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FIXED SUBGROUPS ARE COMPRESSED IN SURFACE GROUPS

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For a compact surface Σ (orientable or not, and with boundary or not) we show that the fixed subgroup, $\mathrm{Fix}\,\mathcal{B}$, of any family $\mathcal{B}$ of endomorphisms of $\pi_1(\Sigma)$ is compressed in $\pi_1(\Sigma)$, i.e., $\mathrm{rk}(\mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(H)$ for any subgroup $\mathrm{Fix}\,\mathcal{B} \leqslant H \leqslant \pi_1(\Sigma)$. On the way, we give a partial positive solution to the inertia conjecture, both for free and for surface groups. We also investigate direct products, G, of finitely many free and surface groups, and give a characterization of when G satisfies that $\mathrm{rk}(\mathrm{Fix}\,\phi) \leqslant \mathrm{rk}(G)$ for every $\phi \in \mathrm{Aut}(G)$.

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1. Introduction

For a finitely generated group G, let $\mathrm{rk}(G)$ denote the rank (i.e., the minimal number of generators) of G. There are lots of research works in the literature about ranks of groups and, in particular, about controlling the rank of the intersection of subgroups of G in terms of their own ranks. An interesting case is when $G = F_r$ is a finitely generated free group (of rank r): more than half a century ago H. Neumann, see [15, 16], conjectured that, for any two nontrivial finitely generated subgroups A and B

of G,

$$\mathrm{rk}(A \cap B) - 1 \leqslant (\mathrm{rk}(A) - 1)(\mathrm{rk}(B) - 1).$$

After several partial results in this direction (for example, G. Tardos [17] showed it assuming either A or B of rank 2), this was finally proved in full generality independently by J. Friedman [7], and by I. Mineyev [14] (see also a simplified version by W. Dicks [4]).

Before this celebrated result was proved it had been shown that, for some special subgroups and in some special situations, one could say even more about their intersections; this specially applies to fixed subgroups. For the case of finitely generated free groups, there is a lot of research concerning properties of the subgroups fixed by automorphisms or endomorphisms; today, the main open problem in this direction is Conjecture 1.4 below. Some of these results had been translated into surface groups, where the situation is similar.

In the present paper, we give a new partial result in the direction of Conjecture 1.4 for finitely generated free groups, we prove some new results for the case of surface groups, and we study the same problems in the family of groups obtained by finitely many direct products of free and surface groups (where, as far as we know, nothing was previously investigated in this direction). In this larger context, the situation is much reacher, and new algebraic phenomena show up.

Let us first establish some notation and review the main results known about fixed subgroups in finitely generated free and surface groups.

For two groups G and H, let us denote the set of homomorphisms from G to H by $\mathrm{Hom}(G, H)$. Also, let us denote the set of endomorphisms of G by $\mathrm{End}(G)$, and the set of automorphisms of G by $\mathrm{Aut}(G)$. For an arbitrary family $\mathcal{B} \subseteq \mathrm{Hom}(G, H)$, the *equalizer* of $\mathcal{B}$ is

$$\mathrm{Eq}\,\mathcal{B} := \{g \in G \mid \beta_1(g) = \beta_2(g), \ \forall \beta_1, \beta_2 \in \mathcal{B}\} \leqslant G.$$

Similarly, for an arbitrary family $\mathcal{B} \subseteq \mathrm{End}(G)$, the *fixed subgroup* of $\mathcal{B}$ is

$$\mathrm{Fix}\,\mathcal{B} := \{g \in G \mid \phi(g) = g, \ \forall \phi \in \mathcal{B}\} = \cap_{\phi \in \mathcal{B}} \mathrm{Fix}\,\phi \leqslant G.$$

Note that, if G is a subgroup of H and $\mathcal{B} \subseteq \mathrm{Hom}(G, H)$ contains the inclusion of G in H then, $\mathrm{Eq}\,\mathcal{B} = \mathrm{Fix}\,\mathcal{B}$.

In [2], Bestvina–Handel solved the famous Scott's conjecture:

Theorem 1.1 (Bestvina–Handel, [2]). *For any automorphism ϕ of a finitely generated free group F,*

$$\mathrm{rk}(\mathrm{Fix}\,\phi) \leqslant \mathrm{rk}(F).$$

Later, Dicks–Ventura [5] introduced the notions of inertia and compressedness, and proved the following stronger result.

Definition 1.2. *Let G be an arbitrary finitely generated group. A subgroup $H \leqslant G$ is* inert *in G if $\mathrm{rk}(K \cap H) \leqslant \mathrm{rk}(K)$ for every (finitely generated) subgroup $K \leqslant G$. A* *subgroup* $H \leqslant G$ *is* compressed in G *if* $\mathrm{rk}(H) \leqslant \mathrm{rk}(K)$ *for every (finitely generated) subgroup* K *with* $H \leqslant K \leqslant G$.

Note that the family of inert subgroups of G is closed under finite intersections (and, in the case of a free group ambient $G = F_r$, even closed under arbitrary intersections by a standard argument on a descending chain of subgroups, see [5, Corollary I.4.13]). Clearly, if A is inert in G, then it is compressed in G and, in particular, $\mathrm{rk}(A) \leqslant \mathrm{rk}(G)$. It is not known in general whether the converse is true for free groups, see [5, Problem 1] (and also [18] as the "compressed-inert conjecture").

The main result in [5] is the following:

Theorem 1.3 (Dicks–Ventura, [5]). *Let* F *be a finitely generated free group, and let* $\mathcal{B} \subseteq \mathrm{End}(F)$ *be a family of injective endomorphisms of* F*. Then, the fixed subgroup* $\mathrm{Fix}\,\mathcal{B}$ *is inert in* F*.*

As far as we know, the same statement for endomorphisms is still an open problem, conjectured by Dicks–Ventura in [5, Problem 2] (see also [18] as the "inertia conjecture").

Conjecture 1.4. *For an arbitrary family of endomorphisms* $\mathcal{B} \subseteq \mathrm{End}(F)$ *of a finitely generated free group* F*,* $\mathrm{Fix}\,\mathcal{B}$ *is inert in* F*.*

Few progress has been done in this interesting direction during the last twenty years: only two later results gave supporting evidence to this conjecture. The first one was given by G. Bergman in [1], where he extended Bestvina–Handel's result to arbitrary families of endomorphisms:

Theorem 1.5 (Bergman, [1]). *Let* F *be a finitely generated free group, and let* $\mathcal{B} \subseteq \mathrm{End}(F)$*. Then,* $\mathrm{rk}(\mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(F)$*.*

The following result from the same paper will also be used in our arguments below (see [1, Corollary 12]).

Theorem 1.6 (Bergman [1]). *Let* $\phi\colon G \twoheadrightarrow H$ *be an epimorphism of free groups with* H *finitely generated. Then, the equalizer of any family of sections of* ϕ *is a free factor of* H*.*

Here, a *section* of $\phi\colon G \twoheadrightarrow H$ is a homomorphism going in the opposite direction, $\sigma\colon H \to G$, and such that $\phi\sigma = Id\colon H \to H$.

Finally, the second evidence towards Conjecture 1.4 was the following result, proved some years later by Martino–Ventura in [12]:

Theorem 1.7 (Martino–Ventura, [12]). *Let* F *be a finitely generated free group, and let* $\mathcal{B} \subseteq \mathrm{End}(F)$*. Then, the fixed subgroup* $\mathrm{Fix}\,\mathcal{B}$ *is compressed in* F*.*

For more details about fixed subgroups in free groups, we refer the interested reader to the survey [18], which covers the history of this line of research up to 2002.

Let us briefly recall now what is known about fixed subgroups in surface groups. A *surface* group is the fundamental group, $G = \pi_1(X)$, of a connected closed (possibly non-orientable) surface X. To fix the notation, we shall denote by Σ_g the closed orientable surface of genus $g \geqslant 0$, and by

$$S_g = \pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

its fundamental group (by convention, $S_0 = \langle \mid \rangle$ stands for the trivial group, the fundamental group of the sphere Σ_0); here, we use the notation $[x, y] = xyx^{-1}y^{-1}$. And for the non-orientable case, we shall denote by $N\Sigma_k$ the connected sum of $k \geqslant 1$ projective planes, and by

$$NS_k = \pi_1(N\Sigma_k) = \langle a_1, a_2, \ldots, a_k \mid a_1^2 \cdots a_k^2 \rangle$$

its fundamental group. Note that, among surface groups, the only abelian ones are $S_0 = 1$ (for the sphere), $S_1 = \mathbb{Z}^2$ (for the torus), and $NS_1 = \mathbb{Z}/2\mathbb{Z}$ (for the projective plane).

It is well known that the Euler characteristic of orientable surfaces is $\chi(\Sigma_g) = 2 - 2g$, and of the non-orientable ones is $\chi(N\Sigma_k) = 2 - k$. Hence, all surfaces have negative Euler characteristic except for the sphere Σ_0, the torus Σ_1, the projective plain $N\Sigma_1$, and the Klein bottle $N\Sigma_2$ (homeomorphic to the connected sum of two projective plains). As can be seen below, many results about automorphisms and endomorphisms will work in general for surfaces with negative Euler characteristic; S_0, S_1, NS_1, and NS_2 will usually present special and exceptional behaviour. We shall refer to the Euler characteristic also from the groups, namely $\chi(S_g) = \chi(\Sigma_g) = 2 - 2g$, and $\chi(NS_k) = \chi(N\Sigma_k) = 2 - k$.

It is also well known that the standard sets of generators for surface groups given above are minimal, i.e., $\mathrm{rk}(S_g) = 2g$ and $\mathrm{rk}(NS_k) = k$; this can be easily seen, for example, by looking at their corresponding abelianizations. Furthermore, the well known Freiheitssatz proved by Magnus, see [11, Theorem 5.1], states that any proper subset of this set of generators forms a free basis of the subgroup they generate.

First results about fixed subgroups of surface groups are due to Jiang–Wang–Zhang, who showed in [10] that $\mathrm{rk}(\mathrm{Fix}\,\phi) \leqslant \mathrm{rk}(G)$, for every endomorphism $\phi \in \mathrm{End}(G)$ of a surface group G with $\chi(G) < 0$. In the recent paper [19], Wu–Zhang extended it to the following results:

Theorem 1.8 (Wu–Zhang, [19]). *Let G be a surface group with $\chi(G) < 0$, and let $\mathcal{B} \subseteq \mathrm{End}(G)$. Then,*

(i) $\mathrm{rk}(\mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(G)$*, with equality if and only if* $\mathcal{B} = \{id\}$*;*
(ii) $\mathrm{rk}(\mathrm{Fix}\,\mathcal{B}) \leqslant \frac{1}{2}\mathrm{rk}(G)$*, if $\mathcal{B}$ contains a non-epimorphic endomorphism;*
(iii) if $\mathcal{B} \subseteq \mathrm{Aut}(G)$*, then* $\mathrm{Fix}\,\mathcal{B}$ *is inert in G.*

For equalizers of sections of homomorphisms from surface groups to free groups, Wu–Zhang gave the following result, see [19, Proposition 4.7]:

Proposition 1.9 (Wu–Zhang, [19]). *Let G be a surface group with $\chi(G) < 0$, and F a finitely generated free group. If $\phi : G \twoheadrightarrow F$ is an epimorphism, and $\mathcal{B}$ is a family of sections of ϕ, then*

$$\mathrm{rk}(\mathrm{Eq}\,\mathcal{B}) \leqslant \mathrm{rk}(F) \leqslant \frac{1}{2}\mathrm{rk}(G).$$

In view of these results it seems reasonable to state the inertia Conjecture 1.4 for surface groups as well.

