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MULTINOMIAL PROBABILISTIC VALUES*

December 2, 2014

Abstract

Multinomial probabilistic values were introduced by one of us in reliability. Here we define them for all cooperative games and illustrate their behavior in practice by means of an application to the analysis of a political problem.

Keywords: (TU) cooperative game, Shapley value, probabilistic value, binomial semivalue.
Math. Subj. Class. (2000): 91A12.

1 Introduction

Weber's general model for assessing cooperative games [14] is based on *probabilistic values*.[1] Every probabilistic value is defined by a set of *weighting coefficients* and allocates, to each player in each game of its domain, a convex sum of the marginal contributions of the player in the game. These allocations can be interpreted as a measure of players' bargaining relative strength. The most conspicuous member of this family (in fact, the inspiring one) is the *Shapley value* [13]. In the present paper we study a subfamily of probabilistic values that we call *multinomial (probabilistic) values*.[2] Technically, their main characteristic is the *systematic generation of the weighting coefficients in terms of a few parameters (one parameter per player)*.

For more than a decade, our research group has been studying *semivalues*, a subfamily of probabilistic values introduced by Dubey et al. [9], characterized by anonymity and including the Shapley value as the only *efficient* member. In the analysis of certain cooperative problems we have successfully used *binomial semivalues*, a monoparametric subfamily defined by Puente [12] that includes the Banzhaf

*Research supported by Grant SGR 2014–435 of the Catalonia Government (*Generalitat de Catalunya*) and Grant MTM 2012–34426 of the Economy and Competitiveness Spanish Ministry.

[1]A family of values axiomatically characterized in [14] by means of linearity, positivity, and the dummy player property.

[2]These values were introduced in reliability by Puente [12] (see also [10]) with the name of "multibinary probabilistic values." They were independently defined by Carreras [3], for simple games only —i.e. as power indices—, in a work on decisiveness (see also [4]) where they were called "Banzhaf α–indices."

value introduced by Owen [11].[3] From this experience, we feel that multinomial values (n parameters, n being the number of players) offer a deal of flexibility clearly greater than binomial semivalues (one parameter), and hence many more possibilities to introduce additional information when evaluating a game.

Probabilistic values provide tools to study not only games *in abstracto* (i.e. from a merely structural viewpoint) but also the influence of *players' personality* on the issue. They are assessment techniques that do not modify the game but only the criteria by which payoffs are allocated. In the multinomial case, a series of parameters are used to cope with different attitudes the players may hold when playing a given game, even if they are not individuals but countries, enterprises, parties, trade unions, or collectivities of any other kind. We will attach to parameter p_i the meaning of *generical tendency of player* i *to form coalitions*, assuming p_i and p_j independent of each other if $i \neq j$.

Summing up, the paper tries to present multinomial values as a consistent alternative or complement to classical values (Shapley, Banzhaf). Tendency profiles can encompass a variety of situations arising from players' attitudes. Thus, multinomial values represent a wide generalization of binomial semivalues, whose monoparametric condition implies a quite limited capability of analysis for such situations. Of course, these situations cannot be analyzed, without modifying the game, by means of the classical values, which are concerned only with the structure of the game.

The organization of the paper is as follows. Section 2 includes a minimum of preliminaries. In Section 3 we present a motivating political problem and discuss some features of probabilistic values. In Section 4, we introduce multinomial values. Finally, Section 5 is devoted to analyze again the political problem, using now multinomial values, in order to emphasize their intuitive meaning, flexibility and usefulness.

2 Preliminaries

Let N be a finite set of *players*, usually denoted as $N = \{1, 2, \ldots, n\}$. A *(TU) cooperative game* in N is a function v that assigns a real number $v(S)$ to each *coalition* $S \subseteq N$, with $v(\emptyset) = 0$. This number is understood as the utility that coalition S can obtain by itself, that is, independently of the remaining players' behavior.

Game v is *monotonic* if $v(S) \leq v(T)$ when $S \subset T \subseteq N$. Players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. Endowed with the natural operations for real–valued functions, $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set of all cooperative games in N becomes a vector space $\mathcal{G}_N$ of dimension $2^n - 1$.

