastic increment ξ . In the follow-up paper, the method will be generalized to cover a model with numerous back stress tensors and models with distortional hardening.

Acknowledgement: The research was supported by the Russian Science Foundation (project number 15-11-20013).

REFERENCES

- [1] Bröker, C. and Matzenmiller, A. On the generalization of uniaxial thermoviscoplasticity with damage to finite deformations based on enhanced rheological models. *Technische Mechanik* (2014) **34**: 142–165.
- [2] Firat, M., Karadeniz, E., Yenice, M. and Kaya, M. Improving the accuracy of stamping analyses including springback deformations. *JMEPEG* (2013) **22**(2): 332–337.

- [3] Helm, D. *Formgedächtnislegierungen, experimentelle Untersuchung, phänomenologische Modellierung und numerische Simulation der thermomechanischen Materialeigenschaften*, Universitätsbibliothek Kassel (2001).
- [4] Helm, D. Stress computation in finite thermoviscoplasticity. *Int. J. Plast.* (2006) **22**: 1699–1721.
- [5] Henann, D.L. and Anand, L. A large deformation theory for rate-dependent elastic-plastic materials with combined isotropic and kinematic hardening. *Int. J. Plast.* (2009) **25**: 1833–1878.
- [6] Laurent, H., Greze, R., Oliveira, M.C., Menezes, L.F., Manach, P.Y., Alves, J.L. Numerical study of springback using the split-ring test for an AA5754 aluminum alloy. *Fin. Elem. Anal. Des.* (2010) **46**(9): 751–759.
- [7] Lion, A. Constitutive modelling in finite thermoviscoplasticity: a physical approach based on nonlinear rheological elements. *Int. J. Plast.* (2000) **16**: 469–494.
- [8] Shutov, A.V. and Ihlemann, J. On the simulation of plastic forming under consideration of thermal effects. *Mat.-wiss. u. Werkstofftech.* (2011) **42**(7): 632–638.
- [9] Shutov, A.V. and Ihlemann, J. Analysis of some basic approaches to finite strain elasto-plasticity in view of reference change. *Int. J. Plast.* (2014) **63**: 183–197.
- [10] Shutov, A.V., Kreißig, R. Finite strain viscoplasticity with nonlinear kinematic hardening: Phenomenological modeling and time integration. *Comput. Methods Appl. Mech. Engrg.* (2008) **197**(21): 2015–2029.
- [11] Shutov, A.V., Kreißig, R. Geometric integrators for multiplicative viscoplasticity: analysis of error accumulation. *Comput. Methods Appl. Mech. Engrg.* (2010) **199**(9): 700–711.
- [12] Shutov, A.V., Landgraf, R. and Ihlemann, J. An explicit solution for implicit time stepping in multiplicative finite strain viscoelasticity. *Comput. Methods Appl. Mech. Engrg.* (2013) **256**: 213–225.
- [13] Shutov, A.V., Pfeiffer S. and Ihlemann, J. On the simulation of multi-stage forming processes: Invariance under change of the reference configuration. *Mat.-wiss. u. Werkstofftech.* (2012) **43**(7): 617–625.
- [14] Shutov, A.V., Silbermann, C.B. and Ihlemann, J. Ductile damage model for metal forming simulations including refined description of void nucleation. *Int. J. Plast.* (2015) <http://dx.doi.org/10.1016/j.ijplas.2015.03.003>

- [15] Silbermann, C.B., Shutov, A.V. and Ihlemann, J. Modeling the evolution of dislocation populations under non-proportional loading. *Int. J. Plast.* (2014) **55**: 58–79.
- [16] Vladimirov, I., Pietryga, M.P. and Reese, S. On the modelling of non-linear kinematic hardening at finite strains with application to springback - Comparison of time integration algorithms. *Int. J. Numer. Meth. Engng* (2008) **75**: 1–28.
- [17] Vladimirov, I.N., Pietryga, M.P., Kiliclar, Y., Tini, V., Reese, S. Failure modelling in metal forming by means of an anisotropic hyperelastic-plasticity model with damage. *Int. J. Damage Mech.* (2014) **23**(8): 1096–1132.
- [18] Zhu, Y., Kang, G. An extended cyclic plasticity model at finite deformations describing the Bauschinger effect and ratchetting behaviour. *13th International Conference on Fracture* (2013).