Conjecture 1.10. *For an arbitrary family of endomorphisms $\mathcal{B} \subseteq \mathrm{End}(G)$ of a surface group G, $\mathrm{Fix}\,\mathcal{B}$ is inert in G.*

The structure of the paper is as follows. We dedicate Section 2 to finitely generated free groups: our main result in this context is Theorem 2.1, which provides an alternative proof for Theorem 1.7, and also gives a partial positive solution to Conjecture 1.4 (see Corollaries 2.2 and 2.3).

We dedicate Section 3 to surface groups: the advantage of our proof of Theorem 2.1 is that it easily translates into surface groups, see Theorem 3.6 our main result in this context. As corollaries, we get some new results giving supporting evidence to Conjecture 1.10: we prove compression for fixed subgroups of arbitrary families of endomorphisms of a surface group (see Corollary 3.7), and we get partial positive solutions to Conjecture 1.10 (see Corollaries 3.8 and 3.9).

Finally, in Section 4 we investigate the same issues (Bestvina–Handel bound, compression, and inertia) both for automorphisms and endomorphism of groups of the form $G = G_1 \times \cdots \times G_n$, where $n \geqslant 0$ and each G_i is either a finitely generated free group or a surface group, $G_i = F_r$, S_g, NS_k for some $r \geqslant 1$, $g \geqslant 1$ or $k \geqslant 1$, respectively. In this context, we give a characterization of those groups G within this family for which $\mathrm{rk}(\mathrm{Fix}\,\phi) \leqslant \mathrm{rk}(G)$ for every $\phi \in \mathrm{Aut}(G)$ (see Theorem 4.8), a necessary condition for those satisfying that $\mathrm{Fix}\,(\phi)$ is compressed in G for every $\phi \in \mathrm{Aut}(G)$ (see Theorem 4.9), and a necessary condition for those satisfying that $\mathrm{Fix}\,(\phi)$ is inert in G for every $\phi \in \mathrm{Aut}(G)$ (see Theorem 4.12).

2. Fixed points in free groups

In this context, we prove the following result for finitely generated free groups. The proof (as well as the proof of later Theorem 3.6) is an adaptation and generalization of the proof of [19, Theorem 1.3] to our situation. We hope this contribution sheds light for the resolution of the full Conjecture 1.4 in the future.

Theorem 2.1. *Let F be a finitely generated free group, let $\mathcal{B} \subseteq \mathrm{End}(F)$ be an arbitrary family of endomorphisms, let $\langle \mathcal{B} \rangle \leqslant \mathrm{End}(F)$ be the submonoid generated by $\mathcal{B}$, and let $\beta_0 \in \langle \mathcal{B} \rangle$ with image $\beta_0(F)$ of minimal rank. Then, for every subgroup $K \leqslant F$ such that $\beta_0(K) \cap \mathrm{Fix}\,\mathcal{B} \leqslant K$, we have $\mathrm{rk}(K \cap \mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(K)$.*

Proof. Suppose F is a finitely generated free group, $\mathcal{B} \subseteq \mathrm{End}(F)$ is an arbitrary family of endomorphisms of F, and $\langle \mathcal{B} \rangle$ is the closure of $\mathcal{B}$ in $\mathrm{End}(F)$ by composition (note also that $Id \in \langle \mathcal{B} \rangle$). Since, for any $\alpha, \beta \in \mathcal{B}$, $\operatorname{Fix} \alpha \cap \operatorname{Fix} \beta \subseteq \operatorname{Fix}(\alpha\beta)$, it is clear that $\operatorname{Fix} \langle \mathcal{B} \rangle = \operatorname{Fix} \mathcal{B}$ and so, the inequality we have to prove does not change when replacing $\mathcal{B}$ to $\langle \mathcal{B} \rangle$. Hence we can assume, without loss of generality, that $\mathcal{B}$ itself is a submonoid of $\operatorname{End}(F)$, i.e., $\langle \mathcal{B} \rangle = \mathcal{B}$.

Now choose $\beta_0 \in \mathcal{B}$ such that

$$\operatorname{rk}(\beta_0(F)) = \min\{\operatorname{rk}(\gamma(F)) \mid \gamma \in \mathcal{B}\}.$$

Thus all elements of $\mathcal{B}$ act injectively on $\beta_0(F)$. Let $\beta_0\mathcal{B} = \{\beta_0\gamma \mid \gamma \in \mathcal{B}\} \subseteq \mathcal{B}$. Since $\beta_0\gamma(\beta_0(F)) \leqslant \beta_0(F)$ we get, by restriction, a family $\beta_0\mathcal{B}|_{\beta_0(F)}$ of injective endomorphisms of the finitely generated free group $\beta_0(F)$,

$$\beta_0\gamma|_{\beta_0(F)} \colon \beta_0(F) \to \beta_0(F).$$

By Theorem 1.3, $\operatorname{Fix}(\beta_0\mathcal{B}) = \operatorname{Fix}(\beta_0\mathcal{B}|_{\beta_0(F)})$ is inert in $\beta_0(F)$ that is, for every $L \leqslant \beta_0(F)$, we have

$$\operatorname{rk}(L \cap \operatorname{Fix}(\beta_0\mathcal{B})) \leqslant \operatorname{rk}(L). \tag{2.1}$$

Now let $K \leqslant F$ be a subgroup such that $\beta_0(K) \cap \operatorname{Fix} \mathcal{B} \leqslant K$; we have to show that $\operatorname{rk}(K \cap \operatorname{Fix} \mathcal{B}) \leqslant \operatorname{rk}(K)$. Let

$$E = \beta_0^{-1}(\beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B})) \leqslant F.$$

By construction, β_0 gives an epimorphism of free groups,

$$\beta_0|_E \colon E \twoheadrightarrow \beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B}),$$

with image being finitely generated. On the other hand, every $\gamma \in \mathcal{B}$ restricts to a section of $\beta_0|_E$, namely

$$\gamma|_{\beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B})} \colon \beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B}) \to E;$$

in fact, for every $x \in \beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B})$, it is clear that $\beta_0\gamma(x) = x$ and so, $\gamma(x) \in E$ and $\beta_0\gamma|_{\beta_0(K)\cap\operatorname{Fix}(\beta_0\mathcal{B})} = Id_{\beta_0(K)\cap\operatorname{Fix}(\beta_0\mathcal{B})}$; in particular, taking $\gamma = Id$, we have $\beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B}) \leqslant E$. Hence, by Theorem 1.6 applied to this family of sections, we obtain that $\operatorname{Eq}(\mathcal{B}|_{\beta_0(K)\cap\operatorname{Fix}(\beta_0\mathcal{B})})$ is a free factor of $\beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B})$. Since this family of sections contains the inclusion of $\beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B})$ into E, we have

$$\begin{aligned}
\operatorname{Eq}(\mathcal{B}|_{\beta_0(K)\cap\operatorname{Fix}(\beta_0\mathcal{B})}) &= \operatorname{Fix}(\mathcal{B}|_{\beta_0(K)\cap\operatorname{Fix}(\beta_0\mathcal{B})}) \\
&= \operatorname{Fix} \mathcal{B} \cap \beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B}) \\
&= \beta_0(K) \cap \operatorname{Fix} \mathcal{B} \\
&= K \cap \operatorname{Fix} \mathcal{B}.
\end{aligned}$$

(For one of the inclusions in the last equality we use our assumption on K; the other one is immediate.) Hence, using equation (2.1) for $L = \beta_0(K)$, we conclude that

$$\operatorname{rk}(K \cap \operatorname{Fix} \mathcal{B}) \leqslant \operatorname{rk}(\beta_0(K) \cap \operatorname{Fix}(\beta_0\mathcal{B})) \leqslant \operatorname{rk}(\beta_0(K)) \leqslant \operatorname{rk}(K),$$

completing the proof. □

As mentioned above, the argument in Theorem 2.1 provides an alternative proof for Theorem 1.7, easier than the one given in [12]: every subgroup K with $\mathrm{Fix}\,\mathcal{B} \leqslant K \leqslant F$ clearly satisfies the hypothesis $\beta_0(K) \cap \mathrm{Fix}\,\mathcal{B} \leqslant K$ and so, we have $\mathrm{rk}(\mathrm{Fix}\,\mathcal{B}) = \mathrm{rk}(K \cap \mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(K)$. This shows compression of $\mathrm{Fix}\,\mathcal{B}$.

Another interesting consequence of Theorem 2.1 is the following partial positive solution to Conjecture 1.4. We hope this helps to its full resolution in the future.

Corollary 2.2. *Let F be a finitely generated free group, let $\mathcal{B} \subseteq \mathrm{End}(F)$ be an arbitrary family of endomorphisms, let $\langle \mathcal{B} \rangle \leqslant \mathrm{End}(F)$ be the submonoid generated by $\mathcal{B}$, and let $\beta_0 \in \langle \mathcal{B} \rangle$ with image $\beta_0(F)$ of minimal rank. Then,* $\mathrm{Fix}\,\mathcal{B}$ *is inert in $\beta_0(F)$. Moreover, if $\beta_0(F)$ is inert in F then* $\mathrm{Fix}\,\mathcal{B}$ *is inert in F as well.*

Proof. The first statement follows directly from Theorem 2.1 as soon as we show that every $K \leqslant \beta_0(F)$ satisfies the condition $\beta_0(K) \cap \mathrm{Fix}\,\mathcal{B} \leqslant K$. Let $x \in \beta_0(K) \cap \mathrm{Fix}\,\mathcal{B}$; this implies that $\beta_0(k) = x = \beta_0(x)$ for some $k \in K$. But both k and x belong to $\beta_0(F)$, where β_0 is injective by the minimality condition in the definition of β_0. Hence, $x = k \in K$.

For the second statement we only need to recall transitivity of the inertia property (if $A \leqslant B \leqslant C$, and A is inert in B, and B is inert in C, then A is inert in C). □

Corollary 2.3. *Let F be a finitely generated free group, and let $\mathcal{B} \subseteq \mathrm{End}(F)$ be an arbitrary family of endomorphisms. If some composition of endomorphisms from $\mathcal{B}$ has image of rank 1 or 2, then* $\mathrm{Fix}\,\mathcal{B}$ *is inert in F.*

Proof. By assumption, the minimal rank of the image of the endomorphisms in $\langle \mathcal{B} \rangle$ is either 1 or 2. Let $\beta_0 \in \langle \mathcal{B} \rangle$ realize such minimum. Then $\beta_0(F)$ is inert in F (by the positive solution to H. Neumann conjecture or, better, by the special case previously proved by Tardos [17]). Hence, by Corollary 2.2, $\mathrm{Fix}\,\mathcal{B}$ is inert in F. □

3. Fixed points in surface groups

Before studying fixed subgroups in surface groups, let us remind the following folklore facts about surface groups which will be used later. A first fact where the assumption of negative Euler characteristic is crucial, is about the center and the centralizer of non-trivial elements in surface groups. The following lemma can be easily deduced from the fact that closed surfaces with negative Euler characteristic admit a hyperbolic metric (see Theorem 1.2 and the subsequent comment in page 22 of [6]).