We also recall that a cooperative game v is a *simple* if it is monotonic, $v(S) = 0$ or 1 for all $S \subseteq N$, and $v(N) = 1$. In this case, the set of *winning coalitions* $W(v) = \{S \subseteq N : v(S) = 1\}$ determines the game. Often v is a *weighted majority game*: there exist a *quota* $q > 0$ and *weights* $w_1, w_2, \ldots, w_n \geq 0$ such that $S \in W(v)$ if and only if $\sum_{i \in S} w_i \geq q$. We denote this fact by setting $v \equiv [q; w_1, w_2, \ldots, w_n]$.

A *value* on $\mathcal{G}_N$ is a map $g : \mathcal{G}_N \to \mathbb{R}^N$, which assigns to every game v a vector $g[v]$ with components $g_i[v]$ for all $i \in N$. The *total power* of value g in v is $\sum_{i \in N} g_i[v]$.

[3] [1], [5], [6], [7] and [8] are samples of our work in this line.

Following [14], given a set of *weighting coefficients* $\{p_S^i : i \in N,\ S \subseteq N\backslash\{i\}\}$, with

$$\text{all } \ p_S^i \geq 0 \qquad \text{and} \qquad \sum_{S \subseteq N\backslash\{i\}} p_S^i = 1 \ \text{ for each } i, \tag{1}$$

the expression

$$\phi_i[v] = \sum_{S \subseteq N\backslash\{i\}} p_S^i[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N \tag{2}$$

defines a *probabilistic value* ϕ on $\mathcal{G}_N$. Notice that, fixing $i \in N$, the p_S^i provide a probability distribution on the set of coalitions $S \subseteq N\backslash\{i\}$. Thus, the payoff that a probabilistic value allocates to every player in each game is a convex sum of all marginal contributions of the player in the game. We quote from [14]:

> "Let player i view his participation in a game v as consisting merely of joining some coalition S and then receiving as a reward his marginal contribution to the coalition. If p_S^i is the probability that he joins coalition S, then $\phi_i[v]$ is his expected payoff from the game."

Among probabilistic values, *semivalues*, introduced by Dubey et al. [9], are characterized in [14] by the fact that all coalitions of a given size share a common weight with regard to all players. Formally: there is a vector $\{p_s\}_{s=0}^{n-1}$ such that $p_S^i = p_s$ for all $i \in N$ and all $S \subseteq N\backslash\{i\}$, where $s = |S|$. Thus

$$\phi_i[v] = \sum_{S \subseteq N\backslash\{i\}} p_s[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N.$$

The weighting coefficients $\{p_s\}_{s=0}^{n-1}$ of any semivalue ϕ satisfy two characteristic conditions, derived from Eq. (1): each $p_s \geq 0$ and $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$.

Among semivalues, the *Shapley value* [13], denoted here by φ and defined by $p_s = 1/\binom{n-1}{s}n$ for all s, is the only *efficient* semivalue, in the sense that its total power for every $v \in \mathcal{G}_N$ is $\sum_{i \in N} \varphi_i[v] = v(N)$. The *Banzhaf value* [11], denoted here by β and defined by $p_s = 1/2^{n-1}$ for all s, is the only semivalue satisfying the *total power property*:

$$\sum_{i \in N} \beta_i[v] = \frac{1}{2^{n-1}} \sum_{S \subseteq N} \sum_{i \notin S} [v(S \cup \{i\}) - v(S)] \quad \text{for every } \ v \in \mathcal{G}_N.$$

The Banzhaf value is also the only semivalue with constant weighting coefficients.

3 A political problem with ideological constraints

To illustrate the notion of probabilistic value we will discuss here a political problem described by a simple game. Besides their interest for modeling political problems, simple games constitute very often a test bed for many cooperative concepts.