Lemma 3.1. *Let G be a surface group with $\chi(G) < 0$. Then its center is trivial, $Z(G) = 1$, and the centralizer of any non-trivial element $1 \neq g \in G$ is infinite cyclic,* $\mathrm{Cen}_G(g) \simeq \mathbb{Z}$.

Remark 3.2. The same result is true for free groups F_r of rank $r \geqslant 2$; this will be crucial for the arguments in Section 4. However, it is no longer true for the remaining groups within the family of finitely generated free and surface groups: $F_0 = S_0 = 1$ is trivial, $F_1 = \mathbb{Z}$, $S_1 = \mathbb{Z}^2$ and $NS_1 = \mathbb{Z}/2\mathbb{Z}$ are abelian, and the *Klein bottle group*,

$$NS_2 = \langle a_1, a_2 \mid a_1^2 a_2^2 = 1\rangle \simeq \langle a, b \mid aba^{-1}b\rangle,$$

is not abelian but has center isomorphic to $\mathbb{Z}$, with generator $a_1^2 = a^2$ (the isomorphism above is given by $a_1 \mapsto a$, $a_2 \mapsto a^{-1}b$); in fact, NS_2 is virtually abelian since $\mathbb{Z}^2 \simeq \langle a^2, b\rangle$ is an index two normal subgroup of NS_2.

The following are basic facts on surface groups, we include the proofs here for completeness (see also [19, Lemma 2.7]).

Lemma 3.3. *Let G be a surface group with $\chi(G) < 0$. Then,*

(i) if $H < G$ is a proper subgroup with $\mathrm{rk}(H) \leqslant \mathrm{rk}(G)$, then H is a free group;
(ii) if $\phi\colon G \to G$ is a non-epimorphic endomorphism, then $\phi(G)$ is a free group of rank $\mathrm{rk}(\phi(G)) \leqslant \frac{1}{2}\mathrm{rk}(G)$;
(iii) the group G is both Hophian and co-Hopfian (i.e., all injective and all surjective endomorphisms of G are, in fact, automorphisms).

Proof. (i) follows easily from the fact that subgroups of G are either free or finite index; and the subgroups $H \leqslant G$ in this last family are again surface groups with negative Euler characteristic and satisfying $\chi(H)/\chi(G) = [G : H] > 1$ hence, $\mathrm{rk}(H) > \mathrm{rk}(G)$.

For (ii), we recall the fact that the *inner rank* of G, i.e., the maximal rank of a free quotient of G, is at most $\frac{1}{2}\mathrm{rk}(G)$ (see Lyndon–Schupp [11, page 52]).

Finally, by a classical result, surface groups are residually finite (see [8] for a short proof) and so, Hophian. On the other hand, if $\phi \in \mathrm{End}(G)$ is injective then the image $\phi(G)$ is isomorphic to G, but $\mathrm{rk}(\phi(G)) \leqslant \mathrm{rk}(G)$; by (i), $\phi \in \mathrm{Aut}(G)$. This completes the proof of (iii). □

Remark 3.4. Again, the situation is different without the hypothesis of negative Euler characteristic. For the case of the torus, $S_1 = \mathbb{Z}^2$ violates (i), (ii) and the co-Hopfianity. For the projective plane, $NS_1 = \mathbb{Z}/2\mathbb{Z}$ satisfies the Lemma but with trivial meaning. And, finally, the Klein bottle group NS_2 violates again (i), (ii) and the co-Hopfianity, since $a \mapsto a^3$, $b \mapsto b$ defines an injective non-surjective endomorphism whose image is isomorphic to NS_2 itself. (Both $\mathbb{Z}^2$ and NS_2 are Hopfian, like all surface groups without exception.)

The proof of Theorem 2.1 works for surface groups as well, with some extra arguments distinguishing whether certain involved subgroups are free or finite index. We reproduce that argument here to highlight these important points. It works for surface groups with negative Euler characteristic; however, for the exceptional ones one can prove the inertia conjecture directly.

Proposition 3.5. *Let G be either $F_0 = S_0 = 1$, or $S_1 = \mathbb{Z}^2$, or $NS_1 = \mathbb{Z}/2\mathbb{Z}$, or NS_2, and let $\mathcal{B} \subseteq \mathrm{End}(G)$ be an arbitrary family of endomorphisms. Then,* $\mathrm{Fix}\,\mathcal{B}$ *is inert in G.*

Proof. For $F_0 = S_0 = 1$ and $NS_1 = \mathbb{Z}/2\mathbb{Z}$ the result is trivially true. In $S_1 = \mathbb{Z}^2$ subgroups satisfy, in general, the implication $H \leqslant K \Rightarrow \mathrm{rk}(H) \leqslant \mathrm{rk}(K)$ so, again, the result is clearly true.

For the Klein bottle group NS_2, Wu–Zhang showed in [19, Example 6.2] that, for every $Id \neq \phi \in \mathrm{End}(NS_2)$, the fixed subgroup $\mathrm{Fix}\,\phi$ is isomorphic to either $\mathbb{Z}^2$, or $\mathbb{Z}$, or 1. Thus, for every $\{Id\} \neq \mathcal{B} \subseteq \mathrm{End}(NS_2)$, we also have $\mathrm{Fix}\,\mathcal{B} \simeq \mathbb{Z}^2, \mathbb{Z}, 1$ and so, $\mathrm{Fix}\,\mathcal{B}$ is inert in NS_2. □

Theorem 3.6. *Let G be a surface group, let $\mathcal{B} \subseteq \mathrm{End}(G)$ be an arbitrary family of endomorphisms, let $\langle \mathcal{B} \rangle \leqslant \mathrm{End}(G)$ be the submonoid generated by $\mathcal{B}$, and let $\beta_0 \in \langle \mathcal{B} \rangle$ with image $\beta_0(G)$ of minimal rank. Then, for every subgroup $K \leqslant G$ such that $\beta_0(K) \cap \mathrm{Fix}\,\mathcal{B} \leqslant K$, we have* $\mathrm{rk}(K \cap \mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(K)$.

Proof. If $\chi(G) \geqslant 0$ then Proposition 3.5 gives us inertia of $\mathrm{Fix}\,\mathcal{B}$ and so, the inequality $\mathrm{rk}(K \cap \mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(K)$ holds for every $K \leqslant G$ without conditions. Let us assume then $\chi(G) < 0$.

As in the proof of Theorem 2.1, we may assume that $\mathcal{B}$ contains the identity and it is closed under composition. If $\mathcal{B}$ consists of epimorphisms, then $\mathcal{B} \subseteq \mathrm{Aut}(G)$, Theorem 1.8(iii) tells us that $\mathrm{Fix}\,\mathcal{B}$ is inert in G, and we are done.

So, let us assume that $\mathcal{B}$ contains at least one non-epimorphic endomorphism; by Lemma 3.3(ii), its image will be a free group. As above, choose $\beta_0 \in \mathcal{B}$ such that $\beta_0(G)$ is a free group of minimal rank. All elements of $\mathcal{B}$ must then act injectively on $\beta_0(G)$. As above, we consider the subset $\beta_0\mathcal{B} = \{\beta_0\gamma \mid \gamma \in \mathcal{B}\} \subseteq \mathcal{B}$, their restrictions to $\beta_0(G)$ give a family $\beta_0\mathcal{B}|_{\beta_0(G)}$ of injective endomorphisms of the free group $\beta_0(G)$, namely $\beta_0\gamma|_{\beta_0(G)} \colon \beta_0(G) \to \beta_0(G)$, and, by Theorem 1.3, $\mathrm{Fix}\,(\beta_0\mathcal{B}) = \mathrm{Fix}\,(\beta_0\mathcal{B}|_{\beta_0(G)})$ is inert in $\beta_0(G)$, i.e., for every $L \leqslant \beta_0(G)$, we have

$$\mathrm{rk}(L \cap \mathrm{Fix}\,(\beta_0\mathcal{B})) \leqslant \mathrm{rk}(L). \tag{3.1}$$

Now let $K \leqslant G$ be a subgroup satisfying $\beta_0(K) \cap \mathrm{Fix}\,\mathcal{B} \leqslant K$; we have to show that $\mathrm{rk}(K \cap \mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(K)$. As above, we consider $E = \beta_0^{-1}(\beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B})) \leqslant G$, β_0 restricts to an epimorphism $\beta_0|_E \colon E \twoheadrightarrow \beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B})$ whose image is free and finitely generated, and every $\gamma \in \mathcal{B}$ restricts to a section of $\beta_0|_E$, namely

$$\gamma|_{\beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B})} \colon \beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B}) \to E.$$

Now, let us distinguish whether $E \leqslant G$ is a free group (like in the proof of Theorem 2.1) or a surface group. In the first case, Theorem 1.6 tells us that $\mathrm{Eq}\,(\mathcal{B}|_{\beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B})})$ is a free factor of $\beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B})$ and so,

$$\mathrm{rk}(\mathrm{Eq}\,(\mathcal{B}|_{\beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B})})) \leqslant \mathrm{rk}(\beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B})).$$

Otherwise, Proposition 1.9 gives us this inequality directly.

Finally, the exact same argument as above shows that $\mathrm{Eq}\,(\mathcal{B}|_{\beta_0(K)\cap \mathrm{Fix}\,(\beta_0\mathcal{B})}) = K \cap \mathrm{Fix}\,\mathcal{B}$, using here our assumption on K. Hence, by equation (3.1) applied to $L = \beta_0(K)$, we conclude

$$\mathrm{rk}(K \cap \mathrm{Fix}\,\mathcal{B}) \leqslant \mathrm{rk}(\beta_0(K) \cap \mathrm{Fix}\,(\beta_0\mathcal{B})) \leqslant \mathrm{rk}(\beta_0(K)) \leqslant \mathrm{rk}(K),$$

completing the proof. □

As a first corollary, we get compression of fixed subgroups of arbitrary families of endomorphisms, the analogous result to Theorem 1.7 for surface groups.

Corollary 3.7. *Let G be a surface group, and let $\mathcal{B} \subseteq \mathrm{End}(G)$. Then, $\mathrm{Fix}\,\mathcal{B}$ is compressed in G.*

Proof. This is a direct consequence of Theorem 3.6, after observing that $\mathrm{Fix}\,\mathcal{B} \leqslant K \leqslant G$ trivially implies $\beta_0(K) \cap \mathrm{Fix}\,\mathcal{B} \leqslant K$. □

Finally, we can also get the corresponding partial positive solution to Conjecture 1.10. We hope this helps to its full resolution in the future.