Example 3.1 We consider a 50–member parliamentary body with $n = 4$ parties and a seat distribution of 21, 18, 7 and 4 seats, respectively. Assuming that voting is ruled by absolute majority and voting discipline holds within each party, the weighted majority game $v \equiv [26; 21, 18, 7, 4]$ describes the formal structure. The family of *minimal* winning coalitions is $W^m(v) = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$, so the family of winning coalitions is

$$W(v) = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.$$

Notice that players 2 and 3 are symmetric in v. The Shapley value yields the following evaluation of the game:[4]

$$\varphi[v] = (5/12,\ 3/12,\ 3/12,\ 1/12) \approx (0.4167, 0.2500, 0.2500, 0.0833).$$

Let us assume that the basic ideological feature is defined by a classical left–to–right axis[5] where the parties can be precisely located as for example in Fig. 1.

left	4	1	2	3	right
0	0.1	0.4	0.6	0.8	1

Fig. 1: Party–distribution on a left–to–right axis

It is clear that the Shapley value strictly represents the relative strength of each party in the game, disregarding the effect, in the coalition formation process, due to the ideological positions of the involved parties. We wish to incorporate this exogenous information to the evaluation of the game by using a suitable probabilistic value.

Any probabilistic value ϕ is defined by a set $\{p^i_S\}$ of weighting coefficients for all $i \in N$ and all $S \subseteq N\backslash\{i\}$. For each $i \in N$, the coefficients $\{p^i_S\}$ must provide a probability distribution on the family of coalitions $S \subseteq N\backslash\{i\}$. In our case ($n = 4$), 32 coefficients p^i_S are needed in principle. However, since the game is simple and hence some marginal contributions vanish, we only have to define p^i_S when i is *crucial* for $S \cup \{i\}$ in v, i.e. when $S \notin W(v)$ but $S \cup \{i\} \in W(v)$ (we will write $S \in C_v(i)$ to denote this fact). This reduces the set to 12 coefficients, namely

$$p^1_{\{2\}}, p^1_{\{3\}}, p^1_{\{2,3\}}, p^1_{\{2,4\}}, p^1_{\{3,4\}}, p^2_{\{1\}}, p^2_{\{1,4\}}, p^2_{\{3,4\}}, p^3_{\{1\}}, p^3_{\{1,4\}}, p^3_{\{2,4\}}, p^4_{\{2,3\}},$$

and the restrictions in choosing these coefficients for each $S \in C_v(i)$ are

$$\text{all } p^i_S \geq 0, \qquad \sum_{S \in C_v(i)} p^i_S \leq 1 \text{ for each } i, \quad \text{and hence all } p^i_S \leq 1.$$

[4]Incidentally, the Banzhaf value gives the same allocations in this game.

[5]A similar scheme could be applied if the relevant notion were nationalism (vs. centralism), as for example in regions like Quebec, Scotland, Padania (Po Valley), Catalonia or the Basque Country. Higher–dimensional ideological spaces might be treated in a similar but more complicated way.

Once the coefficients are chosen, we will simply have, from Eq. (2),

$$\phi_i[v] = \sum_{S \in C_v(i)} p_S^i. \tag{3}$$

Note that (a) $\phi_i[v] \leq 1$ for all i, and (b) the total power is $\sum_{i \in N} \phi_i[v] \leq n$.[6]

Given $\{p_S^i\}$, let $q^i(v)$ be the probability that i joins any coalition $S \notin C_v(i)$, i.e. such that i is not crucial in $S \cup \{i\}$. This is the amount of irrelevant probability that we may leave undefined. Then, from Eq. (3) it follows that $\phi_i[v] = 1 - q^i(v)$. Thus, *the greater is the probability* $q^i(v)$ *the less is the allocation that player* i *will get according to the corresponding probabilistic value.*

How should we take into account the ideological constraints to define a suitable probabilistic value? Well, one could try to combine, in a rather intuitive form, the available information. Therefore a seemingly reasonable possibility could be

$$p^1_{\{2\}} = 0.4, \quad p^1_{\{3\}} = 0.2, \quad p^1_{\{2,3\}} = 0.1, \quad p^1_{\{2,4\}} = 0.2, \quad p^1_{\{3,4\}} = 0.1, \quad p^2_{\{1\}} = 0.6,$$

$$p^2_{\{1,4\}} = 0.1, \quad p^2_{\{3,4\}} = 0.3, \quad p^3_{\{1\}} = 0.4, \quad p^3_{\{1,4\}} = 0.1, \quad p^3_{\{2,4\}} = 0.5, \quad p^4_{\{2,3\}} = 1.$$