Corollary 3.8. *Let G be a surface group, let $\mathcal{B} \subseteq \mathrm{End}(G)$ be an arbitrary family of endomorphisms, let $\langle \mathcal{B} \rangle \leqslant \mathrm{End}(G)$ be the submonoid generated by $\mathcal{B}$, and let $\beta_0 \in \langle \mathcal{B} \rangle$ with image $\beta_0(G)$ of minimal rank. Then, $\mathrm{Fix}\,\mathcal{B}$ is inert in $\beta_0(G)$. Moreover, if $\beta_0(G)$ is inert in G then $\mathrm{Fix}\,\mathcal{B}$ is inert in G as well.*

Proof. The exact same argument as in Corollary 2.2 works here. □

Corollary 3.9. *Let $\mathcal{B} \subseteq \mathrm{End}(NS_3)$ be an arbitrary family of endomorphisms of the surface group NS_3. Then, $\mathrm{Fix}\,\mathcal{B}$ is inert in NS_3.*

Proof. If $\mathcal{B} \subseteq \mathrm{Aut}(NS_3)$, then $\mathrm{Fix}\,\mathcal{B}$ is inert in NS_3 according to Theorem 1.8(iii). Otherwise, Lemma 3.3(iii) and (ii) imply that some $\beta_0 \in \mathcal{B}$ has image being free of minimal rank, and $\mathrm{rk}(\beta_0(NS_3)) \leqslant \lfloor \frac{1}{2}\mathrm{rk}(NS_3) \rfloor = 1$. Then, $\mathrm{Fix}\,\mathcal{B} \leqslant \beta_0(NS_3)$ is also cyclic and so, inert in NS_3. □

Remark 3.10. I. Mineyev claimed (but did not explicitly prove) that Hanna Neumann conjecture also holds for surface groups (see [13, Section 8]). If this were correct, then every rank two subgroup A of a surface group G would be inert in G, and the argument above would also work for ambient surface groups of rank up to five: any family of endomorphisms $\mathcal{B} \subseteq \mathrm{End}(G)$ of a surface group G with $\mathrm{rk}(G) \leqslant 5$ would satisfy that $\mathrm{Fix}\,\mathcal{B}$ is inert in G (i.e., Corollary 3.9 would also be valid for S_2, NS_4 and NS_5). In fact, the case $\mathcal{B} \subseteq \mathrm{Aut}(G)$ is covered by Theorem 1.8(iii); and otherwise, there is some $\beta_0 \in \mathcal{B}$ with free image of rank $\mathrm{rk}(\beta_0(G)) \leqslant \lfloor \frac{1}{2}\mathrm{rk}(G) \rfloor = 2$ and so, by Corollary 3.8, $\mathrm{Fix}\,\mathcal{B}$ is inert in $\beta_0(G)$, which is inert in G.

4. Fixed points in direct products

In the previous sections we considered, separately, finitely generated free groups, and surface groups. Now, we shall study direct products of finitely many such groups (with possible repetitions), i.e., groups of the form $G = G_1 \times \cdots \times G_n$, where $n \geqslant 1$ and each G_i is either a finitely generated free group F_r, $r \geqslant 1$, or an orientable surface group S_g, $g \geqslant 1$, or a non-orientable surface group NS_k, $k \geqslant 1$. For short (and only within the scope of the present preprint), we shall call such G a *product group*. We shall consider automorphisms and endomorphisms of product groups and shall analyze the properties of their fixed subgroups.

We need to begin with some basic algebraic properties of product groups. First note that, for arbitrary groups A and B, $\mathrm{rk}(A\times B) \leqslant \mathrm{rk}(A)+\mathrm{rk}(B)$. Sometimes this inequality is strict (as illustrated with the well-known fact that a direct product of two finite cyclic groups of coprime orders is again cyclic), but in the case of product groups it is always an equality.

Lemma 4.1. *Let $G = G_1 \times \cdots \times G_n$, where each G_i is either a finitely generated free group or a surface group. Then,* $\mathrm{rk}(G) = \mathrm{rk}(G_1) + \cdots + \mathrm{rk}(G_n)$.

Proof. By the form of the G_i's, the torsion part of G_i^{ab} is either trivial or $\mathbb{Z}/2\mathbb{Z}$; furthermore, $\mathrm{rk}(G_i^{\mathrm{ab}}) = \mathrm{rk}(G_i)$. Then,

$$\begin{aligned}\mathrm{rk}(G) \geqslant \mathrm{rk}(G^{\mathrm{ab}}) &= \mathrm{rk}(G_1^{\mathrm{ab}} \times \cdots \times G_n^{\mathrm{ab}}) \\ &= \mathrm{rk}(G_1^{\mathrm{ab}}) + \cdots + \mathrm{rk}(G_n^{\mathrm{ab}}) \\ &= \mathrm{rk}(G_1) + \cdots + \mathrm{rk}(G_n) \\ &\geqslant \mathrm{rk}(G_1 \times \cdots \times G_n) \\ &= \mathrm{rk}(G),\end{aligned}$$

and the two inequalities are equalities. Hence, $\mathrm{rk}(G) = \mathrm{rk}(G_1) + \cdots + \mathrm{rk}(G_n)$. □

Note that the center of a direct product of groups, $A \times B$, is the product of the corresponding centers, $Z(A \times B) = Z(A) \times Z(B)$. Also, the centralizer of an element $(a, b) \in A \times B$ is clearly the direct product of centralizers of its respectives components, $\mathrm{Cen}_{A\times B}(a, b) = \mathrm{Cen}_A(a) \times \mathrm{Cen}_B(b)$; in particular, $\mathrm{Cen}_{A\times B}(a, 1) = \mathrm{Cen}_A(a) \times B$ and $\mathrm{Cen}_{A\times B}(1, b) = A \times \mathrm{Cen}_B(b)$.

Note also that, among our building blocks (namely, F_r for $r \geqslant 1$, S_g for $g \geqslant 1$, and NS_k for $k \geqslant 1$) the only abelian ones are $F_1 = \mathbb{Z}$, $S_1 = \mathbb{Z}^2$ and $NS_1 = \mathbb{Z}/2\mathbb{Z}$. Also, they all have trivial center and infinite cyclic centralizers for non-trivial elements, except for F_1, S_1, NS_1 and NS_2 (see Lemma 3.1 and Remark 3.2). Using this, we can easily deduce how is the center and the centralizer of an arbitrary element in a product group. The following lemma will be crucial in the forthcoming arguments.

Lemma 4.2. *Let $G = G_1 \times \cdots \times G_n$, $n \geqslant 1$, be a product group, where each G_i is* *a finitely generated free or surface group. Then,*

$$\begin{aligned} Z(G)=1 &\iff \text{each } G_i \text{ is free non-abelian, or a surface group with } \chi(G_i)<0 \\ &\iff \forall i=1,\ldots,n,\ G_i\simeq F_r \text{ with } r\geqslant 2,\ \text{or } G_i\simeq S_g \text{ with } g\geqslant 2,\ \text{or} \\ &\quad\ \ G_i\simeq NS_k \text{ with } k\geqslant 3. \end{aligned}$$

Furthermore, in the case $Z(G)=1$, *every* n*-tuple of elements* $(g_1,\ldots,g_n)\in G$ *satisfies* $\mathrm{Cen}_G(g_1,\ldots,g_n)\simeq \widehat{G_1}\times\cdots\times\widehat{G_n}$, *where* $\widehat{G_i}$ *is* G_i *if* $g_i=1$, *or* $\mathbb{Z}$ *if* $g_i\neq 1$.

Lemma 4.2 separates product groups $G=G_1\times\cdots\times G_n$ into three different types, namely: (i) those for which $Z(G_i)\neq 1$ for all $i=1,\ldots,n$, they will be called of *Euclidean* type, and are precisely the groups of the form $G=NS_2^{\ell}\times\mathbb{Z}^p\times(\mathbb{Z}/2\mathbb{Z})^q$, for some integers $\ell,p,q\geqslant 0$; (ii) those for which $Z(G_i)=1$ for all $i=1,\ldots,n$, they will be called of *hyperbolic* type and are the finite direct products of F_r's with $r\geqslant 2$, S_g's with $g\geqslant 2$, and NS_k's with $k\geqslant 3$; and (iii) those mixing the two behaviours. With this language, Lemma 4.2 says that $Z(G)=1$ if and only if G is of hyperbolic type. Euclidean and hyperbolic product groups will play an important role in the subsequent arguments.

For product groups the global group determines the number of components and the components themselves (except for the case of $\mathbb{Z}^2$ being isomorphic to $\mathbb{Z}\times\mathbb{Z}$); see the next proposition for details. This contrasts with the general situation where $\mathbb{Z}\times A\simeq\mathbb{Z}\times B$ does not imply $A\simeq B$, see [9].

Proposition 4.3. *Let* $G=G_1\times\cdots\times G_n$ *and* $H=H_1\times\cdots\times H_m$, $n,m\geqslant 1$, *be two product groups of hyperbolic type (where each* G_i *and* H_j *is a non-abelian free group or a surface group with negative Euler characteristic). Then,* $G\simeq H$ *if and only if* $n=m$ *and* $G_i\simeq H_i$ *up to reordering.*

Proof. The "if" part is obvious.

For the convers, suppose $G\simeq H$ and, by symmetry, $n\leqslant m$. By Lemma 4.2, $Z(G)=Z(H)=1$. Let us prove the implication by induction on n.

For $n=1$, and looking at centralizers, we immediately deduce $m=1$ and $G_1=G\simeq H=H_1$.

So suppose $n\geqslant 2$, assume the implication is true for $n-1$, and let $\phi\colon G\to H$ be an isomorphism. Let $1\neq g\in G_n$ be one of the standard generators for G_n, and consider the element $(1,\ldots,1,g)\in G$. By Lemma 4.2, $\mathrm{Cen}_G(1,\ldots,1,g)\simeq G_1\times\cdots\times G_{n-1}\times\mathbb{Z}$. And, again by Lemma 4.2, $\mathrm{Cen}_H(\phi(1,\ldots,1,g))\simeq H_{j_1}\times\cdots\times H_{j_{m-k}}\times\mathbb{Z}^k$, where $j_1<\cdots<j_{m-k}$ are the positions of the trivial coordinates in $\phi(1,\ldots,1,g)$, and k is its number of non-trivial coordinates. Then, $G_1\times\cdots\times G_{n-1}\times\mathbb{Z}\simeq \mathrm{Cen}_G(1,\ldots,1,g)\simeq\mathrm{Cen}_H(\phi(1,\ldots,1,g))\simeq H_{j_1}\times\cdots\times H_{j_{m-k}}\times\mathbb{Z}^k$. From here, taking centers, we get $k=1$; this means that $\phi(1,\ldots,1,g)$ has only one non-trivial coordinate, say at position j. In principle j depends on g, but a straightforward argument shows it does not (any pair of different generators g,g' from the standard presentation for G_n do not commute to themselves and so, $\phi(1,\ldots,1,g)$ and $\phi(1,\ldots,1,g')$ do not commute either; this implies they both have their

unique non-trivial coordinate at the same position j). Up to reordering the coordinates of H if necessary, this means that ϕ restricts to an injective morphism $\phi|_{G_n}\colon 1\times\cdots\times 1\times G_n\to 1\times\cdots\times 1\times H_m$. And global surjectivity of ϕ together with an argument as above applied to ϕ^{-1} tells us that $\phi|_{G_n}$ is surjective. Hence, $G_n\simeq H_m$.

Finally, factoring out the centers in the above isomorphism between centralizers, we get $G_1\times\cdots\times G_{n-1}\simeq H_1\times\cdots\times H_{m-1}$. By the inductive hypothesis, $n-1=m-1$ (so, $n=m$) and, again up to reordering, $G_i\simeq H_i$ for all $i=1,\dots,n-1$. □

With similar arguments we can describe *all* automorphisms of a product group G with trivial center: up to permuting the coordinates corresponding to isomorphic G_i's, all automorphisms of G will be rectangular, i.e., a product of individual automorphisms of the coordinates. After introducing the necessary notation, we establish this fact formally in the following proposition.

For a product group $G=G_1\times\cdots\times G_n$, we can collect together the coordinates corresponding to isomorphic G_i's and present it in the form $G=G_1^{n_1}\times\cdots\times G_m^{n_m}$, where $n_i\geqslant 1$ and $G_i\not\simeq G_j$ for different $i,j=1,\dots,m$; of course, $n=n_1+\cdots+n_m$. When we need to distinguish between the coordinates of $G_i^{n_i}$, we shall use the notation $G_i^{n_i}=G_{i,1}\times\cdots\times G_{i,n_i}$, where $G_{i,j}=G_i$ for all $j=1,\dots,n_i$. Assuming a small risk of confusion, we shall use both notations simultaneously (the meaning being clear from the context at any time); we shall refer to them as the *global* notation and the *block* notation, respectively.

Given an automorphism of each coordinate, $\phi_i\in\mathrm{Aut}(G_i)$ for $i=1,\dots,n$, their product $\phi=\prod_{i=1}^n\phi_i=\phi_1\times\cdots\times\phi_n\colon G\to G$, $(g_1,\dots,g_n)\mapsto(\phi_1(g_1),\dots,\phi_n(g_n))$, is clearly an automorphism of G; let us refer to such automorphisms of G as the *rectangular* ones. On the other hand, given a permutation $\sigma\in S_{n_i}$ of the set of indices $\{1,\dots,n_i\}$, the automorphism of $G_i^{n_i}$ defined by $(g_1,\dots,g_{n_i})\mapsto(g_{\sigma(1)},\dots,g_{\sigma(n_i)})$ extends to an automorphism of G by fixing the rest of coordinates; abusing notation, we shall denote both of them by σ (so, $\sigma\in S_{n_i}$, $\sigma\in\mathrm{Aut}(G_i^{n_i})$, and $\sigma\in\mathrm{Aut}(G)$). Note that if $\sigma\in S_{n_i}$ and $\tau\in S_{n_j}$ with $i\neq j$, then the corresponding automorphisms $\sigma,\tau\in\mathrm{Aut}(G)$ act on supports with trivial intersection and so they commute, $\sigma\tau=\tau\sigma$.

It is easy to see that some product groups admit non-rectangular automorphisms, even up to permutation. For instance, take $G=F_2\times\mathbb{Z}=\langle a,b\rangle\times\langle c\rangle$ and $\phi\colon G\to G$, $a\mapsto ca$, $b\mapsto b$, $c\mapsto c$, is such an example. However, note that for such a construction to be a well defined automorphism, it is essential that c is a central element in G (in $F_2\times F_2=\langle a,b\rangle\times\langle c,d\rangle$, sending $a\mapsto ca$, $b\mapsto b$, $c\mapsto c$, $d\mapsto d$ does not work because c does not commute with d). The following proposition states that the presence of non-trivial central elements in G are necessary to have non-rectangular automorphisms, i.e., if G is of hyperbolic type then every automorphism of G is rectangular up to permutation.

Proposition 4.4. *Let $G=G_1^{n_1}\times\cdots\times G_m^{n_m}$ be a product group, where $m\geqslant 1$,*

$n_i \geqslant 1$, $G_i \not\simeq G_j$ for $i \neq j$, and each G_i is a free group or a surface group. If G is of hyperbolic type then, for every $\phi \in \mathrm{Aut}(G)$, there exist automorphisms $\phi_{i,j} \in \mathrm{Aut}(G_i)$ and permutations $\sigma_i \in S_{n_i}$, such that

$$\phi = \sigma_1 \circ \cdots \circ \sigma_m \circ \Big(\prod_{i=1}^{m} \prod_{j=1}^{n_i} \phi_{i,j}\Big) = \prod_{i=1}^{m} \Big(\sigma_i \circ \prod_{j=1}^{n_i} \phi_{i,j}\Big).$$

Proof. Let $\phi \in \mathrm{Aut}(G)$. The exact same argument as in the inductive step of the proof of Proposition 4.3 applied to each coordinate shows that, for every $i = 1, \ldots, n$, there exists $j = 1, \ldots, n$ such that ϕ maps bijectively elements of the subgroup $G_i \leqslant G$ to elements of the subgroup $G_j \leqslant G$ (here we are using the global notation). Furthermore, it is clear that $i \mapsto j$ defines a permutation σ of $\{1, \ldots, n\}$. In other words, there exists a permutation $\sigma \in S_n$ and automorphisms $\phi_i \in \mathrm{Aut}(G_i)$ such that $\phi = \sigma \circ (\prod_{i=1}^{n} \phi_i)$.

Finally, ϕ_i is an isomorphism from G_i to $G_{\sigma(i)}$ so, σ must preserve the isomorphism blocks, say $\sigma = \sigma_1 \circ \cdots \circ \sigma_m$ for some $\sigma_i \in S_{n_i}$, $i = 1, \ldots, m$. This concludes the proof (the equality in the statement is expressed in the block notation). □

Now we develop a technical result, and an interesting construction in presence of both Euclidean and hyperbolic type G_i's, providing explicit automorphisms whose fixed subgroups have rank bigger than the ambient rank (so, violating Bestvina–Handel inequality).

Lemma 4.5. *Let A_i, $i = 1, \ldots, n$, be a family of groups such that every $H \leqslant A_i$ satisfies $\mathrm{rk}(H) \leqslant \mathrm{rk}(A_i)$. Then, $\mathrm{rk}(H) \leqslant \mathrm{rk}(A_1) + \cdots + \mathrm{rk}(A_n)$ holds for every $H \leqslant A_1 \times \cdots \times A_n$.*

Proof. Let us do induction on n, the case $n = 1$ being trivial.

Suppose the result is true for $n-1$, and let $H \leqslant A_1 \times \cdots \times A_n$. By induction, the image of H under the canonical projection $\pi\colon A_1 \times \cdots \times A_n \twoheadrightarrow A_1 \times \cdots \times A_{n-1}$ admits a set of at most $\mathrm{rk}(\pi(H)) \leqslant \mathrm{rk}(A_1) + \cdots + \mathrm{rk}(A_{n-1})$ generators. Choosing a preimage in H of each one, and adding to this set a set of generators for $H \cap \ker \pi = H \cap A_n \leqslant A_n$, we obtain a set of generators for H. Hence, $\mathrm{rk}(H) \leqslant \mathrm{rk}(\pi(H)) + \mathrm{rk}(H \cap A_n) \leqslant \mathrm{rk}(A_1) + \cdots + \mathrm{rk}(A_{n-1}) + \mathrm{rk}(A_n)$, as we wanted to prove. □

Corollary 4.6. *Let $G = NS_2^{\ell} \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some integers $\ell, p, q \geqslant 0$. Then any subgroup $H \leqslant G$ satisfies $\mathrm{rk}(H) \leqslant \mathrm{rk}(G) = 2\ell + p + q$.*

Proof. This is a direct consequence of the previous lemma, after showing that every subgroup of NS_2 has rank at most 2. And this is true by the following argument: using the notation from Remark 3.2, it is easy to see that the abelianization short exact sequence for NS_2 is

$$1 \longrightarrow [NS_2, NS_2] = \langle b^2 \rangle \longrightarrow NS_2 \longrightarrow {NS_2}^{\mathrm{ab}} \longrightarrow 1,$$

where $NS_2^{\mathrm{ab}} = \langle \overline{a}, \overline{b} \mid [\overline{a}, \overline{b}], \overline{b}^2 \rangle \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Given $H \leqslant NS_2$, $\pi(H) \leqslant \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and so, $\mathrm{rk}(\pi(H)) \leqslant 2$. And $H \cap [NS_2 : NS_2] = \langle b^{2\eta} \rangle \leqslant \langle b^2 \rangle$ for some $\eta \geqslant 0$ and so, $\mathrm{rk}(H \cap [NS_2 : NS_2]) \leqslant 1$. Hence, $\mathrm{rk}(H) \leqslant 2+1 = 3$. But in the case $\mathrm{rk}(\pi(H)) = 2$, it must be $\pi(H) = \langle \overline{a}^\nu, \overline{b} \rangle$ for some $\nu \geqslant 1$. Hence, choosing preimages in H, say $a^\nu b^{2\alpha}$ and $b^{1+2\beta}$, of $\overline{a}^\nu$ and $\overline{b}$, respectively, we have that $H = \langle a^\nu b^{2\alpha}, b^{1+2\beta}, b^{2\eta} \rangle = \langle a^\nu b^{2\alpha}, b^\tau \rangle$, where $\tau = \gcd(1+2\beta, 2\eta)$. Therefore, in any case $\mathrm{rk}(H) \leqslant 2$, as we wanted to see. □

Proposition 4.7. *Let $G = G_1 \times \cdots \times G_n$, $n \geqslant 1$, be a product group with G_1 containing a non-trivial central element $1 \neq t \in Z(G_1)$, and with $Z(G_2) = 1$. Then, there exists an automorphism $\phi \in \mathrm{Aut}(G)$ such that $\mathrm{rk}(\mathrm{Fix}\,\phi) > \mathrm{rk}(G)$.*

Proof. By Lemma 4.2 and the facts $Z(F_1) = F_1$, $Z(S_1) = S_1$, $Z(NS_1) = NS_1$ and $Z(NS_2) \simeq \mathbb{Z}$, we deduce that the order of $t \neq 1$ is either two or infinite, $o(t) = 2, \infty$. On the other hand, G_2 is either F_r with $r \geqslant 2$, or S_g with $g \geqslant 2$, or NS_k with $k \geqslant 3$.

Suppose $G_2 = F_r = \langle a_1, \ldots, a_r \mid \, \rangle$ with $r \geqslant 2$. Map a_1 to ta_1, and fix all the other standard generators of G. This determines a well defined automorphism $\phi \in \mathrm{Aut}(G)$ sending $w(a_1, \ldots, a_r)$ to $w(ta_1, a_2, \ldots, a_r) = t^{|w|_1} w(a_1, \ldots, a_r)$, where $|w|_1 \in \mathbb{Z}$ is the total a_1-exponent of $w \in G_2$. Hence,

$$\mathrm{Fix}\,\phi = G_1 \times \{w \in G_2 \mid |w|_1 \equiv 0\} \times G_3 \times \cdots \times G_n,$$

where $\equiv$ means equality of integers modulo $o(t)$ (by convention, read $\mathbb{Z}/\infty\mathbb{Z}$ as just $\mathbb{Z}$). Now consider the projection $\pi\colon G_2 \twoheadrightarrow \mathbb{Z}/o(t)\mathbb{Z}$, $w \mapsto |w|_1$. Its kernel, $\ker \pi$, is a normal subgroup of $G_2 = F_r$ of either infinite index (and so, infinitely generated) or of index 2 (and so, $\mathrm{rk}(\ker \pi) = 1 + 2(r-1) = 2r-1$, according to the Schreier index formula for free groups). In both cases, $\mathrm{rk}(\ker \pi) > r = \mathrm{rk}(G_2)$ and, by Lemma 4.1, $\mathrm{rk}(\mathrm{Fix}\,\phi) = \mathrm{rk}(G_1) + \mathrm{rk}(\ker \pi) + \mathrm{rk}(G_3) + \cdots + \mathrm{rk}(G_n) > \mathrm{rk}(G)$. (In the infinite order case, this example was also considered for similar reasons in [3,20].)