However, by using Eq. (2) or, even better, Eq. (3), we have $\phi_i[v] = 1$ for all i, which does not seem a reasonable result. Indeed, the reader will agree that both the structure and the ideological positions distinguish among parties, and also that it is extremely unlikely that the combination of these two ingredients leads to a common relative strength for all parties. Of course, there are infinitely many other possibilities for defining all p_S^i, and in particular it is not necessary that they satisfy, like in our example, $q^i(v) = 0$ for all i, that is,

$$\sum_{S \in C_v(i)} p_S^i = 1 \qquad \text{for all } i.$$

The conclusion is that the weighting coefficients should be carefully chosen, in terms of the information given by Fig. 1 but also in a way as *systematic* as possible. Hence, in Example 5.1 we will apply multinomial values.

4 Multinomial values

We introduce multinomial values following [12] and [10].

Definition 4.1 Set $N = \{1, 2, \ldots, n\}$ and let a *profile* $\mathbf{p} \in [0,1]^n$, that is, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ with $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$, be given. Then the coefficients

$$p_S^i = \prod_{j \in S} p_j \prod_{\substack{k \in N \setminus S \\ k \neq i}} (1 - p_k) \quad \text{for all } i \in N \text{ and } S \subseteq N \setminus \{i\} \tag{4}$$

[6] The numerical example proposed below in this section shows that this bound cannot be improved, since the total power equals n in this example.

(where the empty product, arising if $S=\emptyset$ or $S=N\backslash\{i\}$, is taken to be 1) define (see [10]) a probabilistic value on $\mathcal{G}_N$ that we call the **p**–*multinomial value* $\lambda^{\mathbf{p}}$. Thus,

$$\lambda_i^{\mathbf{p}}[v] = \sum_{S\subseteq N\backslash\{i\}} \prod_{j\in S} p_j \prod_{\substack{k\in N\backslash S\\ k\neq i}} (1-p_k)[v(S\cup\{i\})-v(S)] \quad \text{for all } i\in N \text{ and } v\in\mathcal{G}_N.$$

As was announced in Section 1, we will attach to p_i the meaning of *generical tendency of player i to form coalitions*, and thus we will say that **p** is a *tendency* profile on N. According to Eq. (4), coefficient p_S^i, the probability of i to join S, will depend on the positive tendencies of the members of S to form coalitions and also on the negative tendencies in this sense of the outside players, i.e. the members of $N\backslash(S\cup\{i\})$. Thus, neither p_S^i nor $\lambda_i^{\mathbf{p}}[v]$ will depend on p_i.

Remarks 4.2 (a) For $n=2$ we have $\mathbf{p}=(p_1,p_2)$ and, if $i\neq j$,

$$\lambda_i^{\mathbf{p}}[v] = (1-p_j)[v(\{i\})-v(\emptyset)] + p_j[v(N)-v(\{j\})].$$

Hence, the allocation given by $\lambda^{\mathbf{p}}$ to player i does not depend on p_i but only on p_j. If player j is not greatly interested in cooperating (say, p_j tends to 0), player i's allocation will tend to his individual utility $v(\{i\})$. Instead, if player j is highly interested in cooperating (say, p_j tends to 1), player i's allocation will tend to his marginal contribution to the grand coalition $v(N)-v(\{j\})$.

(b) Whenever, in particular, $p_1=p_2=\cdots=p_n=q$ for some $q\in[0,1]$, coefficients p_S^i reduce, for all $i\in N$, to

$$p_S^i = p_s = q^s(1-q)^{n-s-1} \quad \text{for } s=0,1,\ldots,n-1,$$

where $s=|S|$ and $0^0=1$ by convention in cases $q=0$ and $q=1$. These coefficients $\{p_s\}_{s=0}^{n-1}$ define the *q–binomial semivalue* ψ^q introduced in [12] and, obviously, $\lambda^{\mathbf{p}}=\psi^q$. If, moreover, $q=1/2$ then we obtain $\psi^{1/2}=\beta$, the Banzhaf value.