Now suppose $G_2 = S_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$ with $g \geqslant 2$. Consider the automorphism $\phi \in \mathrm{Aut}(G)$ defined in the same way, namely mapping a_1 to ta_1 and fixing all the other standard generators of G (this determines a well defined automorphism of G because t commutes with b_1). As above, $w(a_1, b_1, \ldots, a_g, b_g)$ maps to $w(ta_1, b_1, \ldots, a_g, b_g) = t^{|w|_1} w(a_1, b_1, \ldots, a_g, b_g)$, where $|w|_1 \in \mathbb{Z}$ is the total a_1-exponent of $w \in G_2$ (which still makes sense because the defining relation in G_2 has total a_1-exponent equal to zero). Hence, as above,

$$\mathrm{Fix}\,\phi = G_1 \times \{w \in G_2 \mid |w|_1 \equiv 0\} \times G_3 \times \cdots \times G_n,$$

where $\equiv$ means equality of integers modulo $o(t)$. The argument proceeds and concludes like above, after proving that the rank of the kernel of $\pi\colon G_2 \twoheadrightarrow \mathbb{Z}/o(t)\mathbb{Z}$, $w \mapsto |w|_1$ is, again, strictly bigger than $\mathrm{rk}(G_2) = 2g$ (note that $\ker \pi$ is either a free group or a surface group again so, Lemma 4.1 still applies). If $o(t) = 2$, this is true because $\ker \pi$ is a subgroup of index two in G_2, and so a surface group of bigger rank. And if $o(t) = \infty$ then $\ker \pi$ is infinitely generated by the following argument: $\ker \pi$ is a subgroup of infinite index in G_2 (and so free), but maximal as a free subgroup: in fact, for every $x \in G_2 \setminus \ker \pi$, we have $[G_2 : \langle \ker \pi, x \rangle] = [\mathbb{Z} : \langle \pi(x) \rangle] = |\pi(x)| < \infty$ and so, $\langle \ker \pi, x \rangle$ is a surface group of Euler characteristic equal to $[G_2 : \langle \ker \pi, x \rangle]\chi(G_2) = |\pi(x)|(2-2g)$ and thus, of rank $2 + |\pi(x)|(2g-2)$. Choosing x appropriately, this rank is arbitrarily big and therefore $\ker \pi$ cannot be finitely generated.

Finally, suppose $G_2 = \langle a_1, a_2, \ldots, a_k \mid a_1^2 \cdots a_k^2 \rangle$ with $k \geqslant 3$. Map a_1 to ta_1, a_2 to $t^{-1}a_2$ and fix all the other standard generators of G (this determines a well defined $\phi \in \mathrm{Aut}(G)$ because t commutes with both a_1 and a_2). Observe that now, because of the form of the defining relation of G_2, the "total a_i-exponent" of an element of $w \in G$ is not well defined. However, the difference of two of them, say $|w|_1 - |w|_2 \in \mathbb{Z}$, it really is; in other words, $G_2 \twoheadrightarrow \mathbb{Z}$, $w \mapsto |w|_1 - |w|_2$, is a well defined morphism from G_2 onto $\mathbb{Z}$. Composing it with the canonical projection, we get an epimorphism $\pi\colon G_2 \twoheadrightarrow \mathbb{Z}/o(t)\mathbb{Z}$ and, since ϕ maps $w(a_1, \ldots, a_k)$ to $w(ta_1, t^{-1}a_2, a_3, \ldots, a_k) = t^{|w|_1 - |w|_2} w(a_1, \ldots, a_k)$, we deduce like in the above cases that

$$\mathrm{Fix}\, \phi = G_1 \times \ker \pi \times G_3 \times \cdots \times G_n.$$

The same argument as above shows that $\mathrm{rk}(\ker \pi) > \mathrm{rk}(G_2)$ and so, $\mathrm{rk}(\mathrm{Fix}\, \phi) > \mathrm{rk}(G)$, completing the proof. □

The construction in Proposition 4.7 is, essentially, the only way to produce automorphisms whose fixed subgroups have rank bigger than the ambient group. This is the contents of the following result, characterizing exactly which product groups satisfy Bestvina–Handel inequality.

Theorem 4.8. *Let $G = G_1 \times \cdots \times G_n$, $n \geqslant 1$, be a product group, where each G_i is a finitely generated free group or a surface group. Then, $\mathrm{rk}(\mathrm{Fix}\, \phi) \leqslant \mathrm{rk}(G)$ for every $\phi \in \mathrm{Aut}(G)$ if and only if G is either of Euclidean or of hyperbolic type.*

Proof. Proposition 4.7 immediately gives us the "only if" part.

For the "if" part, let us distinguish the two situations. If G is of Euclidean type then, by Lemma 4.2, $G = NS_2^{\ell} \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ and, by Corollary 4.6, we are done.

Now assume G of hyperbolic type, let $\phi \in \mathrm{Aut}(G)$, and let us prove that $\mathrm{rk}(\mathrm{Fix}\, \phi) \leqslant \mathrm{rk}(G)$. By Proposition 4.4 (and adopting the block notation for G), there exist automorphisms $\phi_{i,j} \in \mathrm{Aut}(G_i)$ and permutations $\sigma_i \in S_{n_i}$, such that

$$\phi = \prod_{i=1}^{m} \big(\sigma_i \circ \prod_{j=1}^{n_i} \phi_{i,j}\big).$$

Since $\sigma_i \circ (\phi_{i,1} \times \cdots \times \phi_{i,n_i}) \in \mathrm{Aut}(G_i^{n_i})$ for $i = 1, \ldots, m$, and it is clear that

$$\mathrm{Fix}\, \phi = \mathrm{Fix}\big(\sigma_1 \circ (\phi_{1,1} \times \cdots \times \phi_{1,n_1})\big) \times \cdots \times \mathrm{Fix}\big(\sigma_m \circ (\phi_{m,1} \times \cdots \times \phi_{m,n_m})\big),$$

we are reduced to prove the statement for the case $m = 1$.

So, let us reduce ourselves to the situation $G = G_1^n = G_{1,1} \times \cdots \times G_{1,n}$ (with $G_{1,i} = G_1$) and $\phi = \sigma \circ \big(\phi_1 \times \cdots \times \phi_n\big)$, for some $\sigma \in S_n$ and some $\phi_j \in \mathrm{Aut}(G_{1,j})$, $j = 1, \ldots, n$. If $\sigma = Id$ then

$$\mathrm{Fix}\,\phi = \mathrm{Fix}\,(\phi_1 \times \cdots \times \phi_n) = \mathrm{Fix}\,\phi_1 \times \cdots \times \mathrm{Fix}\,\phi_n$$

and so, by Theorems 1.1 and 1.8(i), and using Lemma 4.1, we have

$$\mathrm{rk}(\mathrm{Fix}\,\phi) \leqslant \mathrm{rk}(\mathrm{Fix}\,\phi_1) + \cdots + \mathrm{rk}(\mathrm{Fix}\,\phi_n) \leqslant n\,\mathrm{rk}(G_1) = \mathrm{rk}(G_1^n) = \mathrm{rk}(G)$$

and we are done.

Assume $\sigma \neq Id$ and consider its decomposition as a product of cycles with disjoint supports, $\sigma = \tau_1 \circ \cdots \circ \tau_\ell$ with $\mathrm{supp}(\tau_1) \sqcup \cdots \sqcup \mathrm{supp}(\tau_\ell) = \{1, \ldots, n\}$. Then,

$$\phi = \tau_1 \circ \cdots \circ \tau_\ell \circ \big(\big(\prod_{i \in \mathrm{supp}(\tau_1)} \phi_i\big) \times \cdots \times \big(\prod_{i \in \mathrm{supp}(\tau_\ell)} \phi_i\big)\big) =$$

$$= \big(\tau_1 \circ \prod_{i \in \mathrm{supp}(\tau_1)} \phi_i\big) \times \cdots \times \big(\tau_\ell \circ \prod_{i \in \mathrm{supp}(\tau_\ell)} \phi_i\big),$$

where $\big(\tau_j \circ \prod_{i \in \mathrm{supp}(\tau_j)} \phi_i\big) \in \mathrm{Aut}(\prod_{i \in \mathrm{supp}(\tau_j)} G_{1,i})$. By counting $\mathrm{rk}(\mathrm{Fix}\,\phi)$ as above, we are reduced to the case $\ell = 1$, i.e., we can assume σ to be a cycle of length n.

Changing the notation if necessary, we can assume $\sigma = (n, n-1, \ldots, 1)$. In this situation, our automorphism ϕ has the form

$$\begin{aligned} \phi\colon G_{1,1} \times \cdots \times G_{1,n} &\to G_{1,1} \times \cdots \times G_{1,n} \\ (g_1, \ldots, g_n) &\mapsto \sigma(\phi_1(g_1), \phi_2(g_2), \ldots, \phi_n(g_n)) = \\ &= (\phi_n(g_n), \phi_1(g_1), \ldots, \phi_{n-1}(g_{n-1})). \end{aligned}$$

Note that if $(g_1, \ldots, g_n) \in \mathrm{Fix}\,\phi$ then $g_1 = \phi_n(g_n)$, $g_2 = \phi_1(g_1)$, $\ldots$, $g_n = \phi_{n-1}(g_{n-1})$ and so, $g_1 = \phi_n \cdots \phi_1(g_1)$. Then, it is straightforward to see that

$$\mathrm{Fix}\,\phi = \{\big(g, \phi_1(g), \phi_2\phi_1(g), \ldots, \phi_{n-1} \cdots \phi_1(g)\big) \mid g \in \mathrm{Fix}\,(\phi_n \cdots \phi_1)\}.$$

Finally, by Theorems 1.1 and 1.8(i), we deduce

$$\mathrm{rk}(\mathrm{Fix}\,\phi) = \mathrm{rk}(\mathrm{Fix}\,(\phi_n \cdots \phi_1)) \leqslant \mathrm{rk}(G_1) \leqslant \mathrm{rk}(G_1^n) = \mathrm{rk}(G),$$

concluding the proof. □

The next natural question is to ask which product groups G enjoy the property that every automorphism has compressed (or inert) fixed point subgroup. Of course, they form a subset of those product groups satisfying Bestvina–Handel inequality, described in Theorem 4.8; compression is more restrictive, and inertia even more, as can be seen in the following two results. We cannot give a full characterization but give necessary conditions, and state two conjectures characterizing both properties.