Remark 4.3 An important difference between the Shapley value and any (multinomial or not) probabilistic value is that the former is efficient whereas the latter, in general, is not (for a discussion on efficient probabilistic values, see [14]). For this reason we speak of *relative* strength. Thus, if the allocations given by a multinomial value in a simple game have to be applied to sharing political responsibilities, or simply compared with the Shapley value, a *normalization process* is needed, similar to that of the original Banzhaf power index [2], by defining

$$\overline{\lambda}_i^{\mathbf{p}}[v] = \frac{\lambda_i^{\mathbf{p}}[v]}{\displaystyle\sum_{j\in N}\lambda_j^{\mathbf{p}}[v]} \tag{5}$$

for each $i\in N$ and any $v\in\mathcal{G}_N$ for which this normalization makes sense. In this case, we interpret each $\lambda_i^{\mathbf{p}}[v]$ just as a *relative measure* of the bargaining strength of player i in v and feel justified in using normalized versions.

Notice that the normalization does not work for all games and values. Nevertheless, it works e.g. for any nonnull monotonic game (this includes all simple games) and any value whose total power in the game does not vanish.

5 The political problem revisited

Our model, based on multinomial values, is able to encompass additional information due to ideological constraints. We will discuss here the political problem described in Example 3.1.[7]

Example 5.1 It is worthy of mention that, in Weber's general model, p^i_S may well depend not only on i's interest in forming coalition $S \cup \{i\}$ —which has been the basis for our failed attempt in Example 3.1— but also on the opinion of the members of S as to joining (accepting) i. In other words, *coefficient p^i_S is not simply a choice of i himself.*

Here we will see that using multinomial values offers a reasonable solution to this since, given a profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, the weighting coefficients of the corresponding multinomial value $\lambda^{\mathbf{p}}$ are defined by means of Eq. (4). It also solves the question of defining the weighting coefficients in a systematic way. Thus, it remains only to choose the profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ in terms of Fig. 1.

The usual decision–making procedure in a parliamentary body is as follows. For each issue at stake, there exists a previous status quo Q and a proposal P to modify it. Each member's action reduces to vote for or against P. The simple game that defines the decision rule merely establishes the set of winning coalitions. Thus, proposal P will pass if and only if the members voting for the proposal form a winning coalition. If this is not so, Q will remain in effect.

Following Remark 2.3(c) in [3], an alternative interpretation of the profile in simple games is that *each p_i can be viewed as the probability that member i votes for P.* Since the result of voting is essentially equivalent to forming a coalition (the coalition of members that vote for P), this interpretation of p_i perfectly agrees with that of "tendency to form a coalition" that we are using here.

Step 1. Additional assumption. According to the above paragraph, we will assume that any coalition C represents, in fact, the set of parties that would vote for a given proposal P, and hence we will attach to this coalition C the *ideological degree* μ (such that $0 \le \mu \le 1$) of the proposal P at stake.

Then, it is natural to take p_i as the level of agreement of party i with this ideological degree, i.e.

$$p_i = 1 - |\mu - \mu_i|, \tag{6}$$

where μ_i is the position of party i.

This is a simple but not too radical assumption. If $\mu_i \le \mu$ then p_i can vary between $1-\mu$ and 1, whereas if $\mu \le \mu_i$ then p_i can vary between μ and 1. As extreme cases, $p_i = 0$ if and only if either $\mu = 0$ and $\mu_i = 1$ or $\mu_i = 0$ and $\mu = 1$, and $p_i = 1$ if and only if $\mu = \mu_i$.

[7] As to the additional information given by ideological constraints in politics, it is worthy of mention, at least incidentally, a singular example. In the general elections held in Greece in May 7 and June 17, 2012, the willingness of the parties to form *any* coalition was being, due to Greek economy's dramatic situation, much more decisive than the ideological constraints. Our model might well apply to study this situation. The profile components after May 7 were very low and led to an impasse, whereas they increased after June 17 and gave rise, finally, to a coalition government.