Theorem 4.9. *Let $G = G_1 \times \cdots \times G_n$, $n \geqslant 1$, be a product group, where each G_i is a finitely generated free group or a surface group. If $\mathrm{Fix}\,\phi$ is compressed in G for every $\phi \in \mathrm{Aut}(G)$, then G must be of one of the following forms:*

*(*euc1*)* $G = \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ *for some* $p, q \geqslant 0$*; or*
*(*euc2*)* $G = NS_2 \times (\mathbb{Z}/2\mathbb{Z})^q$ *for some* $q \geqslant 0$*; or*
*(*euc3*)* $G = NS_2 \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})$ *for some* $p \geqslant 1$*; or*
*(*euc4*)* $G = NS_2^\ell \times \mathbb{Z}^p$ *for some* $\ell \geqslant 1$, $p \geqslant 0$*; or*

*(*hyp1*)* $G = F_r \times NS_3^\ell$ *for some* $r \geqslant 2$, $\ell \geqslant 0$*; or*
*(*hyp2*)* $G = S_g \times NS_3^\ell$ *for some* $g \geqslant 2$, $\ell \geqslant 0$*; or*
*(*hyp3*)* $G = NS_k \times NS_3^\ell$ *for some* $k \geqslant 3$, $\ell \geqslant 0$.

Proof. Assume that $\operatorname{Fix}\phi$ is compressed in G for every $\phi \in \operatorname{Aut}(G)$. In particular every $\phi \in \operatorname{Aut}(G)$ satisfies $\operatorname{rk}(\operatorname{Fix}\phi) \leqslant \operatorname{rk}(G)$ and, by Theorem 4.8, G is either of Euclidean type or of hyperbolic type. In the first case, we shall prove that G is specifically of the form (euc1), or (euc2), or (euc3), or (euc4). And in the second case G will be of the form (hyp1), or (hyp2), or (hyp3).

Let us assume G is Euclidean, i.e., $G = NS_2^\ell \times \mathbb{Z}^p \times (\mathbb{Z}/2\mathbb{Z})^q$ for some integers $\ell, p, q \geqslant 0$. The next two paragraphs will prove that "if $\ell \geqslant 2$ then $q = 0$", and "if $\ell, p \geqslant 1$ then $q = 0, 1$". By distinguishing whether $\ell = 0$, or $\ell = 1$, or $\ell \geqslant 2$, these two restrictions in the parameters force G to fall into one of the forms (euc1), or (euc2), or (euc3), or (euc4).

To see that $\ell \geqslant 2$ implies $q = 0$, assume $\ell \geqslant 2$ and $q \geqslant 1$ and let us construct an automorphism of G whose fixed subgroup is not compressed. In this case we have $G = NS_2^2 \times (\mathbb{Z}/2\mathbb{Z}) \times G_4 \times \cdots \times G_n = \langle a, b \mid aba^{-1}b\rangle \times \langle c, d \mid cdc^{-1}d\rangle \times \langle e \mid e^2\rangle \times G_4 \times \cdots \times G_n$. Consider the map $\phi\colon G \to G$, $a \mapsto a$, $b \mapsto be$, $c \mapsto cd$, $d \mapsto d$, $e \mapsto e$, and fixing the rest of generators. It is straightforward to see that ϕ is a well defined automorphism and, using normal forms of elements, its fixed subgroup is $\operatorname{Fix}\phi = \langle a, b^2, c^2, d, e\rangle \times G_4 \times \cdots \times G_n = \langle a, b^2\rangle \times \langle c^2, d\rangle \times \langle e\rangle \times G_4 \times \cdots \times G_n \simeq NS_2 \times \mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z} \times G_4 \times \cdots \times G_n$. By Lemma 4.1, $\operatorname{rk}(\operatorname{Fix}\phi) = 5 + \operatorname{rk}(G_4) + \cdots + \operatorname{rk}(G_n)$. But $\operatorname{Fix}\phi$ is contained in $\langle a, bc, d, e\rangle \times G_4 \times \cdots \times G_n$ (note that conjugating bc by a one gets $b^{-1}c$), which has rank less than or equal to $4 + \operatorname{rk}(G_4) + \cdots + \operatorname{rk}(G_n) < r(\operatorname{Fix}\phi)$; therefore, $\operatorname{Fix}\phi$ is not compressed in G.

And, in order to see that $\ell, p \geqslant 1$ implies $q = 0, 1$, assume $\ell, p \geqslant 1$ and $q \geqslant 2$ and let us construct an automorphism of G whose fixed subgroup is not compressed. So, in the situation $G = NS_2 \times \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2 \times G_5 \times \cdots \times G_n = \langle a, b \mid aba^{-1}b\rangle \times \langle c \mid \ \rangle \times \langle d \mid d^2\rangle \times \langle e \mid e^2\rangle \times G_5 \times \cdots \times G_n$, consider the automorphism $\phi\colon G \to G$ given by $a \mapsto a$, $b \mapsto bd$, $c \mapsto ce$, $d \mapsto d$, $e \mapsto e$, and fixing all elements from $G_5, \ldots, G_n$ (it is straightforward to check that this is well defined, as well as its obvious invers). Now, it is not difficult to see that $\operatorname{Fix}\phi = \langle a, b^2, c^2, d, e\rangle \times G_5 \times \cdots \times G_n = \langle a, b^2\rangle \times \langle c^2\rangle \times \langle d\rangle \times \langle e\rangle \times G_5 \times \cdots \times G_n \simeq NS_2 \times \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2 \times G_5 \times \cdots \times G_n$ which, by Lemma 4.1, has rank $\operatorname{rk}(\operatorname{Fix}\phi) = 5 + \operatorname{rk}(G_5) + \cdots + \operatorname{rk}(G_n)$; but, as in the example above, $\operatorname{Fix}\phi$ is contained in $\langle a, bc, d, e\rangle \times G_5 \times \cdots \times G_n$, which has rank less than or equal to $4 + \operatorname{rk}(G_5) + \cdots + \operatorname{rk}(G_n) < r(\operatorname{Fix}\phi)$ therefore, it is not compressed in G.

Now, let us assume G is hyperbolic, i.e., the direct product of, possibly, several free groups F_r with $r \geqslant 2$, several orientable surface groups S_g with $g \geqslant 2$, and several non-orientable surface groups NS_k with $k \geqslant 3$; it just remains to see that, in this case, at most one of the direct summands is not isomorphic to NS_3 (so, forcing G to fall into the forms (hyp1), or (hyp2), or (hyp3)). We shall prove this by assuming two direct summands in G of the form F_r with $r \geqslant 2$, or S_g with $g \geqslant 2$, or NS_k with $k \geqslant 4$, and constructing an automorphism whose fixed subgroup is not compressed in G.

The free group $F_r = \langle a_1, \ldots, a_r \mid \rangle$, $r \geqslant 2$, admits the automorphism $\phi\colon F_r \to F_r$, $a_1 \mapsto a_1a_2$, $a_2 \mapsto a_2$, $a_i \mapsto a_i$ for $i = 3, \ldots, r$, whose fixed subgroup is $\operatorname{Fix}\phi = \langle a_2, a_1a_2a_1^{-1}, a_3, \ldots, a_r \rangle$. Note that $\operatorname{Fix}\phi \simeq F_r$ and so, $\operatorname{rk}(\operatorname{Fix}\phi) = r = \operatorname{rk}(F_r)$.

The orientable surface group $S_g = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$, with $g \geqslant 2$, admits the automorphism $\phi\colon S_g \to S_g$, $a_1 \mapsto a_1b_1$, $b_1 \mapsto b_1$, $a_2 \mapsto a_2b_2$, $b_2 \mapsto b_2$, $a_i \mapsto a_i$, $b_i \mapsto b_i$ for $i = 3, \ldots, g$, whose fixed subgroup is $\operatorname{Fix}\phi = \langle b_1, a_1b_1a_1^{-1}, b_2, a_2b_2a_2^{-1}, a_3, b_3, \ldots, a_g, b_g \rangle$. Note here that, because of the defining relation for S_g, $\operatorname{Fix}\phi = \langle b_1, a_1b_1a_1^{-1}, b_2, a_3, b_3, \ldots, a_g, b_g \rangle \simeq F_{2g-1}$ (in fact, it is a subgroup of $\langle b_1, a_1, b_2, a_3, b_3, \ldots, a_g, b_g \rangle$, which is free of rank $2g-1$ by Magnus Freiheitssatz, see [11, Theorem 5.1]). Observe that if $g = 1$ then this fixed subgroup is cyclic (because a_1 and b_1 commute in this case), and this is not good for the coming argument.

Finally, the non-orientable surface group $NS_k = \langle a_1, a_2, \ldots, a_k \mid a_1^2a_2^2 \cdots a_k^2 \rangle$ with $k \geqslant 4$, can also be presented as $\langle a, b, c, d, a_5, \ldots, a_k \mid aba^{-1}bcdc^{-1}da_5^2 \cdots a_k^2 \rangle$ (the isomorphism being essentially the one given in Remark 3.2 namely, $a_1 \mapsto a$, $a_2 \mapsto a^{-1}b$, $a_3 \mapsto c$, $a_4 \mapsto c^{-1}d$, $a_i \mapsto a_i$ for $i = 5, \ldots, k$). With this new presentation, it is straightforward to see that NS_k, for $k \geqslant 4$, admits the automorphism $\phi\colon NS_k \to NS_k$, $a \mapsto ab$, $b \mapsto b$, $c \mapsto cd$, $d \mapsto d$, $a_i \mapsto a_i$ for $i = 5, \ldots, k$, whose fixed subgroup is $\operatorname{Fix}\phi = \langle b, aba^{-1}, d, cdc^{-1}, a_5, \ldots, a_k \rangle$. Again note that, because of the defining relation for NS_k, $\operatorname{Fix}\phi = \langle b, aba^{-1}, d, a_5, \ldots, a_k \rangle \simeq F_{k-1}$ (as above, it is a subgroup of $\langle b, a, d, a_5, \ldots, a_k \rangle$, which is free of rank $k-1$). Observe also that $k \geqslant 4$ is crucial at this point because we need two pairs (a, b) and (c, d) to do the trick (with $k = 2$, ϕ makes perfect sense but $aba^{-1} = b^{-1}$ and $\operatorname{Fix}\phi$ is cyclic, which is not good for the coming argument; and with $k = 3$ one could consider the automorphism $a \mapsto ab$, $b \mapsto b$, $c \mapsto c$, whose fixed subgroup is $\langle b, aba^{-1}, c \rangle = \langle b, c \rangle$, not good either for the argument in the next paragraph).

Let us retake now the main argument: suppose G has at least two direct summands of the above form, i.e., $n \geqslant 2$ and both G_1 and G_2 are of the form F_r with $r \geqslant 2$, or S_g with $g \geqslant 2$, or NS_k with $k \geqslant 4$. Denote by $x_1, x_2, \ldots, x_{r_1}$ and $y_1, y_2, \ldots, y_{r_2}$ the standard generators for G_1 and G_2, respectively, where $r_1 = \operatorname{rk}(G_1)$ and $r_2 = \operatorname{rk}(G_2)$. By the previous three paragraphs, we have automorphisms $\phi_1 \in \operatorname{Aut}(G_1)$ and $\phi_2 \in \operatorname{Aut}(G_2)$ such that $\operatorname{Fix}\phi_1 = \langle x_2, x_1x_2x_1^{-1}, /x_3/, x_4, \ldots, x_{r_1} \rangle$ and $\operatorname{Fix}\phi_2 = \langle y_2, y_1y_2y_1^{-1}, /y_3/, y_4, \ldots, y_{r_2} \rangle$ are free on the listed generators, where the notation $/x_3/$ means that we omit this third generator except in the free ambient case $G_1 = F_r$. Hence, for $i = 1, 2$, $\operatorname{rk}(\operatorname{Fix}\phi_i) = s_i$, where $s_i = r_i$ if G_i is free, and $s_i = r_i - 1$ if G_i is a surface group.