Step 2. A particular case. As a matter of illustration, let us take $\mu = 0.5$ for C (see Fig. 2). Then, by Eq. (6),

$$p_1 = 0.9, \qquad p_2 = 0.9, \qquad p_3 = 0.7, \qquad p_4 = 0.6.$$

left	4		**A**	1	**C**	2	**B**	3	right
0	0.1		**0.3**	0.4	**0.5**	0.6	**0.7**	0.8	1

Fig. 2: Coalitions on the left–to–right axis

The weighting coefficients are given by Eq. (4). Precisely,

$$p^1_{\{2\}} = 0.108, \quad p^1_{\{3\}} = 0.028, \quad p^1_{\{2,3\}} = 0.252, \quad p^1_{\{2,4\}} = 0.162,$$

$$p^1_{\{3,4\}} = 0.042, \quad p^2_{\{1\}} = 0.108, \quad p^2_{\{1,4\}} = 0.162, \quad p^2_{\{3,4\}} = 0.042,$$

$$p^3_{\{1\}} = 0.036, \quad p^3_{\{1,4\}} = 0.054, \quad p^3_{\{2,4\}} = 0.054, \quad p^4_{\{2,3\}} = 0.063.$$

To compute $\lambda^{\mathbf{p}}[v]$ we use Eq. (3) and obtain

$$\lambda^{\mathbf{p}}_1[v] = 0.592, \qquad \lambda^{\mathbf{p}}_2[v] = 0.312, \qquad \lambda^{\mathbf{p}}_3[v] = 0.144, \qquad \lambda^{\mathbf{p}}_4[v] = 0.063.$$

These allocations are the result of combining both the strategic position of each party in the game and its ideological relevance in forming a "politically balanced" coalition ($\mu = 0.5$). Notice that the symmetry between parties 2 and 3, reflected by the Shapley value, has been broken by the introduction of ideological constraints since $\lambda^{\mathbf{p}}_2[v] \neq \lambda^{\mathbf{p}}_3[v]$. The total power is $\sum_{i \in N} \lambda^{\mathbf{p}}_i[v] = 1.111$.

Looking at $q^i(v)$ we find

$$q^1(v) = 0.408, \qquad q^2(v) = 0.688, \qquad q^3(v) = 0.856, \qquad q^4(v) = 0.937.$$

These amounts represent the probability wasted by each party in joining coalitions where it is not crucial. For example, party 1 is not crucial in $\{1\}$, $\{1,4\}$ and $\{1,2,3,4\}$, and $q^1(v)$ is therefore the probability that party 1 joins $\emptyset$, $\{4\}$ or $\{2,3,4\}$. This waste of probability is the effect of the choice of p_1 but also of p_2, p_3, p_4.

According to Remark 4.3, the above allocations must be normalized, using Eq. (5), before comparing them with the Shapley value of the game. They are given in Table 1. We also report the allocations corresponding to coalitions A and B described in Fig. 2, with ideological positions $\mu = 0.3$ and $\mu = 0.7$, respectively.

Thus, the (normalized or not) allocations take into account: (a) the relative strength of each party in the game, where $W^m(v) = \{\{1,2\},\{1,3\},\{2,3,4\}\}$; (b) the ideological positions of the parties in the left–to–right axis; and (c) the particular definition of the profile, given by $p_i = 1 - |\mu - \mu_i|$ for each i.

parties	1	2	3	4
$\varphi[v]$	0.4167	0.2500	0.2500	0.0833
(normalized)	$\overline{\lambda}_1^{\mathbf{p}}[v]$	$\overline{\lambda}_2^{\mathbf{p}}[v]$	$\overline{\lambda}_3^{\mathbf{p}}[v]$	$\overline{\lambda}_4^{\mathbf{p}}[v]$
C	0.5329	0.2808	0.1296	0.0567
B	0.5265	0.1407	0.1407	0.1921
A	0.4011	0.3448	0.2294	0.0246

Table 1: Shapley value and normalized allocations relatively to A, B and C

The comparison of these allocations with the Shapley value shows the influence of the ideological positions of the parties when rewarding them, but always *relatively to a particular coalition (i.e., to a proposal) with a given ideological degree.*

Step 3. Arbitrary ideological position. Now we proceed for a general μ. From Eq. (6) we have in Table 2 the expression of the profile in terms of μ.