Finally, consider the automorphism $\phi = \phi_1 \times \phi_2 \times Id \times \cdots \times Id \in \mathrm{Aut}(G)$. It happens that

$$\begin{aligned}\mathrm{Fix}\,\phi &= \langle x_2, x_1x_2x_1^{-1}, /x_3/, \ldots, x_{r_1}\rangle \times \langle y_2, y_1y_2y_1^{-1}, /y_3/, \ldots, y_{r_2}\rangle \times G_3 \times \cdots \times G_n \\ &\simeq F_{s_1} \times F_{s_2} \times G_3 \times \cdots \times G_n.\end{aligned}$$

On one hand, by Lemma 4.1, we have $\mathrm{rk}(\mathrm{Fix}\,\phi) = s_1 + s_2 + \mathrm{rk}(G_3) + \cdots + \mathrm{rk}(G_n)$. On the other hand, since the x_i's commute with the y_i's, $\mathrm{Fix}\,\phi$ is contained in the subgroup

$$H = \langle x_2, y_2, x_1y_1, /x_3/, \ldots, x_{r_1}, /y_3/, \ldots, y_{r_2}\rangle \times G_3 \times \cdots \times G_n \leqslant G,$$

with $\mathrm{rk}(H) \leqslant \mathrm{rk}(\mathrm{Fix}\,\phi) - 1$. Therefore, $\mathrm{Fix}\,\phi$ is not compressed in G, concluding the proof. □

Remark 4.10. We suspect that the implication in Theorem 4.9 is an equivalence. To see this, two implications are missing.

In the hyperbolic case, one should be able to see that the fixed subgroup of any automorphism of a group G of the form (hyp1), or (hyp2), or (hyp3) is compressed in G. Any such automorphism is rectangular up to permutation; and we already know that the fixed subgroup of an automorphism of any component G_i is compressed in G_i. Now $A_i \leqslant G_i$ being compressed in G_i for $i = 1, \ldots, n$, implies that $A = A_1 \times \cdots \times A_n \leqslant G_1 \times \cdots \times G_n = G$ satisfies $r(A) \leqslant r(H)$ for every H of the form $H = H_1 \times \cdots \times H_n \leqslant G$ with $A_i \leqslant H_i$. It remains to study what happens with the non-rectangular subgroups H such that $A \leqslant H \leqslant G$. (The trick used in the proof of Theorem 4.9 to destroy compressedness playing with such non-rectangular subgroups does not work for NS_3.)

In the Euclidean case, we conjecture a little more: the fixed subgroup of any endomorphism of a group G of the form (euc1), or (euc2), or (euc3), or (euc4) is inert in G. This is obviously true for groups of the form (euc1); with a variation of the argument in Corollary 4.6 it is straightforward to see it for groups of the forms (euc2) and (euc3); and it remains to study the case where G is of the form (euc4). A complete proof for this case would require a detailed analysis of the automorphisms and endomorphisms of G (note that every direct summand contributes non-trivially to the center of G and so G admits automorphisms far from being rectangular, even up to permutation). Moreover, the form of endomorphisms must play an important role because these groups, even though looking very close to abelian, they contain subgroups which are not compressed: in fact, following the same idea as in the proof of Theorem 4.9, inside the group $G = NS_2 \times \mathbb{Z} = \langle a, b \mid aba^{-1}b\rangle \times \langle c \mid \rangle$ we have the subgroup $K = \langle a, bc\rangle$, which contains $H = \langle a, b^2, c^2\rangle$. But $\mathrm{rk}(K) = 2$ while $H \simeq NS_2 \times \mathbb{Z}$ and so $\mathrm{rk}(H) = 3$, therefore H is not compressed in G. However, the trick used above to realize H as the fixed subgroup of an automorphism of G made essential use of two elements of order 2, which are not available now (and, in fact, H is not the fixed subgroup of any endomorphism of G, since it is easy to see that in G the only solution to the equation $x^2 = b^2$ is $x = b$ so, any endomorphism fixing b^2 must also fix b).

Conjecture 4.11. *Let $G = G_1 \times \cdots \times G_n$, $n \geqslant 1$, be a product group, where each G_i is a finitely generated free or surface group. Then, $\operatorname{Fix}\phi$ is compressed in G for every $\phi \in \operatorname{Aut}(G)$ if and only if G is of one of the forms (euc1), or (euc2), or (euc3), or (euc4), or (hyp1), (hyp2), or (hyp3).*

As the following result expreses, inertia for the fixed subgroup of every automorphism is an even stronger condition, at least for the hyperbolic case.

Theorem 4.12. *Let $G = G_1 \times \cdots \times G_n$, $n \geqslant 1$, be a product group, where each G_i is a finitely generated free or surface group. If $\operatorname{Fix}\phi$ is inert in G for every automorphism $\phi \in \operatorname{Aut}(G)$, then G is of one of the forms (euc1), or (euc2), or (euc3), or (euc4), or*

*(*hyp1'*) $G = F_r$ for some $r \geqslant 2$; or*
*(*hyp2'*) $G = S_g$ for some $g \geqslant 2$; or*
*(*hyp3'*) $G = NS_k$ for some $k \geqslant 3$.*

Proof. Assume that $\operatorname{Fix}\phi$ is inert in G for every $\phi \in \operatorname{Aut}(G)$. In particular every $\phi \in \operatorname{Aut}(G)$ has $\operatorname{Fix}\phi$ being compressed in G and, by Theorem 4.9, G is of one of the forms (euc1), (euc2), (euc3), (euc4), or (hyp1), (hyp2), or (hyp3). In the Euclidean case, we are done; in the hyperbolic case it just remains to see that $n = 1$.

We shall prove this by assuming G of the form (hyp1), (hyp2), or (hyp3) with $n \geqslant 2$, and constructing an automorphisms of G whose fixed subgroup is not inert in G. In these three situations, we have $G = G_1 \times G_2 \times \cdots \times G_n$ with $n \geqslant 2$, $G_1 = F_r, S_g$, or NS_k, for some $r \geqslant 2$, $g \geqslant 2$, or $k \geqslant 3$, respectively, and $G_2 = NS_3 = \langle c, d, e \mid cdc^{-1}de^2 \rangle$. Consider the automorphism $\phi_2 \colon NS_3 \to NS_3$, $c \mapsto cd$, $d \mapsto d$, $e \mapsto e$, and note that $\operatorname{Fix}\phi_2 \cap \langle c \rangle = 1$ (one can see this, for example, abelianizing). Now consider $\phi = Id \times \phi_2 \times Id \times \cdots \times Id \in \operatorname{Aut}(G)$, and we claim that $\operatorname{Fix}\phi = G_1 \times \operatorname{Fix}\phi_2 \times G_3 \times \cdots \times G_n$ is not inert in G.

Suppose $G_1 = F_r = \langle a_1, \ldots, a_r \mid \, \rangle$ with $r \geqslant 2$. Take $K = \langle ca_1, a_2, \ldots, a_r \rangle \leqslant G$, and consider the projection $\pi \colon F_r \twoheadrightarrow \mathbb{Z}$, $w \mapsto |w|_1$. It is easy to see that $\operatorname{Fix}\phi \cap K = \operatorname{Fix}(Id \times \phi_2) \cap K = (F_r \times \operatorname{Fix}\phi_2) \cap K$ equals $\ker\pi$, which is a normal subgroup of infinite index in F_r, so infinitely generated. In fact, every element in $\ker\pi$ is a word $w(a_1, a_2, \ldots, a_r)$ with $|w|_1 = 0$ and so, $w(a_1, a_2, \ldots, a_r) = c^{|w|_1} w(a_1, a_2, \ldots, a_r) = w(ca_1, a_2, \ldots, a_r) \in K$ since c commutes with all the a_i's; and conversely, if $w(a_1, a_2, \ldots, a_r)v = w'(ca_1, a_2, \ldots, a_r)$ for some $v \in \operatorname{Fix}\phi_2$, then $w(a_1, a_2, \ldots, a_r)v = w'(a_1, a_2, \ldots, a_r)c^{|w'|_1}$ which implies $w = w'$, $v = 1$, and $|w'|_1 = 0$ therefore, $w(a_1, a_2, \ldots, a_r)v \in \ker\pi$.

Suppose now that $G_1 = S_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$ with $g \geqslant 2$. Take $K = \langle ca_1, b_1, a_2, b_2, \ldots, a_g, b_g \rangle \leqslant G$, and consider the projection $\pi \colon S_g \twoheadrightarrow \mathbb{Z}$, $w \mapsto |w|_1$. The same argument as above shows that $\operatorname{Fix}\phi \cap K = \ker\pi$, which in this case is infinitely generated as well, by the argument given in the proof of Proposition 4.7 (case $G_2 = S_g$).

Finally, suppose $G_1 = NS_k = \langle a, b, a_3, \ldots, a_k \mid aba^{-1}ba_3^2 \cdots a_k^2 \rangle$ with $k \geqslant 3$, and consider the projection $\pi \colon NS_k \twoheadrightarrow \mathbb{Z}$, $w \mapsto |w|_1$ (which coincides with that in the proof of Proposition 4.7 case $G_2 = NS_k$, where it is expressed with respect to the other usual presentation of NS_k). The exact same argument as in the previous case shows that $\operatorname{Fix}\phi \cap K = \ker \pi$, which is again infinitely generated. □

A positive solution to Conjectures 1.4 and 1.10, and to that suggested in Remark 4.10, would give a positive solution to the following one.

Conjecture 4.13. *Let $G = G_1 \times \cdots \times G_n$, $n \geqslant 1$, be a product group, where each G_i is a finitely generated free or surface group. Then, the following are equivalent:*

(a) every $\phi \in \operatorname{End}(G)$ satisfies that $\operatorname{Fix}\phi$ is inert in G,
(b) every $\phi \in \operatorname{Aut}(G)$ satisfies that $\operatorname{Fix}\phi$ is inert in G,
(c) G is of the form (euc1), or (euc2), or (euc3), or (euc4), or (hyp1'), or (hyp2'), or (hyp3').

Corollary 4.14. *For product groups, the "compressed-inert" conjecture is false even for fixed subgroups of automorphisms, i.e., there exists a product group G and $\phi \in \operatorname{Aut}(G)$ such that $\operatorname{Fix}\phi$ is compressed in G but not inert in G.*

Proof. Conjectures 4.11 and 4.13 already suggest that such G and ϕ do exist. The easiest example is $G = F_2 \times \mathbb{Z} = \langle a, b \mid \rangle \times \langle c \mid \rangle$ and $\phi\colon G \to G$, $a \mapsto a$, $b \mapsto b$, $c \mapsto c^{-1}$. Clearly, $\operatorname{Fix}\phi = \langle a, b\rangle$ is compressed. But $\operatorname{Fix}\phi \cap \langle ca, b\rangle$ is the normal closure of b in F_2, which is infinitely generated. Hence, $\operatorname{Fix}\phi$ is not inert in G. □

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