μ in:	$[0, 0.1]$	$[0.1, 0.4]$	$[0.4, 0.6]$	$[0.6, 0.8]$	$[0.8, 1]$
p_1	$0.6+\mu$	$0.6+\mu$	$1.4-\mu$	$1.4-\mu$	$1.4-\mu$
p_2	$0.4+\mu$	$0.4+\mu$	$0.4+\mu$	$1.6-\mu$	$1.6-\mu$
p_3	$0.2+\mu$	$0.2+\mu$	$0.2+\mu$	$0.2+\mu$	$1.8-\mu$
p_4	$0.9+\mu$	$1.1-\mu$	$1.1-\mu$	$1.1-\mu$	$1.1-\mu$

Table 2: The profile in terms of parameter μ

Then we get the multinomial value $\lambda^{\mathbf{p}}[v]$ in terms of μ:

$$\lambda_1^{\mathbf{p}}[v] = \begin{cases} -\mu^3 - 2.5\mu^2 + 0.78\mu + 0.448 & \text{if} \quad 0 \leq \mu \leq 0.1, \\ \mu^3 - 1.5\mu^2 + 0.82\mu + 0.432 & \text{if} \quad 0.1 \leq \mu \leq 0.6, \\ -\mu^3 + 3.5\mu^2 - 2.62\mu + 1.128 & \text{if} \quad 0.6 \leq \mu \leq 0.8, \\ \mu^3 - 5.5\mu^2 + 8.02\mu - 2.648 & \text{if} \quad 0.8 \leq \mu \leq 1, \end{cases}$$

and similar expressions for the remaining values $\lambda_i^{\mathbf{p}}[v]$ for $i = 2, 3, 4$.

Finally, if we wish to aggregate these results and obtain a single evaluation of the relative strength of each party in the coalition formation process *in abstracto*, i.e. without prescribing any ideological degree μ to the coalition, it suffices to integrate the multinomial value of each party with respect to μ, thus getting

$$\xi_1[v] = \int_0^1 \lambda_1^{\mathbf{p}}[v] d\mu \approx 0.6333$$

and, similarly,

$$\xi_2[v] \approx 0.3365, \qquad \xi_3[v] \approx 0.2681, \qquad \xi_4[v] \approx 0.1393.$$

Remark 5.2 The normalization procedure may of course be applied also to the single evaluation $\xi[v]$ obtained in Step 3, giving normalized values that sum up to 1:

$$\overline{\xi}_1[v] \approx 0.4598, \qquad \overline{\xi}_2[v] \approx 0.2443, \qquad \overline{\xi}_3[v] \approx 0.1947, \qquad \overline{\xi}_4[v] \approx 0.1012.$$

In the same way as one accepts the Shapley value of the game as an a priori evaluation of the relative strength of each player in the coalition formation bargaining, the values just obtained represent an analogous a priori evaluation of this relative strength *when the political relationships between the parties are taken into account.* The differences between our (normalized) assessment and the mere evaluation of the game provided by the Shapley value are interesting: if $\Delta_i[v] = \overline{\xi}_i[v] - \varphi_i[v]$ and $\overline{\Delta}_i[v] = \Delta_i[v]/\varphi_i[v]$ for all i, then

$$\Delta_1[v] = 0.0431, \qquad \Delta_2[v] = -0.0057, \qquad \Delta_3[v] = -0.0553, \qquad \Delta_4[v] = 0.0179,$$

$$\overline{\Delta}_1[v] = 0.1034, \qquad \overline{\Delta}_2[v] = -0.0228, \qquad \overline{\Delta}_3[v] = -0.2212, \qquad \overline{\Delta}_4[v] = 0.2149.$$

This indicates that the political relationships in this particular game improve party 1 strongly (around 10.34%) and party 4 very strongly (around 21.49%), while they damage party 2 very slightly (around 2.28%) and party 3 very strongly (around 22.12%).

Acknowledgments

The authors wish to thank an editor and two anonymous reviewers for their helpful suggestions, most of which have been incorporated to the text.

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